Pricing Collateralized Swaps

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Abstract

Interest rate swap pricing theory traditionally views swaps as portfolios of forward contracts with net swap payments discounted using the LIBOR curve. Current market practices of marking-to-market and collateralization question this view. Collateralization and marking-to-market affects discounting of swap payments (through altered default characteristics) *and* introduces intermediate cash-flows. This paper provides a theory of swap valuation under collateralization and we find evidence supporting the presence of costly collateral. Using Eurodollar futures rates, we find evidence that swaps are priced above the traditional portfolio of forwards value and below a portfolio of futures value. Moreover, the effect of collateral is time varying. We estimate a term structure model to characterize the cost of collateral and quantify its effect on swap rates.

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1 Introduction

The traditional approach to interest rate swap valuation views swaps as portfolios of forward contracts on the underlying interest rate (Sundaresan (1991), Sun, Sundaresan and Wang (1993) and Duffie and Singleton (1997)). Under specific assumptions regarding the nature of default and the credit risk of the counterparties, Duffie and Singleton (1997) prove that market swap rates are par bond rates of an issuer who remains at LIBOR quality throughout the life of the contract. This result is extremely useful in practice for extracting zero coupon bond prices, for pricing swap derivatives and for econometric testing of spot rate models.

Despite the popularity of the traditional view, market practices bring into question some of the underlying assumptions of the portfolio-of-forwards approach. As the swap market rapidly grew in the 1990s, an increasingly diverse group of counterparties entered the swap market. To mitigate counterparty credit risk, market participants used a number credit enhancements to improve the credit-quality of swap contracts. Arguably, the most important credit enhancement is the posting of collateral in the amount of the current mark-to-market value of the swap contract (ISDA (1999)).[1] Recent high profile events such as the LTCM bailout and the Enron bankruptcy provide a reminder to market participants of the importance of collateralizing over-the-counter (OTC) derivatives transactions and frequent marking of the positions to market.

In this paper, we provide a theory of swap valuation in the presence of bilateral marking-to-market (MTM) and collateralization. MTM and collateralization result in two important departures from the traditional approach for valuing swaps. First,

[1] Clarke (1999) provides a general overview of the various techniques for credit enhancement. For information regarding common market practices, see the ISDA (1998, 2000) or BIS (2001).

MTM and collateralization generate payments by the counterparties between the periodic swap dates. Since these cash payments can induce economic costs/benefits to the payer/receiver (either directly or via an opportunity cost of capital) and are part of the initial swap contract,[2] they must be accounted for when valuing the swap. Second, MTM and collateralization reduce and may eliminate the counterparty credit risk in the swap. In fact, market participants commonly assume collateralized swaps are default-free and it is now common to build models of swap rates assuming that swaps are free of counterparty credit risk (see, e.g., Collin-Dufresne and Solnik (2001) and He (2001)).

Formally, we assume that counterparties post U.S. dollar (USD) cash as collateral and mark the contracts to the market value of the contract. Although cash is certainly not the only form of collateral, it is the most popular form of accepted collateral (ISDA (2000), p. 2).[3] In a discrete-time setting, we show that MTM and collateralization result in intermediate cash flows in the swap contract that appear in the form of a stochastic dividend where dividend rate is the cost of posting collateral. This result is reminiscent of Cox, Ingersoll and Ross (1981) who show that the MTM feature of futures contracts results in stochastic dividends. Futures contracts are marked-to-market daily and variation margin calls are met by cash. This suggests that swap contracts that are collateralized by cash may be more reasonably thought of as a portfolio of futures contracts on LIBOR with an opportunity cost equal to the default-free short interest rate. We pursue this line of reasoning further below.

[2]See, for example the "Credit Support Annex" (ISDA (1994)) to the ISDA Master Swap Agreement.

[3]Cash collateral may be the cheapest form of collateral in many cases because of the large haircuts required for risky securities and the valuation issues involved in determining the market risk in securities posted as collateral.

In continuous-time, we model counterparty credit risk via an exogenous random stopping time which indicates default. Following Duffie and Singleton (1997), we use a default-adjusted instantaneous rate model to price LIBOR rates. We do not assume the occurrence of default in the swap and LIBOR markets are concurrent. We derive closed form solutions (up to ODE's) for market swap rates in an affine setting in the presence of MTM and collateralization.

If there is no cost to posting and maintaining collateral, we show that swaps are priced by discounting net swap payments at the risk-free rate as in He (2001) and Collin-Dufresne and Solnik (2001). The assumption that there is no cost of posting collateral is problematic, however: this would imply that counterparty credit risk can be eliminated at zero cost! This does not square with the fact that swap counterparties and regulators spend non-trivial amounts of time, effort and money to mitigate credit risk. When collateral is costly, it enters as a (negative) convenience yield on the swap, altering the discount rates. In the special case where the cost of collateral equals the default-free instantaneous rate, swap contracts are priced as portfolios of futures contracts on LIBOR instead of forward contracts.

What is the directional effect of MTM and collateralization on swap rates? We argue that swap rates will increase in general as a result of these institutional features. To see this intuitively, consider the swap from the perspective of the receiver of fixed. When floating interest rates go down, the swap will have positive MTM value and the fixed-receiver receives collateral. The return on invested collateral is lower due to the decreased interest rates. Likewise, when rate increases, the swap will have a negative MTM value and the fixed receiver will have to post collateral which is now more costly due to the increased rates. Intuitively, it follows that the receiver of fixed will demand a higher swap rate to compensate for the acceleration of (opportunity) costs implied by collateralization. Formally, we find that under standard assumptions, collateral increases both swap rates and swap spreads.

Empirically, we find strong support for presence of costly collateral using two different independent approaches. First, we use the Eurodollar futures curve to estimate a number of term structure models and provide information about the LIBOR term structure. With the calibrated model, we compute hypothetical swap curves assuming that swaps are priced as portfolios of futures (opportunity cost of collateral is the risk-free rate) and also as par rates (the traditional approach). We find that market swap rates generally lie in between the futures based swap rates and par rates. Moreover, market swap rates move substantially relative to the futures based and par swap rates. This time variation is strong evidence consistent with the presence of costly collateral. Perhaps more significantly, swap rates are closer to the futures based swap curve in periods of market stress such as those in 1997, 1998 and 2000. This finding squares nicely with the observation of increased credit concerns during these years. Independently of our work, Bomfim (2002) performs related experiments and reaches the same conclusions on the relative position of market swap rates.[4] Our results are closely related to Gupta and Subrahmanyam (2000) and Minton (1997) who analyze swap spreads and the literature examining the difference between forward and futures contracts.[5]

[4] In fact, Bomfim (2002) concludes that there is "no statistically significant role for counterparty credit risk in the determination of market swap rates" (p. 33) and argues that this is due to effective counterparty credit risk mitigation using collateralization.

[5] Cox, Ingersoll and Ross (1981), Richard and Sundaresan (1981) and Jarrow and Oldfeld (1981) provide theoretical arguments for the differences and, in the case of interest rate sensitive securities, Sundaresan (1992), Muelbroek (1992) and Grinblatt and Jegadeesh (1996) examine the empirical evidence.

Second, we specify and estimate a dynamic multi-factor term structure to investigate the nature and impact of costly collateral. The risk-free term structure (estimated from Treasuries) is modeled as a two-factor model with a short rate and a time-varying central tendency factor. These factors and the model are the same as those used in Collin-Dufresne and Solnik (2001). The third factor is the spread between instantaneous LIBOR and Treasuries and the final factor is the cost of posting cash collateral. These last two factors are extracted from LIBOR and swap rates and the model is estimated using maximum likelihood.

Estimation results indicate that the implied cost of collateral is generally small, with an in-sample mean of 42 basis points but exhibits significant time-variation. Moreover, the cost of collateral dramatically increases around periods of market stress such as the hedge fund crises in 1998, Y2K concerns in Fall 1999 and the bursting of the dot-com bubble in Spring of 2000. Surprisingly, the cost of collateral is not highly correlated with the factor that captures the instantaneous spread between LIBOR and Treasuries, but is highly correlated with the short term, default-free interest rate. This result reinforces the results obtained from the Eurodollar futures market which indicate that swap are often priced close to a portfolio of LIBOR futures, that is, that the cost of collateral is related to the short term, default-free interest rate.

Our results provide new insight into the determinants of interest rate swap spreads. While standard arbitrage arguments indicate that the swap spread (the difference between swap rates and par yields on similar maturity Treasuries) should be closely related to short-term financing spreads (e.g., LIBOR minus general collateral repo), there is not a strong relationship in observed data.[6] Due to this somewhat puzzling

[6] For example, He (2001) notes that "the volatility we observe in the swap market recently cannot possibly be triggered by the volatility of short-term financing spreads."

result, the literature has turned to investigate the role of credit, interest rate, and liquidity risk premium (see, e.g., He (2001) and Liu, Longstaff and Mandel (2001)). However, the evidence on these risk premium is somewhat mixed. For example, Liu, Longstaff and Mandel (2001) find a significant credit risk premium, but it is negative for much of their sample. Grinblatt (2001) provides an alternative explanation for swap spreads based on a stochastic convenience yield to Treasuries.

Our analysis provides another plausible alternative explanation for the large and volatile swap spreads. Evidence from both the Eurodollar futures market and term structure models indicate that there is a time-varying cost of posting and maintaining collateral and the cost of collateral is related to the short term default-free interest rate. Moreover, costly collateral is a factor that appears only in the swap market and not directly in LIBOR rates, thus it provides a factor that can move swap rates independently of short-term financing spreads. We find that swap rates and therefore swap spreads appear to reflect this cost of collateral. Moreover, our results lend support to the economic intuition which indicates that the cost of posting and maintaining collateral is greatest during period of market stress.

The following section, Section 2, discusses the institutional features of collateralization in the fixed income derivatives market. Section 3 values swaps under collateralization in both discrete and continuous-time. Section 4 uses information in Eurodollar futures to assess the presence of costly collateral. Section 5 introduces and estimates the term structure model to assess the cost of the collateral. Section 6 concludes.

2 Institutional Features of the Swap Market and Collateralization

Over the past decade, the interest rate swap market has grown immensely and is now the largest component of the OTC derivatives market. Across currencies, the Bank for International Settlements reports the notional value of outstanding swaps is now over $71 trillion, with USD denominated swaps accounting for almost $24 trillion (BIS (2003)). Over the 1990s, year-to-year percentage growth rates in interest rate swaps for many currencies averaged in double digits. As a testament to depth and liquidity of the market, on-the-run swap rates have a bid-ask spread that is less than one basis point.

In this section, we provide some background on collateralization of OTC derivatives.[7] Collateralization has always been an important feature of the OTC derivatives market (Litzenberger (1992), p. 838), is currently widespread and has been rapidly growing over the past five to ten years. ISDA (2003) reports that there is currently more than $719 billion worth of collateral insuring OTC derivative obligations and the amount of collateral in circulation increased about 70% from 2001 to 2002. One explanation for the rapid growth of collateral in circulation is the dramatic decrease in interest rates over the past three years. Counterparties who agreed to receive fixed on long dated swaps initiated in the late 1990s are, in expectation, deep in-the-money, and they require the fixed payer to post collateral to secure these obligations.

There are currently about 16,000 collateralized counterparties, an increase of 45% from the number in 2000 (ISDA (2001), p. 10). While institutions collateralize many

[7] Most of the information comes from market surveys by The International Swap and Derivatives Association (ISDA). See, for example, ISDA (1998, 1999, 2000, 2001).

different types of derivatives, ISDA (2001) found that more than 65% of "plan vanilla derivatives, especially interest rate swaps" are collateralized according to the Credit Annex to the Master Swap Agreement. Discussions with market participants indicate that nearly all of the swap transactions at major investment banks are collateralized. Due to the importance of collateral and the size of the collateral programs, new institutions such as SWAPCLEAR have been established with the stated purpose of mitigating the credit exposure in the swap markets through large-scale MTM, collateralization, and netting.

Most of the collateral posted was in the form of USD cash, US government securities or agency securities, although foreign currencies, major index equities and corporate bonds can also be posted. Together, according to ISDA (2000), cash collateral and U.S. government securities cover about 70% of the posted collateral. ISDA (2001) points out that the "broad trend towards greater use of cash continues to gather momentum" (p. 3). Securities whose value changes over time (all collateral except for USD cash) are more difficult to manage with as the receiver must account for the risk that the payer will default and the value of the securities posted might fall below the market value of the swap. Due to this, non-cash collateral is typically subject to a substantial haircut. The key to effective credit risk mitigation is frequent margin calls. ISDA (2001) surveyed market participants and found that at least 74% of survey respondents MTM at least at a daily frequency.

An essential key to the success of collateralization for credit risk mitigation are amendments to the U.S. Bankruptcy Code. These amendments passed in the 1970s and 1980s assign a special status to collateralized derivatives transactions. Unlike any other creditor,[8] a derivatives counterparty receives an exemption from the Bankruptcy

[8]Technically, this statement is not completely correct. In unusual cases, certain other creditors

Code's automatic stay provision and the 90-day preference period. These provisions prevent creditors from seizing the debtor's assets once they declare bankruptcy and may allow the court to recover any transfers from the debtor to creditors in the 90 days prior to bankruptcy. Due to their exemption, counterparties to derivatives transactions can seize any margin or collateral even though the debtor has filed for bankruptcy and its assets are shielded from collection activities by any other creditors. Given this status, there is no concern that the debtor will have legal recourse to recover the collateral or that the creditor will have to participate in legal proceedings.[9]

Collateralization provides a number of private and social benefits (see, BIS (2001)). First, collateralization reduces realized losses conditional upon default. The party who has received collateral keeps the collateral and the maximum loss is now the total exposure minus any posted collateral. Second, collateral reduces regulatory capital requirements. According to the Basle accord, collateralized transactions often generate a zero credit risk weighting, which frees up scarce capital for other purposes. Third, frequent posting and MTM of collateral constrains firms from taking too much leverage, the implications of which were clearly seen in the hedge fund crises of late Summer-Fall 1998. This in principle could reduce the probability that a counterparty would default. Last, using collateral expands the list of potential counterparties as

also have an exemption from the Bankruptcy Code's automatic stay provisions. These include government agents (e.g., local police departments) exercising their regulatory powers and certain Federal agencies (e.g., Housing and Urban Development) with monetary claims against the debtor, for example, subsidized loans to developers of low income housing. These exemptions will rarely, if ever, arise in the context of the bankruptcy of a party to a derivatives transaction. We thank Ed Morrison for pointing this out.

[9]The Financial Institutions Reform, Recovery and Enforcement Act of 1989 ("FIRREA") also implies that collateral posted by commercial banks (who are regulated entities outside of the U.S. Bankruptcy Code) can be seized.

institutions are less concerned about the credit risk of the counterparty provided they are willing to collateralize the transaction. This increases competition in the swap market as lower rated financial institutions can compete with more highly rated firms for transactions. Combined, the increase in counterparties and competition is credited with increasing volume and liquidity in the swap market, providing a market wide social benefit of lower bid-ask spreads and greater competition.

It is important to note that the posting of collateral no matter what or how it is posted entails a cost or, for the other counterparty, generates a benefit. The easiest way to see this is that the receiver of the collateral, when allowed, will typically reuse or rehypothecate the collateral for other purposes. In fact, according to ISDA, 89% of reusable collateral is rehypothecated. Thus firms can use collateral they have received to cover their margin calls. Many firms are net holders of large amounts of collateral. For example, Brandman (2000) reports that Goldman, Sachs & Co. held $6.6 billion in collateral at the end of 1999 and Fannie Mae held $3.1 billion at the end of 2002 (Fannie Mae (2003)). It is reasonable to assume that these firms receive a net benefit from holding these large amounts of cash and securities.

3 Swap Valuation

This section reviews the traditional approach to swap valuation and provides swap valuation with collateralization in discrete and continuous-time.

3.1 Pricing LIBOR rates

The first stage in pricing swap contracts is characterizing the evolution of the floating rate. We focus on interest rate swaps and always assume the floating rate is indexed to six-month LIBOR. To model LIBOR rates, we follow Duffie and Singleton (1997) and use a reduced form specification as opposed to, for example, a structural model of default (see, also Cooper and Mello (1991)). The default-risk adjusted instantaneous spot rate, R_t, is given by $R_t = r_t + \delta_t$, where r_t is the default-free instantaneous rate and δ_t is the spread between instantaneous LIBOR and the default-free rate. In Duffie and Singleton (1997), $\delta_t = \lambda_t h_t$, where h_t is the exogenous hazard process and λ_t is the fractional default loss conditional on default. Intuitively, the probability of default over a short interval Δ is $h_t\Delta$ and $(1 - \lambda_t)$ is the fraction of the market value recovered. This assumes that LIBOR rates can be modeled as a single-defaultable entity even though, in reality, LIBOR is a trimmed average of rates quoted by commercial banks.

At any given point in time, T, the price of a zero coupon bond whose principal is discounted at rate R_t and that matures at time $T + s$ is given by, $P^R(T, s)$, where

$$P^R(T, s) = E_T^{\mathbb{Q}}\left[e^{-\int_T^{T+s} R_t dt}\right].$$

Throughout we use $P^x(T, s)$ to denote the zero coupon bond price discounted at rate x. Given the "LIBOR" bond price, the discretely compounded six-month LIBOR rate is

$$L_6(T) = 2\left[\frac{1}{P^R(T, 6)} - 1\right].$$

Given a model of LIBOR rates, we next outline the traditional portfolio-of-forwards approach for valuing interest rate swaps.

3.2 The Traditional Approach

Assume that there are two counterparties, A and B, and that Party A agrees to pays Party B the fixed swap rate, which we denote by s_0, and Party B pays Party A the floating rate. The floating payment can be either the floating rate at the time of the exchange or it can be the value six months earlier (settled-in-arrears). In certain cases, interpretation of results is cleaner with contemporaneous settlement, so we often use that for simple examples although the differences between the two rates are quite small.[10]

The traditional approach assumes that cash payments between counterparties are made only on the discrete reset dates, in our case, every six months. Since only net cash flows are exchanged, the swap contract formally takes the form of a portfolio forward contracts with a common strike price, the swap rate. Therefore, to value the swap contract using the traditional approach, that is, to find the fixed swap rate, we need only to determine the interest rate used to discount the net swap payments, $L_6(T) - s_0$.

If both counterparties were free of default-risk, the net swap payments would be discounted at the default-free instantaneous rate. In the more general case where the counterparties of default-risky, the discount rate will depend critically on each of the counterparty's credit risk profile. Duffie and Singleton (1997) provide the following set of formal assumptions.

1. They assume that default-risk is exogenous. This implies that λ_t and h_t are exogenous and, more specifically, are not functions of the swap value.

2. They assume that both parties have the same credit quality. This assumption does not appear to be particularly important (see Duffie and Huang (1996)).

[10]Sundaresan (1991) finds that the difference between the fixed swap rate when the floating payments settled-in-arrears and those settled contemporaneously is a fraction of a basis point. In a slightly different setting, Duffie and Huang (1996, p. 931) argue that the differences are negligible.

3. They assume that both swap counterparties have a credit risk which is identical to the average member of the LIBOR panel. Moreover, they assume the counterparties retain this refreshed LIBOR status throughout the life of the swap contract.

4. They assume that the default events and recovery rates in the LIBOR and swap markets are the same. As Duffie and Singleton (1997) note, there is no reason to believe that a default event in an OTC derivatives market will coincide with a default event in the LIBOR (high quality interbank) market.

Together, these four assumptions imply that both counterparties have credit quality equal to the average member of LIBOR which implies that swap payments are discounted at R_t. In the case of a single period swap, the market fixed rate, s_0^R, solves:

$$E_0^{\mathbb{Q}}\left[e^{-\int_0^T R_t dt}\left(s_0^R - L_6\left(T\right)\right)\right] = 0$$

where $L_6\left(T\right)$ can be either six-month LIBOR at time T or six-month LIBOR at time $T-6$ (as mentioned above, the difference is negligible). We use the notation s_0^x to denote the swap rate when the net cash flows are discounted at the rate x. The swap rate is

$$s_0^R = \frac{E_0^{\mathbb{Q}}\left[e^{-\int_0^T R_t dt} L_6\left(T\right)\right]}{P^R\left(0,T\right)} = E_0^{\mathbb{Q}}\left[L_6\left(T\right)\right] + \frac{cov_0^{\mathbb{Q}}\left[e^{-\int_0^T R_t dt}, L_6\left(T\right)\right]}{P^R\left(0,T\right)}. \qquad (1)$$

where $E_0^{\mathbb{Q}}\left[L_6\left(T\right)\right]$ is the futures rate of six-month LIBOR. This shows the close relationship between swap rates and futures rates. Since the covariance between the discount factor $e^{-\int_0^T R_t dt}$ and LIBOR is always negative, swap rates in the traditional approach are less than the associated futures rates.[11]

[11] This "convexity" correction is well known, observation is not new, see, for example, Gupta and Subramanyam (2000).

More generally, the annualized fixed rate in multi-period swap on six-month LIBOR settled-in-arrears is given by:

$$s_0^R = 2 \cdot \frac{1 - P^R(0,T)}{\sum_{j=1}^{2T} P^R\left(0, \frac{j}{2}\right)}.$$

This is the familiar par rate representation of swap rates which is commonly used for yield curve construction, empirical work and pricing swap derivatives. *Throughout, we treat the par representation of swap rates as the benchmark specification.*

3.3 Pricing collateralized swaps: discrete-time

In this section, we provide a discrete-time approach to valuing interest rate swaps subject to collateralization. This model-independent formulation draws on the insights of Cox, Ingersoll and Ross (1981). For reasons that will become clear, we focus initially on a single period swap that is priced at time 0 for the exchange of fixed and floating payments at time T.

Consider first a two-period example with three dates, $t = 0, 1,$ and 2. We assume the swap is MTM at time 1 and fully collateralized by USD cash. At the end of period 2, party A agrees to pay party B a fixed rate and receive the floating rate. We assume it is costly to post collateral and that holding collateral generates a benefit. The costs/benefits are symmetric: the cost to one party is equal to the benefit to the other party. Let s_0 denote the fixed swap rate, $\{S_t\}_{t=0}^2$ be the market value of the swap contract at time t, y_1 is the cost/benefit to posting/receiving the cash collateral at time 1, and L_2 is six-month LIBOR at time 2. We view the cost of posting collateral as an interest rate to keep the analogy with the pricing of futures contracts, where the cost of collateral is the risk-free rate. The mechanics of the swap and collateralization procedure are as follows:

- At time 0, the swap rate, s_0, is set to make the market value of all future cash flows zero: thus, $S_0 = 0$.
- At time 1, assume the market value of the seasoned swap changes, and for simplicity, assume $S_1 > 0$. Party B pays Party A \$$S_1$.
- At time 2, Party A receives a benefit from holding the collateral in the amount of $y_1 S_1$; party B entails a cost of posting collateral in the amount of $y_1 S_1$. The parties net the collateral payment with the exchange of fixed and floating payments, $L_2 - s_0$.

At initiation, the market value of the swap is zero and the market swap rate solves:

$$0 = PV_0 \left[(L_2 - s_0) + S_1 y_1 \right]$$

where we use the notation PV_0 to denote the present value of the cash flows at time 0. We intentionally do not specify what interest rate (default-free or default-risky) is used to discount the cash flows. At this point, note that the swap rate on the collateralized swap is different from the uncollateralized formulation where the swap rate solves

$$0 = PV_0 \left[S_2 \right] = PV_0 \left[L_2 - s_0 \right].$$

This simple example provides the intuition for the more general results in the next section.

From this simple example, there at least three important implications of collateralization. First, MTM and collateralization result in a stochastic dividend, $S_1 y_1$, between contract initiation and the final period. This implies that collateralized swaps are no longer portfolios of forward contracts. The stochastic dividend result is reminiscent of Cox, Ingersoll and Ross (1981) who demonstrate that, due to MTM, futures contracts have stochastic dividends. Second, MTM and collateralization alter the recovery characteristics in the case of default. If Party B defaults on Party A, Party A can keep the collateral posted, S_1. The maximum loss is now $L_2 - s_0 + S_1$. This dramatically reduces any potential losses, conditional on default. Third, as noted in the references earlier, collateralization may reduce the probability that Party B defaults as their leverage has been reduced.

3.4 The impact of collateral on swaps: continuous-time

We now turn to the valuation of swaps in continuous-time. This allows us to use the Duffie and Singleton (1997, 2000) reduced form approach to valuing defaultable securities and to focus on exactly how various assumptions regarding counterparty credit risk, marking-to-market and collateralization effect swap valuation in a formal manner.

We retain Duffie and Singleton's (1997) assumptions regarding default in the LIBOR market, that is, we assume there is a default adjusted short rate, $R_t = r_t + \delta_t$, and that δ_t is exogenous. To value the swap contract with collateral, we only assume that default by a counterparty can be represented by a first jump time, τ, of jump process with a (potentially stochastic) intensity. We let $\mathbf{1}_{[\tau > T]} = 1$ if there is no default by time T (see Bielecki and Rutkowski (2001) for formal definitions and regularity conditions for modeling default with point processes). We do not require any further assumptions on the nature of default by the counterparties. Conditional on default, we assume that there is no recovery in excess of any collateral posted, although it would be easy to model recovery. This is consistent with the legal status of collateral in the U.S. Bankruptcy Code.

Our specification relaxes two of the assumptions in Duffie and Singleton (1997):

(1) that default characteristics and occurrences in the LIBOR and swap market are the same and (2) we do not assume that the counterparties are refreshed and remain at LIBOR quality throughout the life of the swap. Instead we assume that swaps are MTM and collateralized continuously in time. In practice, they are typically marked at least daily with the option to demand additional collateral in the case of large market moves.[12] The amount of collateral posted at time t is $\$C_t$ and we assume that there is a stochastic cost of posting collateral, $\{y_t\}_{t\geq 0}$. We interpret the cost of collateral as an instantaneous interest rate accrual on the principal of C_t. As an example, it is commonly assumed that the cost of collateral when valuing futures contracts is the default-free short rate, r_t. Because of the presence of MTM and collateral, we do not need to make specific assumptions regarding the individual counterparties credit risk profile.[13]

What is the value of a swap in this setting? First, consider the case where it is costless to post and maintain collateral. In this case, the market value of a swap struck at s_0, S_t, is given by the solution of

$$S_t = E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\Phi_T \mathbf{1}_{[\tau > T]} + e^{-\int_t^\tau r_s ds} C_\tau \mathbf{1}_{[\tau \leq T]}\right],$$

where $\Phi_T = L_6(T) - s_0$ and $S_0 = 0$. The first term in the value of the swap, $e^{-\int_t^T r_s ds}\Phi_T$, is the present value of the cash flows conditional on no default and the second component, $e^{-\int_t^\tau r_s ds}C_\tau$, is the present value of the amount received conditional on default occurring at time $\tau \leq T$, C_τ.

In practice the amount of collateral posted is the MTM value of the swap, that

[12] ISDA (2001) indicates that most contracts are marked at least daily, while SWAPCLEAR marks daily with an option to mark more frequently if there are large market movements.

[13] We do assume that cost of collateral is symmetric for the counterparties. This assumption considerably simplifies the analysis.

is, that $C_t = S_t$. This implies that the swap price process solves (for $t < \tau$)

$$S_t = E_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \Phi_T \mathbf{1}_{[\tau > T]} + e^{-\int_t^\tau r_s ds} S_\tau \mathbf{1}_{[\tau \leq T]} \right].$$

Since recovery is full conditional on default, a collateralized swap contract is just a contract with a random termination time. An application of the law of iterated expectations implies that

$$S_t = E_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \Phi_T \right]$$

and it is clear that the swap contract contains no counterparty credit-risk as recovery in the case of default is full. This provides a justification for the formula of He (2001) and Collin-Dufresne and Solnik (2001): *if* the swap is fully MTM and collateralized *and* it is costless to post and maintain collateral, then swaps are discounted at the risk-free rate. This, of course, is counterfactual as it implies that it is costless to remove credit risk. At initiation, the value of the swap is zero, $S_0 = 0$ which implies that

$$s_0^r = \frac{E_0^{\mathbb{Q}} \left[e^{-\int_0^T r_s ds} L_6(T) \right]}{P^r(0,T)} = E_0^{\mathbb{Q}} \left[L_6(T) \right] + \frac{cov_0^{\mathbb{Q}} \left[e^{-\int_0^T r_s ds}, L_6(T) \right]}{P^r(0,T)} \tag{2}$$

where s_0^r is the swap rate when the net payments are discounted at r_t, and $P^r(0,T)$ is the price of a default-free zero at time 0 expiring at time T. The covariance between the risk-free discount factor and swap is typically negative (since $R_t = r_t + \delta_t$).

Next, consider the case with costly collateral. The value of the swap is, for $t < \tau$,

$$S_t = E_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \Phi_T \mathbf{1}_{[\tau > T]} + e^{-\int_t^\tau r_s ds} C_\tau \mathbf{1}_{[\tau \leq T]} \right] + \tag{3}$$

$$E_t^{\mathbb{Q}} \left[\mathbf{1}_{[\tau \leq T]} \int_t^\tau e^{-\int_t^s r_u du} y_s C_s ds + \mathbf{1}_{[\tau > T]} \int_t^T e^{-\int_t^s r_u du} y_s C_s ds \right]. \tag{4}$$

The first term is the discounted value of the final swap payments conditional on no default, the second is the discounted value of the collateral seized conditional on default, the third is the discounted value of the opportunity cost of the collateral up to a default time, and the last term is the discounted value of the collateral conditional on default. If the contract is fully marked-to-market, $C_s = S_s$, the collateralized swap value is

$$S_t = E_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \Phi_T + \int_t^T e^{-\int_t^s r_u du} y_s S_s ds \right].$$

This formula is the familiar stochastic dividend-yield formula and implies that

$$S_t = E_t^{\mathbb{Q}} \left[e^{-\int_t^T (r_s - y_s) ds} \Phi_T \right].$$

At initiation, the value of the swap is zero, $S_0 = 0$ which implies that

$$s_0^{r-y} = \frac{E_0^{\mathbb{Q}} \left[e^{-\int_0^T (r_s - y_s) ds} L_6(T) \right]}{P^{r-y}(0,T)} = E_0^{\mathbb{Q}} \left[L_6(T) \right] + \frac{cov_0^{\mathbb{Q}} \left[e^{-\int_0^T (r_s - y_s) ds}, L_6(T) \right]}{P^{r-y}(0,T)} \tag{5}$$

where s_0^{r-y} is the swap rate when the net payments are discounted at $r_t - y_t$, and

$$P^{r-y}(0,T) = E_t^{\mathbb{Q}} \left[e^{-\int_0^T (r_s - y_s) ds} \right].$$

As in the case of costless collateral, swap contracts are again free of counterparty default-risk via the posting of collateral in the MTM value of the swap contract. However, costly collateral now alters the discount rate and as (5) shows, the impact of collateral will be determined by the covariance of $r_t - y_t$ and LIBOR. The potential impact can be large. To see this, suppose that $y_t = r_t$ which implies that swaps are priced as a portfolio of futures contracts on six-month LIBOR. The differences between futures and forwards is significant and can be large (see, Sundaresan (1991) and Grinblatt and Jegadeesh (1996)). With futures contracts, futures prices are reset continuously and the value of the contract is zero. With a collateralized swap, the swap rate remains fixed until termination of the contract (either by default or expiration). The value of the swap contract is exactly offset by the collateral.

Is it possible to generically order the swap rates under the various assumptions regarding MTM and collateralization? There are four rates of interest: the swap rates of Duffie and Singleton (1997), s_0^R, the swap rates of Collin-Dufresne and Solnik (2001) and He (2001), s_0^r, and the costly collateral swap rates, s_0^{r-y}, and the futures rates. For simplicity in this section, we consider single-period swap rates and do not settle the contracts in arrears. From equations (1),(2), and (5), we see the close relationship between swap rates and futures rates and that the covariance of the discount factors with the swap rates determines the gap between futures and swap rates.

The covariance between $exp\left(\int_0^T R_s ds\right)$ and $L_6(T)$ and the covariance between $exp\left(\int_0^T R_s ds\right)$ and $L_6(T)$ are both negative, which implies that $s_0^r, s_0^R < E_0^{\mathbb{Q}}[L_6(T)]$. Another case that is easy to determine is the case when y_t is a nonrandom function of time, then $s_0^{r-y} = s_0^r$. This results does not carry over to the multi-period case due to a nonlinear affect. In this case, s_0^{r-y} is greater than s_0^r.

Consider the difference between the default-free swap curve, s_0^r, and the default-risky swap curve, s_0^R:

$$s_0^r - s_0^R = \frac{cov_0^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}, L_6(T)\right]}{P^r(0,T)} - \frac{cov_0^{\mathbb{Q}}\left[e^{-\int_0^T R_s ds}, L_6(T)\right]}{P^R(0,T)},$$

If we assume that (which is commonly supported in the data),

$$cov_0^{\mathbb{Q}}\left[e^{-\int_0^T R_s ds}, L_6(T)\right] < cov_0^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}, L_6(T)\right]$$

we have $s_0^r - s_0^R > 0$ (since $P^R < P^r$) . This implies that discounting by r instead of R results in higher swap rates, holding all else equal. Similarly, if

$$cov_0^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}, L_6(T)\right] < cov_0^{\mathbb{Q}}\left[e^{-\int_0^T (r_s - y_s) ds}, L_6(T)\right],$$ [14]

[14] Consider for the condition

$$cov_0^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}, L_6(T)\right] < cov_0^{\mathbb{Q}}\left[e^{-\int_0^T (r_s - y_s) ds}, L_6(T)\right].$$

then

$$E_0^{\mathbb{Q}}\left[L_6\left(T\right)\right] > s_0^{r-y} > s_0^{r} > s_0^{R}.$$

For the models presented below this ordering holds. In the general case of multi-period swaps settled in arrears, we expect this ordering to hold, although the ordering will depend subtly on the relationships between the shocks and levels of the factors that drive these variables.

4 Does Collateral Matter?

While collateral is clearly a contractual feature of interest rate swaps, it is important to demonstrate that collateral matters, that is, that the presence of collateral affects market swap rates. In this section, we use the information embedded in Eurodollar futures rates to examine this issue.

As argued in the previous sections, collateral affects the discounting of swap rates and generally increases swap rates relative to their value using the traditional approach. An obvious way to examine the impact of collateral would be to construct hypothetical swap rates from refreshed LIBOR bond prices, P^R, using the par representation and compare them to market swap rates. If market rates are above the hypothetical swap rates, then collateral matters, i.e., swaps are not discounted at R_t.

A first order approximation to the exponential, $e^x = 1 + x$, implies that the required condition is

$$\begin{aligned} cov_0^{\mathbb{Q}}\left[1-\int_0^T r_s ds, L_6\left(T\right)\right] &< cov_0^{\mathbb{Q}}\left[1-\int_0^T r_s ds+\int_0^T y_s ds, L_6\left(T\right)\right] \\ &= cov_0^{\mathbb{Q}}\left[1-\int_0^T r_s ds, L_6\left(T\right)\right]+cov_0^{Q}\left[\int_0^T y_s ds, L_6\left(T\right)\right] \end{aligned}$$

and thus we have that $s_0^{r-y} > s_0^r$ if $cov_0^{\mathbb{Q}}\left[\int_0^T y_s ds, L_6\left(T\right)\right] > 0$. This is satisfied, if, for example, collateral is positively correlated with the default-free short rate, r_t, or the spread to LIBOR, δ_t.

Unfortunately, refreshed LIBOR bond prices are typically obtained from swap rates assuming the par representation holds, which makes this exercise tautological.

To investigate the validity of the par representation, we need to extract information about the refreshed-LIBOR zero coupon term structure. The best source of information regarding refreshed-quality LIBOR rates is the futures contract on three-month LIBOR, typically known as the Eurodollar futures market. Using the default-adjusted short rate approach of Duffie and Singleton (1997, 2000), the futures rate at time t of a contract that expires at time $T_n > t$ is

$$FUT_{t,T_n} = E_t^{\mathbb{Q}} \left[L_3 \left(T_n \right) \right],$$

where $L_3 (T_n)$ is the three-month LIBOR rate,

$$L_3 \left(T_n \right) = 4 \left[\frac{1}{P^R \left(T_n, T_n + 3 \right)} - 1 \right],$$

$P^R (T_n, T_n + 3)$ is the price of a three-month zero coupon "LIBOR" bond,

$$P^R \left(T_n, T_n + 3 \right) = E_T^{\mathbb{Q}} \left[e^{-\int_{T_n}^{T_n+3} R_s ds} \right],$$

and $R_s = r_s + \delta_s$ is the default-adjusted short rate.

The Eurodollar futures curve has a number of advantages. It provides a "clean" piecewise view of expectations of future LIBOR rates. Each contract embodies the markets expectation of discount rates over a 3-month period. Second, and unlike swap rates, modeling futures rates does not require potentially controversial assumptions regarding the cost of collateral and the credit risk of the contracts. Third, the market is very transparent and liquid. The Eurodollar futures are traded 24 hours a day, on the Chicago Mercantile Exchange, the Singapore Exchange and the LIFFE in London as well as online through the GLOBEX system. The Eurodollar futures market is the most liquid derivatives market in the world in terms of notational dollar volume of daily transactions. Moreover, the contracts trade out to ten years.

The only disadvantage of Eurodollar futures is that, unlike forward contracts, they do not directly provide refreshed-LIBOR zero coupon bond prices. Thus, we must estimate a term-structure model to compute refreshed LIBOR bond prices. Unfortunately, there is no way around this issue. For robustness, we use a number of different term structure models.

We obtained daily close prices for the Eurodollar futures contract from the Chicago Mercantile Exchange from 1994 to 2002. We discard serial month contracts, and use weekly Wednesday close prices. If Wednesday is not available, we use Thursday rates. We use the first 28 quarterly contracts to calibrate the models which corresponds to roughly the first seven years worth of data. We do not use data past seven years to avoid any potential liquidity concerns on the long end of the futures curve.

To calibrate the models, we use the Vasicek (1977) and Cox, Ingersoll, and Ross (1985) models and the Hull and White (1990) calibration procedure. For every day, we compute the parameters that provide the closest fit:

$$\widehat{\Theta}_t = \arg\min \sum_{j=1}^{28} \left\| Fut_{t,T_j}\left(\Theta_t\right) - Fut_{t,T_j}^{Mar} \right\|$$

where $Fut_{t,T_j}\left(\Theta_t\right)$ are the model implied futures rates, Fut_{t,T_j}^{Mar} are the market observed futures rates, and $\|\cdot\|$ is a distant measure. We have used both absolute deviations and squared deviations, and the results reported below use squared deviations. All of the models considered provide an accurate fit to the futures curve and to insure smoothness of the curves, we constrain the parameters from taking extreme values (this is especially important for the interest rate volatility). We repeat this exercise weekly.

We have also used convexity corrections from a proprietary model from an investment bank that uses the information embedded in swaptions to estimate volatility. As implied volatility tends to be higher than historical volatility, the convexity adjustments are even larger and our results stronger. In general, our Vasicek and CIR model convexity corrections are conservative.

Given the calibrated parameters, we compute swap rates assuming the par representation (portfolio of forwards) holds. In addition to the par rates, we also compute swap rates under the assumption that the cost of collateral is the instantaneous default-free rate, r_t. In this case, swaps are priced as a portfolio of futures contracts on six-month LIBOR and the fixed rate, $s_{0,T}^{FUT}$, on a fixed-for-floating swap on six-month LIBOR with semi-annual payments settled-in-arrears solves

$$0 = E_0^{\mathbb{Q}}\left[\sum_{j=1}^{2N}\left(L_6\left((j-1)/2\right) - s_{0,T}^{FUT}\right)\right]$$

which implies that

$$s_{0,T}^{FUT} = \frac{1}{2N}\sum_{j=1}^{2N} E_0^{\mathbb{Q}}\left[L_6\left((j-1)/2\right)\right].$$

Note that we take into account the fact that the dates on which swap payments are exchanged are six-months later than the date on which the floating index is determined.

Tables 1 and 2 provide summary statistics for the Vasicek (1978) and CIR (1985) models. We compute the difference between market swap rates and the hypothetical par rates, $s_{0,T}^{R} - s_{0,T}^{Mar}$ where $s_{0,T}^{Mar}$ is the market observed swap rate for a T-year swap. We report results for five and seven year rates. The results also hold for 3 and 10 year rates, but provide little additional insight and are not reported. We also compute the difference between the swap rates generated by a portfolio of futures on six-month LIBOR: $s_{0,T}^{Fut} - s_{0,T}^{Mar}$. If we have the correct term structure model and swaps are priced as par rates, $s_{0,T}^{R} - s_{0,T}^{Mar}$ should be zero. As noted by Gupta and Subrahmanyam (2000), most term structure models deliver similar convexity corrections and from their Figure 4, we know that the CIR and Vasicek models cover the reasonable range of convexity corrections.

These results indicate that swaps are not always and, in fact, rarely priced as par rates. For example, for the entire 1994-2002 period, the average difference between market and forward rates for seven-year swaps is -0.0969 for the Vasicek model and is significantly different from zero. This implies that, on average, the market swap curve is about nine basis points above par rate curve. Similarly, it is about seven basis points below the futures curve.[15] Taken together, the market swap rates on average lies someone in between the portfolio of forwards and futures formulations. The 5-year swap rates are even more striking: they are not statistically different from the hypothetical futures based swap rates! This is true for both models. In light of our theoretical arguments above, this indicates that, on average, collateral has a significant effect on market swap rates.

Next, note that the position of the market swap rates vis-a-vis the forwards and futures curves changes drastically over time. For example, for the 7-year swap rate, in 1997, 1998 and 2000, the market swap curve was, on average, very close to the portfolio of futures swap rate. For example, in 1998, the market swap rates were within one basis point of $s_{0,T}^{Fut}$ but more than 16 basis points greater than $s_{0,T}^{For}$. It is particularly interesting note that 1997, 1998 and 2000 were all years in which there was significant periods of market stress: the Asian currency crises, the collapse

[15] Note that we have very conservative convexity corrections of less than 20 basis points on average. Larger corrections only move the hypothetical par swap curve down further and make the results even stronger.

Period	5 Year Swaps		7 Year Swaps	
	Forwards	Futures	Forwards	Futures
1994-2002	-0.1043 (0.0031)	-0.0012 (0.0029)	-0.0969 (0.0040)	0.0696 (0.0093)
1994	-0.0972 (0.0077)	-0.0002 (0.0068)	-0.0727 (0.0076)	0.0713 (0.0066)
1995	-0.1256 (0.0094)	-0.0068 (0.0080)	-0.1277 (0.0131)	0.0756 (0.0058)
1996	-0.1143 (0.0069)	-0.0062 (0.0056)	-0.0997 (0.0088)	0.0686 (0.0044)
1997	-0.1714 (0.0070)	-0.0280 (0.0059)	-0.2026 (0.0101)	0.0336 (0.0056)
1998	-0.1430 (0.0074)	-0.0442 (0.0034)	-0.1628 (0.0127)	0.0078 (0.0040)
1999	-0.0720 (0.0070)	0.0137 (0.0070)	-0.0763 (0.0091)	0.0695 (0.0070)
2000	-0.0898 (0.0057)	-0.0348 (0.0075)	-0.0781 (0.0050)	0.0188 (0.0064)
2001	-0.0684 (0.0095)	0.0140 (0.0107)	-0.0215 (0.0088)	0.1024 (0.0105)
2002	-0.0553 (0.0077)	0.0821 (0.0086)	-0.0285 (0.0075)	0.1792 (0.0069)

Table 1: For each Wednesday from 1994 to 2002, we calibrate the parameters of the Vasicek (1978) model to fit the Eurodollar futures curve. Given the calibrated term structure model, we compute hypothetical swap rates under the assumption that swaps are priced via the par representation (portfolio of forwards, discounted at R_t) and as a portfolio of futures contracts. The columns marked forwards (futures) give the difference between the hypothetical swap priced as a portfolio of forwards (futures) and the market swap rate. The standard errors are given to the right of the mean estimates in parenthesis.

Period	5 Year Swaps		7 Year Swaps	
	Forwards	Futures	Forwards	Futures
1994-2002	-0.0666 (0.0028)	-0.0036 (0.0030)	-0.0352 (0.0026)	0.0676 (0.0031)
1994	-0.1025 (0.0077)	0.0031 (0.0068)	-0.0834 (0.0071)	0.0746 (0.0067)
1995	-0.0641 (0.0085)	-0.0133 (0.0079)	-0.0201 (0.0061)	0.0692 (0.0413)
1996	-0.0776 (0.0059)	-0.0096 (0.0079)	-0.0456 (0.0056)	0.0651 (0.0057)
1997	-0.0779 (0.0067)	-0.0292 (0.0061)	-0.0527 (0.0066)	0.0298 (0.0058)
1998	-0.0846 (0.0038)	-0.0508 (0.0038)	-0.0580 (0.0066)	0.0005 (0.0043)
1999	-0.0389 (0.0057)	0.0098 (0.0068)	-0.0166 (0.0054)	0.0660 (0.0069)
2000	-0.0738 (0.0072)	-0.0371 (0.0073)	-0.0480 (0.0059)	0.0148 (0.0061)
2001	-0.0721 (0.0072)	0.0094 (0.0108)	-0.0237 (0.0088)	0.1039 (0.0107)
2002	-0.0078 (0.0117)	0.0859 (0.0111)	0.0312 (0.0078)	0.1855 (0.0068)

Table 2: For each Wednesday from 1994 to 2002, we calibrate the parameters of the Cox, Ingersoll, and Ross (1985) model to fit the Eurodollar futures curve. Given the calibrated term structure model, we compute hypothetical swap rates under the assumption that swaps are priced via the par representation (portfolio of forwards, discounted at R_t) and as a portfolio of futures contracts. The columns marked forwards (futures) give the difference between the hypothetical swap priced as a portfolio of forwards (futures) and the market swap rate. The standard errors are given to the right of the mean estimates in parenthesis.

of LTCM and bursting of the dot-com bubble. The previous results indicate that as collateral becomes more costly, swap rates move closer to futures rates and it is not unreasonable to conjecture that collateral was relatively more costly/important during these periods.

Figure 1 provides a plot of the time series of $s_{0,T}^{Fut} - s_{0,T}^{Mar}$ and $s_{0,T}^{For} - s_{0,T}^{Mar}$. This figure shows the very strong time variation in the relative position of market swap rates vis-a-vis the portfolio of futures and portfolio forwards swap rates. Graphically, and especially for the 7-year rate, it is apparent that market swap rates were very close to the futures based swap curve during periods of market stress in 1997, 1998, and 2000. Together, these results point to the importance of costly collateral and the fact that the cost of posting collateral is time-varying.

Could our results be driven by our choice of data, calibration procedure, or choice of model? This does not appear to be the case. Independently of out work, Bomfim (2002) performs a similar experiment using (1) the entire Eurodollar futures curve (as opposed to the first 28 contracts), (2) convexity adjustments with parameters that are constant over the time (as opposed to varying weekly), (3) information embedded in swaption prices, (4) in addition to the Vasicek (1978) and CIR (1985) models, the Ho and Lee (1986) and a two-factor Gaussian model, and (5) a slightly different procedure for generating the hypothetical futures curve. Bomfim's (2002) goal, analyzing counterparty credit risk during times of market stress, is different than ours, but the conclusions regarding the location of the market swap curve relative to the hypothetical futures and forwards based curves are remarkable similar. Bomfim (2002) finds that the market swap rates for under 5 years are almost identical to the hypothetical swap rates generated via a portfolio of futures argument and, generally that the market swap curve lies in between the hypothetical portfolio of futures and

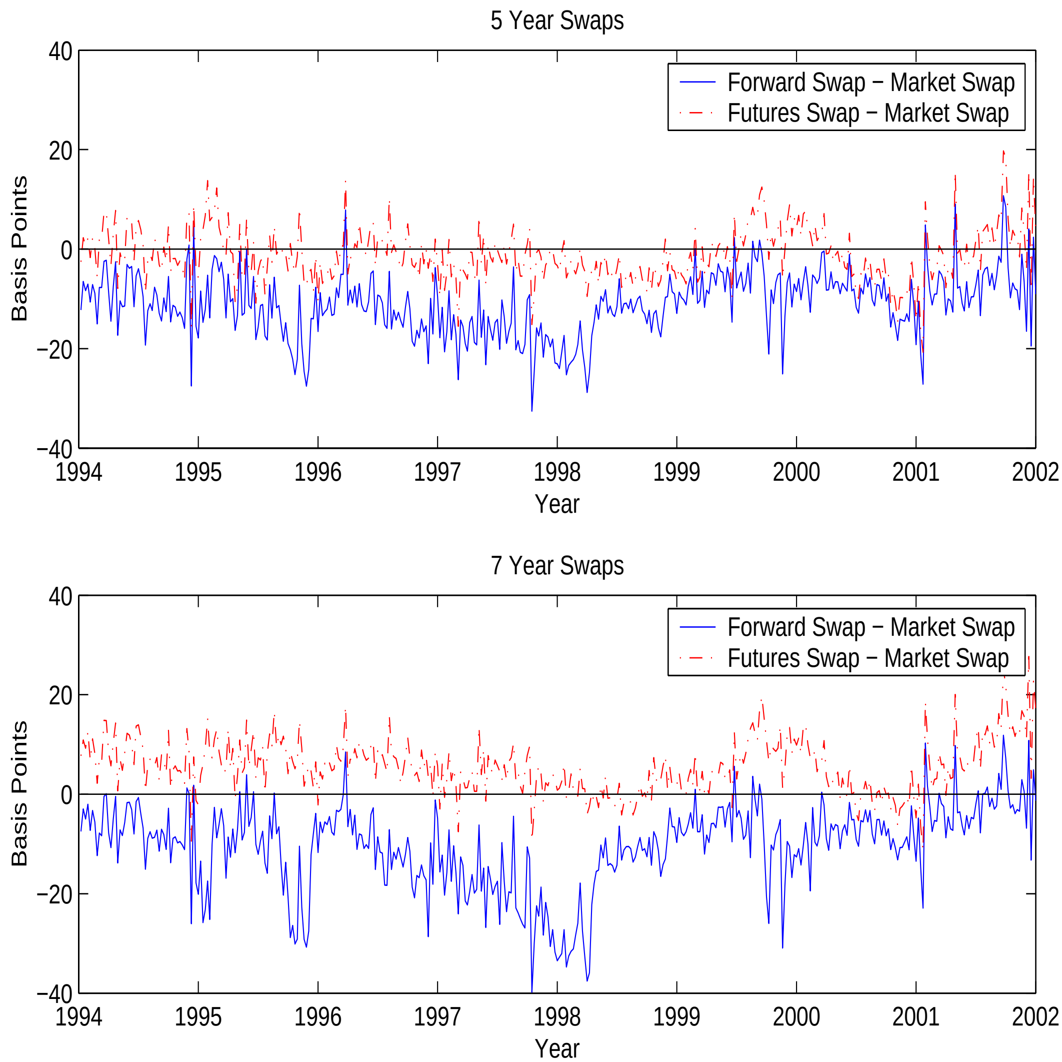

Figure 1: This figure provides weekly time series of the differences between market swap rates and swap curves calibrated from Eurdollar futures using the Vasicek (1978) model and the Hull and White (1990) calibration procedure. The solid line gives the difference between the par rate swap curve and the market swap curve and the dash-dot line gives the difference between the portfolio of futures swap rate and the market swap rates.

portfolio of forwards rates.

Are there alternative explanations that would imply that market swap rates are greater than those implied by the par representation? One alternative is the liquidity explanation of Grinblatt (2001) which relies on the convenience yield generated by holding Treasury securities. The convenience yield is generated by repo specialness of the on-the-run Treasury issuances. Duffie and Singleton (1997) argue in the presence of this liquidity factor, one can adjust the relevant discount rate for swap payments to $\widetilde{R}_t = r_t - l_t + \delta_t$. Since $\widetilde{R}_t < R_t = r_t + \delta_t$, this would be consistent with market swap rates lying above those implied by the par representation. However, the results above indicate that swaps are often priced close to and statistically indistinguishable from the rates implied by a portfolio of futures. In this case, the liquidity based argument would imply that $\widetilde{R}_t = 0$ or that $l_t = r_t + \delta_t$. This is potentially implausible high for a liquidity proxy.[16] For example, consider a 5-year swap. The benchmark 5-year Treasury note which would accrue the convenience yield was typically auctioned monthly in the 1990s. Due to this, at most, the specialness could have accrued for a maximum of 1 month and therefore is likely to be a minor component of swap rates. The same is true for 10-year swap rates, although in this case the liquidity factor could play a slightly larger role as these are auctioned quarterly.

The results in this section are important because they rely only on the information embedded in Eurodollar futures and interest rate swaps. In the following section, we formally estimate a model embedding costly collateral.

[16]For example, when l_t is positive, one would also expect r_t is decrease and δ_t to increase, the exact size of these effects is difficult to predict.

5 What is the cost of collateral?

Given that collateral appears to matter, the next issue is to characterize the cost of collateral. Evidence from the previous section indicates that during certain periods of time swaps are priced as portfolios of futures. In our context, that means that y_t is related to the short term, default-free interest rate, but the effect is time-varying. The purpose of this section is to give an illustrative feel for the time series properties of the cost of collateral in the context of a term structure model. The extracted state variable y_t will allow us to examine whether the cost of collateral implied form the model increases in periods of crisis as one's intuition might suggest. This section is not intended to provide precise characterization of the dynamics of swap spreads, but rather to get a sense of its properties in periods of market stress. He (2001) and Liu, Longstaff and Mandel (2001) provide more general models for explaining swap spreads.

Our analysis in Section 3 indicates that swap rates should be discounted at $r_t - y_t$ which implies that we must model both the default-free term structure and the LIBOR/swap term structure. We use a multi-factor Vasicek (1977) style model with conditionally Gaussian factors. Gaussian specifications are common when modeling swap rates, see, e.g., Collin-Dufresne and Solnik (2001), He (2001) and Liu, Longstaff and Mandel (2001). The default-free term structure is given by:

$$dr_t = k_r\left(\theta_t - r_r\right)dt + \sigma_r dW_t^r\left(\mathbb{P}\right)$$
$$d\theta_t = k_\theta\left(\theta_\theta - \theta_t\right)dt + \sigma_\theta dW_t^\theta\left(\mathbb{P}\right),$$

where, for simplicity, we assume the Brownian motions are independent. The first factor is the default-free instantaneous rate, r_t, and the second factor, θ_t, is the rate to which the short rate mean-reverts, commonly referred to as the time-varying central tendency. Our goal is to have the simplest possible reasonable description of the default-free term structure. We assume there are constant market prices of risk, λ_r and λ_θ, associated with the Brownian shocks.

Our third factor is the instantaneous spread from the default-free rate to LIBOR: $\delta_t = R_t - r_t$ and is modeled as

$$d\delta_t = \left[\kappa_\delta \left(\theta_\delta - \delta_t\right) + \kappa_{\delta,r} r_t + \kappa_{\delta,y} y_t\right] dt + \sigma_\delta dW_t^\delta \left(\mathbb{P}\right).$$

Again, we assume that there is a constant market price of risk associated with W_t^δ, λ_δ. The cost of collateral process is similarly given by:

$$dy_t = \left[\kappa_y \left(\theta_y - y_t\right) + \kappa_{y,r} r_t + \kappa_{y,r} \delta_t + \kappa_{y,\theta} \theta_t\right] dt + \sigma_y dW_t^y \left(\mathbb{P}\right).$$

We use off-diagonal terms in the specification of δ_t and y_t to capture any contemporaneous relations between the variables. For example, we are especially interested in the relation between y_t and δ_t and r_t. From the previous section, we have a strong suggestion that y_t is positively related in levels to r_t and it is also plausible that the cost of collateral is positively related to the short term funding spread, δ_t, as this is a general proxy for default and liquidity concerns. As it is not possible to separately identify correlations and off-diagonal terms, we assume all of the Brownian motions are mutually independent. Part of the motivation for this is that it simplifies our two-stage estimation procedure described below.

With six-month resettlement, the various swap rates are given by:

$$s_0 = \frac{\sum_{j=1}^{2N} E_0^{\mathbb{Q}} \left[e^{-\int_0^{\frac{j}{2}} (r_s - y_s) ds} L_6 \left(\frac{j-1}{2} \right) \right]}{\sum_{j=1}^{2N} E_0^{\mathbb{Q}} \left[e^{-\int_0^{\frac{j}{2}} (r_s - y_s) ds} \right]}$$

$$s_0^r = \frac{\sum_{j=1}^{2N} E_0^{\mathbb{Q}} \left[e^{-\int_0^{\frac{j}{2}} r_s ds} L_6 \left(\frac{j-1}{2} \right) \right]}{\sum_{j=1}^{2N} E_0^{\mathbb{Q}} \left[e^{-\int_0^{\frac{j}{2}} r_s ds} \right]}$$

$$s_0^R = \frac{\sum_{j=1}^{2N} E_0^{\mathbb{Q}} \left[e^{-\int_0^{\frac{j}{2}} R_s ds} L_6 \left(\frac{j-1}{2} \right) \right]}{\sum_{j=1}^{2N} E_0^{\mathbb{Q}} \left[e^{-\int_0^{\frac{j}{2}} R_s ds} \right]}$$

All of these expressions are available in closed form, up to the solution of ordinary differential equations using standard arguments (see Dai and Singleton (2001)).

We estimate our model using the principle of maximum likelihood (see Chen and Scott (1993)). To extract information about both the risk-free rate and the LIBOR/swap market, we use both Treasury (3-month rates and 3, 5, 7 and 10-year par rates) and LIBOR/swap (3-month LIBOR rates and 3, 5, 7 and 10 year swap rates) market data. Table 3 provides the summary statistics of data used. We follow Collin-Dufresne and Solnik (2001) and Liu, Longstaff and Mandel (2001) and use Treasury rates to extract information about the default-free term structure. One could alternatively use either term Federal funds or general collateral repo rates, although these series are seriously polluted with microstructure noise (e.g., settlement Wednesdays).[17]

[17] Even if clean series for the these variables were available, our results would not likely change.

	Treasury		LIBOR/Swap	
	mean	std	mean	std
3-month	4.750	1.510	5.103	1.612
3-year	5.696	1.325	6.152	1.359
5-year	6.022	1.212	6.531	1.252
7-year	6.256	1.150	6.758	1.206
10-year	6.362	1.138	6.968	1.156

Table 3: Summary statistics of interest rate data used for estimation. All series are sampled weekly, on Wednesdays, from 1/2/1990 to 10/29/2002.

The model of the previous section is a four-factor specification and with this many parameters it is difficult to get reliable MLE estimates. To simplify and make more robust the estimation procedure, we follow Duffie, Pedersen, and Singleton (2002) and use a two step procedure. In the first stage, we estimate the two-factor risk-free term structure using time series of 3-month Treasury bill rates and 3, 5, 7 and 10 year par rates. We fit the three-month rate and seven year par rates without error

The reasoning is as follows. Standard models of swap spreads (He (2000)) indicate that swap spreads are, roughly speaking, properly ammortized present discounted values of short term spreads. In our case, we use the LIBOR-Treasury Bill spread, which is about 35 basis points on average. The problem in these models of swap spreads is that the short term spreads are highly volatile and rapidly mean-reverting which implies the same for swap spreads. However, swap spreads tend to be persistent and much larger (60-70 basis points) than the present value of the short-term spreads, the difficulty noted in He (2001) and Liu, Longstaff and Mandel (2001). If we instead we able to use the LIBOR-GC repo short term spread, the problem would be even worse as this series is also rapidly mean-reverting but has a mean of only about 15 basis points. In this case, our collateral factor would likely play an even greater role.

and the three, five and seven year rates with error. We use the parameters and state variables estimated as inputs for the second stage. In the second stage, we take these parameters and the state variables as given and estimate a two factor model for the LIBOR/swap market using 3-month LIBOR and 3, 5, 7 and 10 year swap rates. The two step procedure sacrifices asymptotic statistical efficiency. The informational loss is measured by the information contained in the LIBOR/swap curve regarding the default-free parameter estimates and is likely to be small (see, also Duffie, Pedersen and Singleton (2002)). We constrain the parameter, θ_δ, to be equal to 35 basis points, the in-sample mean of the TED spread.

We invert using 3-month and 7-year rates.[18] The 3-month rates were chosen because they are the shortest maturities for both markets that are free of microstructure noise (see Duffee (1995)). The short maturity provides a clean view of the state variables that characterize the short end default-free and LIBOR curves (r_t and δ_t). The 7-year rates provide a view of the longer end of the yield curve. Alternatively, we could use 10-year rates for inversion results. The results are similar although collateral plays an even more important role as the 10-year swap spread is larger on average than the seven-year swap spread (60 basis points versus 50 basis points) and more volatile.

Table 4 provides maximum likelihood estimates and Figure 2 provides time series of the states extracted from the maximum likelihood estimation procedure. The parameter estimates are largely consistent with prior studies, although a number of them are insignificant. For example, the long-run mean of the time-varying central

[18] Recently, there is a trend for fitting principal components (see, e.g., Dai and Singleton (2003) and Collin-Dufresne, Goldtein, and Jones (2003)). In our case, we do not have zero coupon yields and so we cannot use this methodology to recover model insensitive estimates of the states.

Parameter	Estimate	S.E.	Parameter	Estimate	S.E.
$k_\theta \times 10^2$	9.149	0.975	$\theta_\delta \times 10^3$	3.500	Fixed
$\theta_\theta \times 10^2$	0.962	4.946	$\sigma_\delta \times 10^3$	9.019	0.437
$\sigma_\theta \times 10^2$	1.567	0.070	$\lambda_\delta^Q \times 10^3$	2.150	3.245
$\lambda_\theta^Q \times 10^2$	-6.142	5.145	$k_y \times 10^3$	3.390	2.520
$k_r \times 10$	8.399	0.062	$k_{y,\theta} \times 10$	-0.229	1.304
$\sigma_r \times 10^3$	8.075	0.134	$k_{y,r} \times 10$	2.652	1.442
$\lambda_r^Q \times 10^2$	-1.950	0.337	$k_{y,\delta} \times 10$	-1.384	0.773
k_δ	1.573	0.178	θ_y	-1.268	0.420
$k_{\delta,r} \times 10$	-1.286	0.468	$\sigma_y \times 10^3$	9.088	4.416
$k_{\delta,y} \times 10$	8.953	4.350	$\lambda_y^Q \times 10$	3.328	9.141

Table 4: Two-stage maximum likelihood estimates obtained using weekly Treasury and LIBOR/swap market data from 1/2/1990 to 10/29/2002. The standard errors were calculated using the outer-product of the scores.

tendency process is insignificant as are a number of the market price of risk parameters. Among the interaction terms in the drift, only $\kappa_{y,\theta}$ is clearly insignificant. The average pricing errors on the 3, 5 and 10 year Treasuries were 0.1, 0.9 and 12.1 basis points, respectively and 14.9, 2.8 and 2.2 for the corresponding swap rates. Not surprisingly, all have autocorrelation over 90 percent.

The implied states are very highly correlated with their analogs in the Treasury and LIBOR/swap data. For example, the in-sample means of the implied states are $\overline{r}_t = 4.6$ percent and $\overline{\delta}_t = 34.3$ basis points. For comparison purposes, the three-month Treasury rate had a sample mean of 4.65 percent and the TED spread (three-LIBOR minus three-month Treasury) has a mean 35.2 basis points. Moreover, the correlation between the implied state variable δ_t and the TED spread is 96.5 percent. Due to this, we can safely identify δ_t as the TED spread. The correlation between the δ_t and r_t is 31.1 percent. While standard structural models would imply that this spread should be negative, our positive correlation is not a surprise as the correlation between the TED spread and the three-month Treasury bill rate over the same time period is 37 percent. The mean of the implied cost of collateral is 43 basis points, which is reasonable and is the same order of magnitude as the TED spread.

Of particular interest is the correlation of the implied cost of collateral process, y_t, and the other state variables. The correlation between y_t and r_t is 48.6 percent and y_t and δ_t is -11.0 percent. While the cost of collateral process is much smaller than the short rate, this provocative result points toward the close relationship between cost of collateral and the default-free short term interest rate. This is broadly consistent with the findings in the previous section which showed that discount factor in the swap market is often much lower than R_t or even r_t. Moreover, from the time series plots in Figure 2 we see that the implied cost of collateral increased drastically in the fall of 1998 and remained high for a long time period. This also is consistent with economic intuition: the cost of posting collateral increases dramatically during periods of market stress.

Further support for this hypothesis come from a summary statistics of the TED and 10-year swap spread. The TED spread is, on average, 35 basis points over the sample period and the ten-year swap spread (ten-year swap rate minus ten-year Treasury rate) was 60 basis points. In addition to the large magnitude of the swap spread relative to the TED spread, the correlation between the ten-year swap spread and the TED spread is only 39% over the sample period. The small magnitude of the TED spread makes it difficult to capture the large swap spreads and the lack of correlation between the TED spread and the swap spread makes it difficult for standard models of instantaneous LIBOR rates to capture these empirical regularities. To generate the large spreads, researchers have typically turned to additional factors such as liquidity or large credit or liquidity risk premium. Our cost of collateral argument squares nicely with the high observed correlation (41 percent) between the ten-year swap spread and the three-month Treasury rate.

Our model implies that net swap payments should be discounted at $r_t - y_t$ instead of $r_t + \delta_t$ and we now examine the impact of collateral on the swap curves. The top panel of Figures 3 plots the constant maturity Treasury curve as well as the collateralized and par swap curves computed at estimated parameters and average state variables. The bottom panel of Figure 3 plots the difference between the collateralized and par swap rates, computed at estimated parameters and average state variables. At ten years, the collateralized swap spread is about 10 basis points higher than the par rate swap spread, which is well outside of the bid-ask spread and shows the significant impact of collateral. However, it is important to note that these magnitudes are model and parameter dependent. Figure 4 compares term structures and swap spreads on a day with particularly high 7-year swap spreads, April 11, 2000. The magnitude of the impact is slightly greater, about 15 basis points at 10 years. While significant, these collateral costs are not enough to generate swap rates that are priced as a portfolio of futures. Other components such as liquidity or more general risk premium may be required to generate these larger effects.

These results have a number of implications. First, it is common to use the par representation, in conjunction with market swap rates, to "bootstrap" the LIBOR/swap

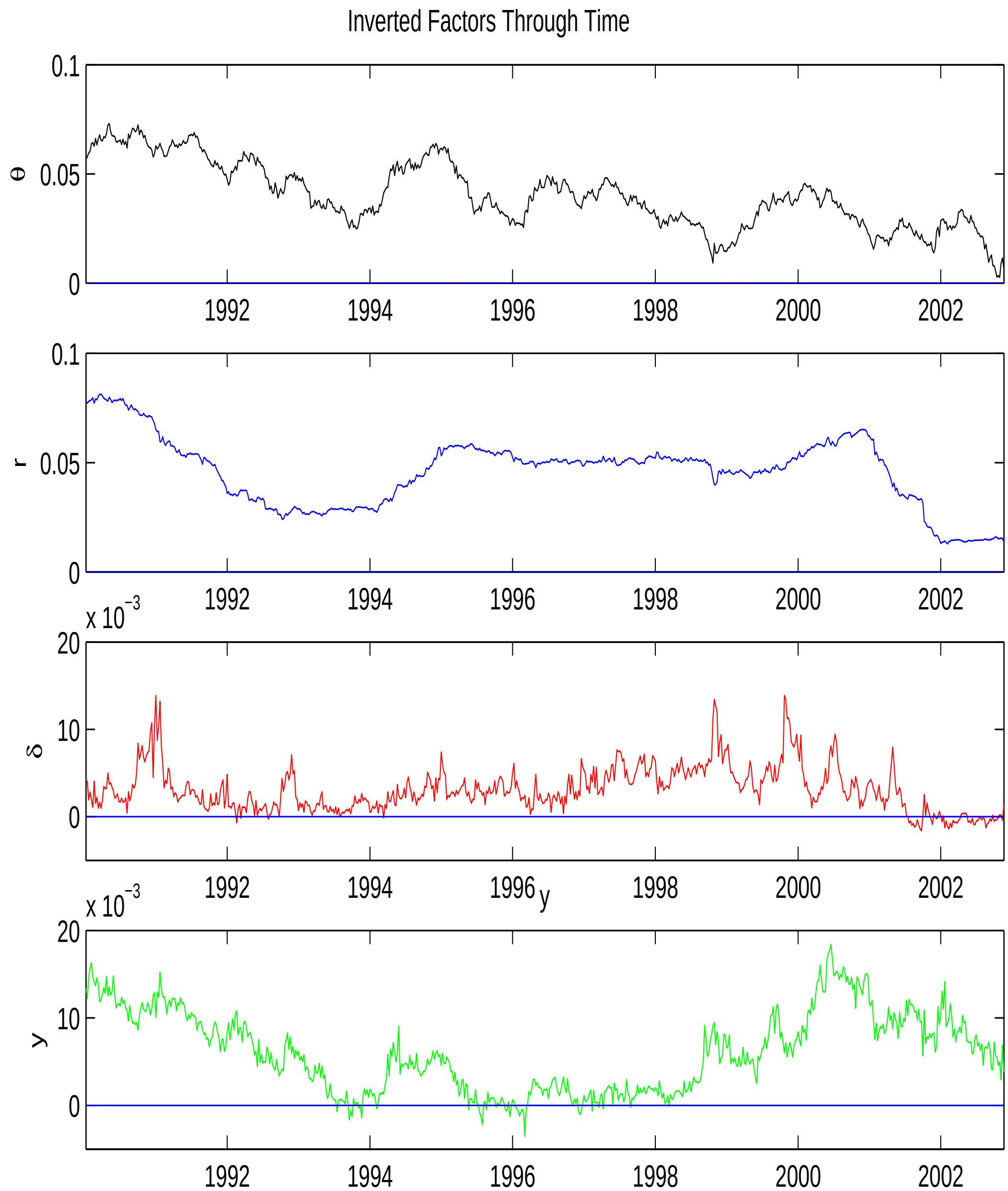

Figure 2: This fiture provides time series of the inverted factors: the central tendency (top panel), the short rate (second panel), the instananeous spread from Treasuries to LIBOR (third panel) and the cost of collateral process (bottom panel).

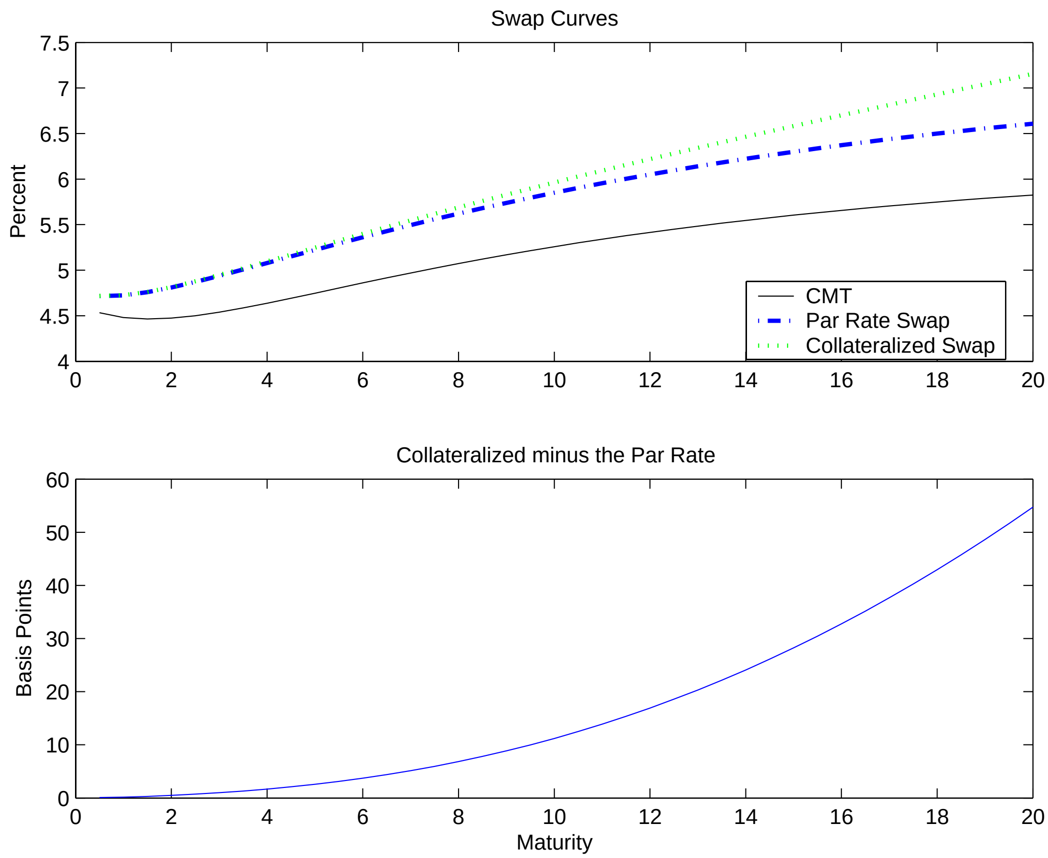

Figure 3: The top panel the Treasury par curve (solid line), the par rate swap curve (dash-dot line) and the collateralized swap curve (dotted line) using the average values for the state variables. The bottom panel displays the difference between collateralized swap rates and swap rates implied by the par representation.

curve and obtain refreshed LIBOR zeroes. Our results indicate that this will generate, in general, negatively biased zero coupon bond rates. Even small differences in these rates are extremely important in practice as these bootstrapped curves are used for pricing derivatives and risk management. Second, it is standard practice to use the par representation to price swaptions, see, for example, Musiela and Rutkowski (Ch. 16). Again, our results indicate that this will generate a directional bias in the swaption prices in the presence of collateralization. Third, it is common to use swap rates to test term structure models, see, e.g., Duffie and Singleton (1997), Dai and Singleton (2000), Collin-Dufresne, Goldstein and Jones (2003) or Piazzesi (2003). In all of these studies, the authors assume the par representation holds. Our results indicate that this will result in biased parameter estimates, although the magnitude is unknown but likely to be small.

6 Conclusions

In this paper, we analyzed the role of collateral in determining market swap rates. Theoretically, we showed that collateralized swaps are free of counterparty default risk and that costly collateral enters as a convenience yield, altering the discounting of net swap payments. Empirically, we find broadly consistent evidence from two independent sources of information, the Eurodollar futures market and the Treasury/LIBOR/swap term structure, which point to the importance of costly collateral. Often, swaps are priced close to portfolios of futures rather than portfolios of forwards discounted at the instantaneous LIBOR rates.

There are a number of important issues that need to be further addressed. First, and foremost is the relationship between the cost of collateral, liquidity and default.

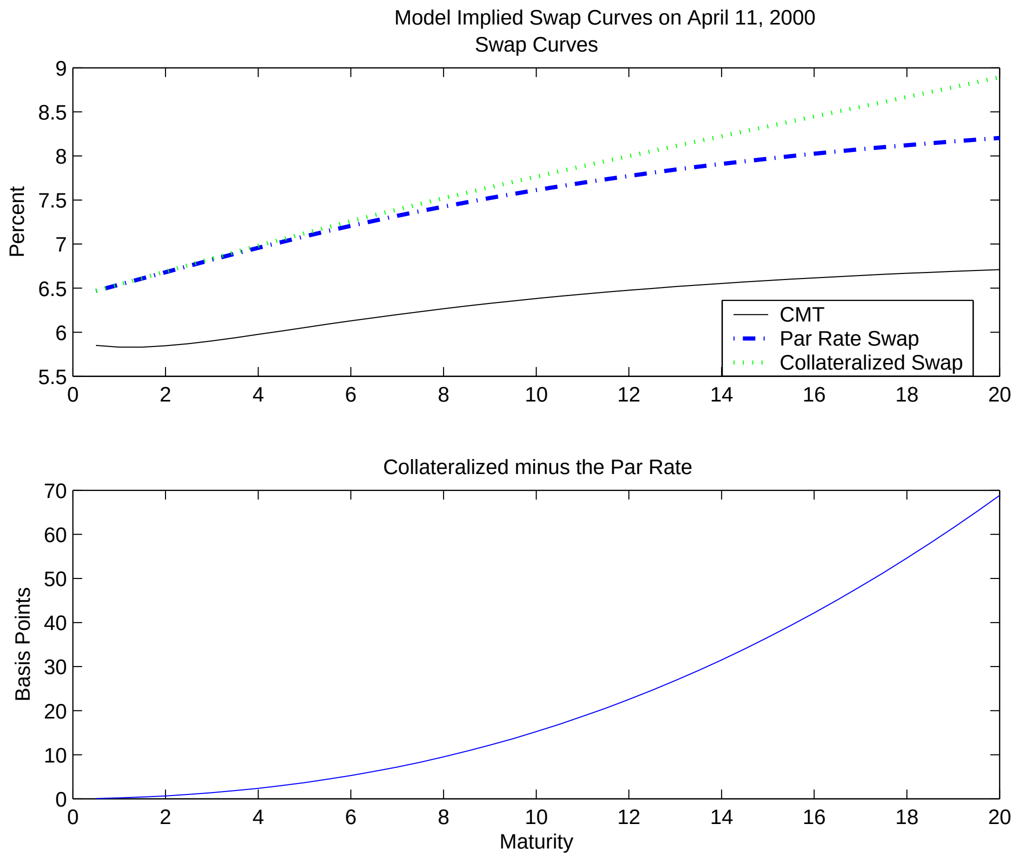

Figure 4: The top panel the Treasury par curve (solid line), the par rate swap curve (dash-dot line) and the collateralized swap curve (dotted line) using the state variables on April 11, 2000, a day with large 7-year swap spreads. The bottom panel displays the difference between collateralized swap rates and swap rates implied by the par representation.

For example, we followed He (2001) and Collin-Dufresne and Solnik (2001) and use Treasuries for the default-free curve, but it might be useful to use alternatives such as the repurchase rates or federal funds rates which might more accurately capture the default-free rate. This would allow us to separately model the liquidity/flight to quality component of Treasuries, the default embedded and LIBOR and may allow us to identify the relative contributions and relationships between liquidity, default and costly collateral. Casual observation implies that in addition to collateral, default and liquidity are important: LIBOR rates are higher than GC repo rate or Fed Funds, which, in turn, are generally higher than Treasury-bill rates. This implies that default (the spread from GC repo to LIBOR) and liquidity (the spread from Treasuries to GC repo) are likely to be significant factors in the Treasury and swap markets.

Second, it is important to consider more general models of the default-free and LIBOR/swap rates. More general default-free and default-adjusted models, such as those in Liu, Longstaff and Mandel (2001) will likely generate more realistic swap spreads and risk premium estimates. Third, in this paper we characterized the impact of collateral on swap rates, but nearly all OTC derivatives are collateralized and MTM. Like swaps, it is common to discount OTC derivatives using the LIBOR curve. Our approach extends in a straightforward manner to handle this case, and contracts that are interest rate sensitive would be particularly sensitive to collateral.

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