An Equilibrium Model of Wealth Distribution*

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Abstract

I present an explicitly solved equilibrium model for the distribution of wealth and income in an incomplete-markets economy. I first propose a self-insurance model with an inter-temporally dependent preference (Uzawa (1968), Lucas and Stokey (1984), and Obstfeld (1990)). I then derive an analytical consumption rule which captures stochastic precautionary saving motive and generates stationary wealth accumulation. Finally, I provide a complete characterization for the equilibrium cross-sectional distribution of wealth and income in closed form by developing a recursive formulation for the moments of the distribution of wealth and income. Using this recursive formulation, I show that income persistence and the degree of wealth mean reversion are the main determinants of wealth-income correlation and relative dispersions of wealth to income, such as skewness and kurtosis ratios between wealth and income.

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1 Introduction

Empirically, labor income and financial wealth are cross-sectionally positively skewed and fat-tailed. Furthermore, wealth is even more skewed and fat-tailed than income. For example, the 1992 Survey of Consumer Finance reports that the top one percent of U.S. households make 15% of total income, but hold 30% of total wealth. Building on Bewley (1986), Aiyagari (1994) and Huggett (1993) provide a framework to analyze the cross-sectional wealth distribution in an equilibrium setting, based on agents' intertemporal optimal consumption-saving decisions. These incomplete-markets models, often referred to as Bewley models, have a large number of *ex ante* identical, but *ex post* heterogeneous infinitely-lived agents who trade a single risk-free asset to partially smooth their consumption over time against stochastic uninsurable labor income shocks. Both goods and asset markets clear. The different realizations of income shocks for different agents imply that the cross-sectional asset holdings and income levels are different for agents. While realizations are different, the cross-sectional distribution of wealth and income remains stable over time. The Bewley model has become the workhorse to understand the equilibrium cross-sectional wealth distribution. Quadrini and Ríos-Rull (1997) summarize both dynastic (infinite horizon) and life-cycle versions of these Bewley models up to late 1990s. Recently, significant progress has been made on generalizing these quantitative Bewley-style models by incorporating more realistic features in order to better explain the highly skewed and fat-tailed empirical wealth distribution. Cagetti and De Nardi (2005b) provide a comprehensive and up-to-date summary of this literature including both the key empirical facts and the performance of various economic models.

In order to characterize the equilibrium wealth distribution, I first construct and then explicitly solve an incomplete-markets consumption-saving model. I follow Uzawa (1968), Lucas and Stokey (1984), and Obstfeld (1990) to assume that the agent whose past consumption is higher has a larger discount rate for his future consumption. This is a convenient and also intuitive way to link the past consumption path with current consumption via inter-temporal dependence. A higher discount rate for the agent when he is richer helps to deliver a stationary wealth process. These are precisely the insights of Uzawa (1968), Epstein (1983), and Obstfeld (1990) in their work on endogenous discounting and growth in deterministic settings. I extend their analysis to a stochastic setting under incomplete markets. The second key assumption is that the agent has constant absolute risk aversion (CARA) utility, following precautionary saving models such as Caballero (1990), Kimball and Mankiw (1989), Merton (1971) and Wang (2003). These modeling choices are partly motivated by analytical tractability. Zeldes (1989) noted in his abstract "no one has derived closed-form solutions for consumption with stochastic labor income and constant relative risk aversion utility."

Unlike typical CARA-utility-based, incomplete-markets consumption models such as Caballero (1991), the newly proposed model generates a *stochastic* precautionary savings demand. This feature comes from the conditional *heteroskedasticity* of the income process, which has rich implications. For example, the process implies that a higher level of income implies a more volatile stream of future incomes (in levels). Therefore, his precautionary saving is larger when his income level is higher. Equivalently stated, the agent consumes less out of his human wealth, the present discounted value of future labor incomes, than out of his financial wealth.[1] Moreover, the model allows for large and unexpected changes in income. I model these movements by embedding jumps into the affine processes.[2] Note that the *stochastic* precautionary savings demand is necessary to generate predictions such as *excess sensitivity* and *excess smoothness*, which are consistent with empirical evidence documented in Flavin (1981) and Campbell and Deaton (1989). The stochastic precautionary savings demand is also predicted in constant-relative-risk-averse (CRRA) utility based models such as Carroll (1997).

To sum up, the agent's decision problem with the following three key features: (*i*) an inter-temporally dependent preference with endogenous discounting; (*ii*) a CARA utility-based precautionary saving model and (*iii*) a conditionally heteroskedastic income process which allows for skewness, kurtosis and large discrete movements (jumps) due to unexpected shocks. The proposed model generates a realistic buffer-stock saving behavior with stochastic precautionary savings demand in an analytically tractable way.

Using the explicitly solved consumption-saving rule, I develop an analytical model of equilibrium wealth distribution. Like Aiyagari (1994) and Huggett (1993), my model has dynastic (infinitely lived) agents whose saving behavior may be described by buffer-stock models. My model also generates the key equilibrium restriction of dynastic Bewley models (e.g. Aiyagari (1994) and Huggett (1993)): The cross-sectional distribution of wealth and income is equal to the long-run stationary distribution of income and wealth for a representative infinitely lived agent. Unlike the numerical methods used in papers such as Aiyagari (1994) and Huggett

[1]See Friedman (1957) for his conjecture on this property of the consumption rule. See Zeldes (1989) for numerical work and Wang (2006) for an analytical model supporting this property of the consumption rule.

[2]Affine processes are widely used in financial economics. See Vasicek (1977), Cox, Ingersoll, Jr., and Ross (1985) and Dai and Singleton (2000) for applications in term structure. See Duffie, Pan, and Singleton (2000) for affine processes and the transform analysis.

(1993), I solve for the equilibrium wealth distribution by developing a closed-form recursive formulation for the moments of the cross-sectional stationary distribution of wealth and income. The analytical tractability comes from (*i*) the linear buffer-stock saving rule and (*ii*) the affine income process. More importantly, my model provides additional economic insights on some determinants of the wealth distribution in dynastic Bewley models. For example, since the individual agent's wealth is proportional to a weighted sum of his past incomes, "averaging" makes the cross-sectional (standardized) wealth[3] smoother than the cross-sectional (standardized) income, *ceteris paribus*. However, for riskier income process (a higher degree of conditional heteroskedasticity), the agent may have a stronger precautionary motive, which tends to make wealth more dispersed than income. The analytical tractability of the model also allows me to show that income persistence and the degree of wealth mean reversion are the main determinants of wealth-income correlation and relative dispersions of wealth to income, such as skewness and kurtosis ratios between wealth and income.

Unfortunately, buffer-stock saving models which are at the core of dynastic Bewley models have difficulties in explaining the saving behavior of the rich. Dynan, Skinner, and Zeldes (2004) find that the rich people (under various definitions) save a larger fraction of their income than the poor, inconsistent with insights based on buffer-stock models. Moreover, buffer-stock dynastic equilibrium models can not generate enough wealth concentration at the right tail of the distribution. These dynastic models place a stringent equilibrium restriction that the cross-sectional distribution must equal a representative dynasty's long-run stationary distribution. In order to ensure a stationary wealth distribution in dynastic buffer-stock-saving-based models, the wealth-rich need to de-cumulate his wealth at a sufficiently high rate to ensure that wealth process mean reverts.[4]

Aiming to improve the quantitative performance of dynastic Bewley models, Quadrini (2000) introduces (endogenous) entrepreneurship, and allows for (*i*) capital market imperfections; (*ii*) (additional) uninsurable entrepreneurial risk; and (*iii*) costly external financing. These features provide additional incentives for entrepreneurs to save.[5] Quadrini (2000) delivers cross-sectional wealth distributions for both entrepreneurs and workers. The economy-wide cross-sectional wealth distribution is a weighted average of those for entrepreneurs and

[3]Standardized wealth is a linear transformation of the cross-sectional wealth: Taking cross-sectional wealth and subtracting its mean, and finally dividing by its standard deviation gives "standardized" wealth.

[4]For example, the marginal propensity to consume out of wealth when the dynasty is rich has to be higher than the interest rate in these models. Otherwise, wealth is not stationary.

[5]Here, entrepreneurship is endogenous. We shall broadly interpret entrepreneurs as both the current ones and those households who plan to take on entrepreneurial activities in the (near) future.

workers. This heterogeneity between entrepreneurs and workers allows room for more concentrated wealth in the economy because the entrepreneurs and workers may have different saving behavior. The key restriction in dynastic Bewley models, the cross-sectional distribution is equal to the long-run stationary distribution for any representative agent, is no longer required. For example, in his model, entrepreneurs are richer on average than workers due to the three features he introduced.[6] However, compared to data, Quadrini (2000) still falls short of generating enough asset holdings by the very richest households.

Both baseline and extended dynastic models ignore the life-cycle dimension of the savings decision. Building on the life-cycle wealth distribution model of Huggett (1996), De Nardi (2004) shows that adding voluntary bequests and inter-generational transmission of human capital help to generate a more skewed and fat-tailed wealth distribution relative to the income distribution. Bequests are a luxury good in her model and thus intuitively, the rich leave more to their children. Moreover, the inter-generational human capital link generates a greater degree of heterogeneity across households and hence induces even more persistent wealth dynamics.

The third class of wealth distribution models contains both dynastic and life-cycle features. Castañeda, Díaz-Giménez, and Ríos-Rull (2003) show that a very risky income process for the richest may generate sufficient wealth dispersion in a life-cycle model with dynastic households. The key intuition is that the rich continues to save at a very high rate due to extremely large uninsurable income shocks. Households may thus accumulate wealth at a very high rate, even when they are rich. Households have different marginal propensities to save during their retirement and working stages. The economy-wide cross-sectional wealth distribution in Castañeda et al. (2003) is a weighted average of the wealth distribution for workers and that for retirees. Given the strong precautionary saving motive (even by the rich worker) and the heterogeneity between workers and retirees, the cross-sectional wealth distribution may thus be more concentrated than those in dynastic Bewley models such as Aiyagari (1994) and Huggett (1993).

In an ambitious and comprehensive paper, Cagetti and De Nardi (2005a) introduce a key friction, imperfect enforcement in the credit markets, into a quantitative life-cycle model with inter-generational altruism, by building on Quadrini (2000). While entrepreneurs have

[6]Krusell and Smith (1998) extend the dynastic models to allow for the dynasty's discount rate to follow a Markov chain process. The stochastic discount rate generates heterogeneity in the dynasty's saving rates over time and hence enough concentration for the wealth distribution. Krusell and Smith (1998) also make a methodological contribution by extending the analysis of wealth distribution to allow for aggregate fluctuations.

higher expected rates of return from their investment opportunities, entrepreneurs also have stronger incentives to save in order to mitigate the credit constraints/collateral requirements. Moreover, the ingredient of *voluntary* bequests also helps to enhance the dispersion of wealth, similar to the intuition in De Nardi (2004). Compared with Quadrini (2000), Cagetti and De Nardi (2005a) obtain a better fit for the richest by endogenizing the firm size distribution. Their model matches well the wealth distributions for both entrepreneurs and non-entrepreneurs. The heterogeneity (workers versus entrepreneurs) plus the higher shadow value of saving for entrepreneurs (due to credit market frictions) generate a large concentration of wealth in the hands of the richest.

Finally, my paper also relates to Benhabib and Bisin (2006). While both papers study the wealth distribution, the focuses and modeling methods of the two papers are rather different. Benhabib and Bisin (2006) study the effects of redistributive fiscal policies on the wealth distribution. I analyze the dispersion of cross-sectional wealth relative to the dispersion of cross-sectional income in a self-insurance setting. Unlike Benhabib and Bisin (2006) which ignores labor income, I use the affine jump diffusion process to model stochastic income, derive the optimal consumption rule under incomplete markets, and then characterize the endogenous joint distribution of income and wealth. While Benhabib and Bisin (2006) allow for bequests in an overlapping generations model of Blanchard (1985) and Yaari (1965), my paper is based on infinitely lived agents. Finally, unlike Benhabib and Bisin (2006) which study both transitions and the steady state, this paper focuses on the stationary cross-sectional equilibrium distribution of income and wealth.

The remainder of the paper is organized as follows. Section 2 describes the setup of the individual agent's optimal consumption problem. Section 3 solves for the optimal consumption and saving rules explicitly. Section 4 computes the joint distribution of income and wealth in closed form and discusses the model-implied properties of the joint distribution. Section 5 concludes. Appendices contain technical details.

2 An Income Fluctuation Problem

An individual agent solves a version of the canonical intertemporal self-insurance problem.[7] He lives forever and receives uninsurable labor income, governed by an exogenously-specified stochastic process. For technical convenience, I cast the model in continuous time. He can borrow or lend at a constant risk-free interest rate that is determined in equilibrium.

[7] Chapter 16 in Ljungqvist and Sargent (2004) provides an introduction to this class of models.

Subsections 2.1 and 2.2 introduce the stochastic income process and the agent's intertemporal preference, respectively.

2.1 Income Process

Empirically, the conditional variance of changes in income increases with the level of income. That is, the labor-income process is conditionally heteroskedastic. Furthermore, income is subject to unexpected large shocks, such as promotions, demotions, and unemployment.[8] These events often happen at low frequencies, but the potential quantitative movements of income may be significant. A natural way to treat these events is with "jumps." Once in a while, with some probability, the agent receives a large "surprise" movement in his income. Moreover, labor income is empirically positively skewed, fat-tailed, and bounded from below.

Motivated by these considerations, I model income using the affine jump-diffusion process by generalizing the income process introduced in Wang (2006). An affine process allows for a monotonic relationship between the conditional variance of changes in labor income and the level of income parsimoniously. This class of stochastic processes is quite flexible for capturing a variety of empirical regularities such as conditional heteroskedasticity, skewness, and kurtosis. In addition, the affine process may capture large and unexpected movements in income by allowing the income process to jump by a random amount at a random time.

A frequently adopted income model postulates that the logarithm of income, rather than the level of income, is a conditionally homoskedastic Markov process.[9] This logarithmic model also implies that the conditional variance of the level of income increases in the level of income.[10] However, the conditionally homoskedastic logarithmic income process does not capture the large and unexpected movements in income, as affine (jump-diffusion) models do. Affine processes are also more convenient to work with, because they allow for closed-form optimal consumption rules as shown in Section 3.[11] Another widely used income process is the autoregressive moving-average (ARMA) process, because of its analytical tractability. Unlike the ARMA process, the affine process is conditionally heteroskedastic, and hence is able to capture empirical regularities, such as positive skewness and excess kurtosis. Moreover, the autoregressive income process is a special case of an affine process, with conditionally

[8] See Jacobson, LaLonde, and Sullivan (1993) for empirical evidence.

[9] See MaCurdy (1982) and Deaton (1991), for example.

[10] If the percentage change of income is conditionally heteroskedastic, then the total change of income must be conditionally heteroskedastic.

[11] There exists no closed-form consumption rule for conditionally homoskedastic logarithmic income process in any precautionary saving model.

homoskedastic income shocks.

Suppose that the income process y is given by the following stochastic differential equation (SDE):

$$dy_t = (\alpha - \kappa y_t)\, dt + \sqrt{l_0 + l_1 y_t}\, dW_t + dZ_t, \quad t \geq 0\,, \quad y_0 \text{ given}, \tag{1}$$

where W is a standard Brownian motion on the real line $\mathbb{R}$, and Z is a pure jump process. For each realized jump, the size of the jump is drawn from a fixed probability distribution ν on $\mathbb{R}$. The intensity at which the jump occurs, $\lambda(y)$, is stochastic and depends upon the underlying income. I further assume that the jump intensity is affine in the level of income, in that

$$\lambda(y) = \lambda_0 + \lambda_1 y, \tag{2}$$

for non-negative coefficients λ_0 and λ_1. That is, the model allows the probability of jumps to be time-varying and to depend on the level of income.[12]

Let δ_j denote the jth moment of the jump size with respect to the jump probability measure ν, in that $\delta_j = \int_{\mathbb{R}} z^j \, d\nu(z)$, for $j \geq 1$. The Laplace transform $\zeta(\,\cdot\,)$ of the jump distribution ν is defined by $\zeta(k) \equiv \int_{\mathbb{R}} e^{kz}\, d\nu(z)$, for any k such that the integral exists. When the expected jump size of income is not zero ($\delta_1 \neq 0$), jumps lead to additional expected changes of income over time, that is, the expected instantaneous rate of change of y is then given by $(\alpha - \kappa y + (\lambda_0 + \lambda_1 y)\, \delta_1)$.

Following Friedman (1957) and Hall (1978), I define human wealth as follows:

Definition 1 *Human wealth h_t at time t is the expected present value of future labor income, discounted at the risk-free interest rate r, given the agent's information set $\mathcal{F}_t$ at time t. That is,*

$$h_t = E_t \left(\int_t^{\infty} e^{-r(s-t)} y_s \, ds \right) , \tag{3}$$

where E_t denotes $\mathcal{F}_t$-conditional expectation.

Equation (3) does not take the riskiness of the income process into account. The interest rate r is assumed to be strictly positive.[13] In order to ensure that human wealth is finite, I

[12] There are two cases to consider in terms of the parameter admissibility. If the Brownian innovations are conditionally homoskedastic ($l_1 = 0$), then the jump intensity is restricted to be constant ($\lambda_1 = 0$), in order to ensure that the jump intensity $\lambda(y)$ is positive, for all possible values of income y. If the Brownian innovations are conditionally heteroskedastic ($l_1 > 0$), then $\lambda_0 l_1 - \lambda_1 l_0 > 0$ is necessary, and the distribution ν supports only positive jumps. Furthermore, $\alpha l_1 + \kappa l_0 > 0$ is required, in order for the instantaneous drift $\mu(\,\cdot\,)$ to be positive, at $y = -l_0/l_1$, the lower boundary.

[13] It is straightforward to provide conditions to support a positive interest rate in equilibrium as shown later in Section 4.1.

also assume that $r + \kappa_y > 0$, where

$$\kappa_y = \kappa - \lambda_1 \delta_1. \tag{4}$$

The parameter κ_y is the effective rate of mean reversion. The second term in (4) incorporates the effect of the jump component on the degree of income persistence. If the income process is given by (1), then human wealth is affine in current income y, in that

$$h_t = \frac{1}{r + \kappa_y} \left(y_t + \frac{\alpha_y}{r} \right) , \tag{5}$$

where κ_y is given in (4), and $\alpha_y = \alpha + \lambda_0 \delta_1$. Note that α_y is the constant component of the drift function for income, and the second term $\lambda_0 \delta_1$ captures the effect due to jumps.

2.2 Agent's Preference

The standard preference assumption in the consumption-saving literature is a time additive separable utility. However, there is substantial amount of work that challenges the expected utility models. For example, Obstfeld (1990) stated that "mathematical convenience, rather than innate plausibility, has always been the main rationale for assuming time-additive preferences in economic modeling." Koopmans (1960) and Koopmans, Diamond, and Williamson (1964) initiated the modern work on generalizing the expected utility to allow for non time-additive, but rather inter-temporally dependent preferences. This class of preferences is often dubbed as *recursive* preferences.

In this paper, I follow this literature on recursive preferences and assume that the agent's time rate of preference is not constant, and instead depends on the agent's past cumulative consumption. Indeed, the recursive preference has been used in many branches of economics. Uzawa (1968) pioneered the use of this inter-temporally dependent discount function in his work on growth. Epstein (1983) uses this recursive utility to study growth in a stochastic setting. Lucas and Stokey (1984) analyzes growth with heterogenous consumers whose preferences are inter-temporally dependent as in Uzawa (1968). Bergman (1985) studies the intertemporal capital asset pricing implications using this recursive (non-time additive) preference. Obstfeld (1990) develops geometric methods to analyze optimal consumption rules with such preferences. Obstfeld and Rogoff (1996) provide a good introduction to this recursive preference in discrete time and in a deterministic setting.[14] I extend the analysis of Obstfeld (1990) in the deterministic setting to the stochastic setting where labor income shocks are uninsurable and hence markets are incomplete.

[14]See Supplement B to Chapter 2 from page 722 to 726.

Formally, I suppose that the agent has the following preference:

$$U(c) = E\left[\int_0^\infty \exp\left(-\int_0^t \beta(c_u)\,du\right) u(c_t)\,dt\right]. \tag{6}$$

I assume a linear relationship for the instantaneous discounting function $\beta(\,\cdot\,)$, in that

$$\beta(c) = \beta_0 + \beta_c\,c. \tag{7}$$

When $\beta_c = 0$, then the agent's utility becomes the standard time-additive separable utility. I follow Obstfeld (1990) and Obstfeld and Rogoff (1996) and assume that the agent becomes more impatient when his past consumption is higher. Because consumption increases in wealth, *ceteris paribus*, equation (7) with a positive β_c implies that the richer agent is more impatient. I will show later that a positive β_c helps to deliver a stationary wealth distribution. Intuitively, a stronger incentive to consume for the richer agent narrows the wealth dispersion over time and hence generates a stationary wealth distribution. If we make the alternative assumption that the agent is more patient when his past consumption is higher($\beta_c < 0$), then the agent's marginal propensity to consume out of wealth is less than the interest rate, as we will show later. As a result, the agent's wealth process is then no longer stationary. In an economy with infinitely lived agents, the cross-sectional wealth distribution is then non-stationary. We rule this situation out by requiring $\beta_c > 0$.

In order to derive an analytical consumption rule under an incomplete markets setting, I follow Merton (1971), Kimball and Mankiw (1989), Caballero (1991) and Wang (2004) to assume that the period utility function is constant absolute risk aversion (CARA), in that $u(c) = -e^{-\gamma c}/\gamma$, with $\gamma > 0$. Although CARA utility lacks wealth effects, it still captures stochastic precautionary saving demand, provided that we use conditionally heteroskedastic income process.

One key prediction of CRRA-utility-based self insurance models such as those of Huggett (1993) and Aiyagari (1994) is that the agent engages in buffer-stock saving in equilibrium. Buffer-stock saving means that the agent aims at a target level of wealth. If his wealth is above the target level, the agent dis-saves. If his wealth is below the target level, he saves. In general, the CRRA utility model predicts that consumption rule is concave (Carroll and Kimball (1996)). However, quantitatively, the consumption rule is approximately linear, provided that the agent's wealth level is not close to zero.[15] The other important prediction of the CRRA-utility-based consumption model is that the agent has a stochastic precautionary savings demand (Carroll (1997)).

[15] See Zeldes (1989) and Deaton (1991).

It is worth noting that our recursive utility with CARA specification also generates (i) the buffer-stock saving behavior and (ii) the stochastic precautionary savings, the same two key features mentioned above for CRRA utility based models. An explicitly-solved optimal consumption rule substantially simplifies the analysis of the cross-sectional distribution of income and wealth. I further exploit the analytical tractability of the explicit consumption rule to derive higher-order moments of the distribution of wealth and income. This recursive formulation completely characterizes the equilibrium distribution, and moreover, provides additional insights on the determinants of the distribution of income and wealth, as we will show later.

As in standard consumption models, I assume that the agent can invest in one risk-free asset. Let x denote the agent's financial wealth. His wealth accumulation is given by:

$$dx_t = (rx_t + y_t - c_t)\, dt, \tag{8}$$

with an initial wealth endowment x_0. The agent's optimization problem is to choose his consumption process c to maximize his utility given in (6), subject to an exogenously specified labor-income process y given in (1), the wealth accumulation equation (8), and the transversality condition $\lim_{t\to\infty} E\left[e^{-rt}|J(x_t, y_t)|\right] = 0$.

3 Optimal Consumption and Saving

In this section, I first derive the consumption rule and discuss the intuition behind the policy rule in Subsection 3.1. Then, I decompose the saving motives implied by the consumption rule in Subsection 3.2, and uses the decomposition result to provide some insights on the determinants of the saving rule.

3.1 Consumption Rule

Let $J(x, y)$ denote the corresponding value function. By a standard argument, the value function $J(x, y)$ solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \sup_{\bar{c}} \left\{ u(\bar{c}) - \beta(\bar{c}) J(x, y) + \mathcal{D}^{\bar{c}} J(x, y) \right\}, \tag{9}$$

where

$$\begin{aligned} \mathcal{D}^{\bar{c}} J(x, y) = {} & (rx + y - \bar{c})\, J_x(x, y) + (\alpha - \kappa y)\, J_y(x, y) + \frac{1}{2}(l_0 + l_1 y)\, J_{yy}(x, y) \\ & + (\lambda_0 + \lambda_1 y)\, E[J(x, y + q) - J(x, y)], \end{aligned} \tag{10}$$

and where q has probability distribution ν. The last term in (10) captures the effect of jumps, and E is taken with respect to the jump distribution ν.

The first-order condition (FOC) for the HJB equation is

$$u'(\bar{c}) = e^{-\gamma\bar{c}} = J_x + \beta_c J. \tag{11}$$

Unlike the FOC for an expected utility agent, the second term $\beta_c J$ on the right side of (11) captures the effect of endogenous discounting $\beta(\,\cdot\,)$ on the agent's tradeoff. The agent who discounts his future more when his past consumption is greater ($\beta_c > 0$), has stronger incentives to consume in order to keep the discount rate for his future consumption from being too high, *ceteris paribus*. The FOC (11) states that the marginal utility of consumption $u'(\bar{c})$ is less than the marginal value of saving J_x at the agent's optimality. (Note that the value function $J(x, y) < 0$, because the felicity function in each period $u(c) = -e^{-\gamma c}/\gamma < 0$ for any consumption level.)

Using the standard guess-and-verify procedure, I derive a linear consumption rule in closed form. The following proposition reports the optimal consumption rule, and Appendix A contains the details involved in the derivations.

Proposition 1 *The optimal consumption rule is affine in financial wealth x and current labor income y, in that, for all t,*

$$c_t^* = \omega\left(x_t + a_y y_t + \bar{a}\right), \tag{12}$$

where

$$\omega = r + \kappa_x, \tag{13}$$

$$\kappa_x = \beta_c/\gamma, \tag{14}$$

$$a_y = \frac{a_h}{r + \kappa_y}, \tag{15}$$

$$\bar{a} = \frac{1}{r}\left[\frac{\beta_0 - r}{\gamma\omega} + \left(\frac{\alpha_y}{r + \kappa_y} - \lambda_0\Gamma(\epsilon)\right)a_h - \frac{1}{2}\Delta_0 a_h^2\right], \tag{16}$$

$$0 = -1 + \left(1 + \lambda_1\Gamma(\epsilon)\right)a_h + \frac{1}{2}\Delta_1 a_h^2, \tag{17}$$

$$\Gamma(\epsilon) = \frac{1}{r + \kappa_y}\left(\delta_1 + \frac{1}{\epsilon}\left(\zeta(-\epsilon) - 1\right)\right), \tag{18}$$

$$\epsilon = \gamma\omega a_y, \tag{19}$$

$$\Delta_i = \frac{\gamma\omega l_i}{(r + \kappa_y)^2}, \quad \text{for } i = 0, 1. \tag{20}$$

Next, I show that the model predicts stochastic precautionary savings, a key feature of incomplete-markets consumption models. The optimal consumption rule (12) may also be expressed in terms of financial wealth x and human wealth h given in (5), in that

$$c_t^* = \omega \left(x_t + a_h h_t - b_0 \right), \tag{21}$$

where

$$b_0 = \frac{1}{r} \left(\frac{1}{2} \Delta_0 a_h^2 + \lambda_0 \Gamma(\epsilon) a_h - \frac{\beta_0 - r}{\gamma \omega} \right). \tag{22}$$

For the purpose of future reference, I call a_h, the ratio between the MPC out of human wealth and that out of financial wealth, *the MPC ratio*. Equation (17) implies that the MPC ratio, a_h is always less than one. This inequality holds strictly, when the conditional variance of labor income depends directly on its level ($l_1 > 0$ or $\lambda_1 > 0$). The intuition is that, with an increase in income, "human" wealth has increased, but its volatility has also increased ($l_1 > 0$); therefore, the prudent agent (Kimball (1990)) increases his consumption out of his "human" wealth less than proportionally. In particular, if the jump intensity is constant ($\lambda_1 = 0$), then (17) specializes to the following quadratic equation:

$$0 = \frac{\Delta_1}{2} a_h^2 + a_h - 1. \tag{23}$$

In general, (23) has two roots, for conditionally heteroskedastic income. I discard the negative root, since it implies a negative MPC out of current income y. The positive root lies between zero and one, and is given by

$$a_h = \frac{2}{\sqrt{1 + 2\Delta_1} + 1}. \tag{24}$$

That the MPC ratio, a_h, is less than one implies that consumption responds less to a unit increase in human wealth than a unit increase in financial wealth, because of the precautionary motive.

I use the following metric to quantify the agent's precautionary savings motive.

Definition 2 *Let c^* be the optimal consumption characterized by (21), given financial wealth x and human wealth h. Let c^p be the corresponding certainty-equivalence ($l_0 = l_1 = 0 = \lambda_0 = \lambda_1$) consumption. Then, the precautionary savings premium is $\pi \equiv c^p - c^*$.*

The precautionary savings premium then is given by

$$\pi_t \;=\; \omega \left[\left(1 - a_h\right) h_t + \frac{1}{2r} \Delta_0 a_h^2 + \frac{\lambda_0}{r} \Gamma(\epsilon) a_h \right] = \frac{\omega}{r} \left(\phi_t + \xi_t \right), \tag{25}$$

where

$$\phi_t = \frac{1}{2}(\Delta_0 + \Delta_1 r h_t)\, a_h^2 = \frac{\gamma \omega a_y^2}{2}\sigma^2(r h_t), \tag{26}$$

$$\xi_t = (\lambda_0 + \lambda_1 r h_t)\,\Gamma(\epsilon) a_h = a_h \Gamma(\gamma \omega a_y)\,\lambda(r h_t). \tag{27}$$

The terms ϕ_t and ξ_t capture the effects of the diffusion risk and the jump risk on the precautionary savings demand, respectively (Note that they are proportional to the conditional variance $\sigma^2(\cdot)$, and the jump intensity $\lambda(\cdot)$, respectively). Recall our earlier discussion on the effect of conditional heteroskedasticity of income ($l_1 > 0$ or $\lambda_1 > 0$) on the *MPC ratio* a_h. When the agent's income is higher, his income risk is bigger. Hence, his precautionary saving demand is higher. However, the precautionary saving does not depend on the agent's wealth level. That is, the precautionary saving is the same for the wealth-rich and the wealth-poor in this model.

Next, I provide a saving decomposition analysis that allows us to better understand the model's implications on saving.

3.2 Saving Decomposition

Let $g(x, y)$ be the optimal saving rule, in that $g(x_t, y_t) = s_t = dx_t/dt$. Substituting the consumption rule (12) into wealth accumulation equation (8) gives

$$g(x, y) = -\kappa_x x + \psi y + \kappa_0, \tag{28}$$

where $\kappa_x = \beta_c/\gamma$, and

$$\psi = 1 - \omega a_y = \frac{r(1 - a_h) + \kappa_y - \kappa_x a_h}{r + \kappa_y}, \tag{29}$$

$$\kappa_0 = -\omega \bar{a}. \tag{30}$$

Under the assumption that the agent's subjective discount rate $\beta(c)$ increases with his consumption ($\kappa_x > 0$), the marginal propensity to save out of financial wealth, $-\kappa_x$ is negative. Intuitively, a higher κ_x makes the agent attach lower values to his future consumption, and hence encourages him to dis-save out of his financial wealth at a higher rate by consuming more now, *ceteris paribus*. Note that the rate κ_x at which he dis-saves out of his financial wealth is independent of his labor income. Now consider the alternative assumption: $\kappa_x < 0$. Then, the MPS out of financial wealth is positive. In such a setting, the agents' incentive to save is stronger. While this certainly helps to generate a concentrated wealth distribution in equilibrium, in dynastic model, it leads to non-stationarity. However, in overlapping generation models with finitely lived agents, a negative κ_x may be desirable in terms of generating a more concentrated wealth distribution than income distribution. This to some extent is in line with the empirical observation that the rich save more as documented by Dynan et al. (2004) and others.

For a general affine income process (1), the savings rate (28) may be decomposed into the three components:

$$s_t^* = \pi_t + f_t - j_t, \tag{31}$$

where

$$f_t = \frac{\kappa_y y_t - \alpha_y}{r + \kappa_y}, \tag{32}$$

$$j_t = \kappa_x (x_t + h_t) + \frac{\beta_0 - r}{\gamma r} = \frac{\beta_0 + \kappa_x \gamma r (x_t + h_t) - r}{\gamma r} = \frac{\beta (r (x_t + h_t)) - r}{\gamma r}, \tag{33}$$

and $\beta(c) = \beta_0 + \beta_c c$ is the stochastic discounting function given in (7). The first term π_t is the precautionary savings and is given in (25).

The second term f_t captures the agent's motive of smoothing consumption over time, even in the absence of stochastic shocks. If the agent's saving $s_t^* = f_t$, then his behavior is completely characterized by the permanent-income hypothesis of Friedman (1957) and the martingale consumption model of Hall (1978). Campbell (1987) dubbed this behavior "savings for a rainy day." This saving component exists in any forward-looking consumption model. It is worth emphasizing that "savings for a rainy day" as defined in Campbell (1987) and here is *unrelated* to precautionary savings and is purely driven by the *expected* changes of income over time. Therefore, the key determinant of saving f_t for rainy days is the persistence of the income process. If $\kappa_y \leq 0$, the agent's income grows over time in expectation, and hence he borrows against his future income ($f_t < 0$). If $\kappa_y > 0$, income is stationary, and then "savings for a rainy day" is

$$f_t = \frac{\kappa_y}{r + \kappa_y} (y_t - \bar{y}), \tag{34}$$

where

$$\bar{y} = \frac{\alpha_y}{\kappa_y} = \frac{\alpha + \lambda_0 \delta_1}{\kappa - \lambda_1 \delta_1} \tag{35}$$

is the long-run mean of income. Equivalently, and more intuitively, I may express (34), using "human" wealth, as

$$f_t = y_t - r h_t = \kappa_y (h_t - \bar{h}), \tag{36}$$

where the long-run mean $\bar{h}$ of human wealth is simply equal to the perpetuity of long-run mean $\bar{y}$ of income, in that $\bar{h} = \bar{y}/r$. When $h_t > \bar{h}$, the agent's human wealth is higher than his long-run mean $\bar{h}$, and hence he rationally saves a portion κ_y of his human wealth in excess of long-run mean $\bar{h}$ in anticipation of future "rainy" days.

The last term j_t captures the dissavings due to impatience. This part of the saving reflects the agent's intertemporal motives of smoothing consumption, even when his income is deterministic. Note that the effect of randomness of income on consumption is fully absorbed into precautionary savings demand π_t. As a result, the dissavings due to impatience must be affine in $(x_t + h_t)$, the sum of financial and human wealth. Unlike the time-additive separable CARA utility, this term j_t is stochastic and increases in the agent's "total" wealth $(x_t + h_t)$. With $\beta_c > 0$, the agent's dissaving increases with his "total" wealth $(x_t + h_t)$. This induces a mean reverting wealth process and hence stationary wealth distribution. This term j_t captures the intuition that richer agents are more impatient and hence dis-save more.

Having derived and analyzed the optimal consumption and saving rules, I next study the implications on the equilibrium cross-sectional distribution of wealth and income.

4 Equilibrium Distribution of Income and Wealth

I provide a recursive and analytical approach to characterize the joint distribution of wealth and income. I first provide a description of equilibrium and the determination of the equilibrium interest rate.[16] Then, I solve the cross-sectional distribution of wealth and income in closed form by using the equilibrium restriction that (i) the cross-sectional distribution of wealth and income and (ii) the long-run stationary distribution of individual's wealth and income are the same. The individual's long-run stationary distribution may be solved from his joint wealth and income evolution dynamics. I further show that the cross-sectional wealth is less skewed and less fat-tailed than the cross-sectional income in these models, due to buffer-stock saving behavior.

4.1 Equilibrium

Consider a continuum of individual agents, whose preferences and income (endowment) are introduced in Section 2 in an equilibrium setting. These agents are *ex ante* identical, but *ex post* generally different in both asset holdings and income. Since we are interested in

[16] See Ljungqvist and Sargent (2004) for extensive textbook treatment on these Bewley-style equilibrium wealth distribution models. The equilibrium description is related to Aiyagari (1994) and Huggett (1993).

the steady state cross-sectional distribution, we choose the initial distribution of wealth and income to be the steady state one. We thus will have a stationary economy where the individual agents' wealth and income move over time, and the cross-sectional distribution and aggregate quantities remain invariant at all times. The equilibrium environment is a stationary pure-exchange economy with a fixed supply of a risk-free asset, which the agent uses as the saving instrument. The equilibrium interest rate is determined by market clearing. The insights delivered in this paper may be also obtained in a production economy used in Aiyagari (1994) by introducing a neoclassical production function and thus using capital as the saving instrument.

We now provide a formal description of equilibrium.

Definition 3 *A stationary equilibrium is an interest rate r, an optimal saving rule $g(x,y)$, and a stationary cross-sectional distribution $\Phi(x,y)$ of wealth and income for which*

- *the saving rule $g(x,y)$ is optimal from the individual agent's perspective;*
- *the stationary distribution $\Phi(x,y)$ is implied by the stationary distribution of income and the optimal saving rule $g(x,y)$;*
- *The risk-free asset market clears at all time, in that*

$$\int\int \Phi(x,y)g(x,y)\,dx\,dy = 0\,. \tag{37}$$

Without loss of generality, I may normalize the initial total endowment of the risk-free asset to zero, as in Huggett (1993). That is, I consider a pure-exchange environment. An equilibrium production economy similar to Aiyagari (1994) may also be constructed.[17]

Next, I sketch out the equilibrium implications on the saving rule (28). Specifically, I show that the saving rule (28) implies that the precautionary saving demand is stochastic, and the wealth process is stationary. Moreover, the saving rule captures the intuition that wealth is a weighted average of past incomes.

[17]Alternatively, we may also support a net positive supply of assets in the economy. The economy will have the stationary equilibrium, provided that the initial cross-sectional distribution of wealth endowment is consistent with the steady-state cross-sectional distribution of wealth and income. One way to think about positive aggregate wealth in an exchange economy is to view these positive initial wealth endowment as the ownership endowment of a tree, which drops a continuous flow of dividend at a constant rate, which is the risk-free rate. Details are available upon request.

4.2 Stochastic Precautionary Saving and Stationary Wealth

In equilibrium, the aggregate saving (in flow terms) is zero. Plugging (28) into the market equilibrium condition (37) gives

$$-\kappa_x \bar{x} + \psi \bar{y} + \kappa_0 = 0\,. \tag{38}$$

The above equation thus defines the equilibrium interest rate.[18] Using (38), I may write the agent's saving rule (28) (evaluated at the equilibrium interest rate) as follows:

$$s_t = -\kappa_x \left(x_t - \bar{x}\right) + \psi \left(y_t - \bar{y}\right)\,, \tag{39}$$

where $\bar{x}$ is the average wealth and $\bar{y}$ is the average income. Because the economy is normalized with a unit measure of agents, aggregate quantities such as wealth and income are the same as the corresponding average quantities. Because all agents are *ex ante* identical in the model, $\bar{x}$ may be naturally interpreted as the target wealth. Note that agents discount future more when their wealth is higher ($\beta_c > 0$). Hence, the wealth process is stationary as seen from a negative marginal propensity to save (MPS) out of wealth ($\kappa_x > 0$), consistent with the notion of buffer stock.[19]

The saving rule (39) has two intuitive features: (i) wealth serves as a *buffer*, fluctuating around a finite "target" level $\bar{x}$; (ii) the agent saves out of his current income if his income is larger than the average ($y_t > \bar{y}$), and otherwise dissaves. These two key properties of the saving rule are precisely captured by the two saving components: the precautionary saving term π_t and dissaving due to impatience term j_t, respectively. The *stochastic* precautionary saving π_t reflects the conditional *heteroskedasticity* of the income process, which makes the MPC ratio less than unity ($a_h < 1$). Recall that richer agents have stronger demand for dissaving due to a greater magnitude of impatience as seen from j_t given in (33).

The linear equilibrium saving rule (39) substantially simplifies the analysis of the dispersion of the agent's endogenous wealth distribution relative to his stationary income distribu-

[18]The equilibrium market clearing condition is analogous to that in standard Bewley models (Aiyagari (1994)).

[19]By contrast, if the agent's impatience decreases in consumption ($\beta_c < 0$), then the agent with a higher level of consumption is more patient, and accumulates more assets. This implies that the wealth process is not stationary, and suggests that wealth inequality widens up without bound over time in an infinite-horizon equilibrium model.

tion. Equation (39) implies that the associated stationary wealth process x satisfies[20]

$$x_t - \bar{x} = \psi \int_{-\infty}^{t} e^{-\kappa_x (t-s)} \left(y_s - \bar{y}\right) ds. \tag{40}$$

Without loss of generality, we choose current time to be 0 and suppose that the agent's wealth distribution have reached steady state. We may then re-write (40) as follows (with $t = 0$ and $u = -s$):

$$x_0 - \bar{x} = \frac{\psi}{\kappa_x} \int_{0}^{\infty} wt(u) \left(y_{-u} - \bar{y}\right) du, \tag{41}$$

where the weight $wt(u)$ for $(y_{-u} - \bar{y})$ is given by $wt(u) = \kappa_x e^{-\kappa_x u}$ and sums to unity, in that $\int_0^\infty wt(u)du = 1$. Equation (41) states that the deviation of long-run wealth from its mean, $(x_0 - \bar{x})$ is proportional to a weighted sum of $(y_{-u} - \bar{y})$, the deviation of his past income y_{-u} from its long-run mean $\bar{y}$. First, the weight $wt(u)$ for the income deviation from its long-run mean, $(y_{-u} - \bar{y})$ decays exponentially in "time distance" u at the rate of κ_x. For the agent whose discount rate increases more with his past consumption (a higher β_c), the agent dis-saves at a higher rate $\kappa_x = \beta_c/\gamma$. As a result, a "stronger" averaging effect induces a lower wealth dispersion, *ceteris paribus.* On the other hand, when the income risk increases with the level of income (conditional heteroskedasticity), the agent's precautionary motive will induce the agent to save more out of income, *ceteris paribus.* This suggests a higher MPS ψ out of income, which in turn implies a more dispersed wealth distribution relative to income. This can been seen from (41) where wealth in excess of its long-run mean, $(x_0 - \bar{x})$, is a constant multiple ψ/κ_x of $\int_0^\infty wt(u) \left(y_{-u} - \bar{y}\right) du$, with the multiple increasing in the MPS ψ. In Section 4.3, we will return to these two opposing effects: (i) dispersion reduction of stationary wealth relative to stationary income due to buffer stock saving ($\kappa_x > 0$) and (ii) high precautionary saving induced by a risky (conditionally heteroskedastic) income process.[21]

In order to deepen our insights, I next provide an analytical characterization for the cross-sectional stationary distribution of wealth and income.

4.3 Cross Sectional Distribution

Let $\begin{pmatrix} X & Y \end{pmatrix}^T$ be the random vector that has the cross-sectional stationary distribution $\Phi(x, y)$ of wealth and income, and $\mu = \begin{pmatrix} \mu_X & \mu_Y \end{pmatrix}' = \begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix}'$ be the corresponding first moment.

[20] Using the definition of saving $s_t = \dot{x}_t$, we apply integration by parts to the formula (39) and obtain

$$d[e^{\kappa_x t} \left(x_t - \bar{x}\right)] = \psi \left(y_t - \bar{y}\right) e^{\kappa_x t} dt.$$

Integrating the above from $-\infty$ to current time t yields (40).

[21] I am grateful to the referee for suggesting this line of discussions.

Note that μ_X is equal to the target wealth $\bar{x}$ and μ_Y is equal to the long-run income $\bar{y}$ of a representative individual. Let $M_{i,j}$ be the moment of the joint distribution of income and wealth, defined below:

$$M_{i,j} \equiv E\left[(X-\mu_X)^i\,(Y-\mu_Y)^j\right]\,, \quad \text{for } \; i,j = 0,1,\cdots\,. \tag{42}$$

Infinite-horizon and stationarity assumptions together imply that the stationary cross-sectional distribution is the same as the individual's steady-state distribution. Using this equivalence, I compute the cross-sectional distribution of wealth and income, by working with the dynamics of individual income and wealth. Let w and v denote de-meaned wealth and de-meaned income, respectively, in that $w_t = x_t - \mu_X$, and $v_t = y_t - \mu_Y$.

The joint dynamics of de-meaned wealth and de-meaned income (w_t and v_t) may be written as

$$d\begin{pmatrix} w_t \\ v_t \end{pmatrix} = \left[\begin{pmatrix} -\kappa_x & \psi \\ 0 & -\kappa \end{pmatrix}\begin{pmatrix} w_t \\ v_t \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{\alpha} \end{pmatrix}\right] dt + \begin{pmatrix} 0 \\ \sqrt{\bar{l}_0 + l_1 v_t} \end{pmatrix} dW_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dZ_t, \tag{43}$$

where $\bar{\alpha} = \alpha - \kappa\mu_Y$, and $\bar{l}_0 = l_0 + l_1\mu_Y$. The de-meaned income process v jumps at a stochastic intensity $\bar{\lambda}_0 + \lambda_1 v$, where $\bar{\lambda}_0 = \lambda_0 + \lambda_1\mu_Y$.

Appendix B supplies the details for the complete characterization of the cross moments $M_{i,j}$ defined in (42) for any non-negative integers i and j. It shows that the cross moment $M_{i,j}$ is a linear combination of lower-order moments $M_{i,k}$, for $0 \le k \le j-1$, and the cross moment $M_{i-1,j+1}$, in that

$$M_{i,j} = \frac{1}{\mathcal{K}_{ij}}\left(P_1(j)\,M_{i,j-1} + P_2(j)\,M_{i,j-2} + \sum_{n=3}^{j} P_n(j)\,M_{i,j-n} + i\psi\,M_{i-1,j+1}\right), \tag{44}$$

where $\mathcal{K}_{ij}$, $P_1(j)$, $P_2(j)$, and $P_n(j)$ are given by (B.3), (B.4), (B.5), and (B.6). Since (44) also applies to $M_{i-1,j+1}$ and $M_{i,k}$, it is straightforward to conclude that $M_{i,j}$ may be written as a linear combination of the cross-sectional income moments $M_{0,k}$, for $0 \le k \le i+j$. This result is intuitive and useful. It states that the cross moment $M_{i,j}$ whose total order is $i+j$ is given by a weighted sum of income moments up to the order $(i+j)$. As a special case of this result ($j = 0$), the wealth moment of order i is only related to the income moments up to order i, but does not depend on any income moments higher than order i.

Having developed the general recursive formulation of moments for the joint distribution, I now analyze some specific implications of the model.

Corollary 1 *The wealth moments are related to the moments of the joint wealth and income as follows:*

$$M_{i,0} = \frac{\psi}{\kappa_x} M_{i-1,1}, \tag{45}$$

$$M_{i,1} = \frac{i\psi}{i\kappa_x + \kappa_y} M_{i-1,2}. \tag{46}$$

An immediate implication of the above two equations is

$$M_{i,0} = \frac{(i-1)\psi^2}{\kappa_x[(i-1)\kappa_x + \kappa_y]} M_{i-2,2}, \quad i \geq 2. \tag{47}$$

Corollary 1 imposes a set of linear testable restrictions between the stationary moment $M_{i,0}$ of wealth and the cross-moment $M_{i-1,1}$, for any $i \geq 2$. Using Corollary 1, I show that the mean-reversion rates of income and wealth (κ_x and κ), and the MPS ψ out of income determine the correlation coefficient ρ and the variance ratio σ_X^2/σ_Y^2 between wealth and income.

Corollary 2 *The variance ratio σ_X^2/σ_Y^2 and the correlation coefficient ρ between income and wealth are given by*

$$\frac{\sigma_X^2}{\sigma_Y^2} = \frac{\psi^2}{\kappa_x(\kappa_x + \kappa_y)}, \tag{48}$$

$$\rho = \sqrt{\frac{\kappa_x}{\kappa_x + \kappa_y}} = \sqrt{\frac{1}{1+\eta}}, \tag{49}$$

respectively, where

$$\eta = \kappa_y/\kappa_x. \tag{50}$$

The marginal propensity ψ to save out of income is in general positive. This implies a positive correlation between income and wealth: $\rho > 0$. Equation (49) implies that the correlation coefficient ρ decreases with η. A more persistent (lower κ_y) income process implies that current income is more correlated with the past income. Since wealth is accumulated out of past income, this suggests a higher correlation between current income and wealth, *ceteris paribus.*

The variance ratio σ_X^2/σ_Y^2 is a natural measure of cross-sectional wealth dispersion relative to cross-sectional income. Clearly, the MPS ψ out of income plays a crucial role in determining this variance ratio. When income shocks are more conditionally heteroskedastic (a higher l_1 or λ_1), the MPS ψ is higher. The greater incentive to accumulate wealth will in turn make wealth more volatile than income, giving rise to a higher variance ratio σ_X^2/σ_Y^2. However, the variance ratio does not capture higher-order effects such as *relative* skewness and *relative* fat-tails of the wealth distribution relative to the income distribution. One way to capture these results is to use the following metric:

Definition 4 *A measure of relative wealth-to-income inequality is*

$$J_i \equiv \frac{E\left[\left(X-\mu_X\right)/\sigma_X\right]^i}{E\left[\left(Y-\mu_Y\right)/\sigma_Y\right]^i} = \frac{M_{i,0}/\sigma_X^i}{M_{0,i}/\sigma_Y^i}. \tag{51}$$

Trivially by construction, $J_2 = 1$. The coefficients J_3 and J_4 are the wealth-to-income skewness and wealth-to-income kurtosis ratios, respectively. These *relative* dispersion measures have controlled for the difference between the variance of wealth σ_X^2 and the variance of income σ_Y^2, because these measures are based on "standardized" wealth, $\left(X-\mu_X\right)/\sigma_X$ and "standardized" income $\left(Y-\mu_Y\right)/\sigma_Y$, where "standardized" random variables refer to those with zero mean and unity variance.

Using the variance ratio (48), we may write "standardized" stationary wealth as follows:

$$\frac{x_0-\mu_X}{\sigma_X} = \sqrt{1+\eta}\int_0^\infty wt(u)\left(\frac{y_{-u}-\mu_Y}{\sigma_Y}\right)du, \tag{52}$$

where $\eta = \kappa_y/\kappa_x$. Equation (52) implies that the MPS ψ does not directly affect the distribution of "standardized" stationary wealth. This is because a higher MPS ψ out of income increases both the variance and higher order moments for stationary wealth in such a way that it does not affect the moments for the "standardized" wealth directly. Equation (52) states that the standardized wealth is proportional to a weighted average of past "standardized" incomes, with a multiple larger than unity. On one hand, the multiple $\sqrt{1+\eta}$ being larger than unity indicates that the "standardized" wealth is more dispersed than the weighted average of past "standardized" income, *ceteris paribus*. Moreover, a higher degree of mean reversion for income (a higher κ_y and hence a higher $\eta = \kappa_y/\kappa_x$) implies a larger multiple. On the other hand, averaging past "standardized" incomes that are mean reverting further reduces dispersion for wealth. Therefore, the moments for the standardized wealth may be either larger or small than the (corresponding) moments for the standardized income, depending on whether the "averaging" effect is stronger than the "multiple" effect or not. The net effect of mean reversion on the relative dispersion of wealth to income is indeterminate in general.

I next present a few example economies that provide explicit solutions to the skewness and kurtosis moments, using the general recursive formulation (44) for the moments of the joint distribution. We show that stationary cross-sectional (standardized) wealth is less skewed and less fat-tailed than stationary cross-sectional (standardized) income because the "averaging" effect turns out to be stronger than the "multiple" effect in the example economies studied below.

4.4 A Gaussian Model of Income

If the income process (1) is conditionally homoskedastic without jumps, namely, an Ornstein-Uhlenbeck process, then (43) is simplified to a first-order SDE, in that

$$d\begin{pmatrix} w_t \\ v_t \end{pmatrix} = \begin{pmatrix} -\kappa_x & \psi \\ 0 & -\kappa \end{pmatrix}\begin{pmatrix} w_t \\ v_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma_0 \end{pmatrix} dW_t, \tag{53}$$

where $\bar{\alpha} = 0$ by equilibrium restriction. Note that the MPS ψ out of income is given by $\psi = (\kappa - \kappa_x)/(r + \kappa)$. Because the bi-variate process of income and wealth is conditionally homoskedastic and Gaussian, the *stationary* distribution $\Phi(x, y)$ of wealth and income is bi-variate normal.[22] The joint normality implies that both the marginal distribution of wealth and that of income are Gaussian. Corollary 2 gives the variance ratio and correlation between wealth and income, which then completely characterize the joint income-wealth distribution.

Obviously, this example provides a counter-factual prediction on the cross sectional joint distribution, as neither cross-sectional wealth nor income is normally distributed empirically. However, it gives a good benchmark against which we may think about the determinants of the skewness and fat-tails of the wealth distribution.

4.5 A Conditionally Homoskedastic Jump-Diffusion Model of Income

One simple way to incorporate skewness and kurtosis into income is to generalize an Ornstein-Uhlenbeck income process as in (53) by adding a conditionally homoskedastic jump component. That gives a special case of (1) with $l_1 = \lambda_1 = 0$.

Appendix B.1 shows that the skewness and excess kurtosis ratios are given by

$$\frac{S_X}{S_Y} = \frac{2\sqrt{(1+\eta)^3}}{(2\eta+1)(\eta+2)} \leq 1, \tag{54}$$

$$\frac{K_X}{K_Y} = \frac{3(1+\eta)}{(1+3\eta)(3+\eta)} \leq 1, \tag{55}$$

respectively. Therefore, $J_3 = S_X/S_Y \leq 1$ and $J_4 = (K_X + 3)/(K_Y + 3) \leq 1$, for all $\eta > 0$. This states that cross-sectionally, wealth is less skewed and less fat-tailed than income, in this self-insurance-based equilibrium model. As we noted earlier, the MPS ψ out of income does not matter for either the skewness ratio J_3 or the kurtosis ratio J_4. This is because the MPS ψ does not matter for the distribution of "standardized" stationary wealth.

[22]See Karatzas and Shreve (1991), or Appendix D in Duffie (2001).

This jump model implies that $\psi = (\kappa - \kappa_x)/(r + \kappa)$, and thus predicts a constant precautionary savings demand. A positively-skewed jump distribution ν implies a positively-skewed income distribution, which endogenously generates a positively-skewed wealth distribution. Empirically, both income and wealth are positively skewed. This model captures the intuition that a skewed income distribution may lead to a skewed wealth distribution.

4.6 A Conditionally Heteroskedastic Model of Income

Although Subsection 4.5 generates a model of skewed and fat-tailed wealth distribution, the associated optimal consumption rule predicts a constant precautionary savings demand. However, in general, precautionary saving is stochastic and depends on the level of wealth and income. Next, I propose a model that generates stochastic precautionary saving by using a conditionally heteroskedastic income process, a special case of (1), in that

$$dy_t = \kappa\,(\theta - y_t)\,dt + \sigma\sqrt{y_t}\,dW_t. \tag{56}$$

Since the conditional variance of changes in income is proportional to the level of income, the precautionary savings demand induces a *lower* MPC out of human wealth than out of financial wealth.[23] The stationary distribution for the income process (56) is a Gamma distribution. Appendix B.2 shows that the skewness and kurtosis ratios are given by

$$\frac{S_X}{S_Y} = \frac{2\sqrt{1+\eta}}{2+\eta} \leq 1, \tag{57}$$

$$\frac{K_X}{K_Y} = \frac{5\eta + 6}{(3+\eta)(2+\eta)} \leq 1. \tag{58}$$

Therefore, $J_3 = S_X/S_Y \leq 1$ and $J_4 = (K_X + 3)/(K_Y + 3) \leq 1$. This again confirms our intuition that cross-sectionally, wealth is less skewed and less fat-tailed than income.

In this section, I have provided a complete characterization for the equilibrium cross-sectional distribution of wealth and income by developing a recursive formulation for the moments of the joint distribution of wealth and income. Using the recursive formulation, I have illustrated that cross-sectional wealth is less skewed and less fat-tailed than cross-sectional income by working out the details for several model economies.

[23] We impose the parametric restriction $2\kappa\theta \geq \sigma^2$ to ensure that the boundary of zero income is never visited in finite time, and therefore, income always stays positive. See Feller (1951) and Cox, Ingersoll, Jr., and Ross (1985).

5 Conclusions

This paper develops an incomplete markets consumption-saving model and then derives the equilibrium cross-sectional distribution of wealth and income in closed form. It first proposes a general income process, known as an affine (jump diffusion) process, which allows for both conditional heteroskedasticity of income changes and jumps in income. This paper then derives an explicit buffer-stock saving rule for agents with inter-temporally dependent preferences as in Uzawa (1968) and Obstfeld (1990). The consumption model has the desirable property of stochastic precautionary savings.

This paper then provides an analytical solution for the stationary distribution of wealth and income in a heterogeneous agent (Bewley-style) economy, by exploiting the analytically tractable but also rich "affine" structure of the model. The analytical feature of the model allows us to show that the individual agent's wealth is a weighted average of past incomes. In equilibrium, the individual's long-run steady-state distribution is equal to the cross-sectional distribution. The impatience (increasing with past consumption) helps to generate a cross-sectional distribution for (standardized) wealth that is smoother than the cross-sectional distribution for (standardized) income. While income volatility naturally feeds into the higher order moments of wealth, the persistence of income shocks and the mean reversion of wealth (induced by the impatience assumption) are the main determinants of cross-sectional wealth-income correlation and the relative dispersion of "standardized" wealth to "standardized" income, such as the relative skewness and the relative kurtosis (fat-tails) of the cross-sectional wealth to the cross-sectional income. I develop these new insights and results on wealth distribution by providing an analytical recursive formulation for the moments of the joint distribution of income and wealth. This recursive formulation of the moments completely characterizes the joint distribution of wealth and income. The analytical approach developed here provides a complementary perspective to the existing literature towards the understanding of equilibrium wealth distribution.

Appendices

A Proof of Proposition 1

This appendix derives the optimal consumption rule (12) reported in Proposition 1. I conjecture that the value function $J(x,y)$ takes an exponential-affine form:

$$J(x,y) = -\frac{1}{\gamma\omega}\exp\left[-\gamma\omega\left(x + a_y y + a_0\right)\right], \tag{A.1}$$

where ω, a_y, and a_0 are constant coefficients to be determined. The first-order condition (11) implies that

$$\bar{c} = \omega\left(x + a_y y + \bar{a}\right), \tag{A.2}$$

where

$$\bar{a} = -\frac{1}{\gamma\omega}\log\left(1 - \frac{\beta_c}{\gamma\omega}\right) + a_0. \tag{A.3}$$

Plugging the implied values for $\bar{c}$, J_x, J_y, and J_{yy} into (9) leaves

$$\begin{aligned} 0 = &-\frac{1}{\gamma}\left(1 - \frac{\beta_c}{\gamma\omega}\right) + \left[\beta_0 + \beta_c\omega\left(x + a_y y + \bar{a}\right)\right]\frac{1}{\gamma\omega} + \left[(r-\omega)x + \left(1 - \omega a_y\right)y - \omega\bar{a}\right] \\ &+ \left(\alpha - \kappa y\right)a_y + \frac{1}{2}\left(l_0 + l_1 y\right)\times\left(-\gamma\omega a_y^2\right) - \left(\lambda_0 + \lambda_1 y\right)\frac{1}{\gamma\omega}E\left[e^{-\gamma\omega a_y q} - 1\right]. \end{aligned} \tag{A.4}$$

Because the above equality holds for any wealth x and income y, we thus have

$$0 = \beta_c\omega\frac{1}{\gamma\omega} + (r-\omega), \tag{A.5}$$

$$0 = \frac{\beta_c\omega a_y}{\gamma\omega} + \left(1 - \omega a_y\right) - \kappa a_y - \frac{\gamma\omega a_y^2 l_1}{2} - \frac{\lambda_1}{\gamma\omega}\left[\zeta(-\gamma\omega a_y) - 1\right], \tag{A.6}$$

$$0 = -\frac{1}{\gamma}\left(1 - \frac{\beta_c}{\gamma\omega}\right) + \frac{\beta_0 + \beta_c\omega\bar{a}}{\gamma\omega} - \omega\bar{a} - \frac{\gamma\omega a_y^2 l_0}{2} - \frac{\lambda_0}{\gamma\omega}\left[\zeta(-\gamma\omega a_y) - 1\right], \tag{A.7}$$

where $\zeta(\,\cdot\,)$ is the Laplace transform of the jump distribution. Simplifying (A.5), (A.6) and (A.7) gives the form of the optimal consumption rule, reported from (12) to (20).

B Calculating Cross Moments

This appendix characterizes the cross-sectional distribution of income and wealth by computing the moments. I calculate all moments $M_{i,j}$, by exploiting the recursive structure in the drift function for the dynamics of $\{w_t^i v_t^j\}$, for non-negative integers i and j.

Under technical regularity conditions, Ito's formula implies that the drift function $D_t(i,\, j)$ of $w_t^i v_t^j$ is

$$\begin{aligned} D_t(i,j) =\; & i w_t^{i-1} v_t^j \left(-\kappa_x w_t + \psi v_t\right) + j w_t^i v_t^{j-1} \left(\bar{\alpha} - \kappa v_t\right) + \frac{1}{2} j \left(j-1\right) w_t^i v_t^{j-2} \left(\bar{l}_0 + l_1 v_t\right) \\ & + \left(\bar{\lambda}_0 + \lambda_1 v_t\right) w_t^i E\left[\left(v_t + q\right)^j - v_t^j\right], \end{aligned} \tag{B.1}$$

where q is the pure jump size drawn from the distribution ν, and E is taken with respect to the jump distribution. Using the binomial expansion, (B.1) implies that

$$\begin{aligned} D_t(i,j) =& -\left(i\kappa_x + j\kappa\right)\left(w_t^i v_t^j\right) + j\bar{\alpha}\left(w_t^i v_t^{j-1}\right) + \frac{l_1}{2} j(j-1)\left(w_t^i v_t^{j-1}\right) \\ & + \frac{\bar{l}_0}{2} j(j-1)\left(w_t^i v_t^{j-2}\right) + i\psi(w_t^{i-1} v_t^{j+1}) + \left(\bar{\lambda}_0 + \lambda_1 v_t\right)\sum_{n=0}^{j-1}\binom{j}{n}\delta_{j-n}\left(w_t^i v_t^n\right) \\ =& -\mathcal{K}_{ij}(w_t^i v_t^j) + P_1(j)\left(w_t^i v_t^{j-1}\right) + P_2(j)\left(w_t^i v_t^{j-2}\right) + \sum_{n=3}^{j} P_n(j)\left(w_t^i v_t^{j-n}\right) \\ & + i\psi(w_t^{i-1} v_t^{j+1}), \end{aligned} \tag{B.2}$$

where

$$\mathcal{K}_{ij} = i\kappa_x + j\kappa_y, \tag{B.3}$$

$$P_1(j) = j\left(\bar{\alpha} + \bar{\lambda}_0\delta_1\right) + \binom{j}{2}\left(l_1 + \lambda_1\delta_2\right) = \binom{j}{2}\left(l_1 + \lambda_1\delta_2\right), \tag{B.4}$$

$$P_2(j) = \binom{j}{2}\left(\bar{l}_0 + \bar{\lambda}_0\delta_2\right) + \binom{j}{3}\lambda_1\delta_3, \tag{B.5}$$

$$P_n(j) = \binom{j}{n}\bar{\lambda}_0\delta_n + \binom{j}{n+1}\lambda_1\delta_{n+1}, \quad 3 \le n \le j, \tag{B.6}$$

and $\kappa_y = \kappa - \lambda_1\delta_1$. Equation (B.4) follows $\bar{\alpha} + \bar{\lambda}_0\delta_1 = \alpha_y - \kappa_y\mu_Y = 0$, using the steady-state stationarity condition $\mu_Y = \alpha_y/\kappa_y$. The parameter $\mathcal{K}_{ij}$ may be viewed as the rate of mean reversion for $\{w_t^i v_t^j\}$. It is equal to the sum of income and wealth mean reversion rates, multiplied by their corresponding power.

Equation (B.2) implies that

$$\begin{aligned} E\left(w_t^i v_t^j\right) &= e^{-\mathcal{K}_{ij}t} E\left(w_0^i v_0^j\right) + \int_0^t e^{-\mathcal{K}_{ij}(t-s)} Q(j)\, ds \\ &= e^{-\mathcal{K}_{ij}t} E\left(w_0^i v_0^j\right) + \frac{1}{\mathcal{K}_{ij}}\left(1 - e^{-\mathcal{K}_{ij}t}\right) Q(j), \end{aligned} \tag{B.7}$$

where

$$Q(j) = P_1(j) E\left(w_s^i v_s^{j-1}\right) + P_2(j) E\left(w_s^i v_s^{j-2}\right) + \sum_{n=3}^{j} P_n(j) E\left(w_s^i v_s^{j-n}\right) + i\psi E\left(w_s^{i-1} v_s^{j+1}\right),$$

and E is taken with respect to the cross-sectional stationary distribution $\Phi(x,\, y)$. Taking the limit $t \to \infty$ in (B.7) gives (44).

B.1 Moments Calculations for Section 4.5

The second, third, and fourth centered moments of stationary income are

$$\sigma_Y^2 = M_{02} = \frac{\sigma_0^2 + \lambda_0\delta_2}{2\kappa}, \quad M_{03} = \frac{\lambda_0\delta_3}{3\kappa}, \quad M_{04} = \frac{\lambda_0\delta_4}{4\kappa} + 3M_{02}^2.$$

Equation (47) implies that

$$M_{30} = \frac{2\psi^2}{\kappa_x(2\kappa_x + \kappa)} M_{12} = \frac{2\psi^3}{\kappa_x(2\kappa_x + \kappa)(\kappa_x + 2\kappa)} M_{03}, \tag{B.8}$$

where the second equality follows from

$$M_{12} = \frac{\psi}{\kappa_x + 2\kappa} M_{03}, \tag{B.9}$$

using (44). The skewness S_X of stationary wealth is then given by

$$S_X \equiv \frac{M_{30}}{\sigma_X^3} = \frac{2\sqrt{\kappa_x(\kappa + \kappa_x)^3}}{(2\kappa_x + \kappa)(\kappa_x + 2\kappa)} S_Y, \tag{B.10}$$

where the skewness S_Y of stationary income is given by

$$S_Y \equiv \frac{M_{03}}{\sigma_Y^3} = \frac{2\sqrt{2\kappa}}{3} \frac{\lambda_0\delta_3}{\left(\sigma_0^2 + \lambda_0\delta_2\right)^{3/2}}. \tag{B.11}$$

Similarly, (47) implies that

$$M_{40} = \frac{3\psi^2}{\kappa_x(3\kappa_x + \kappa)} M_{22}, \tag{B.12}$$

where applying (44), (45), and (46) leaves

$$M_{22} = \frac{1}{2(\kappa_x + \kappa)} \left[\left(\sigma_0^2 + \lambda_0\delta_2\right) \frac{\psi^2}{\kappa_x(\kappa_x + \kappa)} M_{02} + 2\psi M_{13} \right], \tag{B.13}$$

$$M_{13} = \frac{1}{\kappa_x + 3\kappa} \left[3\left(\sigma_0^2 + \lambda_0\delta_2\right) \frac{\psi}{\kappa_x + \kappa} M_{02} + \psi M_{04} \right], \tag{B.14}$$

$$M_{21} = \frac{2\psi}{2\kappa_x + \kappa} M_{12} = \frac{2\psi^2}{(2\kappa_x + \kappa)(\kappa_x + \kappa)} M_{03}. \tag{B.15}$$

Therefore,

$$M_{40} = \frac{3\lambda_0\delta_4\psi^4}{4\kappa_x\kappa(3\kappa_x + \kappa)(3\kappa + \kappa_x)(\kappa_x + \kappa)} + 3M_{20}^2. \tag{B.16}$$

The excess kurtosis K_X of wealth is then given by

$$K_X \equiv \frac{M_{40}}{M_{20}^2} - 3 = \frac{3\kappa_x(\kappa + \kappa_x)}{(\kappa + 3\kappa_x)(3\kappa + \kappa_x)} K_Y, \tag{B.17}$$

where the excess kurtosis K_Y of income is

$$K_Y = \frac{M_{04}}{M_{02}^2} - 3 = \frac{\lambda_0\delta_4\kappa}{(\sigma_0^2 + \lambda_0\delta_2)^2} \geq 0. \tag{B.18}$$

B.2 Moments Calculations for Section 4.6

The stationary distribution for (56) is Gamma with a rate parameter $\lambda = 2\kappa/\sigma^2$ and a scale parameter $\nu = 2\kappa\theta/\sigma^2$, in that

$$f_\infty(y) = \frac{\lambda}{\Gamma(\nu)} (\lambda y)^{\nu-1} e^{-\lambda y}, \tag{B.19}$$

where $\Gamma(\,\cdot\,)$ is the Gamma function. The second, third and fourth centered moments of stationary income are

$$M_{02} = \frac{\sigma^2\theta}{2\kappa}, \quad M_{03} = \frac{\sigma^4\theta}{2\kappa^2}, \quad M_{04} = \frac{3\sigma^4\theta}{4\kappa^3}\left(\kappa\theta + \sigma^2\right) = \frac{3\sigma^6\theta}{4\kappa^3} + 3M_{02}^2.$$

Equation (47) implies that

$$M_{30} = \frac{2\psi^2}{\kappa_x(2\kappa_x+\kappa)} M_{12} = \frac{2\psi^3}{\kappa_x(2\kappa_x+\kappa)(\kappa_x+\kappa)} M_{03}, \tag{B.20}$$

where the second equality follows from (44), in that

$$M_{12} = \frac{\sigma^2 M_{11} + \psi M_{03}}{\kappa_x + 2\kappa} = \frac{1}{\kappa_x + 2\kappa}\left(\sigma^2 \frac{\psi}{\kappa_x+\kappa}\frac{\sigma^2\theta}{2\kappa} + \psi\frac{\sigma^4\theta}{2\kappa^2}\right) = \frac{\psi}{\kappa_x+\kappa} M_{03}.$$

The skewness S_X of stationary wealth is then given by

$$S_X \equiv \frac{M_{30}}{\sigma_X^3} = \frac{2\sqrt{\kappa_x(\kappa+\kappa_x)}}{2\kappa_x+\kappa} S_Y, \tag{B.21}$$

where the skewness of stationary income is $S_Y = m_{03}/\sigma_Y^3 = \sqrt{2\sigma^2}/\sqrt{\kappa\theta}$.

Similarly, (47) implies that the fourth moment M_{40} of wealth is given by

$$M_{40} = \frac{3\psi^2}{\kappa_x(3\kappa_x+\kappa)} M_{22}, \tag{B.22}$$

where applying (44), (45), and (46) leaves

$$M_{22} = \frac{1}{2(\kappa_x+\kappa)}\left(\sigma^2 M_{21} + \sigma^2\theta M_{20} + 2\psi M_{13}\right), \tag{B.23}$$

$$M_{13} = \frac{1}{\kappa_x+3\kappa}\left(3\sigma^2 M_{12} + 3\sigma^2\theta M_{11} + \psi M_{04}\right) = \frac{\psi}{\kappa_x+\kappa} M_{04}, \tag{B.24}$$

$$M_{21} = \frac{2\psi}{2\kappa_x+\kappa} M_{12} = \frac{2\psi^2}{(2\kappa_x+\kappa)(\kappa_x+\kappa)} M_{03}. \tag{B.25}$$

Together, the fourth moment of wealth is given by

$$M_{40} = \frac{3\psi^4\sigma^6\theta(5\kappa+6\kappa_x)}{4\kappa_x\kappa^3(3\kappa_x+\kappa)(2\kappa_x+\kappa)(\kappa_x+\kappa)^2} + 3M_{20}^2. \tag{B.26}$$

The excess kurtosis K_X of wealth is then given by

$$K_X \equiv \frac{M_{40}}{M_{20}^2} - 3 = \frac{\kappa_x(5\kappa+6\kappa_x)}{(3\kappa_x+\kappa)(2\kappa_x+\kappa)} K_Y, \tag{B.27}$$

where the excess kurtosis of income is $K_Y = M_{04}/\sigma_Y^4 - 3 = 3\sigma^2/\kappa\theta$.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics 109*(3), 659–684.

Benhabib, J. and A. Bisin (2006). The distribution of wealth and redistributive policies. Manuscript, New York University.

Bergman, Y. Z. (1985). Time preference and capital asset pricing model. *Journal of Financial Economics 14*, 145–59.

Bewley, T. (1986). Stationary monetary equilibrium with a continuum of independently fluctuating consumers. In W. Hildenbrand and A. Mas-Colell (Eds.), *Contributions to Mathematical Economics: In Honor of Gerard Debreu*, pp. 79–102. New York: North-Holland.

Blanchard, O. J. (1985, April). Debt, deficits, and finite horizons. *Journal of Political Economy 93*(2), 223–247.

Caballero, R. J. (1990). Consumption puzzles and precautionary savings. *Journal of Monetary Economics 25*(1), 113–136.

Caballero, R. J. (1991). Earnings uncertainty and aggregate wealth accumulation. *American Economic Review 81*(4), 859–871.

Cagetti, M. and M. De Nardi (2005a). Entrepreneurship, frictions, and wealth. Federal Reserve Bank of Chicago, WP 2005-9, *Journal of Political Economy*, forthcoming.

Cagetti, M. and M. De Nardi (2005b). Wealth inequality: data and models. Federal Reserve Bank of Chicago, WP 2005-10, *Macroeconomic Dynamics*, forthcoming.

Campbell, J. Y. (1987). Does saving anticipate declining labor income? An alternative test of the permanent income hypothesis. *Econometrica 55*(6), 1249–1273.

Campbell, J. Y. and A. Deaton (1989). Why is consumption so smooth? *Review of Economic Studies 56*, 357–74.

Carroll, C. D. (1997). Buffer-stock saving and the life cycle/permanent income hypothesis. *Quarterly Journal of Economics 112*(1), 1–55.

Carroll, C. D. and M. S. Kimball (1996, July). On the concavity of the consumption function. *Econometrica 64*(4), 981–992.

Castañeda, A., J. Díaz-Giménez, and J.-V. Ríos-Rull (2003). Accounting for the U.S. earnings and wealth inequality. *Journal of Political Economy 111*(4), 818–857.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*, 385–407.

Dai, Q. and K. J. Singleton (2000). Specification analysis of affine term structure models. *Journal of Finance 55*, 1943–78.

De Nardi, M. (2004). Wealth inequality and intergenerational links. *Review of Economic Studies 71*, 743–768.

Deaton, A. (1991). Saving and liquidity constraints. *Econometrica 59*(5), 1221–1248.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (Third ed.). Princeton: Princeton University Press.

Duffie, D., J. Pan, and K. J. Singleton (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica 68*, 1343–1376.

Dynan, K., J. Skinner, and S. P. Zeldes (2004). Do the rich save more? *Journal of Political Economy 112*(2), 397–444.

Epstein, L. G. (1983). Stationary cardinal utility and optimal growth under uncertainty. *Journal of Economic Theory 31*, 133–152.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics 54*, 173–182.

Flavin, M. (1981). The adjustment of consumption to changing expectations about future income. *Journal of Political Economy 89*, 974–1009.

Friedman, M. (1957). *A Theory of the Consumption Function*. Princeton: Princeton University Press.

Hall, R. E. (1978). Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence. *Journal of Political Economy 86*(6), 971–987.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control 17*(5-6), 953–969.

Huggett, M. (1996). Wealth distribution in life-cycle economies. *Journal of Monetary Economics 38*, 469–494.

Jacobson, L. S., R. J. LaLonde, and D. G. Sullivan (1993). Earnings losses of displaced workers. *American Economic Review 83*, 685–709.

Karatzas, I. and S. Shreve (1991). *Brownian Motion and Stochastic Calculus* (Second ed.). New York: Springer-Verlag.

Kimball, M. S. (1990). Precautionary saving in the small and in the large. *Econometrica 58*(1), 53–73.

Kimball, M. S. and N. G. Mankiw (1989). Precautionary saving and the timing of taxes. *Journal of Political Economy 97*, 863–79.

Koopmans, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica 28*, 287–309.

Koopmans, T. C., P. Diamond, and R. Williamson (1964). Stationary utility and time perspective. *Econometrica 32*, 82–100.

Krusell, P. and A. A. Smith (1998, October). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy 106*(5), 867–896.

Ljungqvist, L. and T. J. Sargent (2004). *Recursive Macroeconomic Theory* (Second ed.). Cambridge, MA: MIT Press.

Lucas, R. E. and N. L. Stokey (1984). Optimal growth with many consumers. *Journal of Economic Theory 32*, 139–171.

MaCurdy, T. E. (1982). The use of time series processes to model the error structure of earnings in a longitudinal data analysis. *Journal of Econometrics 18*, 83–114.

Merton, R. C. (1971). Optimal consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory 3*, 373–413.

Obstfeld, M. (1990). Intertemporal dependence, impatience, and dynamics. *Journal of Monetary Economics 26*, 45–75.

Obstfeld, M. and K. Rogoff (1996). *Foundations of International Macroeconomics.* Cambridge, MA: MIT Press.

Quadrini, V. (2000). Entrepreneurship, saving and social mobility. *Review of Economic Dynamics 3*, 1–40.

Quadrini, V. and J.-V. Ríos-Rull (1997). Understanding the U.S. distribution of wealth. *Federal Reserve Bank of Minneapolis Quarterly Review 21*(2), 22–36.

Uzawa, H. (1968). Time preference, the consumption function, and optimal asset holdings. In J. N. Wolfe (Ed.), *Value, Capital, and Growth: Papers in honour of Sir John Hicks*, pp. 485–504. Edinburgh: Edinburgh University Press.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics 5*, 177–188.

Wang, N. (2003). Caballero meets Bewley: The permanent-income hypothesis in general equilibrium. *American Economic Review 93*(3), 927–936.

Wang, N. (2004). Precautionary saving and partially observed income. *Journal of Monetary Economics 51*(8), 1645–1681.

Wang, N. (2006). Generalizing the permanent-income hypothesis: Revisiting Friedman's conjecture on consumption. *Journal of Monetary Economics 53*(4), 737–752.

Yaari, M. E. (1965, April). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies 32*(2), 137–150.

Zeldes, S. P. (1989). Optimal consumption with stochastic income: Deviations from certainty equivalence. *Quarterly Journal of Economics 104*(2), 275–298.