Investment under Uncertainty and Time-Inconsistent Preferences*

Steven R. Grenadier† and Neng Wang‡

July 19, 2005

Abstract

The real options framework has been used extensively to analyze the timing of investment under uncertainty. While standard real options models assume that agents possess a constant rate of time preference, there is substantial evidence that agents are very impatient about choices in the short-term, but are quite patient when choosing between long-term alternatives. We extend the real options framework to model the investment timing decisions of entrepreneurs with such time-inconsistent preferences. Two opposing forces determine investment timing: while evolving uncertainty induces entrepreneurs to defer investment in order to take advantage of the option to wait, their time-inconsistent preferences motivate them to invest earlier in order to avoid the time-inconsistent behavior they will display in the future. We find that the precise trade-off between these two forces depends on such factors as whether entrepreneurs are sophisticated or naive in their expectations regarding their future time-inconsistent behavior, as well as whether the payoff from investment occurs all at once or over time. We extend the model to consider equilibrium investment behavior for an industry comprised of time-inconsistent entrepreneurs. Such an equilibrium involves the dual problem of entrepreneurs playing dynamic games against competitors as well as against their own future selves.

Keywords: irreversible investment, hyperbolic discounting, time inconsistency, real options.

JEL classification: G11, G31, D9

*We thank Gur Huberman, David Laibson, Ulrike Malmendier, Chris Mayer, Tano Santos, Tom Sargent, Mike Woodford, and Wei Xiong for helpful comments.

†Graduate School of Business, Stanford University, Stanford, CA 94305 and National Bureau of Economic Research, Cambridge, MA, USA. Email: sgren@stanford.edu. Tel.: 650-725-0706.

‡Columbia Business School, 3022 Broadway, Uris Hall 812, New York, NY 10027. Email: neng.wang@columbia.edu; Tel.: 212-854-3869.

1 Introduction

Since the seminal work of Brennan and Schwartz (1985) and McDonald and Siegel (1986), the real options approach to investment under uncertainty has become an essential part of modern economics and finance.[1] In this paper, we consider a particularly well-suited application of the real options framework: the investment decision of an entrepreneur. The skills, experience and luck of the entrepreneur have endowed him with an investment opportunity in a risky project.[2] Essentially, the real options approach posits that the opportunity to invest in a project is analogous to an American call option on the investment project. Thus, the timing of investment is economically equivalent to the optimal exercise decision for an option.

In the standard real options framework it is assumed that agents have a constant rate of time preference. Thus real options models typically assume that rewards are discounted exponentially. Such preferences are time-consistent in that an entrepreneur's preference for rewards at an earlier date over a later date is the same no matter when he is asked. However, virtually every experimental study on time preferences suggests that the assumption of time-consistency is unrealistic.[3] When two rewards are both far away in time, decision makers act relatively patiently (e.g., they prefer two apples in 101 days, rather than one apple in 100 days). But when both rewards are brought forward in time, decision makers act more impatiently (e.g., they prefer one apple today, rather than two apples tomorrow). Laibson (1997) models such time-varying impatience with quasi-hyperbolic discount functions, where the discount rate declines as the horizon increases.[4] Such preferences are also termed "present-biased" preferences by O'Donoghue and Rabin (1999a).

This paper merges two important strands of research: the real options approach that emphasizes the benefits of waiting to invest in an uncertain environment, and the literature on hyperbolic preferences where decision makers face the difficult problem of making optimal

[1]The application of the real options approach to investment is quite broad. Brennan and Schwartz (1985) use an option pricing approach to analyze investment in natural resources. McDonald and Siegel (1986) provided the standard continuous-time framework for analysis of a firm's investment in a single project. Majd and Pindyck (1987) enrich the analysis with a time-to-build feature. Dixit (1989) uses the real option approach to examine entry and exit from a productive activity. Titman (1985) and Williams (1991) use the real options approach to analyze real estate development. Grenadier (1996, 2002) and Lambrecht and Perraudin (2003) extend real options to a game-theoretic environment.

[2]We assume that this investment option is non-tradable and its payoff cannot be spanned by existing assets. This could be due to the fact that the option's value emanates from the special skills of the entrepreneur. Similarly, lack of tradability could also be due to asymmetric information in that the sale of the asset would result in a "lemons" problem.

[3]For example, see Thaler (1981), Ainslie (1992) and Loewenstein and Prelec (1992).

[4]Applications of quasi-hyperbolic preferences are now quite extensive. For some examples, see Barro (1999) for an application to the neoclassical growth model, O'Donoghue and Rabin (1999b) for a principal agent model, DellaVigna and Malmendier (2004) for contract design between firms and consumers, and Luttmer and Mariotti (2003) for asset pricing in an exchange economy.

choices in a time-inconsistent framework.[5] On the one hand, standard real options models imply a large option value of waiting: typical parameterizations in the literature show that investment should not occur until the payoff is at least double the cost. On the other hand, time-inconsistent preferences provide an incentive to hurry investment in order to avoid suboptimal decisions made in the future. Our model can show precisely how these two opposing forces interact.[6]

We find it reasonable to believe that entrepreneurs (such as an individual or a small private partnership) are more prone to time-inconsistent behavior than firms. Consistent with this, Brocas and Carrillo (2004) assume that entrepreneurs have hyperbolic preferences. Similarly, DellaVigna and Malmendier (2004) assume that individuals are time-inconsistent, but that firms (with whom the individuals contract) are rational and time-consistent. Presumably there is something about the organization of a firm and its delegated, professional management that mitigates or removes the time-inconsistency from the firm's actions. Of course, little research has been done to precisely identify which individuals or institutions are more prone to time-inconsistency. The classic real option example of commercial real estate development may be particularly apt for this entrepreneurial setting. The development of commercial real estate is analogous to an American call option on a building, where the exercise price is equal to the construction cost. Williams (2001) states that land (both improved and unimproved) is primarily held and developed by noninstitutional investors (such as individuals and private partnerships), rather than by institutional investors. Such developers are often termed "merchant builders" who construct buildings (generally standardized, conventional properties) and then sell them to institutional investors.

As is standard in models of time-inconsistent decision making, such problems are envisioned as the outcome of an "intra-personal game" in which the same individual entrepreneur is represented by different players at future dates. That is, a "current self" formulates an optimal investment timing rule taking into account the investment timing rules chosen by "future selves." Essentially, the time-inconsistent investment problem is solved by jointly

[5]While we are assuming that entrepreneurs apply hyperbolic discounting to cash flows, nothing substantive would change if we instead assumed that entrepreneurs applied hyperbolic discounting to consumption, but where the entrepreneur is liquidity constrained. Being liquidity constrained, the entrepreneur must wait until the option is exercised and cash is obtained before consuming. Prelec and Lowenstein (1997) provide a numerical example of discounting cash flows in the spirit of a real options formulation. It is also worth noting that much of the experimental evidence on time-inconsistent discounting deals with individuals discounting cash payoffs, rather than consumption streams (e.g., Thaler, 1981).

[6]In a different setting, O'Donoghue and Rabin (1999a) also address some of the issues analyzed in this paper. Their paper looks at the choice of an individual with present-biased preferences as to when to take an action. However, their model is deterministic, and thus doesn't involve any of the issues of option timing that are endemic in the framework of investment under uncertainty.

solving two interconnected value functions: the current self's value function and the future selves' value function. To solve this intra-personal game in a continuous-time stochastic environment, we employ the continuous-time model of quasi-hyperbolic time preferences in Harris and Laibson (2004).

The literature on decision making under time-inconsistent preferences proposes two potential assumptions about the strategies chosen by future selves, both of which are considered in this paper. One assumption is that entrepreneurs are "naive" in that they assume that future selves will act according to the preferences of the current self, and is the approach followed by Akerlof (1991). The naive entrepreneur holds a belief (that proves incorrect) that his current self can commit future selves to act in a time-consistent manner. This assumption is in keeping with behavioral beliefs about over-confidence (in the ability to commit). An alternative assumption is that entrepreneurs are "sophisticated" in that they correctly anticipate time-varying impatience, and thus assume that future selves will choose strategies that are optimal for future selves, despite being sup-optimal from the standpoint of the current self. This very rational assumption is in the tradition of subgame perfect game-theoretic equilibrium, and is the approach followed by Laibson (1997). In our model, we will analyze investment timing under both assumptions, and determine the impact of such behavioral assumptions on investment timing strategies.

We find that when the standard real options model is extended to account for time-inconsistent preferences, investment occurs earlier than in the standard, time-consistent framework. Consider our previous example of real estate development. If such merchant builders have time-inconsistent preferences, they may accept lower returns from development in order to protect themselves against the suboptimal development choices of their future selves. Note that the earlier exercise of commercial real estate development options may be a contributor to the tendency for developers to overbuild. In fact, some observers have blamed merchant builders for causing overbuilding in office markets.[7]

The extent of this rush to invest depends on whether the time-inconsistent entrepreneur is sophisticated or naive. Specifically, we find that the naive entrepreneur rushes his investment less than does the sophisticated entrepreneur. Since the naive entrepreneur (falsely) believes that his future selves will invest according to his current wishes, he is not fearful of taking advantage of the option to wait. However, the sophisticated entrepreneur correctly anticipates that his future selves will invest in a manner that deviates from his current preferences. This puts pressure on the sophisticated entrepreneur to extinguish his option to wait

[7]For example, in an April 4, 2001 article in Barron's, merchant builders were accused of contributing to oversupply in suburban office markets.

earlier, so as to mitigate some of the costs of allowing future selves to take over the investment decision. In a sense, if one views the time-consistent solution as somehow "optimal," the naive entrepreneur's false belief in the ability to commit to an investment strategy actually helps the entrepreneur get closer to optimality; self-delusion is somehow preferable to true self-awareness.[8]

The model is extended to deal with the case in which option exercise leads to a series of cash flows rather than a lump sum payoff. Again, we assume the right to this series of future cash flows is non-tradable, for the same reasons as discussed for the lump sum payoff setting. We show that the implications on investment timing for the flow payoff case are much different from the lump sum payoff case. For the case of flow payoff, both the naive and sophisticated hyperbolic entrepreneurs invest later than the time-consistent entrepreneur. Going back to our real estate development example, suppose that the developer continues to hold the completed property and obtains cash flows from leasing the property. Such developers are termed "portfolio developers" (as distinct from merchant builders), and often build specialized properties that take advantage of their operating skills. For example, the portfolio developer may be best able to attract and retain tenants with highest willingness to pay, or keep the operating costs at the lowest level. Given the implications of the model, portfolio developers would be expected to be more cautious than merchant builders, and contribute less to bursts of overbuilding activity.

The intuition for why hyperbolic entrepreneurs wait longer before exercising than time-consistent entrepreneurs for the case with flow payoffs is as follows. While the time-consistent entrepreneur simply discounts the perpetual flow payments to obtain an equivalent lump sum payoff value, the hyperbolic entrepreneur discounts the payments received by future selves at a higher discount rate. Therefore, hyperbolic discounting lowers the present value of future flow payoffs obtained from exercise, and hence increases the entrepreneur's incentive to wait, *ceteris paribus*. While it remains true that hyperbolic entrepreneurs have an incentive to exercise before their future selves (particularly sophisticated entrepreneurs), we shall find that the previously mentioned effect dominates.

We later move beyond the analysis of a single entrepreneur's strategy and look at the equilibrium properties of investment. That is, how does equilibrium investment in an industry comprised of hyperbolic entrepreneurs compare with one comprised of time-consistent entrepreneurs? Clearly, this is empirically relevant, and a problem that is somewhat of a

[8]There is no agreed upon metric for welfare analysis for people with time-inconsistent preferences. However, O'Donoghue and Rabin (1999a) model welfare losses as deviations from long-run utility, where long-run utility is the time-consistent solution.

technical challenge.[9] Specifically, we look at the case of a perfectly competitive industry where entrepreneurs choose rational expectations equilibrium investment strategies, using a framework similar to Leahy (1993), where price-taking entrepreneurs contemplate investing in project with perpetual flow payments. We show that the equilibrium implications for economies with time-inconsistent entrepreneurs are fundamentally different from those for economies with time-consistent entrepreneurs. It is noteworthy that agents are playing both an interpersonal and intrapersonal game: they play a game against other entrepreneurs as well as future selves.

The remainder of the paper is organized as follows. Section 2 describes the underlying model, and provides the solution for the benchmark time-consistent case. Section 3 derives and analyzes the optimal investment strategy of the naive entrepreneur. Section 4 derives and analyzes the optimal investment strategy of the sophisticated entrepreneur. Section 5 extends the model to include the case of investments that yield a series of cash flows. Section 6 considers the implications of our model in an equilibrium setting, and Section 7 concludes.

2 Model Setup

2.1 The Investment Opportunity

Consider the setting for a standard irreversible investment problem.[10] The entrepreneur possesses an opportunity to invest in a project. The investment option is assumed to be non-tradable.[11] Let X denote the payoff value process of the underlying project. Assume that the project payoff value evolves as a geometric Brownian motion process:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dB_t, \tag{1}$$

where α is the instantaneous conditional expected percentage change in X per unit time, σ is the instantaneous conditional standard deviation per unit time, and dB is the increment of a standard Wiener process. Investment at any time costs I. The lump sum payoff from investment at time t is then given by $X(t) - I$. The entrepreneur is free to choose the moment of exercise of his investment option.

[9] While in a very different context, Luttmer and Mariotti (2003) model an equilibrium of a discrete-time exchange economy with hyperbolic discount factors.

[10] See Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994).

[11] Non-tradability may be justified on any of several grounds. For example, the option's value may be contingent upon the unique skills of the entrepreneur; the option may have little or no value in the hands of another entrepreneur. In addition, the entrepreneur may have private information about the option that cannot be credibly conveyed to outside purchasers, and hence a "lemons" problem may result. We also assume that the investment payoffs are not spanned by existing assets.

2.2 Entrepreneur's Time Preferences

We assume that the entrepreneur is risk neutral, but dispense with the standard assumption of exponential discounting. In order to reflect the empirical pattern of declining discount rates, Laibson (1997) adopts a discrete-time discount function to model quasi-hyperbolic preferences. Time is divided into two periods: the present period, and all future periods. Payoffs in the current period are discounted exponentially with the discount rate ρ. Payoffs in future periods are first discounted exponentially with the discount rate ρ and then further discounted by the additional factor $\delta \in (0, 1]$. For example, a dollar payment received at the end of the first period is discounted at the rate ρ and is thus worth $e^{-\rho}$ today, but a payment received at the end of the n^{th} period is worth $\delta e^{-\rho n}$ today, for all $n > 1$.

To see the time-inconsistency implications of such time preferences, consider the choice between investing at time n to receive a payment of P_n and investing at time $n+1$ to receive a payment of P_{n+1}. From the perspective of an entrepreneur at time 0, this represents a choice between $\delta e^{-\rho n} P_n$ and $\delta e^{-\rho(n+1)} P_{n+1}$. Thus, they would prefer receiving P_n at time n over receiving P_{n+1} at time $n+1$ if and only if $P_n > e^{-\rho} P_{n+1}$. Therefore, when viewed over a long horizon, intertemporal trade-offs are determined by the exponential discounting factor ρ. Now, consider the same entrepreneur's decision at time $n-1$. At that point, the entrepreneur views the payoff at time n as occurring in the current period. Thus, at time $n-1$ the same entrepreneur now faces a choice between $e^{-\rho} P_n$ and $\delta e^{-2\rho} P_{n+1}$, and would prefer receiving P_n at time n over receiving P_{n+1} at time $n+1$ if and only if $P_n > \delta e^{-\rho} P_{n+1}$. Therefore, when viewed over a short horizon, the entrepreneur is more impatient, as intertemporal trade-offs are determined by both the exponential discounting factor ρ and the additional discount factor $\delta < 1$. Therefore, the agent at time 0 will view the relative choice between these two future investment timing choices differently than he will at time $n-1$. While the entrepreneur at time 0 would like to commit his future selves to adopt his preference orderings, he is unable to do so.

We follow Harris and Laibson (2004) to model hyperbolic discounting using a continuous-time formulation. We modify the previous formulation to allow each period to have a random period of time. Each self controls the exercise decision in the "present" but also cares about the utility generated by the exercise decisions of future selves. As in Harris and Laibson (2004), the "present" may last for a random duration of time. Let t_n be the calendar time of "birth" for self n. Then, $T_n = t_{n+1} - t_n$ is the lifespan for self n. For simplicity, we assume that the lifespan is exponentially distributed with parameter λ. Stated in another way, the birth of future selves is modeled as a Poisson process with intensity λ. That is, we may imagine a clock ticking with probability $\lambda\Delta t$ over a small time interval Δt, into the indefinite future. Before the clock ticks, we call the entrepreneur self 0. After the clock ticks for the first time, self 0 ends with the birth of self 1. When the clock ticks for the n^{th} time at time t_n, self n is born.

Given this stochastic arrival process for future selves, the quasi-hyperbolic discounting formulation discussed earlier easily applies. Specifically, in addition to the standard discounting at the constant rate ρ, the current self values payoffs obtained after the birth of future selves by an additional discounting factor $\delta \leq 1$. Let $D_n(t,s)$ denote self n's intertemporal discount function: self n's value at time t of \$1 received at the future time s. We thus have

$$D_n(t,s) = \begin{cases} e^{-\rho(s-t)} & \text{if} \quad s \in [t_n,\, t_{n+1}) \\ \delta e^{-\rho(s-t)} & \text{if} \quad s \in [t_{n+1},\, \infty) \end{cases}, \tag{2}$$

for $s > t$ and $t_n \leq t < t_{n+1}$. The magnitude of the parameter δ (along with the magnitude of the intensity parameter λ) determines the degree of the entrepreneur's time-inconsistency. After the death of self n and the birth of self $(n+1)$, the entrepreneur will use the discount function $D_{n+1}(t,s)$ to evaluate his investment project.

Let τ denote the (random) stopping time at which the entrepreneur exercises his investment option. Suppose that at time t the entrepreneur is self n. The entrepreneur chooses the investment time τ to solve the following optimization problem:

$$\max_{\tau \geq t} \quad E_t \left[D_n(t,\tau) \left(X(\tau) - I \right) \right], \tag{3}$$

where E_t denotes the entrepreneur's conditional expectation at time t. The current self's belief about his future selves' investment strategies matters significantly in how the current self formulates his investment decision.

2.3 The Time-Consistent Benchmark (The Standard Real Options Case)

As a benchmark, we briefly consider the case in which payoffs are discounted at the rate ρ. That is, the hyperbolic preference parameter δ is set equal to one. Alternatively, time-consistent discounting can be obtained if there are no arrivals of future selves (by setting the jump intensity λ to 0). Let $V(X)$ denote the entrepreneur's value function and X^* be his optimal investment threshold. Using standard arguments (i.e., Dixit and Pindyck, 1994), $V(X)$ solves the differential equation:

$$\frac{1}{2}\sigma^2 X^2 V''(X) + \alpha X V'(X) - \rho V(X) = 0, \quad X \leq X^*. \tag{4}$$

Equation (4) is solved subject to appropriate boundary conditions. These boundary conditions serve to ensure that an optimal exercise strategy is chosen:

$$V(X^*) = X^* - I, \tag{5}$$

$$V'(X^*) = 1, \tag{6}$$

$$V(0) = 0. \tag{7}$$

The first boundary condition is the value-matching condition. It simply states that at the moment the option is exercised, the payoff is $X^* - I$. The second boundary condition is the smooth-pasting or high-contact condition. (See Merton, 1973, for a discussion of the high-contact condition.) This condition ensures that the exercise trigger is chosen so as to maximize the value of the option. The third boundary condition reflects the fact that $X = 0$ is an absorbing barrier for the underlying project value process.[12]

The solutions for the value function $V(X)$ and the exercise trigger X^* are respectively given by

$$V(X) = \begin{cases} \left(\frac{X}{X^*}\right)^{\beta_1}(X^* - I), & \text{for } X < X^*, \\ X - I, & \text{for } X \geq X^*, \end{cases} \tag{8}$$

and

$$X^* = \frac{\beta_1}{\beta_1 - 1} I, \tag{9}$$

where β_1 is the positive root of the fundamental quadratic equation[13] and is given by

$$\beta_1 = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\rho\sigma^2}\right] > 1. \tag{10}$$

We now turn to the entrepreneurs' investment decisions when they have time-inconsistent preferences.

3 The Naive Entrepreneur

First consider the case of a naive entrepreneur who makes investment decisions under the false belief that future selves will act in the interest of the current self. This assumption of naivete was first proposed by Strotz (1956), and has been analyzed in Akerlof (1991) and O'Donoghue and Rabin (1999a, 1999b), among others. Naivete is consistent with empirical evidence

[12] This absorbing barrier condition will apply to all of our valuation equations. To avoid repetition, we shall refrain from listing it in future boundary conditions. Nevertheless, we ensure that it always holds.

[13] The fundamental quadratic equation is $\sigma^2\beta(\beta - 1)/2 + \alpha\beta - \rho = 0$.

on 401(k) investment (Madrian and Shea, 2001), task completion (Ariely and Wertenbroch (2002)) and health club attendance (DellaVigna and Malmendier (2003)).

The current self, self 0, has preferences $D_0(t,s)$, as specified in (2). Specifically, the current self discounts payoffs during his lifetime with the discount function $e^{-\rho t}$ for $t < t_1$, and discounts payoffs received by future selves with the discount function $\delta e^{-\rho t}$, for $t \geq t_1$. Given the time-inconsistent preferences, future self 1 will have the discount function $D_1(t,s)$, future self 2 will have the discount function $D_2(t,s)$, and so on. Since the naive entrepreneur (mistakenly) believes that all future selves will act as if their discount function remains unchanged at $D_0(t,s)$, we may effectively view the naive entrepreneur as acting as if he can commit his future selves to behave according to his current preferences. Of course, in our model there is no actual commitment mechanism and thus the naive entrepreneur's optimistic beliefs will prove incorrect.

Consider the naive entrepreneur's investment opportunity. At any time prior to the arrival of his future self, he may exercise the option and receive the net payoff $X - I$. However, if the future self arrives prior to the option being exercised, the current self receives what is known as a continuation value: the present value of the payoff determined by the decisions of future selves. Let $N_c(X)$ denote the continuation value function for the naive entrepreneur. We claim that the continuation value function for the naive entrepreneur equals $\delta V(X)$, where $V(X)$ is the value function for time-consistent entrepreneurs and is given in (8). To see the intuition behind this argument, note that the naive entrepreneur mistakenly believes that his future selves discount all future payoffs by the discount function $\delta e^{-\rho t}$. Since the multiplicative constant δ simply lowers all payoffs by the same proportion, the current self believes that future selves will act as time-consistent entrepreneurs who discount at the constant rate ρ. Therefore, the naive current self falsely foresees a continuation value of $\delta V(X)$, and believes that all future selves will exercise at the time-consistent trigger X^*.

Let $N(X)$ denote the naive entrepreneur's value function, and X_{Naive} be the optimal investment threshold at which the current self exercises. By the standard arguments in real options analysis,[14] $N(X)$ solves the following differential equation:

$$\frac{1}{2}\sigma^2 X^2 N''(X) + \alpha X N'(X) - \rho N(X) + \lambda \left[N_c(X) - N(X)\right] = 0, \quad X \leq X_{Naive}, \tag{11}$$

where $N_c(X) = \delta V(X)$. The last term in (11) states that the naive entrepreneur's value function $N(X)$ is equal to the continuation value function $N_c(X)$, upon the arrival of the future self, which occurs at the intensity λ. Equation (11) is solved subject to the following

[14]See Dixit and Pindyck (1994), Chapter 4, Section 1.1 for a derivation of the equilibrium differential equation for mixed processes with both Poisson and diffusion components.

standard value-matching and smooth-pasting conditions:

$$N(X_{Naive}) = X_{Naive} - I, \tag{12}$$

$$N'(X_{Naive}) = 1, \tag{13}$$

respectively. We assume for the moment that $X_{Naive} < X^*$, and will later verify this conjecture. Solving (11) subject to boundary conditions (12) and (13) yields the following value function and the exercise trigger:

$$N(X) = \frac{\beta_1 - 1}{\beta_2 - \beta_1}(X^* - X_{Naive})\left(\frac{X}{X_{Naive}}\right)^{\beta_2} + \delta\left(\frac{X}{X^*}\right)^{\beta_1}(X^* - I), \tag{14}$$

$$X_{Naive} = \frac{1}{\beta_2 - 1}\left[\beta_2 I + (\beta_2 - \beta_1)\,\delta\left(\frac{X_{Naive}}{X^*}\right)^{\beta_1}(X^* - I)\right], \tag{15}$$

where β_1 is given in (10), and β_2 is given by[15]

$$\beta_2 = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2(\rho + \lambda)\sigma^2}\right] > \beta_1. \tag{16}$$

The naive entrepreneur's exercise trigger X_{Naive} solves a simple implicit function (15). We next show that the naive entrepreneur exercises earlier than the time-consistent entrepreneur, verifying the assumption made above.

Proposition 1 *The naive entrepreneur exercises earlier than the time-consistent entrepreneur, in that* $X_{Naive} < X^*$.

The intuition is straightforward. Beyond the standard exponential discounting, the current self values the payoff obtained from exercise decisions by future selves less than had he exercised himself. Therefore, this δ factor provides an extra incentive for the current self to exercise before the future selves arrive. Therefore, the current self with hyperbolic discounting preference believes that he has a less valuable option to wait than a time-consistent entrepreneur does, and thus exercises the investment option earlier than the time-consistent entrepreneur.

It is important to emphasize the "irrational" expectations of the naive entrepreneur. When formulating his optimal exercise trigger X_{Naive}, he truly believes that his future selves will exercise at the time-consistent trigger X^*. However, once the future self arrives, the future self becomes a current self and also mistakenly believes that its future selves will exercise at X^*.

[15] β_2 is the positive root of the fundamental quadratic equation: $\sigma^2\beta(\beta-1)/2 + \alpha\beta - (\rho+\lambda) = 0$.

We now turn to the case of the sophisticated entrepreneur, who correctly realizes that his preferences are time-inconsistent and also knows that he cannot commit to a pre-determined investment timing strategy.

4 The Sophisticated Entrepreneur

Unlike the naive entrepreneur, the sophisticated entrepreneur correctly foresees that his future selves will act according to their own preferences. That is, self n makes his decision based on self n's preferences, fully anticipating that all future selves will do likewise. This leads to time-inconsistency in the policy rule. That is, self n and self $(n+1)$ do not agree on the optimal investment timing strategy.

As we will see, the solution for the sophisticated entrepreneur is non-trivial. For illustrative purposes, we will begin this section with the simple case of a sophisticated entrepreneur with just three selves: the current self will live for two more periods. We then move on to the more complicated case of the entrepreneur with any finite number of selves N. This is analogous to the general case of an entrepreneur with a finite lifespan. Finally, we consider the more analytically tractable case in which the entrepreneur has an infinite number of future selves.

4.1 A Model with Three Selves

The case of a sophisticated entrepreneur with three selves is the simplest one for bringing out the intuition of solving the time-inconsistent investment timing problem. Self 0 is the current self. In each (small) time period Δt, self 1 is born with probability $\lambda \Delta t$. Similarly, after the birth of self 1, self 1 will be replaced in each period Δt with probability $\lambda \Delta t$ by self 2. Self 2 will then live forever after. We solve this problem by backward induction.

Self 2's Problem

First, consider the optimization problem from self 2's perspective. Since there are no more future selves, self 2 faces a simple exponential discounting case. Thus, self 2 will invest at the time-consistent threshold X^*, and will have value function $V(X)$, as derived in Section 2.3. Denoting self 2's trigger value and value function by $X_{S,2}$ and $S_2(X)$, respectively, where "S" signifies "sophisticated," we thus have:

$$S_2(X) = V(X) = \left(\frac{X}{X^*}\right)^{\beta_1} (X^* - I), \quad X \leq X^*, \tag{17}$$

$$X_{S,2} = X^* = \frac{\beta_1}{\beta_1 - 1} I. \tag{18}$$

Self 1's Problem

Self 1 formulates his optimal exercise trigger $X_{S,1}$, taking into account that his future self will exercise at the trigger $X_{S,2} = X^*$, if his future self has the opportunity to exercise the option. However, because of self 1's hyperbolic time preferences, he values the payoff obtained from the exercise decision by self 2 at only δ of its future value. Self 1's problem is thus mathematically identical to that of the naive entrepreneur, solved in Section 3. However, note that while the naive entrepreneur in Section 3 has false beliefs, the self 1 of the sophisticated entrepreneur has rational beliefs.

Using the result in Section 3, we may write self 1's option value $S_1(X)$ as follows:

$$S_1(X) = N(X) = \frac{\beta_1 - 1}{\beta_2 - \beta_1} (X^* - X_{S,1}) \left(\frac{X}{X_{S,1}} \right)^{\beta_2} + \delta \left(\frac{X}{X^*} \right)^{\beta_1} (X^* - I) \, , \tag{19}$$

for $X \leq X_{S,1}$ and where the optimal trigger strategy solves the implicit function given by

$$X_{S,1} = \frac{1}{\beta_2 - 1} \left[\beta_2 I + (\beta_2 - \beta_1) \, \delta \left(\frac{X_{S,1}}{X^*} \right)^{\beta_1} (X^* - I) \right] = X_{Naive}. \tag{20}$$

Note that $X_{S,1} < X_{S,2}$, as demonstrated in Proposition 1.

Self 0's Problem

Now, we turn to the optimization problem for self 0. Self 0 will choose his optimal exercise trigger $X_{S,0}$, knowing that selves 1 and 2 will exercise at the triggers, $X_{S,1}$ and $X_{S,2}$, respectively. Due to self 0's hyperbolic preferences, in addition to discounting future cash flows at the rate ρ, he will further discount cash flows obtained from exercise decisions by either selves 1 or 2 by the additional factor δ.

Let $S_1^c(X)$ denote the continuation value function for self 0, self 0's valuation of the proceeds of exercise occurring after the arrival of self 1. The continuation value function $S_1^c(X)$ has a recursive formulation. If self 1 is alive when his trigger $X_{S,1}$ is reached, then the option is exercised, and its payoff to self 0 is $\delta \, (X_{S,1} - I)$. If instead self 2 arrives before $X_{S,1}$ is reached, than self 0's continuation value evolves into self 1's continuation value, $S_2^c(X)$, where $S_2^c(X) = \delta V(X)$. Thus $S_1^c(X)$ solves the following differential equation:

$$\frac{1}{2}\sigma^2 X^2 S_1^{c\prime\prime}(X) + \alpha X S_1^{c\prime}(X) - \rho S_1^c(X) + \lambda \left[\delta V(X) - S_1^c(X) \right] = 0, \quad X \leq X_{S,1}, \tag{21}$$

where the value-matching condition is given by

$$S_1^c(X_{S,1}) = \delta \, (X_{S,1} - I) \, . \tag{22}$$

Note that we only have the value-matching condition, not the smooth-pasting condition for the continuation value function $S_1^c(X)$. This is intuitive since solving the continuation value function $S_1^c(X)$ does not involve an optimality decision. The value-matching condition simply follows from the continuity of the continuation value function. Solving (21) and (22) jointly gives

$$S_1^c(X) = \delta\left(\frac{X}{X^*}\right)^{\beta_1}(X^* - I) + \delta\left[X_{S,1} - I - \left(\frac{X_{S,1}}{X^*}\right)^{\beta_1}(X^* - I)\right]\left(\frac{X}{X_{S,1}}\right)^{\beta_2}, \qquad (23)$$

for $X \leq X_{S,1}$.

Self 0 maximizes his value function $S_0(X)$, by taking his continuation value function $S_1^c(X)$ computed in (23) as given and choosing his investment threshold value $X_{S,0}$. Using the standard principle of optimality, we have the following differential equation for self 0's value function:

$$\frac{1}{2}\sigma^2 X^2 S_0''(X) + \alpha X S_0'(X) - \rho S_0(X) + \lambda\left[S_1^c(X) - S_0(X)\right] = 0, \quad X \leq X_{S,0}. \qquad (24)$$

Equation (24) is solved subject to the following value-matching and smooth-pasting conditions:

$$S_0(X_{S,0}) = X_{S,0} - I, \qquad (25)$$

$$S_0'(X_{S,0}) = 1. \qquad (26)$$

The solution for self 0's value function is

$$S_0(X) = \delta\left(X^* - I\right)\left(\frac{X}{X^*}\right)^{\beta_1} + G_{0,0}X^{\beta_2} + G_{0,1}X^{\beta_2}\log X\,, \quad X \leq X_{S,0}. \qquad (27)$$

Self 0's exercise trigger $X_{S,0}$ is the solution to the implicit equation

$$X_{S,0} = \frac{\beta_2}{\beta_2 - 1}I + \left(\frac{\beta_2 - \beta_1}{\beta_2 - 1}\right)\delta\left(\frac{X_{S,0}}{X^*}\right)^{\beta_1}(X^* - I) - \frac{G_{0,1}}{\beta_2 - 1}X_{S,0}^{\beta_2}, \qquad (28)$$

where

$$G_{0,1} = -\frac{\lambda}{\alpha + (2\beta_2 - 1)\sigma^2/2}\,\delta\left[X_{S,1} - I - \left(\frac{X_{S,1}}{X^*}\right)^{\beta_1}(X^* - I)\right]\left(\frac{1}{X_{S,1}}\right)^{\beta_2},$$

$$G_{0,0} = X_{S,0}^{-\beta_2}\left[X_{S,0} - I - \delta\left(X^* - I\right)\left(\frac{X_{S,0}}{X^*}\right)^{\beta_1} - G_{0,1}X_{S,0}^{\beta_2}\log\left(X_{S,0}\right)\right].$$

We will show later that each self will exercise at a lower trigger than its future selves, in that $X_{S,0} < X_{S,1} < X_{S,2}$. The intuition is clear by using the backward induction argument. First, self 2 will live forever, so he has time-consistent preferences and will exercise at the time-consistent trigger X^*. Self 1, however, faces a different option exercise problem. He knows that if self 2 arrives before he exercises, he will ultimately receive only the fraction δ of the payoff from self 2's exercise decision. Thus, self 1 has a less valuable option to wait than self 2, since the longer he waits, the greater the chance that self 2 will arrive and provide a lowered payoff. Thus, self 1 exercises earlier than self 2. Finally, the same argument holds for self 0. If self 1 arrives before self 0 exercises, he will receive only the fraction δ of the payoff value from self 1's investment decision. Thus, self 0 has a lower option value to wait than self 1, and hence exercises at a trigger lower than does self 1.

4.2 The Sophisticated Entrepreneur with Any Finite Number of Selves

In this subsection, we consider the general case of a sophisticated entrepreneur with any finite number of selves. Self 0 is followed by self 1, who is followed by self 2, all the way through self N. Just as in the case of three selves, one can solve the model by backward induction. Given self $(n+1)$ through self N's exercise triggers, self n can formulate his optimal exercise strategy, discounting any future self's exercise proceeds by the additional factor δ. Let $S_n(X)$ be the value function for self n and $S^c_{n+1}(X)$ denote the continuation value function for self n, consistent with the notations used in analysis for the three-self case.

We will only present an outline of the derivation. A full derivation of the results appears in the appendix. Importantly, we will derive a recursive formula for the value function of each self along with their optimal exercise triggers. This will also pave the way for the more analytically tractable case with an infinite number of selves.

First consider self N's problem. Since self N is the final self, he faces the standard time-consistent option exercise problem. Therefore, self N's value function $S_N(X)$ is equal to the time-consistent entrepreneur's value function $V(X)$ and self N's exercise trigger $X_{S,N}$ is also equal to the time-consistent entrepreneur's exercise trigger X^*. The solution for the penultimate self, self $(N-1)$, is also easily obtained. As discussed in the previous subsection, the penultimate sophisticated entrepreneur faces mathematically the same problem as the naive entrepreneur. Thus, the value function $S_{N-1}(X)$ for self $(N-1)$, the continuation value function $S^c_N(X)$ for self $(N-1)$, and the exercise trigger $X_{S,N-1}$ chosen by self $(N-1)$ are given by $S_{N-1}(X) = N(X)$, $S^c_N(X) = \delta V(X)$, and $X_{S,N-1} = X_{Naive}$, respectively, where these formulas for the naive entrepreneurs are derived in Section 3.

For $n \leq N-2$, self n's value function and exercise strategy may also be solved by backward induction. Similar to the three-self case analysis, the continuation value function $S^c_{n+1}(X)$, which is self n's valuation of the payoffs from exercise occurring after the arrival of self $(n+1)$, satisfies the following differential equation:

$$\frac{1}{2}\sigma^2 X^2 S^{c\prime\prime}_{n+1}(X)+\alpha X S^{c\prime}_{n+1}(X)-\rho S^c_{n+1}(X)+\lambda\left[S^c_{n+2}(X)-S^c_{n+1}(X)\right]=0,\ X \leq X_{S,n+1}, \quad (29)$$

where the value-matching condition is given by

$$S^c_{n+1}(X_{S,n+1}) = \delta\left(X_{S,n+1} - I\right). \tag{30}$$

As in the three-self case, only the value-matching condition, not the smooth-pasting condition, applies to the continuation value function $S^c_{n+1}(X)$. The recursive relationship starts with the known solutions $X_{S,N-1} = X_{Naive}$ and $S^c_N(X) = \delta V(X)$. The solutions for the continuation value functions $S^c_n(X)$ for $n = 1, \cdots, N-1$ are presented in the appendix. Here we take the triggers $X_{S,n+1}$ as given, when computing the continuation value function. These triggers are obtained as part of the solution for the entrepreneur's value maximization.

We now formulate the recursive relationship for the entrepreneur's value function $S_n(X)$ using the following differential equation:

$$\frac{1}{2}\sigma^2 X^2 S''_n(X) + \alpha X S'_n(X) - \rho S_n(X) + \lambda\left[S^c_{n+1}(X) - S_n(X)\right] = 0, \quad X \leq X_{S,n}, \tag{31}$$

where the value-matching and smooth-pasting conditions are given by

$$S_n(X_{S,n}) = X_{S,n} - I, \tag{32}$$

$$S'_n(X_{S,n}) = 1. \tag{33}$$

The solutions for the value functions $S_n(X)$ for $n = 0, 1, \cdots, N$ are presented in the appendix. Most importantly, however, are the optimal exercise triggers chosen by each of the selves. The optimal exercise trigger for self n, satisfies the recursive formula

$$X_{S,n} = \frac{\beta_2}{\beta_2 - 1} I + \left(\frac{\beta_2 - \beta_1}{\beta_2 - 1}\right)\delta\left(\frac{X_{S,n}}{X^*}\right)^{\beta_1}(X^* - I) - \frac{1}{\beta_2 - 1}\sum_{i=1}^{N-n-1} i C_{n,i}\, X^{\beta_2}_{S,n}\left(\log X_{S,n}\right)^{i-1}, \tag{34}$$

for $n = 0, 1, \cdots, N-2$, and where the triggers $X_{S,N}$ and $X_{S,N-1}$ are equal to X^* and X_{Naive}, respectively. The constants $C_{n,i}$ are defined in the appendix.

The following proposition demonstrates that each self's trigger value is lower than that of its future self. That is, $X_{S,0} < X_{S,1} < ... < X_{S,N}$. This makes intuitive sense since the time-inconsistency problem will be greater for the earlier selves, as earlier selves have a greater number of future selves whose decisions may detrimentally influence earlier selves' value functions.

Proposition 2 *$X_{S,n}$ is increasing in n.*

For the case of a finite number of selves, we can now easily prove that the sophisticated entrepreneur will exercise earlier than the naive entrepreneur, who in turn will invest earlier than the time-consistent entrepreneur. This is summarized in the following proposition.

Proposition 3 *For the sophisticated entrepreneur with a finite number of selves N, $X_{S,0} < X_{Naive} < X^*$.*

For the sophisticated entrepreneur, each additional future self introduces an extra layer of potentially detrimental exercise behavior from the standpoint of the current self's utility, magnifying the problem of time-inconsistency. In an effort to avoid the detrimental effect of future selves' exercise decisions, the current self finds it optimal to exercise earlier than he otherwise would, in order to lessen the chance of failing to exercise prior to the arrival of his future selves. This will be discussed in greater detail in Section 4.4.

4.3 The Sophisticated Entrepreneur with an Infinite Number of Selves

We have so far fixed the number of selves to a finite number. Although we have delivered the intuition on the effect of hyperbolic discounting on investment decision via the finite N-self model, the model solution may be substantially simplified by proceeding to the case with a countably infinite number of selves. For a fixed number of selves N, we have shown that *(i)* self N chooses the time-consistent investment trigger X^* and *(ii)* the investment trigger for self n is lower than the investment trigger for self $(n+1)$. Given the monotonicity of the investment trigger and the fact that all investment triggers are positive, we may conjecture that the investment trigger for self 0 converges to the steady-state limiting investment trigger, when the total number of selves N goes to infinity.

When we have infinite number of selves, the sophisticated entrepreneur faces the same time-invariant option exercising problem, for any self n. That is, the sophisticated entrepreneur's optimization problem does not depend on n. The stationary solution will involve searching for a fixed-point to the investment exercise problem.[16] Specifically, suppose that all stationary future selves exercise at the trigger X_S. Then, X_S will represent the (intra-personal) equilibrium investment trigger if the current self's optimal exercise trigger, conditional on the fact that future selves will exercise at X_S, is also X_S.

Before solving for the intra-personal equilibrium exercise trigger, we consider the current self's exercise strategy conditional on an assumed future self exercise trigger. Let $\hat{X}$ denote the conjectured exercise trigger by the future selves. Let $\Phi(\hat{X})$ denote the entrepreneur's optimal exercise trigger, as a function of $\hat{X}$, the conjectured exercise trigger chosen by his future selves.

We solve the entrepreneur's investment trigger by working backwards. Let $S(X;\hat{X})$ and $S_c(X;\hat{X})$ denote the entrepreneur's value function and the continuation value function, re-

[16]We here exclusively focus on the most natural Markov perfect equilibrium, in which all selves exercise at the same trigger. However, it is conceivable that other equilibria may exist.

spectively, conditioning on the conjectured exercise trigger $\hat{X}$ chosen by his future selves. As in the previous analysis, first consider the entrepreneur's continuation value function $S_c(X;\hat{X})$. Since all future selves are conjectured to exercise at the same trigger, $\hat{X}$, the continuation value function is therefore given by δ times the present value of receiving the payoff value $\hat{X} - I$, when the entrepreneur exercises at the trigger $\hat{X}$. Using the standard present value analysis with stopping time (Dixit and Pindyck (1994)), we thus have

$$S_c(X;\hat{X}) = \begin{cases} \delta \left(\frac{X}{\hat{X}}\right)^{\beta_1} \left(\hat{X} - I\right), & \text{for } X < \hat{X}, \\ \delta\,(X - I)\,, & \text{for } X \geq \hat{X}. \end{cases} \tag{35}$$

Having derived the continuation value function $S_c(X;\hat{X})$, we now turn to the sophisticated entrepreneur's investment optimization problem. Using the standard argument, we have

$$\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 S(X;\hat{X})}{\partial X^2} + \alpha X \frac{\partial S(X;\hat{X})}{\partial X} - \rho S(X;\hat{X}) + \lambda \left[S_c(X;\hat{X}) - S(X;\hat{X})\right] = 0, \tag{36}$$

for $X \leq \hat{X}$. The differential equation (36) is solved subject to the following value-matching and smooth-pasting conditions

$$S\left(\Phi(\hat{X});\hat{X}\right) = \Phi(\hat{X}) - I, \tag{37}$$

$$\frac{\partial S\left(\Phi(\hat{X});\hat{X}\right)}{\partial X} = 1. \tag{38}$$

We may obtain the intra-personal equilibrium sophisticated exercise trigger, X_S, by substituting the continuation value function $S_c(X;\hat{X})$ given in (35) into the differential equation (36), applying boundary conditions (37) and (38), and solving for the value function $S(X;\hat{X})$ and the exercise trigger $\Phi(\hat{X})$. We may then impose the intra-personal equilibrium condition that all selves exercise at the same trigger: $\Phi(X_S) = X_S$. Define the intra-personal equilibrium value function $S(X;X_S) \equiv S(X)$, we thus obtain the solution of the stationary sophisticated entrepreneur problem:

$$S(X) = \left[\delta \left(\frac{X}{X_S}\right)^{\beta_1} + (1-\delta)\left(\frac{X}{X_S}\right)^{\beta_2}\right](X_S - I)\,, \tag{39}$$

$$X_S = \frac{\beta_1 \delta + \beta_2 (1-\delta)}{(\beta_1 - 1)\delta + (\beta_2 - 1)(1-\delta)}\, I. \tag{40}$$

Note that the value of the sophisticated entrepreneur's option is equal to a weighted average of two time-consistent present value functions, $\left(\frac{X}{X_S}\right)^{\beta_1}(X_S - I)$ and $\left(\frac{X}{X_S}\right)^{\beta_2}(X_S - I)$, where the weights are δ and $(1-\delta)$, respectively. Both present value functions represent the value to a time-consistent entrepreneur of receiving the exercise payoff of $(X_S - I)$ when the payoff value X reaches the trigger X_S. However, the first present value uses the discount rate ρ with the implied option parameter β_1, and the second uses the discount rate $(\rho + \lambda)$ with the implied option parameter β_2.

The numerator of the sophisticated trigger X_S is a weighted average of β_1 and β_2, with δ and $(1 - \delta)$ as respective weights. Similarly, the denominator weights $(\beta_1 - 1)$ and $(\beta_2 - 1)$, with δ and $(1 - \delta)$, respectively. Obviously, we have

$$\frac{\beta_2}{\beta_2 - 1} I < X_S < \frac{\beta_1}{\beta_1 - 1} I \equiv X^*. \tag{41}$$

That is, the equilibrium trigger strategy must lie in between (i) the investment threshold X^* for the time-consistent entrepreneur with discount rate ρ and (ii) the investment threshold for the time-consistent entrepreneur with discount rate $(\rho + \lambda)$. We may easily show that the entrepreneur's investment threshold decreases with the degree of time-inconsistency, in that $\partial X_S / \partial \delta > 0$. Just as in the case with finite selves, the sophisticated entrepreneur invests earlier than the naive entrepreneur. Proposition 4 demonstrates this timing result that is the stationary case analog to Proposition 3.

Proposition 4 *The sophisticated entrepreneur in the stationary case exercises earlier than the naive entrepreneur, who in turn exercises earlier than the time-consistent entrepreneur, in that* $X_S < X_{Naive} < X^*$.

4.4 Discussion

Propositions 3 and 4 demonstrate that time-inconsistent entrepreneurs invest earlier than time-consistent entrepreneurs. Moreover, the sophisticated time-inconsistent entrepreneur invests even earlier than the naive time-inconsistent entrepreneur. In this section we discuss these results and their implications.

The first fundamental result is the precise trade-off between the benefits of waiting to invest and the increased impatience driven by time-inconsistent discounting. In our intertemporal stochastic setting, as is well-known from real options theory, an entrepreneur holds a valuable option to wait. This option to wait is what drives the time-consistent entrepreneur to exercise when the option is sufficiently in the money, as embodied by the distance between X^* and I. Now, when we introduce the time-varying impatience driven by time-inconsistent preferences, we then have a force that counteracts the benefits of waiting for uncertainty to resolve itself. This counteracting force is caused by the current self's motivation to exercise before the future selves take control of the exercise decision, because the payoff to the current self from future exercise is discounted by the factor δ in addition to the conventional exponential discounting. Therefore, the lowered value of the option to wait induces time-inconsistent entrepreneurs to exercise earlier than the time-consistent entrepreneur. time-inconsistency reduces, but does not eliminate, the option value of waiting ($I < X_S < X_{Naive} < X^*$).

The second fundamental result is the distinction between sophisticated and naive entrepreneurs. Sophisticated entrepreneurs invest even earlier than naive entrepreneurs. The intuition is relatively simple. While naive entrepreneurs are optimistic in that they incorrectly forecast that their future selves will behave according to their current preferences, sophisticated entrepreneurs correctly forecast that their future selves will invest suboptimally relative to their current preferences. The realistic pessimism of sophisticated entrepreneurs compels them to invest earlier than naive entrepreneurs, so as to lessen the probability that future selves will take over the investment decision and invest suboptimally. This result is referred to by O'Donoghue and Rabin (1999a) as the "sophistication effect." The fact that sophisticated entrepreneurs are concerned about the suboptimal timing decisions of future selves further erodes the value of their option to wait relative to that of naive entrepreneurs.

Figure 1 plots the option values for the time-consistent, naive and sophisticated entrepreneurs. For each type of entrepreneur, the option value smoothly pastes to the project's net payoff value, $(X - I)$, at the entrepreneur's exercise trigger. For each value of X prior to exercise, the vertical distance between the option value and the payoff value measures the value of the option to wait. Note that at all levels of X prior to exercise, the time-consistent entrepreneur has the most valuable option to wait, followed by the naive entrepreneur and then the sophisticated entrepreneur.

[Insert Figure 1 here.]

5 An Extension: The Flow Payoff Case

While some real world examples may fit in the lump sum payoff setting that we have analyzed, there are other situations under which the investment payoffs are given in flows over time. For time-consistent entrepreneurs, the lump sum and the flow payoff cases are equivalent after adjusting for discounting. However, we show that this seemingly minor alteration generates fundamentally different predictions about investment decisions and provide new economic insights, when entrepreneurs have time-inconsistent preferences.

In the flow payoff case, after the entrepreneur irreversibly exercises his investment option at some stopping time τ, he obtains a perpetual stream of flow payments $\{p(t) : t \geq \tau\}$. Here, the payoffs are assumed to be non-tradable for the same reason as for the lump sum payoff case treated earlier. For example, the flow payoffs may be contingent on the unique skills of the entrepreneur, or there may be moral hazard or adverse selections issues that can undermine the selling of the cash flow stream. Assume that the flow payoff process p follow a geometric Brownian motion process:

$$dp(t) = \alpha p(t)dt + \sigma p(t)dB_t, \tag{42}$$

where we assume $\alpha < \rho$ for convergence. The entrepreneur thus will evaluate the investment project and choose his investment time optimally based on his hyperbolic discounting preference.

Unlike the lump sum case in which the net payoff value upon option exercise is simply given by $(X - I)$, the payoff value for the flow case depends on the entrepreneur's time preferences. Let $M(p)$ denote the present value of the future cash flows. Using the hyperbolic discounting function given in (2), we have

$$M(p) = E\left[\int_0^T e^{-\rho t}p(t)dt + \int_T^\infty \delta e^{-\rho t}p(t)dt\right] = \gamma\frac{p}{\rho - \alpha}, \tag{43}$$

where

$$\gamma = \frac{\rho + \delta\lambda - \alpha}{\rho + \lambda - \alpha} \leq 1, \tag{44}$$

and where T has an exponential distribution with mean $1/\lambda$, and the expectation is taken over the joint distribution of T and $p(t)$.[17] Therefore, the net present value of the payoff from exercise is $M(p) - I$.

If the entrepreneur has time-consistent preferences ($\delta = 1$ or $\lambda = 0$), then the present value is given by $M(p) = p/(\rho - \alpha)$, the standard result. When the entrepreneur has time-inconsistent preferences, the present value $M(p)$ of the flow payoffs is less than that for the time-consistent entrepreneur, in that $\gamma < 1$. A stronger degree of time-inconsistency (manifested by a lower δ or a higher λ) implies a lower present value $M(p)$ as seen in (43). Unlike the lump-sum payoff case, the time-inconsistency not only lowers the option value of waiting, but also reduces the project's payoff value $M(p)$ upon option exercise. Since both

[17] In order for the entrepreneur's problem to make sense in the flow payoff setting, we must restrict the parameter region to ensure the existence of an intra-personal equilibrium. Specifically, it must be the case (in equilibrium) that the current self receives a greater payoff from exercising himself than he would receive from having a future self exercise. In the lump sum payoff case of Section 4.3, this was obvious, since the payoff to the current self was $(X - I)$ if he exercised, while the payoff was $\delta\,(X - I)$ if a future self exercised. In the flow payoff setting it may not always hold. The payoff to the current self is $\gamma\frac{p}{\rho-\alpha} - I$ if he exercises, and $\delta(\frac{p}{\rho-\alpha} - I)$ if a future self exercises. We will ensure that a solution exists by restricting the parameters such that $\gamma\frac{p_s}{\rho-\alpha} - I > \delta \quad \frac{p_s}{\rho-\alpha} - I$, or equivalently $p_s > \frac{(\rho-\alpha)(1-\delta)}{\gamma-\delta}I = (\rho + \lambda - \alpha)\,I$, where p_s is the sophisticated equilibrium trigger that appears in (65). Note that this will also ensure the existence of a naive solution, since Proposition 6 demonstrates that the naive trigger is greater than the sophisticated trigger.

the option value and the project payoff values are lowered by hyperbolic discounting, *a priori*, the time-inconsistent entrepreneur may invest either earlier or later than a time-consistent entrepreneur when his payoffs are given in flow terms.

5.1 The Time-Consistent Entrepreneur

First consider the benchmark case in which all cash flow payoffs are discounted at the constant rate ρ. Let $v(p)$ denote the entrepreneur's value function and p^* be his optimal investment threshold to be determined. By standard arguments, the value function $v(p)$ solves the following differential equation:

$$\frac{1}{2}\sigma^2 p^2 v''(p) + \alpha p v'(p) - \rho v(p) = 0, \quad p \leq p^*, \tag{45}$$

subject to the following value-matching and smooth-pasting conditions:

$$v(p^*) = \frac{p^*}{\rho - \alpha} - I, \tag{46}$$

$$v'(p^*) = \frac{1}{\rho - \alpha}. \tag{47}$$

The value function $v(p)$ is given by

$$v(p) = \begin{cases} \left(\frac{p}{p^*}\right)^{\beta_1} \left(\frac{p^*}{\rho-\alpha} - I\right), & p < p^*, \\ \frac{p}{\rho-\alpha} - I, & p \geq p^*, \end{cases} \tag{48}$$

and the investment exercise trigger p^* is

$$p^* = \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha)\, I. \tag{49}$$

It is immediate to note that the investment threshold expressed in the present value term for the flow payoff case, $p^*/(\rho - \alpha)$, is equal to X^*, the investment threshold for the corresponding lump sum payoff case. This equivalence no longer holds when the entrepreneur has time-inconsistent preferences. We next analyze the time-inconsistent entrepreneur's investment decision when the payoffs are given in flows.

5.2 The Naive Entrepreneur

Now consider the case in which the entrepreneur naively assumes that future selves will behave according to his current preferences. Following the same procedure as in the lump sum payoff case, we first compute the continuation value function and then solve for the value function and the investment trigger.

As in the lump sum payoff case, the naive entrepreneur falsely believes that future selves will exercise at the time-consistent trigger p^*. Using the same argument as the one for the naive entrepreneur with lump sum payoffs, the naive entrepreneur's continuation value function $n_c(p)$ is thus given by

$$n_c(p) = \begin{cases} \delta\left(\frac{p}{p^*}\right)^{\beta_1}\left(\frac{p^*}{\rho-\alpha} - I\right), & \text{if } p < p^*, \\ \delta\left(\frac{p}{\rho-\alpha} - I\right), & \text{if } p > p^*. \end{cases} \tag{50}$$

For the lump sum payoff case, time-inconsistency only lowers the option value of waiting, not the project payoff value upon option exercise. For the flow payoff case, we have shown that the project payoff value $M(p)$ is also lowered by time-inconsistent preferences. It is thus conceivable that hyperbolic discounting may have a stronger effect on the project payoff value than on the option value of waiting. If so, the net effect of hyperbolic discounting on investment may lead to a further delayed investment compared with the benchmark with time-inconsistent preferences. This intuition is consistent with the result in O'Donoghue and Rabin (1999). In their paper, they show that if the benefits are more distant, the agent may procrastinate.

Motivated by these considerations, we conjecture and then later verify that the investment trigger for the naive entrepreneur is larger than the time-consistent investment trigger p^*. Note that the continuation value function $n_c(p)$ given in (50) differs depending on whether p is larger or smaller than p^*. Since we conjecture that the naive entrepreneur's exercise trigger p_{naive} is larger than p^*, we thus naturally need to divide the regions for p into two and compute the corresponding value functions jointly.

Let $n_l(p)$ and $n_h(p)$ denote the naive entrepreneur's value function $n(p)$ for $p < p^*$ and $p \geq p^*$ regions, respectively. Let p_{naive} denote the selected exercise trigger by the naive entrepreneur. As stated earlier, we conjecture and then verify $p_{naive} > p^*$.

First consider the higher region $p \geq p^*$. Following the same argument as in the lump sum payoff case, the value function $n_h(p)$ satisfies:

$$\frac{1}{2}\sigma^2 p^2 n_h''(p) + \alpha p n_h'(p) - \rho n_h(p) + \lambda\left[\delta\left(\frac{p}{\rho-\alpha} - I\right) - n_h(p)\right] = 0, \quad p \geq p^*, \tag{51}$$

where we have used the continuation value function given in (50) in the higher region. The general solution for $n_h(p)$ is thus given by

$$n_h(p) = A_h p^{v_2} + B_h p^{\beta_2} + \frac{\lambda\delta}{(\rho-\alpha)(\rho+\lambda-\alpha)}p - \frac{\lambda\delta I}{\rho+\lambda}, \tag{52}$$

where the coefficients A_h and B_h are to be determined, and v_2 is the negative root of a fundamental quadratic equation[18] and is given by

$$v_2 = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) - \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2(\rho+\lambda)\sigma^2}\right] < 0\,. \tag{53}$$

Finally, the general solution for the value function $n_h(p)$ given in (52) is solved with the following standard value-matching and smooth-pasting conditions:

$$n_h(p_{naive}) = M(p_{naive}) = \gamma\frac{p_{naive}}{\rho-\alpha} - I, \tag{54}$$

$$n_h'(p_{naive}) = M'(p_{naive}) = \frac{\gamma}{\rho-\alpha}. \tag{55}$$

Now consider the lower region $p < p^*$. Based on our conjecture $p_{naive} > p^*$, the naive entrepreneur will not invest in the lower region. By the standard argument, the value function $n_l(p)$ for the lower region satisfies:

$$\frac{1}{2}\sigma^2 p^2 n_l''(p) + \alpha p n_l'(p) - \rho n_l(p) + \lambda\left[\delta\left(\frac{p}{p^*}\right)^{\beta_1}\left(\frac{p^*}{\rho-\alpha} - I\right) - n_l(p)\right] = 0, \quad p < p^*, \tag{56}$$

where we have used the continuation value function for the lower region given in (50). The general solution for $n_l(p)$ is thus given by

$$n_l(p) = \delta\left(\frac{1}{p^*}\right)^{\beta_1}\left(\frac{p^*}{\rho-\alpha} - I\right)p^{\beta_1} + B_l p^{\beta_2}\,, \tag{57}$$

where B_l is a constant to be determined. Finally, we now need to provide boundary conditions for $n_l(p)$, which connect $n_l(p)$ with $n_h(p)$ at the boundary p^*. We require that the value function $n(p)$ is continuously differentiable at p^* (see Dixit (1993), Section 3.8), in that

$$n_l(p^*) = n_h(p^*), \tag{58}$$

$$n_l'(p^*) = n_h'(p^*). \tag{59}$$

We solve the naive entrepreneur's investment trigger p_{naive}, and the three undetermined coefficients A_h, B_h, B_l appearing in the value functions $n_h(p)$ and $n_l(p)$ jointly as shown in the appendix.

We also prove that the naive entrepreneur will invest later than the time-consistent entrepreneur, in that $p^* < p_{naive}$. Therefore, we have verified the presumption for our solution methodology sketched out here.

Proposition 5 *For the flow payoff case, the naive entrepreneur invests later than the time-consistent entrepreneur, in that* $p^* < p_{naive}$.

[18]The fundamental quadratic equation is $\sigma^2\beta(\beta-1)/2 + \alpha\beta - (\rho+\lambda) = 0$. Note that β_2 is the positive root of the same quadratic equation.

5.3 The Sophisticated Entrepreneur

Now, consider the flow payoff case for the sophisticated entrepreneur. For analytical tractability, we analyze the case with an infinite number of selves. However, nothing substantive would change if we instead modeled the case with a finite number of selves, as we have done earlier for the case with lump sum payoffs.

The intra-personal equilibrium trigger for sophisticated entrepreneurs with flow payoffs represents the solution to a fixed-point problem. In a stationary intra-personal equilibrium, the current self's optimal exercise trigger, conditional on an assumed trigger for future selves, must be the same as that of future selves. Let $\hat{p}$ denote the current self's conjectured trigger chosen by future selves. Let $s(p;\hat{p})$ and $s_c(p;\hat{p})$ denote the value function and the continuation value function, respectively, conditioning on the conjectured trigger $\hat{p}$ of future selves.

We first calculate the continuation value function $s_c(p;\hat{p})$. Since all future selves exercise at the same trigger $\hat{p}$ in the stationary setting, using the present value argument, we may compute the continuation value $s_c(p;\hat{p})$ as follows:

$$s_c(p;\hat{p}) = \begin{cases} \delta\left(\frac{p}{\hat{p}}\right)^{\beta_1}\left(\frac{\hat{p}}{\rho-\alpha} - I\right), & \text{for } p < \hat{p}, \\ \delta\left(\frac{p}{\rho-\alpha} - I\right), & \text{for } p \geq \hat{p}. \end{cases} \tag{60}$$

Let $\varphi(\hat{p})$ denote the sophisticated entrepreneur's optimal exercise trigger, expressed as a function of the current self's conjectured investment trigger $\hat{p}$ by future selves. Using the continuation value function $s_c(p)$, we may write the sophisticated entrepreneur's value function as follows:

$$\frac{1}{2}\sigma^2 p^2 \frac{\partial^2 s(p;\hat{p})}{\partial p^2} + \alpha p \frac{\partial s(p;\hat{p})}{\partial p} - \rho s(p;\hat{p}) + \lambda\left[s_c(p;\hat{p}) - s(p;\hat{p})\right] = 0, \quad p \leq \hat{p}, \tag{61}$$

where the value-matching and smooth-pasting conditions are given by

$$s\left(\varphi\left(\hat{p}\right);\hat{p}\right) = M\left(\varphi\left(\hat{p}\right)\right) = \gamma\frac{\varphi(\hat{p})}{\rho-\alpha} - I, \tag{62}$$

$$\frac{\partial s\left(\varphi\left(\hat{p}\right);\hat{p}\right)}{\partial p} = M'\left(\varphi\left(\hat{p}\right)\right) = \frac{\gamma}{\rho-\alpha}. \tag{63}$$

Let p_s denote the intra-personal equilibrium sophisticated exercise trigger. The equilibrium condition requires that all selves of the entrepreneur exercise at the same trigger, in that $\varphi\left(p_s\right) = p_s$. Let $s(p)$ denote the intra-personal equilibrium value function, in that $s(p;p_s) \equiv s(p)$. Solving the differential equation (61) subject to the boundary conditions (62)-(63) and imposing the equilibrium conditions gives the following equilibrium value function $s(p)$ and the equilibrium exercise trigger p_s for the sophisticated entrepreneur:

$$s(p) = \delta\left(\frac{p}{p_s}\right)^{\beta_1}\left(\frac{p_s}{\rho-\alpha}-I\right)+\left[\gamma\frac{p_s}{\rho-\alpha}-I-\delta\left(\frac{p_s}{\rho-\alpha}-I\right)\right]\left(\frac{p}{p_s}\right)^{\beta_2}, \quad (64)$$

$$p_s = (\rho-\alpha)\frac{\beta_1\delta+\beta_2(1-\delta)}{(\beta_2-1)\gamma-(\beta_2-\beta_1)\delta}I. \quad (65)$$

Having analyzed the exercise triggers for the time-consistent, naive and sophisticated entrepreneurs, we now may state the following proposition.

Proposition 6 *For the case with flow payoffs, the naive entrepreneur exercises later than the sophisticated entrepreneur, who exercises later than the time-consistent entrepreneur, in that* $p_{naive} > p_s > p^*$.

5.4 Discussion

As demonstrated by Propositions 5 and 6, the flow payoff case provides very different results from the lump sum payoff case. This result is due to the interaction of two conflicting forces for the flow payoff case. First, as we know from the case with lump-sum payoffs, hyperbolic discounting increases the desire to exercise earlier, as this allows the entrepreneur to protect himself from the "sub-optimal" investment decision of future selves. Second, for the case with flow payoffs, the hyperbolic entrepreneur actually receives a "lower" present value $M(p)$ for the flow payoffs than would a time-consistent agent. This is apparent from the γ parameter that enters the payoff value $M(p)$. This lowered payoff from the current self's exercise motivates the hyperbolic entrepreneur to wait longer before exercising, to justify the investment cost I. We show that the second effect dominates the first effect.[19]

Figure 2 plots the option values for the time-consistent, naive and sophisticated entrepreneurs. Also plotted is the net present values (upon immediate exercise) for the time-consistent entrepreneur, $p/(\rho-\alpha)-I$, and for the time-inconsistent entrepreneurs, $M(p)-I$. For each value of p prior to exercise, the vertical distance between the option value and the payoff value measures the value of the option to wait. Because the time-inconsistent entrepreneur values the project payoff less than the time-consistent entrepreneur ($\gamma < 1$), the time-inconsistent entrepreneur naturally has weaker incentives to exercise the investment option than the time-consistent entrepreneur. Thus, the time-consistent entrepreneur invests

[19] If instead of using an infinite horizon for the cash flows we moved to a finite horizon T, then we would find for a particular finite horizon the two effects would exactly offset each other. That is, there exists a T^* in the flow payment case such that for $T = T^*$ the sophisticated and time-consistent entrepreneurs would exercise at the same time. For $T < T^*$ the sophisticated entrepreneur would exercise earlier than the time-consistent entrepreneur, and for $T \geq T^*$ the sophisticated entrepreneur would exercise later than the time-consistent entrepreneur.

earlier than the time-inconsistent entrepreneur, whether naive or sophisticated. Now we turn to the comparison between the sophisticated and naive entrepreneurs.

As in the lump sum payoff case, the sophisticated entrepreneur invests earlier than the naive entrepreneur. The sophisticated entrepreneur has a greater desire to invest earlier than the naive entrepreneur so as to protect himself against the behavior of future selves due to his belief that his future selves will not behave in his own interest. Therefore, the value of the option to wait for the sophisticated entrepreneur is lower due to the fact that its future selves will exercise at a suboptimal exercise trigger (from the vantage of the current self). For both the lump sum and flow payoff cases, the sophisticated entrepreneur invests earlier than the naive entrepreneur does. This "sophistication effect" is dubbed by O'Donoghue and Rabin (1999) in their analysis of deterministic task completion. Figure 2 confirms our intuition.

[Insert Figure 2 here.]

6 The Interaction of Time-Inconsistent Entrepreneurs: The Case of Competitive Industry Equilibrium

In this section we model the perfectly competitive equilibrium outcome when the industry is comprised of sophisticated hyperbolic entrepreneurs. It is an equilibrium extension of the flow payoff case of Section 5, where the entrepreneurs acted as monopolists. The competitive equilibrium framework that we use is similar to that of Leahy (1993) and Dixit and Pindyck (1994). The key contribution of this section is the extension of the equilibrium to the case with time-inconsistent entrepreneurs.

Consider an industry comprised of a large number of entrepreneurs. Each entrepreneur has the option to irreversibly undertake a single investment by paying an up-front investment cost of I at chosen time τ. Upon investment, the project yields a stream of stochastic (profit) flow of $\{p(s) : s \geq \tau\}$ forever.[20] The industry is perfectly competitive, in that each unit of output is small in comparison with industry supply, $Q(t)$. Thus, each entrepreneur acts as a price taker. The equilibrium price is determined by the condition equating industry supply and demand. Each entrepreneur takes as given the stochastic process of price p. In the rational expectations equilibrium, this conjectured price process will indeed be the market clearing price.

The price of a unit of output is given by the industry's inverse demand curve

$$p(t) = \theta(t) \cdot D\left(Q(t)\right), \tag{66}$$

[20] Without loss of generality and for the simplicity reason, we assume no variable costs of production, and thus the process p represents cash flow process, as in Section 6.

where $D'(Q) < 0$ and $\theta(t)$ is a multiplicative shock and is given by following the geometric Brownian motion process:

$$d\theta(t) = \alpha\theta(t)dt + \sigma\theta(t)dB_t. \tag{67}$$

Over an interval of time in which no entry takes place, $Q(\,\cdot\,)$ is fixed, and thus the price process p evolves as follows:

$$dp(t) = \alpha p(t)dt + \sigma p(t)dB_t. \tag{68}$$

Given the multiplicative shock specification of the demand curve in (66), entry by new entrepreneurs causes the price process to have an upper reflecting barrier. Thus, in this simple setting, each price taking entrepreneur will take the process (68) with an upper reflecting barrier as given. In the rational expectations equilibrium, the entry response by entrepreneurs who assume such a process will lead precisely to the supply process that equates supply and demand.[21]

6.1 Equilibrium with Time-Consistent Entrepreneurs

As a benchmark, consider an industry comprised of time-consistent entrepreneurs. Conjecture that the equilibrium entry will be at the trigger p^*_{eq}, and thus in equilibrium the price process will have an upper reflecting barrier at p^*_{eq}. Consider the value of an active entrepreneur, one that has already paid the entry cost and is producing output. Let $G(p)$ denote the value of an active entrepreneur. By the standard argument, $G(p)$ satisfies the equilibrium differential equation:

$$\frac{1}{2}\sigma^2 p^2 G''(p) + \alpha p G'(p) - \rho G(p) + p = 0, \quad p \leq p^*_{eq}. \tag{69}$$

The impact of the reflecting barrier necessitates the boundary condition[22]:

$$G'(p^*_{eq}) = 0. \tag{70}$$

Similarly, let $F(p)$ denote the value of an inactive entrepreneur, its value prior to investing. By the standard argument, $F(p)$ satisfies the following differential equation:

$$\frac{1}{2}\sigma^2 p^2 F''(p) + \alpha p F'(p) - \rho F(p) = 0, \quad p \leq p^*_{eq}. \tag{71}$$

[21]While we solve for the equilibrium comprised of sophisticated entrepreneurs, we do not construct an equilibrium for the case of naive entrepreneurs. This is due to the problematic nature of defining an equilibrium for naive entrepreneurs. While the literature on naive hyperbolic preferences provides a well-defined notion of a current self's expectations regarding future selves' behavior, there is no standard assumption regarding what naive entrepreneurs forecast for others' current and future selves. For example, do naive entrepreneurs believe that other entrepreneurs possess self control, or do they believe that only they themselves possess self control? The implications for either assumption make for a very complex equilibrium.

[22]See Malliaris and Brock (1982, p. 200).

The inactive entrepreneur's investment trigger is determined by value-matching and smooth-pasting conditions. In equilibrium, the entry trigger must equal the conjectured reflecting barrier p_{eq}^*, in that

$$F\left(p_{eq}^*\right) = G(p_{eq}^*) - I, \tag{72}$$

$$F'\left(p_{eq}^*\right) = G'(p_{eq}^*). \tag{73}$$

The solution to this equilibrium system is:

$$F(p) = 0, \tag{74}$$

$$G(p) = -\frac{I}{\beta_1 - 1}\left(\frac{p}{p_{eq}^*}\right)^{\beta_1} + \frac{p}{\rho - \alpha}, \quad p \leq p_{eq}^*, \tag{75}$$

$$p_{eq}^* = p^* = \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha)\, I, \tag{76}$$

with a price process governed by a geometric Brownian motion process (68) with a reflecting barrier at p^*.

The equilibrium is clearly very intuitive. Free entry ensures that the value of an inactive entrepreneur is zero. The value of an active entrepreneur is equal to the present value of future cash flows, where the reflecting barrier ensures that the value of an active entrepreneur at entry is equal to the cost of entry, $G(p_{eq}^*) = I$. Finally, as has been demonstrated by Leahy (1993) and others, the exercise trigger for a perfectly competitive industry equals the monopolist trigger p^*. The intuition is that the reflecting barrier has two exactly opposing effects: it lowers the value of the payoff from exercise (since the future cash flow is capped at the barrier), while it also lowers the option value of waiting.

6.2 Equilibrium with Sophisticated Entrepreneurs

We now consider the equilibrium for an industry, which is comprised of time-inconsistent entrepreneurs with sophisticated beliefs. Conjecture that the equilibrium entry will occur at the trigger p_{eq}, and thus in equilibrium the price process will have an upper reflecting barrier at p_{eq}. Consider the value of an active entrepreneur, one that has already paid the entry cost and is producing output. Let $g(p)$ and $g_c(p)$ denote the value function and the continuation value function of an active entrepreneur, respectively.

First consider the solution for the continuation value function, $g_c(p)$. Following the same argument used earlier, $g_c(p)$ satisfies the following differential equation:

$$\frac{1}{2}\sigma^2 p^2 g_c''(p) + \alpha p g_c'(p) - \rho g_c(p) + \delta p = 0, \quad p \leq p_{eq}, \tag{77}$$

subject to the boundary condition at the upper reflecting barrier p_{eq}:

$$g_c'(p_{eq}) = 0. \tag{78}$$

Solving (77) subject to (78) gives the solution for the continuation value:

$$g_c(p) = -\frac{\delta}{\rho-\alpha}\frac{p_{eq}}{\beta_1}\left(\frac{p}{p_{eq}}\right)^{\beta_1} + \frac{\delta p}{\rho-\alpha}, \quad p \le p_{eq}. \tag{79}$$

Now we turn to the solution of the value function $g(p)$ for an active entrepreneur. By the standard argument, $g(p)$ satisfies the following differential equation:

$$\frac{1}{2}\sigma^2 p^2 g''(p) + \alpha p g'(p) - \rho g(p) + p + \lambda[g_c(p) - g(p)] = 0, \quad p \le p_{eq}, \tag{80}$$

where $g_c(p)$ is the continuation value given in (79). The impact of the reflecting barrier necessitates the boundary condition:

$$g'(p_{eq}) = 0. \tag{81}$$

Just as in the case of the previously derived equilibrium with time-consistent entrepreneurs, free-entry will ensure that the value of an active entrepreneur will equal the cost of investment at the entry trigger[23]:

$$g(p_{eq}) = I. \tag{82}$$

The equilibrium value function $g(p)$ and the investment trigger for an active entrepreneur are thus given by

$$g(p) = \frac{1}{\beta_2}\frac{p_{eq}}{\rho-\alpha}(\delta-\gamma)\left(\frac{p}{p_{eq}}\right)^{\beta_2} - \frac{\delta}{\rho-\alpha}\frac{p_{eq}}{\beta_1}\left(\frac{p}{p_{eq}}\right)^{\beta_1} + \gamma\frac{p}{\rho-\alpha}, \quad p \le p_{eq}, \tag{83}$$

$$p_{eq} = \frac{\beta_2\beta_1}{\gamma(\beta_2-1)\beta_1 - \delta(\beta_2-\beta_1)}(\rho-\alpha)I. \tag{84}$$

In the following proposition, we show that the competitive equilibrium trigger for sophisticated entrepreneurs is greater than that for time-consistent entrepreneurs. That is, industries comprised of time-consistent entrepreneurs will have more rapid growth than industries comprised of sophisticated entrepreneurs with time-inconsistent preferences.

Proposition 7 *In the flow payoff case, the competitive equilibrium trigger p_{eq} for sophisticated entrepreneurs is greater than the competitive equilibrium trigger p^*_{eq} for time-consistent entrepreneurs, in that $p_{eq} > p^*_{eq}$.*

[23] While we could explicitly derive the value of an inactive sophisticated firm, with free-entry this value will always equal zero.

Given that in the flow payoff case the investment trigger for the sophisticated entrepreneur is greater than that for the time-consistent entrepreneur, it is not surprising that sophisticated entrepreneurs also procrastinate, compared with time-consistent entrepreneurs in equilibrium. Sophisticated entrepreneurs must discount the flow payments from entry received by future selves by an additional factor δ, reducing the net payoff values occuring after exercise and thus decreasing their incentives to invest.

While in the time-consistent equilibrium it is the case that the monopoly and competitive equilibrium triggers coincide (that is, $p^*_{eq} = p^*$), this is not the case for sophisticated entrepreneurs. In the following proposition we demonstrate that the sophisticated equilibrium trigger is below the sophisticated monopoly trigger. This is an interesting result, since it demonstrates that the Leahy (1993) result on the equivalence between the monopoly and competitive equilibrium triggers does not survive the extension to time-inconsistent preferences. The key reason for this is due to the fact that in equilibrium, the time-inconsistent entrepreneur competes both interpersonally (against competitors) and intrapersonally (against future selves), while the time-consistent entrepreneur competes only interpersonally.

Proposition 8 *The competitive equilibrium trigger p_{eq} for sophisticated entrepreneurs is lower than the monopoly trigger p_s for sophisticated entrepreneurs, in that $p_{eq} < p_s$.*

The intuition for this result is as follows. As in Leahy (1993), with time-consistent agents, competitive equilibrium introduces two offsetting changes to the monopoly entrepreneur problem. First, equilibrium competition places an upper bound on cash flows (through the reflecting barrier). This effect, taken by itself, makes exercise less valuable and pushes the equilibrium entry trigger above the monopoly trigger. Second, the free-entry condition of equilibrium eliminates the value of the option to wait. This effect, taken by itself, pushes the equilibrium entry trigger below the monopoly trigger. For the case of time-consistent agents, these two effects precisely cancel each other out, leading to an equilibrium trigger equal to the monopoly trigger. Now, with the sophisticated time-inconsistent agent, the second effect dominates. Recall that in the flow payment case, the value of the sophisticated entrepreneur's option to wait is greater than that for the time-consistent entrepreneur due to the increased discounting of future cash flows. Therefore, the impact of the free-entry condition's elimination of the option to wait has a greater impact for the sophisticated entrepreneur, leading the equilibrium trigger to be below that of the monopoly trigger.

7 Conclusion

This paper extends the real options framework to account for time-inconsistent preferences. Entrepreneurs need to formulate their investment decisions taking into account their beliefs about the behavior of their future selves. This sets up a conflict between two opposing forces: the desire to take advantage of the option to wait, and the desire to invest early to avoid allowing future selves to take over the investment decision. We find that the precise trade-off between these two forces depends on such factors as whether entrepreneurs are sophisticated or naive in their expectations regarding their future time-inconsistent behavior, as well as whether the payoff from investment occurs all at once or over time. We extend the model to consider equilibrium investment behavior for an industry comprised of time-inconsistent entrepreneurs. Equilibrium involves the dual problem of agents playing dynamic games against competitors as well as against their own future selves.

Several further extensions of the model would prove interesting. First, the model could be extended to account for intermediate cases between the extremes of perfectly naive or perfectly sophisticated entrepreneurs. While the naive entrepreneur is fully unaware of his future self-control problems, the sophisticated entrepreneur is fully aware of his future self-control problems. O'Donoghue and Rabin (2001) provide a model of partial naivete, where an agent is aware of his future self-control problems, but underestimates its degree of magnitude. Second, this paper provides results for both the monopolist and perfectly competitive settings. This could be extended to the case of oligopolistic equilibrium in the manner of Grenadier (2002). Finally, the equilibrium could be further extended to account for industries made up of both time-consistent and time-inconsistent entrepreneurs.

Appendices

A Proofs

Proof of Proposition 1. We search for the fixed point solution for $f(x) = x$, where

$$f(z) = \frac{1}{\beta_2 - 1}\left[\beta_2 I + (\beta_2 - \beta_1)\,\delta\left(\frac{z}{X^*}\right)^{\beta_1}(X^* - I)\right]. \tag{A.1}$$

It is immediate to note that $f(z)$ is increasing and convex in z. Moreover,

$$\begin{aligned} f(X^*) &= \frac{1}{\beta_2 - 1}\left[\beta_2 I + (\beta_2 - \beta_1)\,\delta\,(X^* - I)\right] \\ &< \frac{1}{\beta_2 - 1}\left[\beta_2 I + (\beta_2 - \beta_1)(X^* - I)\right] = X^*, \end{aligned} \tag{A.2}$$

where the last equality follows from simplification. Since $f(0) = \frac{\beta_2}{\beta_2 - 1} I > 0$, $f(X^*) < X^*$, and $f'(z) > 0$, there exists a unique $X_{Naive} < X^*$ such that $f(X_{Naive}) = X_{Naive}$. ■

Proof of Proposition 2. We use the method of mathematical induction. First, we verify that $X_{S,N-1} < X_{S,N}$, $S^c_{N-1}(X) < S^c_N(X)$, and $S_{N-1}(X) < S_N(X)$, by using their analytical expressions. Now suppose $X_{S,n} < X_{S,n+1}$, $S^c_n(X) < S^c_{n+1}(X)$ and $S_n(X) < S_{n+1}(X)$ hold for some $1 \leq n \leq N-1$. Our objective is then to show $X_{S,n-1} < X_{S,n}$, $S^c_{n-1}(X) < S^c_n(X)$ and $S_{n-1}(X) < S_n(X)$ hold for the same n. By the logic of induction, we have then completed the proof.

Consider the differential equation (31) and boundary conditions (32)-(33) for the value function $S_n(X)$. We my view $S_n(X)$ as the value of an asset with a dividend flow payment of $\lambda S^c_{n+1}(X)$, and a terminal payout of $X - I$ at the first passage time to a trigger value $X_{S,n}$ determined by the smooth-pasting optimality condition. This asset is thus an American option that promises a dividend payout while unexercised. A similar characterization can be made for the value function $S_{n-1}(X)$. The only difference is that the dividend flow payment for the asset with value $S_{n-1}(X)$ is $\lambda S^c_n(X)$, which is lower than the dividend flow payment $\lambda S^c_{n+1}(X)$ for the asset with value $S_n(X)$ following the previous conjecture. Comparing two American options where one has a higher dividend payment than the other while unexercised, we know that the former one with higher dividend will be exercised later, *ceteris paribus*. Therefore, $X_{S,n-1} < X_{S,n}$. As a result, the option value for the one with lower dividend payment will be smaller, in that $S_{n-1}(X) < S_n(X)$.

Now, consider the continuation value function $S^c_n(X)$. From the differential equation (29) and the boundary condition (30), we may view $S^c_n(X)$ as the value of an asset a dividend flow payment of $\lambda S^c_{n+1}(X)$, (discounted at the rate of $\rho + \delta$), and a terminal value of $\delta\,(X_{S,n} - I)$ at the first moment the given trigger value $X_{S,n}$ is reached. This is very similar to the payouts for the asset with value $S_n(X)$: it has the same dividend flow payments, but a different terminal payout which is discounted by δ. We can express the asset value $S_n^c(X)$ as δ times the asset $S_n(X)$, plus the present value of the dividend flow $(1-\delta)\lambda S_{n+1}^c(X)$ until the time trigger $X_{S,n}$ is reached. Similarly, we can express the asset value $S_{n-1}^c(X)$as δ times the asset value $S_{n-1}(X)$, plus the present value of the dividend flow $(1-\delta)\lambda S_n^c(X)$ until the time trigger $X_{S,n-1}$ is reached. From this decomposition, we can see that asset value $S_n^c(X)$ dominates asset value $S_{n-1}^c(X)$ as follows. First, we have shown in the above that $\delta S_n(X) > \delta S_{n-1}(X)$. Second, by assumption we have that $S_{n+1}^c(X) > S_n^c(X)$ and $X_{S,n+1} > X_{S,n}$, the present value of receiving $(1-\delta)\lambda S_{n+1}^c(X)$ until the trigger $X_{S,n+1}$ is reached is greater than the present value of receiving $(1-\delta)\lambda S_n^c(X)$ until the trigger $X_{S,n}$ is reached. Therefore, we may conclude that $S_n^c(X) > S_{n-1}^c(X)$. ■

Proof of Proposition 3. *For the sophisticated entrepreneur, $X_N = X^*$, and $X_{N-1} = X_{Naive}$. From Proposition 1, $X_{Naive} < X^*$. From Proposition 2, $X_{S,n}$ is increasing in n, and thus $X_{S,0} < X_{S,N-1} = X_{Naive}$. Therefore, we have $X_{S,0} < X_{Naive} < X^*$.* ■

Proof of Proposition 4. Since Proposition 1 has shown $X_{Naive} < X^*$, it is thus sufficient to show that $X_S < X_{Naive}$. Define

$$f(x;a) = -x + \left[\frac{\beta_2}{\beta_2 - 1} I + \left(\frac{\beta_2 - \beta_1}{\beta_2 - 1}\right) \delta \left(\frac{x}{a}\right)^{\beta_1} (a - I)\right], \quad x \le a. \tag{A.3}$$

By construction, X_S solves $f(x;X_S) = 0$, in that $f(X_S;X_S) = 0$, and X_{Naive} solves $f(x;X^*) = 0$, in that $f(X_{Naive};X^*) = 0$. Let $x(a)$ denote the solution to (A.3), in that $f(x(a);a) = 0$. By the implicit function theorem, we have

$$\frac{dx(a)}{da} = -\frac{f_a(x(a);a)}{f_x(x(a);a)}. \tag{A.4}$$

Equation (A.3) implies $f_a(x;a) > 0$ for $a \le X^*$, and $f_{xx}(x;a) > 0$. Evaluating $f_x(x;a)$ at the boundary $x = a$ gives

$$\begin{aligned} \frac{d}{dx} f(a;a) &= -1 + \delta \left(1 - \frac{I}{a}\right) \beta_1 \left(\frac{\beta_2 - \beta_1}{\beta_2 - 1}\right) \\ &< -1 + \delta \left(1 - \frac{\beta_1 - 1}{\beta_1}\right) \beta_1 \left(\frac{\beta_2 - \beta_1}{\beta_2 - 1}\right) \\ &= -\frac{1}{(\beta_2 - 1)} (\beta_2 - 1)(1 - \delta) + (\beta_1 - 1)\delta] < 0, \end{aligned} \tag{A.5}$$

where the inequality follows from $a \le X^*$. Jointly, $f_{xx}(x;a) > 0$ and $f_x(a;a) < 0$ imply that $f_x(x;a) < 0$ for $x \le a$. Thus, we have $x'(a) < 0$. Since $X^* > X_S$, we may then conclude that $X_S < X_{Naive}$.

Proof of Propositions 5 and 6. First, we show $p_s > p^*$. Re-arranging the terms in p_s and p^* gives

$$\frac{\beta_2}{\beta_2 - 1}(\rho + \lambda - \alpha) > \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha). \tag{A.6}$$

Define the functional mapping from the discount rate ρ to the parameter β using the following familiar fundamental quadratic:

$$\frac{\sigma^2}{2}\beta(\rho)(\beta(\rho) - 1) + \alpha\beta(\rho) - \rho = 0. \tag{A.7}$$

Therefore, to prove (A.6) is equivalent to show

$$\frac{d}{d\rho}\left(\frac{\beta(\rho)}{\beta(\rho) - 1}(\rho - \alpha)\right) = \frac{d}{d\rho}\left(\frac{\sigma^2}{2}\beta(\rho)^2 + \alpha\beta(\rho)\right) = \left(\sigma^2\beta(\rho) + \alpha\right)\frac{d\beta(\rho)}{d\rho} > 0, \tag{A.8}$$

where the first equality uses (A.7). Since

$$\frac{d\beta(\rho)}{d\rho} = \left[\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\rho\right]^{-1/2} > 0, \tag{A.9}$$

we thus have shown $p_s > p^*$.

We now show $p_{naive} > p_s$. The four boundary conditions in (58), (59), (54), and (55) can be written as:

$$\begin{aligned}
A_h p^{*v_2} + B_h p^{*\beta_2} + \frac{\lambda\delta}{(\rho-\alpha)(\rho+\lambda-\alpha)}p^* - \frac{\lambda\delta I}{\rho+\lambda} &= \delta\left(\frac{p^*}{\rho-\alpha} - I\right) + B_l p^{*\beta_2} \\
v_2 A_h p^{*v_2} + \beta_2 B_h p^{*\beta_2} + \frac{\lambda\delta}{(\rho-\alpha)(\rho+\lambda-\alpha)}p^* &= \beta_1\delta\left(\frac{p^*}{\rho-\alpha} - I\right) + \beta_2 B_l p^{*\beta_2} \\
A_h p_{naive}^{v_2} + B_h p_{naive}^{\beta_2} + \frac{\lambda\delta}{(\rho-\alpha)(\rho+\lambda-\alpha)}p_{naive} - \frac{\lambda\delta I}{\rho+\lambda} &= \gamma\frac{p_{naive}}{\rho-\alpha} - I \\
v_2 A_h p_{naive}^{v_2} + \beta_2 B_h p_{naive}^{\beta_2} + \frac{\lambda\delta}{(\rho-\alpha)(\rho+\lambda-\alpha)}p_{naive} &= \gamma\frac{p_{naive}}{\rho-\alpha}.
\end{aligned} \tag{A.10}$$

Simplification of the above four equations gives

$$(\beta_2 - v_2)A_h p^{*v_2} = -\frac{\lambda\delta(\beta_2 - 1)}{(\rho-\alpha)(\rho+\lambda-\alpha)}p^* + \frac{\beta_2\lambda\delta I}{\rho+\lambda} + (\beta_2 - \beta_1)\delta\left(\frac{p^*}{\rho-\alpha} - I\right), \tag{A.11}$$

$$(\beta_2 - v_2)A_h p_{naive}^{v_2} = -\frac{\lambda\delta(\beta_2 - 1)}{(\rho-\alpha)(\rho+\lambda-\alpha)}p_{naive} + \frac{\beta_2\lambda\delta I}{\rho+\lambda} + (\beta_2 - 1)\gamma\frac{p_{naive}}{\rho-\alpha} - \beta_2 I. \tag{A.12}$$

First, we show $A_h > 0$ by demonstrating that the right side of (A.11) is positive. It is sufficient to show

$$\frac{\beta_1}{\beta_1 - 1}\left(\frac{\rho-\alpha}{\rho}\right) > \left(\frac{\rho+\lambda-\alpha}{\rho+\lambda}\right)\frac{\beta_2}{\beta_2 - 1}. \tag{A.13}$$

Let

$$k(\rho) = \frac{\beta(\rho)}{\beta(\rho) - 1}\left(\frac{\rho-\alpha}{\rho}\right) = 1 + \frac{\sigma^2/2}{\sigma^2(\beta(\rho) - 1)/2 + \alpha}, \tag{A.14}$$

where the second equality uses the fundamental quadratic (A.7). Therefore, we have $k'(\rho) < 0$ since $\beta'(\rho) > 0$. Hence, we have proved (A.13), and $A_h > 0$.

Define the function $h(p)$ as

$$h(p) = (\beta_2 - v_2) A_h p^{v_2}. \tag{A.15}$$

Note that $h(p)$ is a decreasing and convex function, with $h(0) = \infty$ and $h(\infty) = 0$. The left sides of (A.11) and (A.12) are equal to $h(p^*)$ and $h(p_{naive})$, respectively. The right sides of (A.11) and (A.12) are respectively $k_1(p^*)$, and $k_2(p_{naive})$, where

$$k_1(p) = -\frac{\lambda\delta(\beta_2 - 1)}{(\rho-\alpha)(\rho+\lambda-\alpha)} p + \beta_2 \frac{\lambda\delta I}{\rho+\lambda} + (\beta_2 - \beta_1)\delta\left(\frac{p}{\rho-\alpha} - I\right), \tag{A.16}$$

$$k_2(p) = -\frac{\lambda\delta(\beta_2 - 1)}{(\rho-\alpha)(\rho+\lambda-\alpha)} p + \beta_2 \frac{\lambda\delta I}{\rho+\lambda} + (\beta_2 - 1)\gamma\frac{p}{\rho-\alpha} - \beta_2 I. \tag{A.17}$$

Moreover, p_s is the unique solution for $k_1(p) = k_2(p)$. Note that $k_1(p)$ is decreasing ($k_1'(p) < 0$) with $k_1(0) > 0$; and $k_2(p)$ is increasing ($k_2'(p) > 0$) with $k_2(0) < 0$.

Define $w(p) = h(p) - k_1(p)$. We know that $w(0) = \infty$, $w(p^*) = 0$, $w(\infty) < 0$, and $w''(p) > 0$. Thus, p^* must be a unique root of $w(p)$. This implies that the graph of $h(p)$ must be tangent to the line of $k_1(p)$ at their point of intersection, p^*.

Using the properties of the curve h and the lines k_1 and k_2, we can see graphically from Figure 3 that the tangency point p^* must be to the left of p_s, where k_1 intersects k_2, since $p^* < p_s$. Finally, p_{naive} must be greater than p_s as $h(p)$ will intersect k_2 at a point to the right of p_s. Therefore, $p^* < p_s < p_{naive}$. ■

Proof of Proposition 7. The inequality $p_{eq} > p_{eq}^*$ may be equivalently written as $\beta_1\beta_2\varphi(\lambda) > 0$, where

$$\varphi(\lambda) = \lambda + \frac{\rho-\alpha}{\beta_2} - \frac{\rho+\lambda-\alpha}{\beta_1}. \tag{A.18}$$

Note that β_2 depends on λ. We now show that $\varphi(\lambda) > 0$. We have

$$\begin{aligned}
\varphi'(\lambda) &= (\beta_1 - 1)\left[\frac{1}{\beta_1} - \frac{1}{\beta_2}\left(\frac{1}{2}\sigma^2\beta_2^2 + \rho + \lambda\right)^{-1}\left(\frac{\sigma^2}{2}\beta_1 + \alpha\right)\right] \\
&> (\beta_1 - 1)\left[\frac{1}{\beta_1} - \frac{1}{\beta_2}\left(\frac{1}{2}\sigma^2\beta_1 + \rho + \lambda\right)^{-1}\left(\frac{\sigma^2}{2}\beta_1 + \alpha\right)\right] \\
&> (\beta_1 - 1)\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) > 0,
\end{aligned} \tag{A.19}$$

using $\rho + \lambda > \alpha$, and $\beta_2 > \beta_1$. With $\varphi(0) = 0$, we thus have $\varphi(\lambda) > 0$ and $p_{eq} > p_{eq}^*$. ■

Proof of Proposition 8. The inequality $p_{eq} < p_s$ can be written as

$$\frac{\beta_2\beta_1}{\gamma\left(\beta_2-1\right)\beta_1-\delta\left(\beta_2-\beta_1\right)}(\rho-\alpha)I < \frac{\beta_1\delta+\beta_2\left(1-\delta\right)}{\left(\beta_2-1\right)\gamma-\left(\beta_2-\beta_1\right)\delta}(\rho-\alpha)I\,, \tag{A.20}$$

or equivalently as $\varphi(\lambda) > 0$, where $\varphi(\lambda)$ defined in (A.18) is shown to be positive in the proof of Proposition 7. ■

B Solutions Details for the Sophisticated Entrepreneur with Any Finite Number of Selves

Here we provide the solution details for the sophisticated entrepreneur's optimization problem for the case with any finite number of selves, analyzed Subsection 4.2. We first verify the recursion for the continuation value function $S_{n+1}^c(X)$ and then for the value function $S_n^c(X)$.

Solving for the continuation value function $S_{n+1}^c(X)$. Consider self n's continuation value function $S_{n+1}^c(X)$. For notational convenience, let $n = N-(j+1)$. We conjecture that $S_{n+1}^c(X) = S_{N-j}^c(X)$ is given by

$$S_{N-j}^c(X) = \delta\left(\frac{1}{X^*}\right)^{\beta_1}\left(X^*-I\right)X^{\beta_1} + \sum_{i=0}^{j-1} C_{N-j,i}\left(\log X\right)^i X^{\beta_2}, \tag{B.1}$$

for $j = 1, 2, \cdots, N-1$, where the coefficients $C_{N-j,i}$ are to be determined later. Since the solution details for self $(N-1)$ and self N are already provided in the text, here we start the induction from self $(N-2)$.

We first show that (B.1) gives the correct continuation value function $S_{N-1}^c(X)$ for self $(N-2)$. Using the same analysis as in Section 4.1 for the three-self model, we show that the continuation value function S_{N-1}^c for self $(N-2)$ satisfies the conjecture (B.1), where

$$C_{N-1,0} \;=\; \delta\left[X_{S,N-1} - I - \left(\frac{X_{S,N-1}}{X^*}\right)^{\beta_1}\left(X^*-I\right)\right]\left(\frac{1}{X_{S,N-1}}\right)^{\beta_2}, \tag{B.2}$$

and $X_{S,N-1} = X_{Naive}$, the naive entrepreneur's exercise trigger given in (15).

Using the induction logic, we next postulate that the continuation value function $S_{N-j}^c(X)$ for self $(N-(j+1))$ takes the form of (B.1), and then verify that the continuation value function $S_{N-(j-1)}^c(X)$ for self $(N-j)$ also takes the form of (B.1). We substitute the conjectured continuation value function (B.1) and its first two derivatives into (29), sort terms by $X^{\beta_2}\left(\log X\right)^k$ for each k, and then set the coefficients for each term $X^{\beta_2}\left(\log X\right)^k$, where

$k = 0, 1, \cdots, j-1$. After doing some algebraic work, we have the following relationship for the coefficients:

$$
\begin{aligned}
0 &= \frac{\sigma^2}{2}\left[\left(2\beta_2 - 1\right)(k+1)\, C_{N-j,k+1} + (k+2)(k+1)\, C_{N-j,k+2}\right] + \alpha\,(k+1)\, C_{N-j,k+1} \\
&\quad + \lambda C_{N-j+1,k}, \qquad \text{for } k = 0, 1, \cdots, j-1
\end{aligned}
\tag{B.3}
$$

Note that terms involving X^{β_1} automatically satisfy the valuation equation (29).

Let

$$
\eta = -\left(\frac{\sigma^2}{2}\left(2\beta_2 - 1\right) + \alpha\right)^{-1} = -\left(\frac{\beta_2}{\rho + \lambda + \ \sigma^2 \beta_2^2/2}\right), \tag{B.4}
$$

where the second equality follows from (A.7). Equation (B.3) thus may be written as

$$
C_{N-j,k+1} = \eta\left[\frac{\sigma^2}{2}(k+2)\, C_{N-j,k+2} + \frac{\lambda C_{N-j+1,k}}{k+1}\right]. \tag{B.5}
$$

Note that $C_{N-j,k} = 0$ for $k \geq j$ (by the conjecture (B.1) and the fact $C_{N-1,1} = 0$). Solving the recursion gives the following formula for $C_{N-j,k}$; $1 \leq k \leq j-1$,

$$
C_{N-j,k} = \frac{\lambda}{k}\left[\eta \sum_{n=0}^{j-k-2}\left(\frac{\sigma^2 \eta}{2}\right)^n C_{N-j+1,k+n} \Pi_{m=0}^{n}\,(k+m) + \eta C_{N-j+1,k-1}\right], \tag{B.6}
$$

for $k = 1, 2, \cdots, j-1$. We may solve for $C_{N-j,0}$ from the value-matching condition (30) for the continuation value function:

$$
\delta\left(\frac{1}{X^*}\right)^{\beta_1}(X^* - I)\, X_{N-j}^{\beta_1} + \sum_{i=0}^{j-1} C_{N-j,i}\left(\log X_{N-j}\right)^i X_{N-j}^{\beta_2} = \delta\left(X_{N-j} - I\right), \tag{B.7}
$$

where the exercise trigger X_{N-j} for self $(N-j)$, is obtained by maximizing the value function $S_{N-j}(x)$ for self $(N-j)$. This is to which we now turn.

Solving for the value function $S_n(X)$. We conjecture that the value function $S_{N-(j+1)}(X)$ for self $n = N-(j+1)$ is given by

$$
S_{N-(j+1)}(X) = \delta\left(\frac{1}{X^*}\right)^{\beta_1}(X^* - I)\, X^{\beta_1} + \sum_{i=0}^{j} G_{N-(j+1),i}\left(\log X\right)^i X^{\beta_2}, \tag{B.8}
$$

for $j = 0, 1, \cdots, N-1$, where the coefficients $G_{N-(j+1),i}$ are to be determined later.

First, Section 4.2 shows that the value function $S_{N-1}(X)$ for self $(N-1)$ is given by $S_{N-1}(X) = N(X)$, where $N(X)$ is the naive entrepreneur's value function given in (14). Therefore, conjecture (B.8) applies to value function $S_{N-1}(X)$ for self $(N-1)$, with

$$
G_{N-1,0} = \frac{\beta_1 - 1}{\beta_2 - \beta_1}\left(X^* - X_{Naive}\right)\left(\frac{1}{X_{Naive}}\right)^{\beta_2}. \tag{B.9}
$$

Similar to the analysis for the continuation value function, we next postulate that the value function $S_{N-j}(X)$ of self $(N-j)$ takes the form of (B.8), and then verify that the value function $S_{N-(j-1)}(X)$ of self $(N-j)$ also satisfies (B.8). We substitute the conjectured value function (B.1), value function $S_n(X)$ and its first two derivatives into (31), sort terms by $X^{\beta_2}(\log X)^k$ for each k, and then set the coefficients, for each term $X^{\beta_2}(\log X)^k$, where $k = 0, 1, \cdots, j$, to zero. After doing some algebraic work, we have the following relationship for the coefficients:

$$\begin{aligned} 0 &= \frac{\sigma^2}{2}\left[(2\beta_2 - 1)(k+1)\,G_{N-(j+1),k+1} + (k+2)(k+1)\,G_{N-(j+1),k+2}\right] \\ &\quad + \alpha(k+1)\,G_{N-(j+1),k+1} + \lambda C_{N-j,k}, \qquad \text{for } k = 0, \cdots, j \end{aligned} \tag{B.10}$$

Note that terms involving X^{β_1} automatically satisfy the valuation equation (31).

Equation (B.10) thus may be written as

$$G_{N-(j+1),k+1} = \eta\left[\frac{\sigma^2}{2}(k+2)\,G_{N-(j+1),k+2} + \frac{\lambda C_{N-j,k}}{k+1}\right], \tag{B.11}$$

where η is given in (B.4). Solving the above recursion (B.11) gives

$$G_{N-(j+1),k} = \frac{\lambda}{k}\left[\frac{1}{2}\sigma^2\eta^2\sum_{n=0}^{j-k-1}\eta^n C_{N-j,k+n}\Pi_{m=0}^{n}(k+m) + \eta C_{N-j,k-1}\right], \tag{B.12}$$

for $1 \leq k \leq j$. Finally, we solve $X_{N-(j+1)}$ and $G_{N-(j+1),0}$ by using the value-matching and smooth-pasting conditions for the value function. The investment trigger $X_{S,n}$ for self n is given in (34), and

$$G_{N-(j+1),0} = C_{N-(j+1),0} + X_{N-(j+1)}^{-\beta_2}(1-\delta)\left(X_{N-(j+1)} - I\right). \tag{B.13}$$

References

[1] Ainslie, G. W., 1992, "Picoeconomics" Cambridge University Press, Cambridge, UK.

[2] Akerlof, G., 1991, "Procrastination and obedience," *American Economic Review* 81, 1-19.

[3] Ariely, D., and K. Wertenbroch, 2002, "Procrastination, deadlines, and performance: Self-control by precommitment," *Psychological Science* 13, 219-224.

[4] Barro, R. J., 1999, "Ramsey meets Laibson in the neoclassical growth model," *Quarterly Journal of Economics* 114, 1125-52.

[5] Brennan, M. J., and E. Schwartz, 1985, "Evaluating natural resource investments," *Journal of Business* 58, 135-157.

[6] Brocas I., and J. D. Carrillo, 2004, "Entrepreneurial boldness and excessive investment," *Journal of Economics and Management Strategy* 13, 321-350.

[7] DellaVigna, S., and U. Malmendier, 2003, "Overestimating self-control: Evidence from the health club industry," working paper, Stanford University.

[8] DellaVigna, S., and U. Malmendier, 2004, "Contract design and self-control: Theory and evidence," *Quarterly Journal of Economics* 119, 353-402.

[9] Dixit, A. K., 1989, "Entry and exit decisions under uncertainty," *Journal of Political Economy* 97, 620-638.

[10] Dixit, A. K., 1993, "The art of smooth pasting," Vol. 55 in *Fundamentals of Pure and Applied Economics*, eds. Lesourne, J., and H. Sonnenschein, Harwood Academic Publishers, Chur, Switzerland.

[11] Dixit, A. K., and R. S. Pindyck, 1994, "Investment under uncertainty," Princeton University Press, Princeton, New Jersey.

[12] Gose, J., 2001, "Investors prefer urban office space to suburban (again)," *Barron's* April 4, 2001, 53.

[13] Grenadier, S. R., 1996, "The strategic exercise of options: Development cascades and overbuilding in real estate markets," *Journal of Finance* 51, 1653-79.

[14] Grenadier, S. R., 2002, "Option Exercise Games: An application to the equilibrium investment strategies of firms," *Review of Financial Studies* 15, 691-721.

[15] Harris, C., and D. Laibson, 2001, "Hyperbolic discounting and consumption," *Proceedings of the 8th World Congress of the Econometric Society*, Forthcoming.

[16] Harris, C., and D. Laibson, 2004, "Instantaneous gratification," working paper, Harvard University.

[17] Laibson, D., 1997, "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics* 112, 443-77.

[18] Lambrecht, B. M., and W. R. Perraudin, 2003, "Real options and preemption under incomplete information," *Journal of Economic Dynamics and Control* 27, 619-643.

[19] Leahy, J. V., 1993, "Investment in competitive equilibrium: The optimality of myopic behavior," *Quarterly Journal of Economics* 108, 1105-33.

[20] Loewenstein, G., and D. Prelec, 1992, "Anomalies in intertemporal choice: Evidence and an interpretation," *Quarterly Journal of Economics* 57, 573-598.

[21] Luttmer E. G. J. and T. Mariotti, 2003, "Subjective discounting in an exchange economy," *Journal of Political Economy* 111, 959-989.

[22] Madrian, B. C., and D. Shea, 2001 "The power of suggestion: Inertia in 401(k) participation and savings behavior," *Quarterly Journal of Economics* 116, 1149-1187.

[23] Majd, S., and R. Pindyck, 1987, "Time to build, option value, and investment decision," *Journal of Financial Economics* 18, 7-28.

[24] Malliaris, A. G., and W. A. Brock, 1982, "Stochastic methods in economics and finance," North-Holland, New York.

[25] McDonald, R., and D. Siegel, 1986, "The value of waiting to invest," *Quarterly Journal of Economics* 101, 707-727.

[26] Merton, R. C., 1973, "Theory of rational option pricing," *Bell Journal of Economics and Management Science* 4, 141-183.

[27] O'Donoghue, T., and M. Rabin, 1999a, "Doing it now or later," *American Economic Review* 89, 103-24.

[28] O'Donoghue, T., and M. Rabin, 1999b, "Incentives for procrastinators," *Quarterly Journal of Economics* 114, 769-816.

[29] O'Donoghue, T., and M. Rabin, 2001, "Choice and procrastination," *Quarterly Journal of Economics* 116, 121-160.

[30] Prelec, D., and G. Loewenstein, 1997, "Beyond time discounting," *Marketing Letters* 8, 97-108.

[31] Strotz, R. H., 1956, "Myopia and inconsistency in dynamic utility maximization," *Review of Economic Studies* 23, 165-80.

[32] Thaler, R., 1981, "Some empirical evidence on dynamic inconsistency," *Economics Letters* 8, 201-207.

[33] Titman, S., 1985, "Urban land prices under uncertainty," *American Economic Review* 75, 505-514.

[34] Williams, J. T., 1991, "Real estate development as an option," *Journal of Real Estate Finance and Economics* 4, 191-208.

[35] Williams, J. T., 2001, "Agency, ownership, and returns on real assets," working paper, Professors Capital.

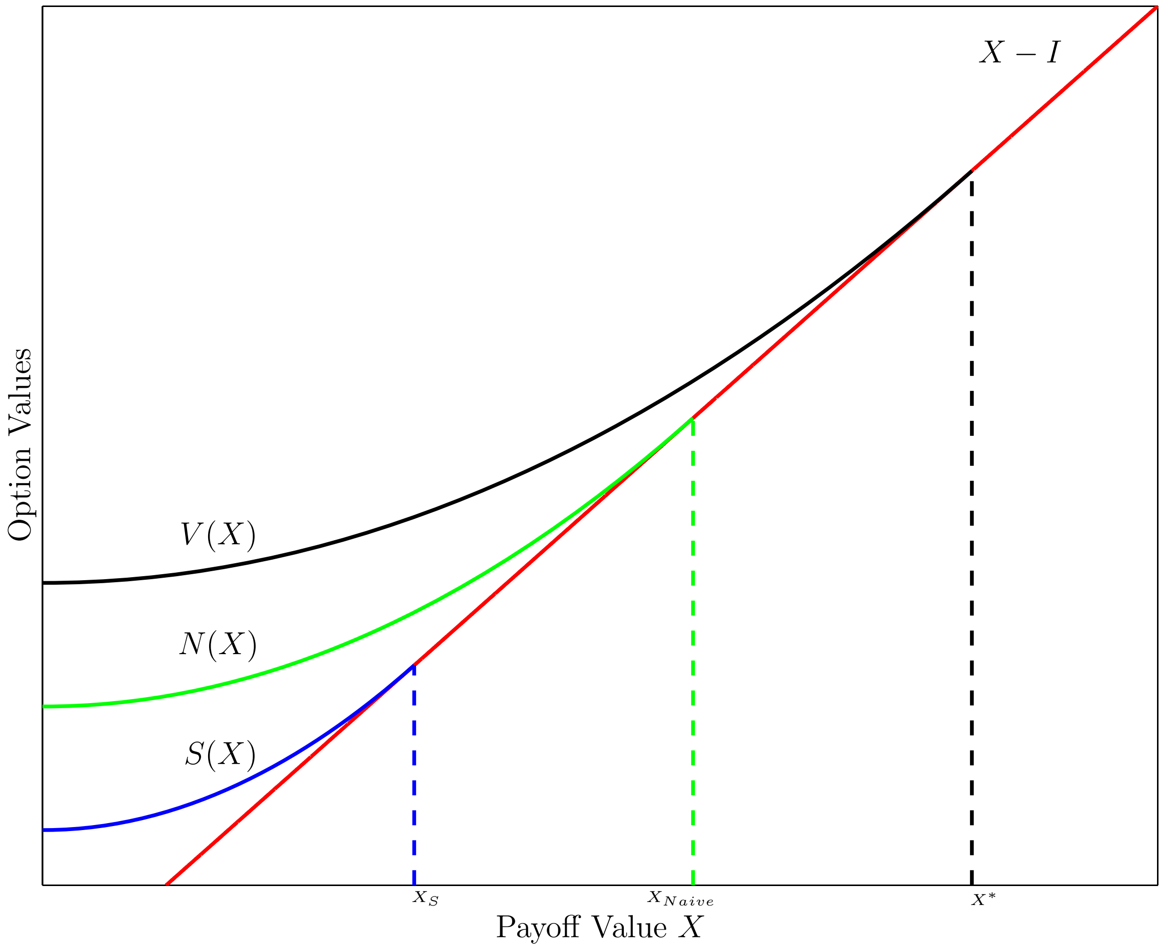

Figure 1: The Impact of Time-Inconsistent Preferences on the Option to Wait. *This graph plots the option values of the time-consistent, naive, and sophisticated entrepreneurs, denoted as $V(X)$, $N(X)$, and $S(X)$, respectively. The vertical distance between an option value and the investment payoff value, $X - I$, represents the value of the option to wait. At all points prior to exercise, the time-consistent entrepreneur has the most valuable option to wait, followed by the naive entrepreneur and then the sophisticated entrepreneur.*

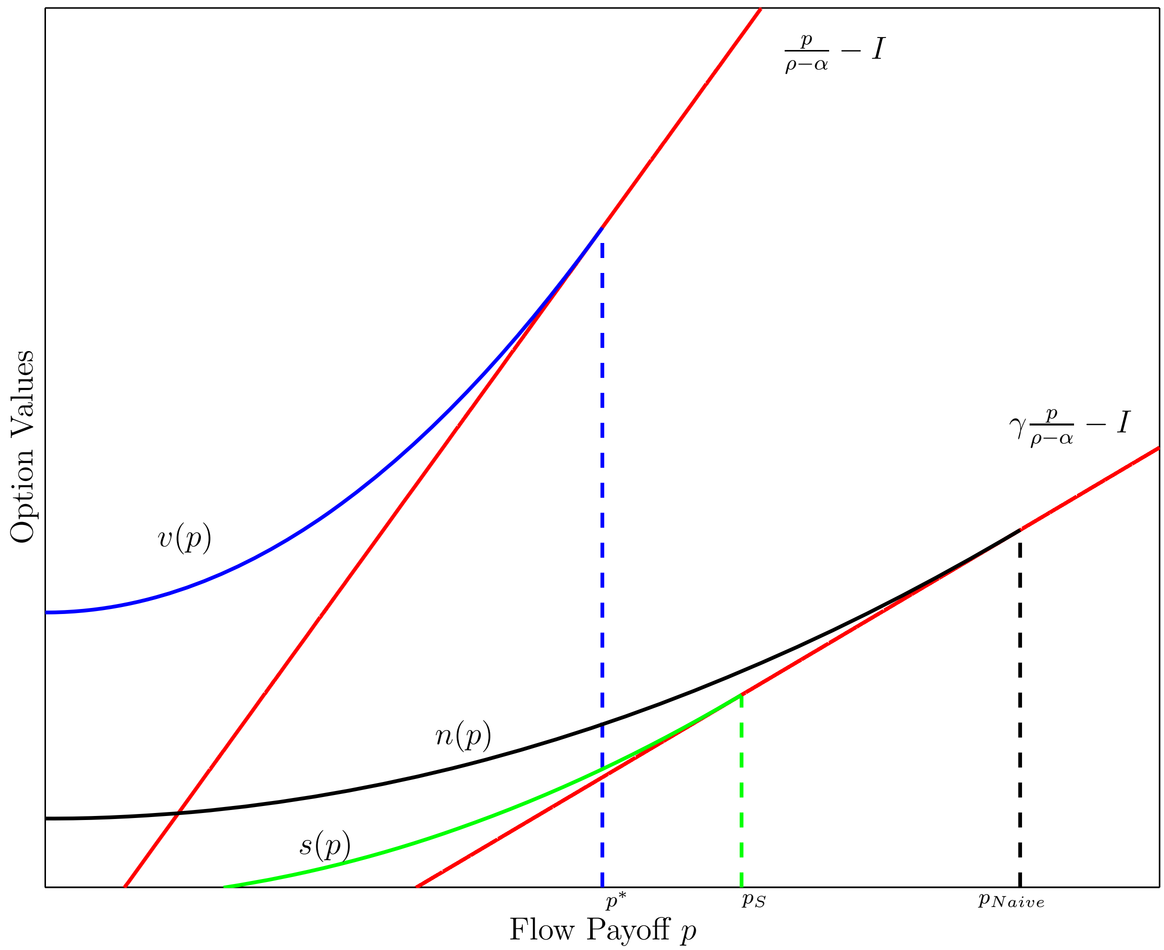

Figure 2: The Impact of Time-Inconsistent Preferences on the Option to Wait for the Case of Flow Payoffs. *For the case of flow payoffs, this graph plots the option values of the time-consistent, naive, and sophisticated entrepreneurs, denoted as $v(p)$, $n(p)$, and $s(p)$, respectively. For the time-consistent entrepreneur, the investment payoff value is $p/(\rho-\alpha)-I$, while for the naive and sophisticated entrepreneurs the investment payoff value is $\gamma p/(\rho-\alpha)-I$. The vertical distance between an option value and the investment payoff value represents the value of the option to wait.*

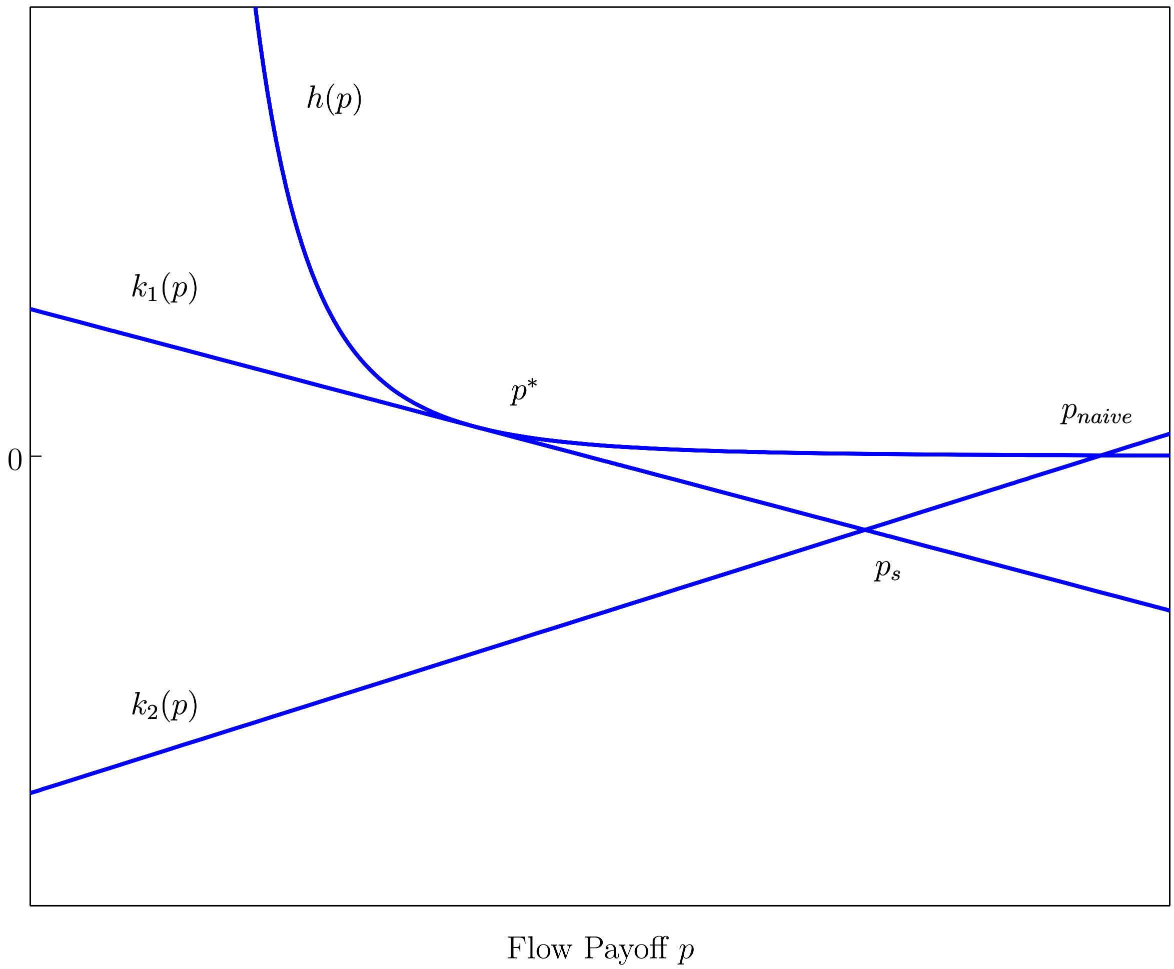

Figure 3: Relative orderings of p^*, p_s, and p_{naive}. The curves $k_1(p)$ and $h(p)$ intersect at the point p^*. The curves $k_1(p)$ and $k_2(p)$ intersect at the point p_s, where we see that $p_s > p^*$. The curves $k_2(p)$ and $h(p)$ intersect at the point p_{naive}, where we see that $p_{naive} > p_s$.