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Investment, consumption, and hedging under incomplete markets ☆

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Abstract

Entrepreneurs often face undiversifiable idiosyncratic risks from their business investments. We extend the standard real options approach to an incomplete markets environment and analyze the joint decisions of business investments, consumption/savings, and portfolio selection. For a lump-sum investment payoff and an agent with a sufficiently strong precautionary savings motive, an increase in volatility can accelerate investment, contrary to the standard real options analysis. When the agent can trade the market portfolio to partially hedge against investment risk, the systematic volatility is compensated via the standard CAPM argument, and the idiosyncratic volatility generates a private equity premium. Finally, when the investment payoff is a series of flows, the agent's

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idiosyncratic risk exposure alters both the implied option value and the implied project value, causing a reversal of the results in the lump-sum payoff case.

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1. Introduction

Real investment activities play a fundamental role in the economy. A real investment typically has three important characteristics. First, it is often partially or completely irreversible. Second, its future rewards are uncertain. Finally, the investment time is to some extent flexible. In the last three decades, a voluminous literature has developed that aims to study the implications of these three characteristics for the real investment decision.[1] A key insight of this literature is to view an investment decision as an American-style call option, where "American style" refers to the flexibility of choosing the time of option exercise. Based on this analogy and the seminal contributions to option pricing by Black and Scholes (1973) and Merton (1973), we can apply financial option theory to the irreversible investment decision. This real options approach to investment has become a workhorse in modern economics and finance.

The real options approach relies on one of the following assumptions: (i) the real investment opportunity is tradable; (ii) its payoff can be spanned by existing traded assets; or (iii) the agent is risk neutral. However, these assumptions are violated in many applications. For example, consider entrepreneurial activities. Entrepreneurs combine their skills with their business investment opportunities and ideas to generate economic profits. While entrepreneurs might have valuable projects, these projects might not be freely traded or their payoffs might not be spanned by existing assets because of liquidity restrictions or the lack of liquid markets. These capital market imperfections could be due to moral hazard, adverse selection, transactions costs, or contractual restrictions. Thus, investment opportunities can have substantial undiversifiable idiosyncratic risks. Entrepreneurs' well-being depends heavily on the outcome of their investments. Moreover, entrepreneurs' attitudes towards risk should play an important role in determining their interdependent consumption, savings, portfolio selection, and investment decisions.[2]

While entrepreneurial activities have other important dimensions such as how much to invest and how to finance the investment project, we focus on the investment timing aspect of entrepreneurial activities. We extend the standard real options approach to analyze the implications of uninsurable idiosyncratic risk for this decision, using entrepreneurship

[1] Arrow (1968) and Bernanke (1983) are among the early contributions on irreversible investment. For early stochastic continuous-time models, see Brennan and Schwartz (1985), McDonald and Siegel (1986), Pindyck (1988),and Bertola and Caballero (1994). Abel and Eberly (1994) provide a unified model of (incremental) investment under uncertainty. Dixit and Pindyck (1994) provide a textbook treatment of important contributions to this literature.

[2] There is a fast-growing literature on empirical evidence for entrepreneurship. See Gentry and Hubbard (2004), Heaton and Lucas (2000), and Moskowitz and Vissing-Jorgensen (2002), among others.

as a motivating example. We use a utility maximization framework in which an agent chooses his consumption and portfolio allocations, and also undertakes an irreversible investment.

To facilitate the discussions of our model and results, consider real estate development as an example. The value of vacant land can be viewed as the option value of developing the real estate (Titman, 1985; Williams, 1991; Grenadier, 1996, provide more recent analysis on the real options approach to real estate development.). Suppose that a land owner also knows the best use of his land. For example, the owner has superior knowledge about local market conditions and knows the most profitable property to construct. However, the owner cannot sell this yet-to-be-developed property without incurring a significant value discount due to moral hazard, adverse selection, or his inalienable human capital. Therefore, it might be of interest for the owner to keep the land (option) and to be the developer even though owning the land exposes himself to uninsurable idiosyncratic risks of the underlying asset. That is, the option to obtain the highest value of development (via the ‘‘best’’ use of land) are not tradable due to frictions such as moral hazard or the inalienability of human capital. It is worth noting that land is primarily held by noninstitutional investors such as individuals and private partnerships (Williams, 2001). In addition, individuals and private partnerships are subject to more undiversifiable idiosyncratic risks than are institutional investors like pension fund firms and life insurance companies.

While a real estate entrepreneur owns the land and will choose when to build the property, he can either sell the property or continue to manage the property after developing it. Of course, choosing to sell or manage the property is another decision. We assume that this decision is exogenous in order to focus on the effect of idiosyncratic risks on the development decision.[3] When the entrepreneur pays the construction cost and sells the property upon the completion of development, he receives a lump-sum sale price. We call this situation the lump-sum payoff case. Alternatively, the real estate entrepreneur can be not only the developer but also the manager. The entrepreneur might be the most qualified manager if he can locate the tenants with the highest willingness to pay and maintain the property at the lowest operating costs. Therefore, it could still make economic sense for the developer to manage the property after construction is complete, even though he will face additional undiversifiable idiosyncratic property risks after development. In this setting, the developer receives a perpetual stream of uninsurable rental payments (in excess of operating expenses) from managing the property after development. We call this scenario the flow payoff case.

Standard real options analysis (under complete markets) assumes that an agent can fully diversify the idiosyncratic property risks. One can then take the risk-adjusted present value of future cash flows as the market sale value, and thus there is no distinction between the lump-sum and flow payoff scenarios. However, when the investment opportunity is not tradable and not spanned by existing traded assets, the standard replicating and no-arbitrage argument does not apply. We thus follow the certainty-equivalent approach in

[3]We can extend our model to endogenize the sale/no sale decision. Essentially, in the sale situation the bidder with the highest valuation of the property is someone else with a comparative advantage in management. This fits reasonably well into the description of merchant builders. The no-sale scenario corresponds to the case where the developer is also the best manager in that he can find the tenants with the highest willingness to pay and manage the property with the lowest operating expenses.

the literature on the pricing of nontraded assets to value cash flows by analyzing the entrepreneur's utility maximization problem.[4]

We show that the lump-sum and flow payoff cases deliver different economic predictions due to the effect of uninsurable idiosyncratic shocks. Hence, the equivalence between these two cases (under complete markets) no longer holds. Moreover, when the entrepreneur can partially hedge against project risk by trading a risky financial asset such as the market portfolio, the total volatility can be decomposed into idiosyncratic and systematic volatility. The ability to hedge reduces the entrepreneur's precautionary savings demand because idiosyncratic volatility is lowered, which naturally has implications for the investment timing decision. For the convenience of illustrating the effects of incomplete markets on investment timing in an intuitive way, we proceed with our analysis and develop insights in a pedagogical way by working out four models, which are sorted by both the timing of the payoffs and the menus of financial assets as follows:

	Risk-free asset only	Risk-free & risky financial assets
Lump-sum payoff	Model I	Model II
Flow payoff	Model III	Model IV

We start with Model I, the lump-sum payoff case. We first analyze the effect of risk attitude. We show that a stronger precautionary savings motive lowers the certainty-equivalent wealth associated with the investment opportunity, which is also the implied option value. Thus, risk aversion speeds up investment.

We next turn to the effect of risk. An important prediction of our model is that the idiosyncratic project volatility has two opposing effects on the implied option value and hence on the investment timing decision. On one hand, the standard real options model states that volatility increases the option value due to its asymmetric convex payoff. On the other hand, idiosyncratic volatility lowers the certainty-equivalent wealth and consumption because of the entrepreneur's precautionary savings motive and the interdependence of consumption and investment under incomplete markets. Hence, the net effect of volatility on the option value is ambiguous. When the entrepreneur has a sufficiently strong precautionary savings motive or the idiosyncratic volatility is sufficiently large, the precautionary savings effect can dominate the standard option effect. If volatility does not directly affect the investment payoff as in the lump-sum payoff case (for example, via sale to diversified buyers such as real estate investment trust investors in our real estate example), then idiosyncratic volatility under incomplete markets encourages the entrepreneur to invest earlier, contrary to the standard real options analysis. Going back to our real estate development example, our model predicts that the entrepreneur will exercise his development option early when he is exposed to uninsurable idiosyncratic shocks to his investment opportunity, particularly if he plans to sell his property upon the completion of construction. The entrepreneur's urge to avoid the certainty-equivalent

[4]See Carpenter (1998), Detemple and Sundaresan (1999), Hall and Murphy (2000), and Kahl, Liu, and Longstaff (2003), among others, on nontraded asset valuation such as employee stock options; see Section 2 for further discussions.

wealth discount due to idiosyncratic shocks encourages him to invest earlier, ceteris paribus.

We now turn to Model II in order to understand the effect of hedging for the lump-sum payoff case. When the entrepreneur can partially hedge against project risk by trading a risky asset such as the market portfolio, total volatility can be decomposed into idiosyncratic and systematic volatility. As a result, the entrepreneur's precautionary savings demand (due to idiosyncratic volatility) is mitigated, which in turn makes the investment option more valuable, ceteris paribus. With the lump-sum payoff case, the payoff (after investment) can be independent of idiosyncratic volatility (for example, via sale to diversified buyers), and the model then predicts that the entrepreneur invests sooner under incomplete markets than under complete markets, because the option value is lower in the presence of idiosyncratic shocks. Intuitively, exercising the investment option allows the agent to exit from incomplete markets.

We finally turn to Models III and IV to analyze investment payoffs given in flow terms. In our previous real estate development example, these two cases correspond to developer also managing the real estate after its completion. Because the developer still faces undiversifiable idiosyncratic risk from the payoff stream after exercising the investment option, this payoff stream has a lower certainty equivalent wealth lower than it would if the payoff stream were marketable. Intuitively, unlike the lump-sum payoff case, exercising the investment option does not allow the entrepreneur to exit the incomplete markets setting when the payoffs are given in flow terms. Thus, the previously discussed precautionary savings effect influences not only the certainty-equivalent option value but also the certainty-equivalent value of the project payoffs after exercising the investment option. Because of this additional effect on investment payoffs, many results obtained in the lump-sum payoff case are reversed.

Building on the insights behind the Black-Merton-Scholes analysis, we show that hedging demand under incomplete markets also increases with the investment option delta (Delta is defined as the change in the investment option value for a unit increase in the underlying project payoff value). Since the option delta increases in the underlying project payoff value, our model predicts that the developer's hedging demand increases when his development option gets closer to being "in the money." With regard to the consumption savings literature, we extend the standard incomplete markets analysis to allow the agent to endogenously determine the timing of his income process. We show that volatility not only has a negative effect on consumption (via the precautionary savings motive under incomplete markets), but also a positive option-value-induced wealth effect due to the endogeneity of the income timing choice. Hence, consumption might respond positively to an increase in volatility of the investment project if the option value induced wealth effect is stronger, ceteris paribus.

Three recent papers—Henderson (2005) and Hugonnier and Morellec (2005, 2007)—are related to ours. Henderson (2005) assumes that the agent maximizes expected wealth at the time of investment. Hugonnier and Morellec (2007) assume that the manager trades off his incentives to exercise an option under incomplete markets prematurely in order to reduce the idiosyncratic risk exposure against the cost of increasing the likelihood of a control challenge. Hugonnier and Morellec (2005) apply a concave utility function to both the payoff and the interest cost of investment, and show that risk aversion further delays investment compared with the standard real options prediction. While these papers study real options models under incomplete markets, they do not study the consumption/savings decision and its interaction with investment and portfolio decisions. As in our lump-sum payoff case, the first two papers show that market incompleteness encourages an agent to exercise the investment option sooner. Unlike these two papers, we show that investment can be delayed due to market incompleteness when investment payoffs are delivered over time rather than in one lump sum.

The remainder of the paper proceeds as follows. Section 2 analyzes Model I, which is a self-insurance model when the payoff from real investment is given in a lump sum. Section 3 presents the solution of Model II, which generalizes Model I to allow for hedging. Section 4 studies Models III and IV, where the real investment payoffs are given in flows. Section 5 analyzes the robustness of the results. Section 6 concludes. Technical details are relegated to appendices.

2. A self-insurance model with a lump-sum payoff (Model I)

This section provides a model that allows us to develop intuition for how the agent's attitude towards risk affects his investment decisions when he cannot fully insure himself against the idiosyncratic shocks from investment. In order to achieve this objective in the simplest possible setting, we integrate a canonical consumption/savings model with a standard real-options-based irreversible investment model.[5]

2.1. Model setup

Time is continuous and the horizon is infinite. There is a single perishable consumption good (the numeraire). The agent derives utility from a consumption process C according to

$$\mathrm{E}\left[\int_0^\infty e^{-\beta t} U(C_t)\,\mathrm{d}t\right], \tag{1}$$

where U is an increasing and concave function and $\beta > 0$ is his discount rate. For expositional convenience, we assume that β is equal to r, the risk-free interest rate. It is straightforward to extend our analysis to allow for differences between the agent's subjective discount rate and the interest rate, but no additional insight will be gained for the issue that we are after.

The agent owns the rights to an investment project and can undertake this project irreversibly at some endogenously chosen time τ. Note that the investment time τ is stochastic from today's perspective. The investment costs $I > 0$. The agent pays this cost only at the investment time τ. This cost is financed from the agent's own wealth. If there is a shortage of funds, the agent can borrow at the risk-free rate r. In order to focus on the effect of market incompleteness in the simplest possible setting, we do not consider borrowing constraints or costly external financing. Instead, we impose the conventional transversality condition for the agent to rule out Ponzi games. After the agent exercises the investment option at time τ, the project generates a lump-sum payoff X_τ. We also assume that the payoff process X is governed by an arithmetic Brownian motion process,

$$\mathrm{d}X_t = \alpha_x\,\mathrm{d}t + \sigma_x\,\mathrm{d}Z_t, \quad X_0 \text{ given}, \tag{2}$$

[5]See Leland (1968) for early studies on precautionary savings. See Zeldes (1989), Caballero (1991b), and Deaton (1991) for dynamic incomplete markets consumption models. See Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994) for standard real options models.

where α_x and σ_x are positive constants and Z is a standard Brownian motion.[6] This process implies that payoffs can take negative values. We interpret negative values as losses. We choose the arithmetic Brownian motion process purely for analytical convenience and in line with further analysis in Section 4 when the payoffs are given in flow terms over time. We obtain essentially the same insights with a geometric Brownian motion payoff process (Section 5).

As discussed earlier, investing in the project is analogous to exercising a perpetual American call option, in the sense that the agent has the right but not the obligation to invest at some future time of his choosing. Importantly, unlike financial options, the underlying asset for the real option cannot be traded in the market. For example, the building (the underlying asset in the real estate development example) before it is erected is not traded in the market. If we further assume that existing financial assets do not completely span the payoffs for the underlying asset (the building), then we cannot apply the dynamic replication argument in standard option pricing theory such as the Black-Merton-Scholes model. In this section, the only financial asset available for the agent to trade and to smooth his consumption is the risk-free asset. Hence, the agent inevitably bears project risk, which is undiversifiable.

Let $\{W_t : t \geqslant 0\}$ denote a wealth process. Then the wealth dynamics are given by

$$\mathrm{d}W_t = (rW_t - C_t)\,\mathrm{d}t, \quad W_0 \text{ given.} \tag{3}$$

That is, the agent accumulates wealth at the rate of $(rW_t - C_t)$, the difference between the interest income rW_t and consumption rate C_t. At investment time τ, the agent pays the investment cost I and obtains the lump-sum payoff X_τ, and hence his wealth is raised by the amount $(X_\tau - I)$; $W_\tau = W_{\tau-} + X_\tau - I$, where $W_{\tau-}$ and W_τ denote the agent's wealth just before and immediately after the agent exercises the investment option, respectively. The agent's optimization problem is to choose both his investment timing strategy τ and consumption process C to maximize his utility given in (1) subject to (3) and a transversality condition specified later.

2.2. Optimality conditions

We solve the agent's decision problem by working backwards using dynamic programming. We consider first the problem after the agent exercises the investment option. In this case, the agent's optimization problem is a standard deterministic consumption savings problem without income. Let $V^0(w)$ be the corresponding value function. By a standard argument, $V^0(w)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rV^0(w) = \max_{c \in \mathbb{R}} \; U(c) + (rw - c)V^0_w(w), \tag{4}$$

subject to the transversality condition $\lim_{T\to\infty} e^{-rT}|V^0(W_T)| = 0$. Under the deterministic setting, the agent's consumption is constant over time and is equal to the annuity value rw of his wealth, and therefore, his wealth remains constant at w at all times. This is the familiar consumption-smoothing result. This result follows from two steps: (i) the equality

[6] Unlike the often adopted geometric Brownian motion process, the specification in (2) proves more convenient within our setup. Wang (2006) derives a closed-form consumption-saving rule using affine processes and exponential utility.

between the agent's discount rate and the interest rate implies that the marginal utility is constant at all times ($U'(C_t) = U'(C_s)$); and (ii) the strict concavity of the utility function further implies that $C_t = C_s$. It is immediate to conclude that the value function is thus given by $V^0(w) = U(rw)/r$.

We next consider the case before the option is exercised. It is worth noting that the agent's value function depends on both his wealth w and the current value x of his investment opportunity. Let $V(w,x)$ denote the corresponding value function. The standard dynamic programming argument implies that $V(w,x)$ satisfies the following HJB equation:

$$rV(w,x) = \max_{c \in \mathbb{R}} U(c) + (rw - c)V_w(w,x) + \alpha_x V_x(w,x) + \frac{\sigma_x^2}{2} V_{xx}(w,x). \tag{5}$$

The above HJB equation is similar to an asset pricing equation. It states that the agent chooses his consumption optimally by setting the return $rV(w,x)$ of his value function to equal the sum of his instantaneous utility $U(c)$ and the total expected changes of his value function (due to the change in wealth and also in the investment opportunity).

We now specify boundary conditions. First, the no-bubble condition $\lim_{x\to-\infty} V(w,x) = V^0(w)$ must be satisfied. This condition states that when the investment payoff goes to negative infinity, the agent will never exercise the investment option and his value function is equal to that without the investment option. Next, as is standard in the optimal stopping problems, at the instant of investment, the following value-matching condition must hold:

$$V(w,x) = V^0(w + x - I). \tag{6}$$

This equation implicitly defines an investment boundary $x = \overline{x}(w)$. In general, this boundary $\overline{x}(w)$ depends on the agent's wealth level w. Finally, because this boundary is chosen optimally, the following smooth-pasting condition is satisfied[7]:

$$\left.\frac{\partial V(w,x)}{\partial x}\right|_{x=\overline{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial x}\right|_{x=\overline{x}(w)}, \tag{7}$$

$$\left.\frac{\partial V(w,x)}{\partial w}\right|_{x=\overline{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial w}\right|_{x=\overline{x}(w)}. \tag{8}$$

The first smooth-pasting (7) states that the marginal change in the investment opportunity has the same marginal effect on the agent's value functions just before and immediately after exercising the option. Similarly, the second smooth-pasting (8) states that the marginal effect of wealth must be the same on the agent's value functions just before and immediately after exercising the option. Unlike the conventional irreversible investment models (Dixit and Pindyck, 1994), here the agent's wealth enters as an additional state variable, which gives rise to the second smooth-pasting (8).

2.3. Model solution for CARA utility

We have now formulated the agent's optimization problem as a combined control (consumption) and stopping (investment) problem, which is generally difficult to solve. Our objective is to understand the economic effects of uninsurable idiosyncratic risk and

[7]See, for example, Krylov (1980), Dumas (1991) and Dixit and Pindyck (1994).

the attitude towards risk on the investment and consumption decisions. In order to achieve this objective in the simplest possible way, we assume that the agent has the utility function $U(c) = -e^{-\gamma c}/\gamma$, where the parameter $\gamma > 0$ is the coefficient of absolute risk aversion (CARA). It is also equal to the coefficient of absolute prudence $-U'''(c)/U''(c)$, which captures the precautionary savings motive (Kimball, 1990). By adopting the utility specification, we derive intuitive semi-closed-form solutions that greatly simplify our analysis. While CARA utility does not capture the wealth effect, we emphasize that the main results and insights of this paper (the effect of uninsurable idiosyncratic shocks on investment timing) do not rely on the particular choice of this utility function. As we will see below, the driving force of the paper is the precautionary savings effect, which can be captured by utility functions with convex marginal utility such as CARA. While power utility is more commonly used in economics, this utility will complicate our analysis substantially since it will lead to a much harder two dimensional free-boundary problem.

First, we note that complete consumption smoothing after investment and CARA utility jointly imply that the value function after investment is given by

$$V^0(w) = -\frac{1}{\gamma r}\exp(-\gamma r w). \tag{9}$$

Next, we conjecture that the value function before the option exercise takes the following form:

$$V(w,x) = -\frac{1}{\gamma r}\exp[-\gamma r(w + G(x))], \tag{10}$$

where $G(x)$ is a function to be determined. One can interpret $G(x)$ as the certainty-equivalent wealth derived from the agent's investment opportunity. Specifically, we follow the consumption literature to define certainty-equivalent wealth as the value w_{ce} satisfying the equation $V^0(w + w_{ce}) = V(w,x)$; that is, the agent is indifferent between receiving stochastic income in the future and a total current wealth level of $(w + w_{ce})$. Using the explicit functional forms of $V^0(w)$ and $V(w,x)$, we have $w_{ce} = G(x)$.

The boundary conditions (6)–(8) and the additive separability of wealth w and certainty-equivalent wealth $G(x)$ in the exponent of the value function $V(w,x)$ indicate that the investment boundary is flat, in that $\overline{x}(w)$ is independent of wealth w. This property substantially simplifies our analysis. The following proposition summarizes the solution to the agent's combined consumption and investment problem.

Proposition 1. *The agent exercises the investment option the first time the process X hits the threshold $\bar{x}$ from below. After exercising the option, the agent's value function and consumption rule are given by* (9) *and $\overline{c}(w) = rw$, respectively. Before exercising the option, his value function and consumption rule are, respectively, given by* (10) *and*

$$\overline{c}(w,x) = r(w + G(x)), \tag{11}$$

where $(G(x), \bar{x})$ is the solution to the following free-boundary problem:

$$rG(x) = \alpha_x G'(x) + \frac{\sigma_x^2}{2}G''(x) - \frac{\gamma r \sigma_x^2}{2}G'(x)^2, \tag{12}$$

subject to the no-bubble condition $\lim_{x \to -\infty} G(x) = 0$, *and the boundary conditions*

$$G(\bar{x}) = \bar{x} - I, \tag{13}$$

$$G'(\bar{x}) = 1. \tag{14}$$

Moreover, G is increasing.

We now analyze the intuition behind this proposition and discuss its implications.

2.4. Interdependence of investment and consumption

As in the standard real options approach, the agent trades off between holding the investment option to obtain an implied option value of waiting and exercising this option to obtain investment payoffs. The key to our analysis is to derive the implied option value. We show below that, unlike the standard real options approach, risk aversion and consumption play an important role in the determination of the option value under incomplete markets.

Implied option value. Proposition 1 demonstrates that the certainty-equivalent wealth $G(x)$ solves a free-boundary problem (12)–(14). These equations are similar to, but different from, the valuation equations and boundary conditions in the standard real option models of McDonald and Siegel (1986) and Dixit and Pindyck (1994). Based on this similarity, we interpret x as the project value and the certainty-equivalent wealth $G(x)$ as the implied option value of investing in the underlying project. More formally, we follow the literature on the pricing of nontraded assets by defining the implied option value Q of the project as the solution to the equation $V(w - Q, x) = V^0(w)$; that is, the agent is indifferent between having no investment opportunity and paying the price Q to obtain the investment opportunity. Given the functional form of V^0 and V in (9) and (10), we see that $Q = G(x)$.

The two interpretations of $G(x)$—the certainty-equivalent wealth and the implied option value—are the same in our setup. This is due to the absence of the wealth effect under CARA utility. We will thus use certainty-equivalent wealth (from the consumption literature perspective) and implied option value (from the investment literature perspective) interchangeably throughout the remainder of the paper.

Proposition 1 nests the standard (risk neutral) real options problem as a special case. Setting $\gamma = 0$ in Eq. (12) enables us to derive the following explicit solutions for the option value $G(x)$ and the investment threshold $\bar{x}$:

$$G(x) = \frac{1}{\lambda_0} e^{\lambda_0 (x - \bar{x})}, \quad \text{for } x \leqslant \bar{x} \text{ and } \bar{x} = I + \frac{1}{\lambda_0}, \tag{15}$$

where $\lambda_0 = -\sigma_x^{-2}\alpha_x + \sqrt{\sigma_x^{-4}\alpha_x^2 + 2r\sigma_x^{-2}}$ for $\sigma_x > 0$, and $\lambda_0 = r/\alpha$ for $\sigma_x = 0$. It is straightforward to verify that both the option value $G(x)$ and the investment threshold $\bar{x}$ increase in the volatility σ_x of the payoff. These are the main results of the real options literature. The agent can capture the upside gains by investing and limit the downside losses by simply waiting until the option is sufficiently "in the money." This asymmetric convex payoff generates the positive effect of volatility on the option value and investment threshold.

The main difference between our model and the standard (risk neutral) real options model is that option value $G(x)$ depends not only on the parameters describing the risk-free

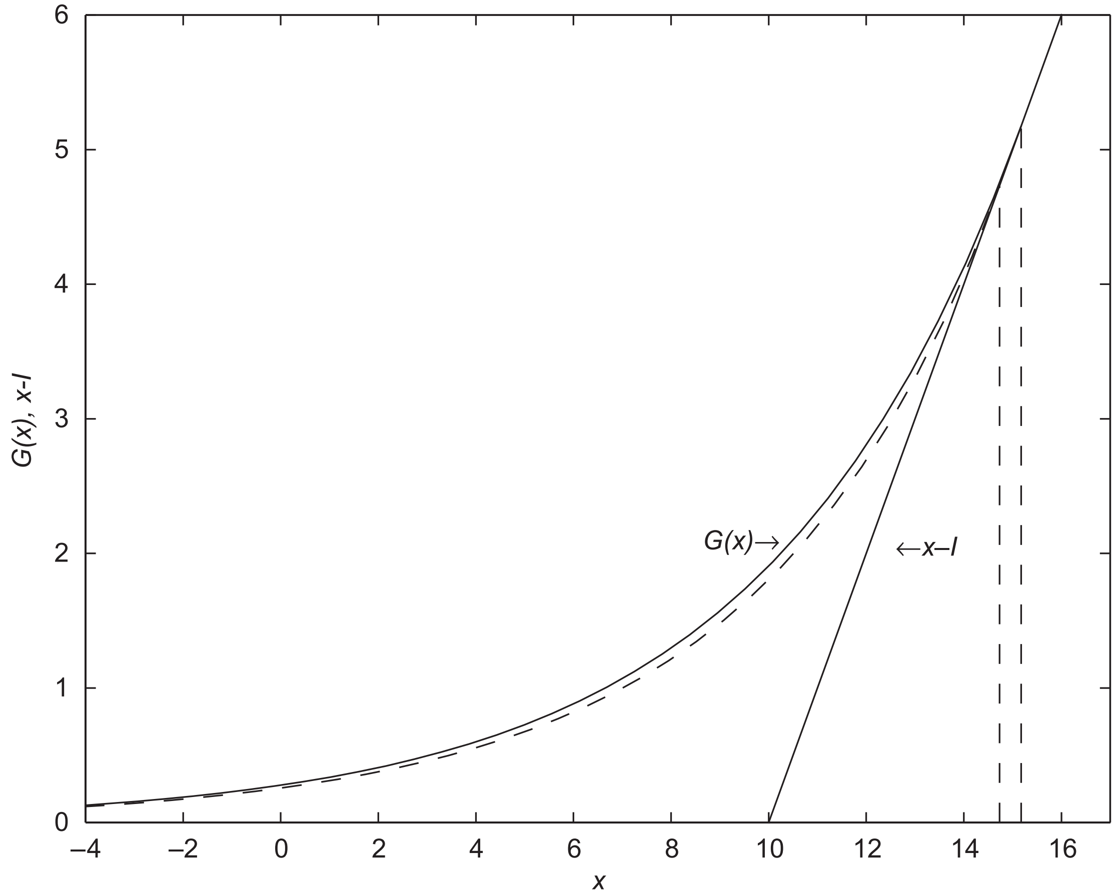

Fig. 1. *Implied option value* $G(x)$. This figure plots the functions $x - I$ and $G(x)$ for the model in Section 2. The parameter values are set as follows: interest rate $r = 2\%$, drift $\alpha_x = 0.1$, volatility $\sigma_x = 20\%$, and investment cost $I = 10$. The solid curve is for the risk aversion parameter $\gamma = 1$, and the dashed curve is for the risk aversion parameter $\gamma = 25$.

rate r, drift α_x, and volatility σ_x but also on the agent's precautionary savings motive. The latter dependence captures the notion that the agent's risk attitude matters not only for consumption decisions but also for investment decisions when markets are incomplete. The last nonlinear term on the right side of (12) captures the agent's precautionary savings motive. It confirms the intuition that the implied option value $G(x)$ is lower when the precautionary motive is stronger, ceteris paribus. Since the project payoff value x does not depend on the agent's risk attitude, the net effect of an increase in γ is to encourage earlier investment. Fig. 1 plots the implied option value $G(x)$ versus the value of the underlying investment opportunity x for two values of γ. Note that the payoff line $(x - I)$ is independent of risk aversion γ. The figure clearly illustrates that the investment threshold decreases with the agent's precautionary savings motive or risk aversion γ.

Investment threshold. To gain further intuition, we use the asymptotic approximation method to compute approximate solutions for the implied option value $G(x)$ and the investment threshold $\bar{x}$.[8] We expand the option value $G(x)$ and the investment threshold $\bar{x}$ to the first order of σ_x^2, in that $G(x) \approx G_0(x) + G_1(x)\sigma_x^2$ and $\bar{x} \approx \bar{x}_0 + \delta_1\sigma_x^2 \equiv \bar{x}_1$. Plugging

[8] See Judd (1998). Also see Kogan (2001) who applies this method to solve an irreversible (incremental) investment model.

these expansions in (12)–(14), we show in the appendix that $\bar{x}_0 = I + \alpha_x/r$ and

$$\bar{x}_1 = \bar{x}_0 + \left(\frac{1}{\alpha_x} - \gamma\right)\frac{\sigma_x^2}{2}. \tag{16}$$

This solution indicates that, to a first-order approximation with respect to σ_x^2, a stronger precautionary savings motive (higher γ) lowers the investment threshold, consistent with our earlier discussions based on the nonlinear ODE (12) and the boundary conditions (13)–(14).

The above approximate solution also helps us to understand the effect of volatility on the investment threshold. An increase in volatility σ_x has two opposing effects. On one hand, higher volatility increases the option value and hence encourages waiting, as in standard real options models. On the other hand, an increase in σ_x also raises the precautionary savings demand and thus lowers the certainty equivalent wealth $G(x)$, and hence lowers the threshold, ceteris paribus. Both effects are reflected in the last term on the right side of (16). When γ is sufficiently small, the option effect dominates the precautionary savings effect. Thus, an increase in volatility σ_x raises the implied option value and delays investment, consistent with the predictions in the standard real options models. By contrast, when γ is sufficiently large, the precautionary savings effect can dominate the option effect. Therefore, an increase in σ_x lowers the certainty-equivalent wealth $G(x)$, and hence encourages the agent to exercise his option sooner, contrary to the standard real options result.

Finally, we use numerical solutions to confirm our intuition. We apply the projection method detailed in the Appendix to solve the free-boundary problem characterized by (12)–(14). We find that, for a small σ_x, our preceding approximate solution is very close to the "true" solution delivered by the projection method. For a large range of parameter values, Fig. 2 plots the investment threshold as a function of volatility σ_x and the risk aversion γ. This figure demonstrates that our preceding results and intuition extend to general parameter values.

Consumption. We now turn to the agent's consumption policies. After exercising the option, the agent solves a deterministic consumption-smoothing problem. As noted earlier, his wealth remains constant and consumption is equal to the interest income at all times. Before exercising the option, the agent's consumption rule (11) is given by the annuity value of the sum of his financial wealth w and his certainty-equivalent wealth $G(x)$.

Even though the agent does not receive payoff x before exercising the option, he rationally anticipates that he will exercise his investment option sometime in the future. Thus, the future investment payoff matters not only for his future consumption but also for his current consumption. Our model captures the forward-looking consumption smoothing intuition in an incomplete markets setting with endogenous stochastic income.

The standard intuition in the consumption literature is that volatility lowers consumption because of the precautionary savings motive. Here, we show that consumption can potentially increase in volatility because the option effect can dominate the precautionary savings effect on $G(x)$. This effect is not present in the consumption literature, because almost all models in that literature take stochastic income as *exogenously* given and hence rule out the option effect of income volatility on consumption.

In summary, the uninsurable idiosyncratic risk alters results in the standard real options and consumption literature. When idiosyncratic risk is large or the precautionary savings motive is strong, the option value and the investment threshold can decrease in volatility,

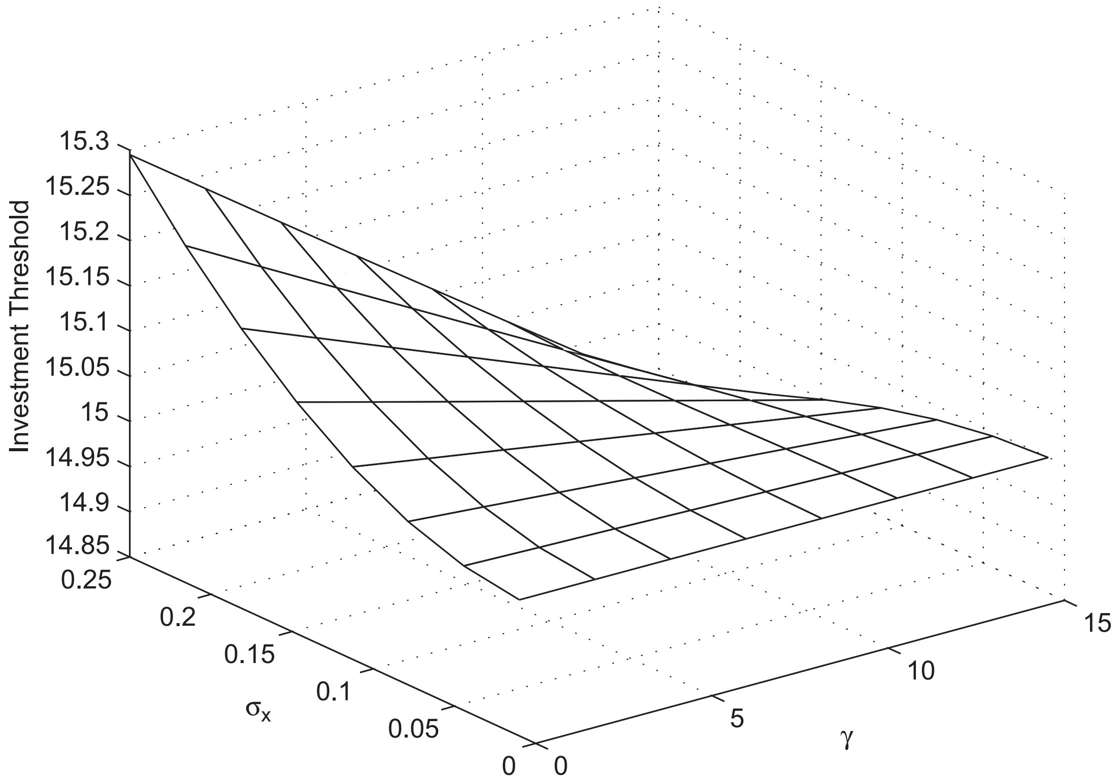

Fig. 2. *Investment threshold, risk aversion, and project volatility.* This figure plots the investment threshold at varying levels of the risk aversion parameter γ and volatility σ_x for the lump sum payoff case. Other parameter values are set as follows: interest rate $r = 2\%$, drift $\alpha_x = 0.1$, and investment cost $I = 10$.

contrary to the standard real option results. Therefore, applying real options analysis and ignoring the consumption-smoothing motive in settings where idiosyncratic risk is likely to matter, such as entrepreneurial investments, is potentially misleading and incorrect.

3. Lump-sum payoff with hedging opportunities (Model II)

In the previous section, the agent can trade only a risk-free asset to partially insure himself against project risk. We now generalize the setting by allowing the agent to trade a risky asset to partially hedge against project risk. We can interpret this financial asset as the market portfolio. Unlike Model I where all risks are idiosyncratic and uninsurable, investing in the risky asset allows the agent to partially hedge and hence separate systematic volatility from idiosyncratic volatility. We will show that these two volatilities play *different* roles in determining the option value and the exercising decisions. Our analysis nests the standard complete-markets analysis as a special case.

3.1. Setup

Let $\{P_t : t \geqslant 0\}$ denote the risky asset's price process and assume that the return is governed by the following process:

$$dP_t/P_t = \mu_e\,dt + \sigma_e\,dB_t, \tag{17}$$

where μ_e and σ_e are positive constants, and B is a standard Brownian motion correlated with the Brownian motion Z, which drives the innovations of the project payoff as given in (2). Let $\rho \in [-1, 1]$ be the correlation coefficient between the return on the risky asset and the agent's project payoff, and let $\eta = (\mu_e - r)/\sigma_e > 0$ denote the Sharpe ratio of the market portfolio.

One can alternatively rewrite the observed payoff process $\{X_t : t \geqslant 0\}$ given in (2) as follows:

$$\mathrm{d}X_t = \alpha_x \,\mathrm{d}t + \rho\sigma_x \,\mathrm{d}B_t + \varepsilon_x \,\mathrm{d}\widetilde{B}_t, \tag{18}$$

where B and $\widetilde{B}$ are two independent standard Brownian motions, and

$$\varepsilon_x = \sqrt{1-\rho^2}\,\sigma_x. \tag{19}$$

One can think of B as the Brownian motion describing the systematic (market) risk, and thus $\rho\sigma_x$ is the systematic component of volatility for the project payoff. One can then interpret $\widetilde{B}$ as the Brownian motion describing idiosyncratic project risk, and thus ε_x is idiosyncratic volatility. A higher absolute value of the correlation coefficient $|\rho|$ implies that systematic volatility has a larger weight, ceteris paribus.

Let π_t be the amount allocated to the risky asset at time t, measured in units of the consumption good. The agent's problem is to choose a consumption process C, a portfolio allocation rule π, and an investment timing strategy τ to maximize his utility (1) subject to his wealth dynamics:

$$\mathrm{d}W_t = (rW_t + \pi_t(\mu_e - r) - C_t)\,\mathrm{d}t + \pi_t\sigma_e \,\mathrm{d}B_t, \quad W_0 \text{ given}. \tag{20}$$

Similar to Section 2, the agent's wealth jumps immediately after he invests, in that $W_\tau = W_{\tau-} + X_\tau - I$, where $W_{\tau-}$ and W_τ are his wealth immediately before and after his investment at time τ, respectively. Note that (20) is the same both before and after the option exercise.

We use the same dynamic programming method as in Section 2 to solve the agent's problem and summarize the results below.

Proposition 2. *The agent exercises the investment option the first time the process X hits the threshold $\bar{x}$ from below. After exercising the option, the optimal consumption and portfolio rules are given by*

$$\bar{c}(w) = r\left(w + \frac{\eta^2}{2\gamma r^2}\right), \tag{21}$$

$$\bar{\pi}(w) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r}. \tag{22}$$

Before exercising the option, the optimal consumption and portfolio rules are given by

$$\bar{c}(w, x) = r\left(w + G(x) + \frac{\eta^2}{2\gamma r^2}\right), \tag{23}$$

$$\bar{\pi}(w, x) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r} - \frac{\rho\sigma_x}{\sigma_e}G'(x), \tag{24}$$

where $(G, \bar{x})$ is the solution to the following free-boundary problem:

$$rG(x) = (\alpha_x - \rho\sigma_x\eta)G'(x) + \frac{\sigma_x^2}{2}G''(x) - \frac{\gamma r\varepsilon_x^2}{2}G'(x)^2, \tag{25}$$

subject to the no-bubble condition $\lim_{x\to-\infty} G(x) = 0$, *and also the boundary conditions*

$$G(\bar{x}) = \bar{x} - I, \tag{26}$$

$$G'(\bar{x}) = 1. \tag{27}$$

Moreover, G is increasing.

We next discuss the implications of this proposition and analyze the role of hedging.

3.2. Undiversifiable idiosyncratic risk and implied option value

Similar to the self-insurance model in Section 2, we can interpret $G(x)$ either as the certainty-equivalent wealth or as the implied option value. Before discussing the option value $G(x)$, we first sketch out the standard complete-markets model when the idiosyncratic risk is fully diversifiable. Let $\Phi(x)$ denote the option value under complete markets. Given complete markets, standard finance theory implies that the option value and the investment threshold are independent of preferences. Indeed, we can apply the martingale method to rewrite the dynamic budget constraint as a static Arrow-Debreu budget constraint.[9] Appendix B shows that $\Phi(x)$ satisfies the following differential equation:

$$r\Phi(x) = (\alpha_x - \rho\sigma_x\eta)\Phi'(x) + \frac{\sigma_x^2}{2}\Phi''(x), \tag{28}$$

and the boundary conditions $\lim_{x\to-\infty} \Phi(x) = 0$, $\Phi(x^*) = x^* - I$, and $\Phi'(x^*) = 1$.

Eq. (28) resembles a standard valuation equation in dynamic asset pricing models (see, e.g., Duffie, 2001). After correcting for risk, traded securities such as the option earn the risk-free rate of return r, as seen from the left side of (28). The right side of (28) gives the instantaneous expected changes in the option value with respect to the underlying asset value x. The risk correction is reflected by the drift change from α_x to $(\alpha_x - \rho\sigma_x\eta)$ in the first term on the right side of (28). This risk correction can be obtained from a CAPM argument and is consistent with standard dynamic asset pricing theories, which state that only systematic risk demands a premium.

We turn to the differential equation (25) for the option value $G(x)$. Re-writing (25) gives

$$rG(x) = \left(\alpha_x - \rho\sigma_x\eta - \frac{\gamma r\varepsilon_x^2}{2}G'(x)\right)G'(x) + \frac{\sigma_x^2}{2}G''(x). \tag{29}$$

First, we note that the standard convexity effect of volatility on option value depends on the total volatility σ_x, which is reflected by the last term in (29), and is the same as in the complete markets setting (see (28)). Also similar to the differential equation (28) for $\Phi(x)$, the change of drift from α_x to $(\alpha_x - \rho\sigma_x\eta)$ in the first term on the right side of (29) accounts for the effect of systematic risk on valuation, the standard CAPM argument. Importantly, unlike the differential equation (28) for $\Phi(x)$, the third component in the bracket of the drift term on the right side of (29), $\gamma r\varepsilon_x^2 G'(x)/2$, reflects the effect of idiosyncratic risk on the implied option value $G(x)$. We call this term the *idiosyncratic* risk premium.

[9]See Cox and Huang (1989), and Karatzas, Lehoczky, and Shreve (1987) on the martingale method. This method and the dynamic programming method deliver the same solution. See Duffie (2001) for a textbook treatment.

Intuitively, when idiosyncratic risks cannot be fully diversified, the agent naturally demands a higher risk premium for a larger idiosyncratic volatility ε_x, ceteris paribus. A more prudent agent (with a larger coefficient of risk aversion γ) also demands a higher risk premium. Finally, a higher option delta $G'(X)$ indicates that the option value is more sensitive to changes in the underlying investment opportunity set and hence requires a higher idiosyncratic risk premium. Moskowitz and Vissing-Jorgensen (2002) find that the private equity premium is low in the U.S. given the amount of idiosyncratic risk that entrepreneurs face. While our model is not designed to address this quantitative private equity premium issue, our model responds to urgent needs to develop theories that capture the role of idiosyncratic risk in the interdependent consumption, investment, and portfolio choices for entrepreneurs, as suggested by Gentry and Hubbard (2004), Heaton and Lucas (2000), and Moskowitz and Vissing-Jorgensen (2002).

We now turn to the effects of idiosyncratic volatility ε_x and the risk aversion coefficient γ on the investment threshold $\bar{x}$. First, note that as in the self-insurance model of Section 2, the payoffs upon option exercise are given by $(x - I)$. Hence, neither idiosyncratic volatility nor risk aversion matters for the project payoff values. Second, a direct comparison between (28) and (29) implies that a larger idiosyncratic volatility ε_x or a higher risk aversion coefficient γ lowers the option value $G(x)$, holding systematic risk constant. Taking the two effects together, we can conclude that a higher idiosyncratic volatility ε_x and a larger risk aversion coefficient γ lower the investment threshold $\bar{x}$, ceteris paribus. This result also implies that the agent hastens investment under incomplete markets relative to complete markets since the solution for the latter is effectively obtained by setting $\gamma = 0$.

3.3. Consumption and portfolio rules

The consumption rule (21) and the portfolio rule (22) after the option exercise are solutions to the standard Merton-style consumption-portfolio choice problem with CARA utility (Merton, 1969). After exercising the option, the agent has no more hedging demand since the lump-sum project payoff has been realized at exercise. Eq. (22) gives the standard mean-variance efficient rule for CARA utility. The agent's ability to invest in the risky asset to take advantage of the risk premium makes him better off relative to the self-insurance setting in Section 2. This is reflected by $\eta^2/(2\gamma r^2)$, the second term in the consumption rule (21).

Next, consider the agent's consumption decision before the option exercise. Eq. (23) states that the agent's consumption is equal to the annuity value of the sum of three terms: (i) financial wealth w, (ii) certainty-equivalent wealth $G(x)$, and (iii) the constant $\eta^2/(2\gamma r^2)$. The forward-looking agent rationally finances a certain fraction of his current consumption via the certainty equivalent wealth $G(x)$ for his investment opportunity. Moreover, investing in the risky asset makes him better off and yields a higher current consumption, ceteris paribus. This is reflected by the third component in the consumption rule (23), similar to the argument for the after-investment consumption rule (21).

We now turn to the agent's portfolio rule (24) before investment. In addition to the standard Merton mean-variance term, the agent also has a hedging demand, because his investment project payoff is correlated with the market portfolio. First, hedging demand is greater when the degree of correlation $|\rho|$ is higher, the standard and well-known result. Second, the portfolio rule (24) suggests that hedging demand is greater when $G'(x)$, the option Δ, is higher. This result is less known, but is intuitive. Before the investment decision is made, the agent holds a valuable option on a non-tradable underlying asset. Hence, the agent naturally hedges more against the fluctuations of the option value of his investment, if this option value is more sensitive to the change of the underlying asset (a higher option delta), ceteris paribus.

4. Models with flow payoffs (Models III and IV)

While some real-world examples fit the lump-sum payoff setting that we have just analyzed, there are many situations under which the investment payoffs are given as cash flows over time, rather than as a lump-sum payment. We emphasize that unlike the lump-sum payoff case where the project payoff is exogenously given, one has to derive the implied value or the certainty-equivalent value of the cash flows by solving the agent's consumption decision after the option exercise. Intuitively, idiosyncratic volatility also lowers the implied project value or the certainty-equivalent wealth after option exercise. Hence, the overall impact of idiosyncratic volatility on the investment decision and implied option value is less obvious. Indeed, we show that the predictions for the flow payoff case can be reversed compared to those for the lump-sum payoff case.

In the flow payoff case, after the agent irreversibly exercises his investment option at some time τ, he obtains a perpetual stream of payoffs $\{Y_t : t \geqslant \tau\}$. Assume that the flow payoff process Y is governed by an arithmetic Brownian motion process:

$$\mathrm{d}Y_t = \alpha_y \,\mathrm{d}t + \sigma_y \,\mathrm{d}Z_t, \quad Y_0 \text{ given}, \tag{30}$$

where α_y and σ_y are positive constants and Z is a standard Brownian motion. As will be clear below, the arithmetic Brownian motion process allows us to obtain explicit solutions after investment so that the problem before investment is easier to analyze. Using a geometric Brownian motion process to model the cash flow process will complicate the analysis without adding many new insights.

We present our analysis in three subsections. First, we analyze Model III, the flow payoff self-insurance case where the agent can trade only a risk-free asset and hence all risk is idiosyncratic. We then allow the agent to trade a market portfolio to partially hedge against the flow payoff risk and hence to separate idiosyncratic volatility from systematic volatility, similar to Model II of Section 3. Finally, we discuss the empirical implications of the models in both the lump-sum and flow payoff cases.

4.1. Self-insurance with flow payoffs (Model III)

When the agent can trade only a risk-free asset, the agent's wealth $\{W_t : t \geqslant 0\}$ after the option exercise $(\tau \leqslant t)$ evolves according to

$$\mathrm{d}W_t = (rW_t + Y_t - C_t)\,\mathrm{d}t. \tag{31}$$

This equation resembles that in a standard incomplete-markets consumption-savings model with a stream of labor income $\{Y_t : t \geqslant \tau\}$. At investment time τ, the agent pays the cost I and hence wealth is lowered from $W_{\tau-}$, the level just prior to investment, to W_τ, the level immediately after the option exercise, in that $W_\tau = W_{\tau-} - I$. Before exercising the option $(0 \leqslant t < \tau)$, the agent does not receive flow payoffs and thus his wealth evolves according to (3) as in the lump-sum case. The agent's decision problem is to choose both an investment timing strategy τ and a consumption process C so as to maximize his utility (1) subject to wealth accumulation equations (31) and (3) and a transversality condition specified in the Appendix.

We solve the agent's decision problem backward by dynamic programming. Let $J(w, y)$ be the value function after the option exercise. Unlike the lump-sum payoff case, the payoff value y is an additional state variable for J. By the standard argument, $J(w, y)$ satisfies the following HJB equation:

$$rJ(w, y) = \max_{c \in \mathbb{R}} \; U(c) + (rw + y - c)J_w(w, y) + \alpha_y J_y(w, y) + \frac{\sigma_y^2}{2} J_{yy}(w, y), \tag{32}$$

subject to a usual transversality condition. Let $V(w, y)$ denote the value function before the option exercise. Note that we use the same notation for the value function before investment as that for the lump-sum payoff case. Similar to Section 2, $V(w, y)$ satisfies the following HJB equation:

$$rV(w, y) = \max_{c \in \mathbb{R}} \; U(c) + (rw - c)V_w(w, y) + \alpha_y V_y(w, y) + \frac{\sigma_y^2}{2} V_{yy}(w, y). \tag{33}$$

We now briefly discuss the boundary conditions for the flow payoff case and relate then to the lump-sum payoff case analyzed earlier. Similar to the lump-sum payoff case, the no-bubble condition $\lim_{y \to -\infty} V(w, y) = V^0(w)$ must be satisfied. Similar to, but different from the lump-sum payoff case, we have the following value-matching condition:

$$V(w, y) = J(w - I, y). \tag{34}$$

This equation determines an investment boundary $\overline{y}(w)$. Moreover, the agent's optimality further requires the following smooth-pasting conditions to hold:

$$\left.\frac{\partial V(w, y)}{\partial y}\right|_{y=\overline{y}(w)} = \left.\frac{\partial J(w - I, y)}{\partial y}\right|_{y=\overline{y}(w)}, \tag{35}$$

$$\left.\frac{\partial V(w, y)}{\partial w}\right|_{y=\overline{y}(w)} = \left.\frac{\partial J(w - I, y)}{\partial w}\right|_{y=\overline{y}(w)}. \tag{36}$$

These smoothing-pasting conditions are both similar to and different from those for the lump-sum case, because the cash flow payoff y enters as an additional state variable even after the agent makes the investment.

We use a procedure similar to that in Section 2 to solve the above problem and then show that the investment threshold $\bar{y}(w)$ is independent of wealth w for CARA utility agents.

Proposition 3. *The agent exercises the investment option the first time the process Y hits the threshold $\bar{y}$ from below. After exercising the option, the optimal consumption rule is given by*

$$\overline{c}(w, y) = r(w + f(y)), \tag{37}$$

where $f(y)$ is given by

$$f(y) = \left(\frac{y}{r} + \frac{\alpha_y}{r^2}\right) - \frac{\gamma\sigma_y^2}{2r^2}. \tag{38}$$

Before exercising the option, the optimal consumption rule is given by

$$\overline{c}(w, y) = r(w + g(y)), \tag{39}$$

where $(g, \bar{y})$ *is the solution to the following free-boundary problem*:

$$rg(y) = \alpha_y g'(y) + \frac{\sigma_y^2}{2} g''(y) - \frac{\gamma r \sigma_y^2}{2} g'(y)^2, \tag{40}$$

subject to the no-bubble condition $\lim_{y \to -\infty} g(y) = 0$ *and the boundary conditions*

$$g(\bar{y}) = f(\bar{y}) - I, \tag{41}$$

$$g'(\bar{y}) = f'(\bar{y}) = \frac{1}{r}. \tag{42}$$

Moreover, g *is increasing*.

Comparing Propositions 1 and 3, we see that the valuation equation for the implied option value is similar. However, the consumption rule after investment and the boundary conditions are different. We next analyze the implications of these differences.

4.1.1. *Implied project value and consumption*

When the payoffs are given in terms of cash flows over time, the agent continues to face undiversifiable idiosyncratic cash flow risk after exercising his investment option. Therefore, idiosyncratic risk lowers both the implied option value and also the certainty-equivalent project payoff value. After option exercise, the agent's optimization problem is a standard incomplete-markets consumption savings problem with stochastic income $\{Y_t : t \geqslant \tau\}$. Because of the CARA utility and arithmetic Brownian motion process specifications, we are able to derive the explicit expression for the consumption rule given in (37)–(38).[10]

To understand the consumption rule (37), we define human wealth $h(y)$ as the present discounted value of all investment cash flows following Friedman (1957) and Hall (1978). For the arithmetic Brownian motion income process (30), this gives

$$h(y) \equiv \mathrm{E}\left(\int_0^\infty e^{-rt} Y_t \,\mathrm{d}t \,\middle|\, Y_0 = y \right) = \frac{y}{r} + \frac{\alpha_y}{r^2}. \tag{43}$$

Note that this traditional definition of human wealth does not incorporate the effect of risk. Using $h(y)$, we can rewrite the consumption rule given in (37) and (38) as follows:

$$\bar{c}(w, y) = r\left(w + h(y) - \frac{\gamma \sigma_y^2}{2r^2} \right). \tag{44}$$

When $\gamma = 0$ or $\sigma_y = 0$, consumption is the annuity value of the sum of financial wealth w and human wealth $h(y)$, in that $\bar{c}(w, y) = r(w + h(y))$. This is Friedman's permanent-income hypothesis. In terms of a time series, this implies that consumption is a martingale, in that $C_t = E_t(C_{t+1})$, or Hall's random walk consumption model.

Importantly, when the agent has a precautionary savings motive ($\gamma > 0$), a precautionary savings demand arises in the presence of uninsurable idiosyncratic shocks. This demand after the option exercise is reflected by the term $\gamma \sigma_y^2/(2r^2)$ in (38). We can interpret $f(y)$ as the certainty-equivalent (risk-adjusted) human wealth or the implied project value,

[10] Caballero (1991b) derives this consumption rule in discrete time. Wang (2004) extends Caballero (1991b) to more general bivariate income processes with partial observability and hence increases precautionary savings demand due to estimation risk. Wang (2003) shows that general equilibrium restriction eliminates the precautionary savings demand in Caballero (1991b).

following essentially the same analysis in Section 2. Since $f(y) = h(y) - \gamma\sigma_y^2/(2r^2)$, the certainty-equivalent human wealth $f(y)$ decreases in the risk aversion coefficient γ and also in income volatility σ_y. This differs from the lump-sum payoff case where option exercise gives a complete exit from incomplete markets and hence the precautionary savings motive and volatility do not affect the value of payoffs from exercising.

Now consider consumption before investment. Eq. (39) implies that the rational forward-looking agent finances his consumption partially out of his future payoffs from his real investment opportunities. More formally, consumption is given by the annuity value of the sum of financial wealth w and $g(y)$, before the investment is made. Following our analysis in the lump-sum payoff case, we can interpret $g(y)$ as the certainty-equivalent wealth for the investment opportunity before investment is made, or equivalently, the implied option value on the investment opportunity. We next turn to the analysis of $g(y)$.

4.1.2. *Implied option value and investment threshold*

The implied option value $g(y)$ and the investment threshold $\bar{y}$ are determined jointly by the differential equation (40) and the corresponding boundary conditions (41) and (42). The differential equation (40) is similar to its counterpart (12) for the lump-sum payoff case. However, the boundary conditions for the flow payoff case are different from those for the lump-sum payoff case in Proposition 1 in that the agent values the stream of payoffs after option exercise with the certainty-equivalent wealth $f(y)$ given in (38). These boundary conditions jointly suggest that the investment threshold is determined by trading off between the option value of waiting $g(y)$ and the certainty-equivalent wealth $f(y)$ for the stochastic income stream (after netting out the fixed investment cost I).

Unlike the lump-sum payoff case, the total payoff volatility σ_y and the precautionary savings motive also lower the implied project value $f(y)$, because the agent is exposed to idiosyncratic shocks after making his investment decision, and hence values the cash flow at a value lower than $h(y)$, the present discounted value of his future income.

We can analyze the impact of the risk aversion coefficient γ and income volatility σ_y on the investment threshold $\bar{y}$ via the approximation method. We approximate $g(y)$ and $\bar{y}$ simultaneously to the first order of σ_y^2. We then obtain the approximate investment threshold:

$$\bar{y}_1 = \bar{y}_0 + \frac{1}{2\alpha_y}\sigma_y^2, \tag{45}$$

where $\bar{y}_0 = rI$ is the exactly solved investment threshold in the deterministic case ($\sigma_y = 0$). Therefore, to the first-order approximation, the investment threshold $\bar{y}_1$ increases in volatility σ_y, and the agent's risk attitude does not affect investment timing. This prediction is thus qualitatively the same as in the standard real options models to the first order.

The intuition for this result is as follows. In the flow payoff case, the agent receives a stream of uninsurable income after the option exercise. Therefore, the agent's precautionary savings motive lowers both the implied project value $f(y)$ and the implied option value $g(y)$. It turns out that the precautionary savings motive has offsetting effects on $g(y)$ and $f(y)$ to the first-order approximation. Thus, there is little impact on investment timing since the investment threshold $\bar{y}$ is determined by the relative magnitudes of the implied option value $g(y)$ and the project payoff $f(y)$. This result differs from the lump-sum payoff case where the precautionary savings motive only affects the implied option value, not the project payoff value. As a result, the investment threshold is lowered by the agent's precautionary savings motive to the first-order approximation in the lump-sum payoff case. In contrast, exercising the option does not eliminate the effect of uninsurable idiosyncratic shocks when payoffs are given in flow terms over time.

To further understand the impact of the agent's precautionary motive γ on the investment decision, we use the second-order approximation with respect to σ_y^2 and obtain the following approximate investment threshold:

$$\bar{y}_2 = \bar{y}_1 + \frac{1}{\alpha_y^2}\left(\gamma - \frac{r}{2\alpha_y}\right)\sigma_y^4, \tag{46}$$

where $\bar{y}_1$ is given in (45). Eq. (46) indicates that, to the second-order approximation, the investment threshold *increases* in γ, *opposite* to the prediction for the lump-sum payoff case. While the precautionary savings effect is present both before and after the option exercise as argued earlier, the precautionary savings effect, to the second-order approximation, has a larger impact on $f(y)$ than on $g(y)$. The intuition is as follows. Before exercising the option, the agent can decide when to invest in the risky investment. While volatility has offsetting effects on the implied option value $g(y)$ and the implied project value $f(y)$ to the first order, the additional flexibility of timing the investment decision at the margin implies that the precautionary savings effect is stronger after exercising the option than before. This line of argument suggests that an increase in the precautionary savings motive γ lowers $f(y)$ more than $g(y)$, thereby delaying the exercise of the option. We emphasize that the effect of γ on the investment decision is of the second order.

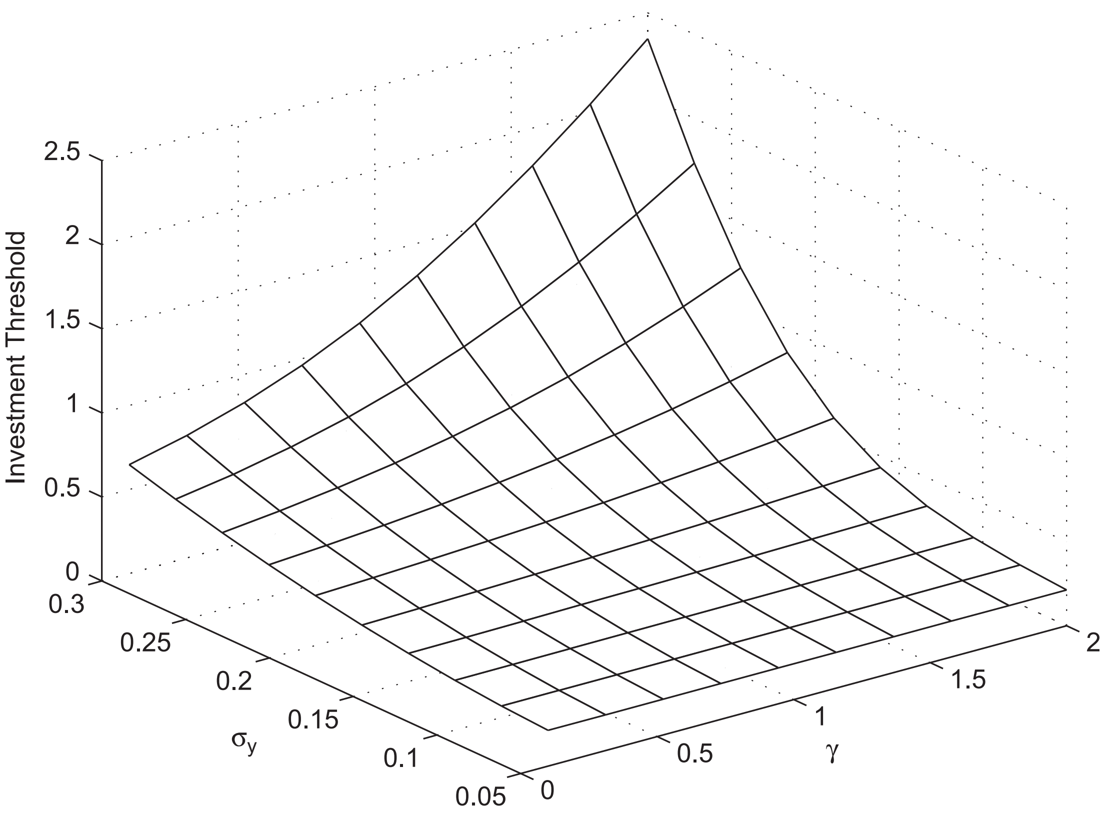

Fig. 3. *Investment threshold, risk aversion, and project volatility.* This figure plots the investment threshold at varying levels of the risk aversion parameter γ and volatility σ_y for the flow payoff case. Other parameter values are set as follows: interest rate $r = 2\%$, drift $\alpha_y = 0.1$, and investment cost $I = 10$.

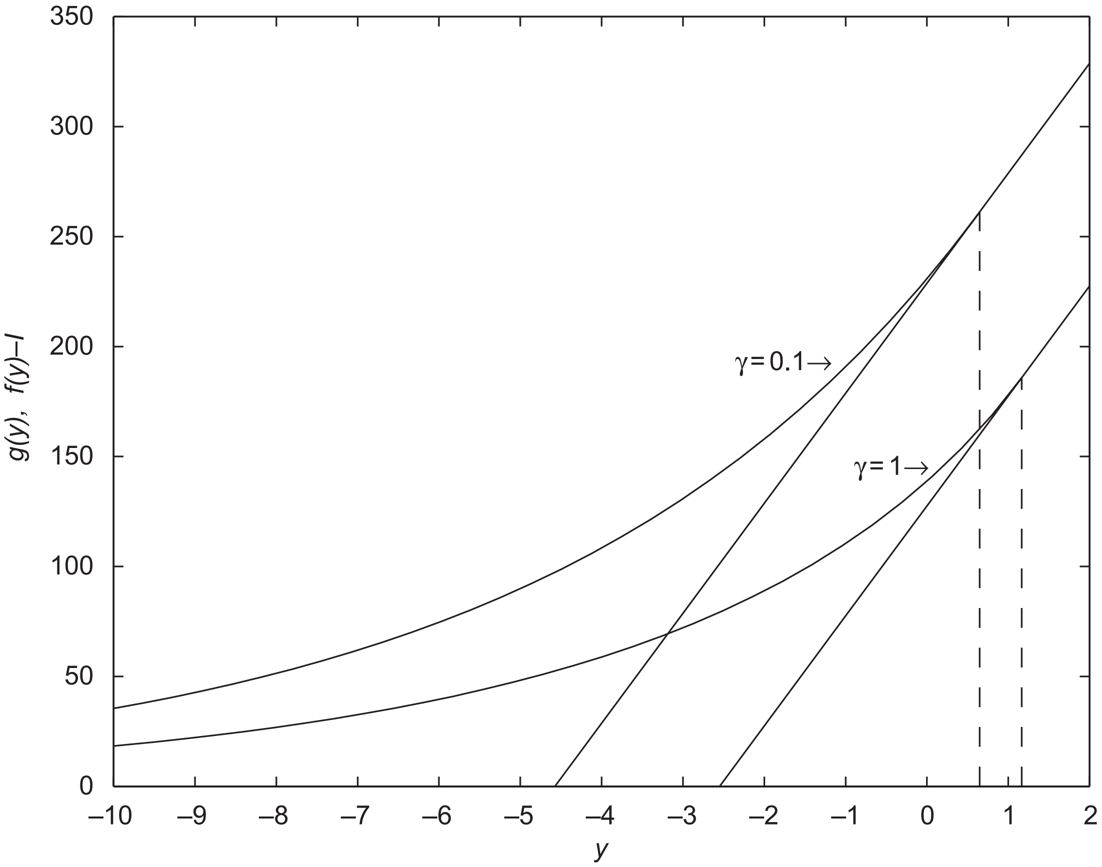

Fig. 4. *Impact of changes in risk aversion* γ *in the flow payoff case*. This figure plots the impact on $g(y)$ and $f(y)$ when the value of γ is increased from 1 to 2. Other parameter values are set as follows: interest rate $r = 2\%$, drift $\alpha_y = 0.1$, volatility $\sigma_y = 30\%$, and investment cost $I = 10$.

Finally, we use numerical solutions to conduct further analysis. Fig. 3 plots the investment threshold as a function of volatility σ_y and the parameter γ. This figure confirms our preceding approximation results. Moreover, it illustrates that the effects of volatility σ_y on the investment threshold are stronger when the agent is more precautionary, i.e., when γ is higher.

Fig. 4 illustrates the effect of changes in γ. An increase in γ raises precautionary savings both after and before the option exercise, thereby lowering both the implied project value $f(y)$ and the implied option value $g(y)$. This figure confirms our earlier analysis that $f(y)$ is lowered more than $g(y)$, so that the agent delays exercising the investment option.

4.2. Flow payoff with hedging opportunities (Model IV)

We now turn to the flow payoff case with hedging. Based on our previous analysis, we anticipate that the model contains the following two features: (i) the hedging opportunity allows the separation of idiosyncratic volatility and systematic volatility, and hence captures the different effects of these two forms of volatility on the investment and consumption decisions; and (ii) the flow payoff implies that idiosyncratic volatility continues to matter after option exercise and hence lowers the certainty-equivalent payoff value, similar to the self-insurance model for the flow payoff case.

Let π_t denote the amount allocated in the risky asset with returns given in (17) at time t. As in Section 3, we can denote ε_y as the idiosyncratic volatility, in that

$$\varepsilon_y = \sqrt{1-\rho^2}\sigma_y. \tag{47}$$

We can rewrite the observed flow payoff process $\{Y_t : t \geqslant 0\}$ given in (30) as follows:

$$\mathrm{d}Y_t = \alpha_y\,\mathrm{d}t + \rho\sigma_y\,\mathrm{d}B_t + \varepsilon_y\,\mathrm{d}\widetilde{B}_t, \tag{48}$$

where B describes the systematic (market) risk and $\widetilde{B}$ describes the idiosyncratic project risk.

Before the agent exercises the investment option at time τ, his wealth accumulation is the same as (20). After time τ, his wealth evolves as follows:

$$\mathrm{d}W_t = [rW_t + \pi_t(\mu_e - r) + Y_t - C_t]\,\mathrm{d}t + \pi_t\sigma_e\,\mathrm{d}B_t. \tag{49}$$

Note that the flow payoff Y appears in (49), not in (20). As before, the agent's wealth immediately after his investment W_τ is given by $W_\tau = W_{\tau-} - I$, where $W_{\tau-}$ denotes his wealth level just prior to his investment at time τ. The following proposition characterizes the solution.

Proposition 4. *The agent exercises the investment option the first time the process Y hits the threshold $\bar{y}$ from below. After exercising the option, the optimal consumption and portfolio rules are given by*

$$\bar{c}(w,y) = r\left(w + f(y) + \frac{\eta^2}{2\gamma r^2}\right), \tag{50}$$

$$\bar{\pi}(w,y) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r} - \frac{\rho\sigma_y}{\sigma_e r}, \tag{51}$$

where $f(y)$ is given by

$$f(y) = \left(\frac{1}{r}y + \frac{\alpha_y - \rho\sigma_y\eta}{r^2}\right) - \frac{\gamma\varepsilon_y^2}{2r^2}. \tag{52}$$

Before exercising the option, the optimal consumption and portfolio rules are given by

$$\bar{c}(w,y) = r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2}\right), \tag{53}$$

$$\bar{\pi}(w,y) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r} - \frac{\rho\sigma_y}{\sigma_e}g'(y), \tag{54}$$

where $(g,\bar{y})$ is the solution to the following free-boundary problem:

$$rg(y) = (\alpha_y - \rho\sigma_y\eta)g'(y) + \frac{\sigma_y^2}{2}g''(y) - \frac{\gamma r\varepsilon_y^2}{2}g'(y)^2, \tag{55}$$

subject to the no-bubble condition $\lim_{y\to-\infty} g(y) = 0$, *and the boundary conditions*

$$g(\bar{y}) = f(\bar{y}) - I, \tag{56}$$

$$g'(\bar{y}) = f'(\bar{y}) = \frac{1}{r}. \tag{57}$$

Moreover, g is increasing.

As in the previous subsection, we interpret $f(y)$ as the implied project value and $g(y)$ as the implied option value. Unlike the lump-sum payoff model with hedging opportunities in Section 3, hedging affects not only the implied option value $g(y)$ but also the implied project value $f(y)$. In particular, hedging lowers the agent's exposure to idiosyncratic volatility from σ_y to $\varepsilon_y = \sqrt{1-\rho^2}\sigma_y$. Thus, the precautionary savings demand after option exercise is reduced from $\gamma\sigma_y^2/(2r^2)$ to $\gamma\varepsilon_y^2/(2r^2)$. In addition, the portfolio rule (51) after option exercise consists of the standard mean-variance term and the hedging demand term.[11]

To compare with the complete-markets solution, we assume that the agent can trade an additional risky asset to diversify the idiosyncratic risk as in Section 3.2. Appendix B shows that the market value of the investment option satisfies the differential equation

$$r\Psi(y) = (\alpha_y - \rho\eta\sigma_y)\Psi'(y) + \frac{\sigma_y^2}{2}\Psi''(y) \tag{58}$$

and the boundary conditions $\lim_{y\to-\infty}\Psi(y) = 0$, $\Psi(y^*) = F(y^*) - I$ and $\Psi'(y^*) = 1/r$, where

$$F(y) = \frac{1}{r}y + \frac{\alpha_y - \rho\eta\sigma_y}{r^2} \tag{59}$$

is the market value of the cash flow process Y. Eqs. (58) and (59) reveal that both the option value and the project value under complete markets are independent of preferences and effectively are the solutions in (52) and (55) for $\gamma = 0$. In addition, both values are higher than under incomplete markets. Similar to our analysis in Section 4.1.2, the net effect of incomplete hedging on investment timing depends on the relative magnitudes of the changes in the implied option value and the project value. Similar to the insights from the self-insurance model with flow payoffs, the impact of idiosyncratic shocks on the project value is greater than on the option value to the second order. Thus, unlike in the lump-sum payoff case analyzed in Section 3.2, incomplete hedging raises the investment threshold and delays investment, compared to the complete-markets benchmark. This result demonstrates that the timing of payoffs matters for the investment decision under incomplete markets, which is different from a complete-markets setting where the timing of payoffs does not matter as shown in Appendix B.

4.3. *Empirical implications*

Our analysis has empirical implications. For example, Model I (self-insurance with a lump-sum payoff) suggests that unlike a standard real options analysis, a positive investment-uncertainty relation may potentially arise for entrepreneurial activities when idiosyncratic risk is sufficiently large. Thus, we must be cautious in interpreting some conflicting results found in empirical studies.[12] Our analysis also suggests that the

[11] See Svensson and Werner (1993), Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), Viceira (2001), Heaton and Lucas (2000), Davis and Willen (2002), and Campbell and Viceira (2002, Chapter 6), among others, on dynamic consumption and portfolio choice when an investor is endowed with nontraded stochastic income.

[12] See Quigg (1993), Berger, Ofek, and Swary (1996), Leahy and Whited (1996), and Moel and Tufano (2002) for empirical works. See Caballero (1991a) for a theoretical analysis.

investment behavior of undiversified individuals is different from that of well-diversified individuals or institutions. In particular, risk attitude plays an important role under incomplete markets. Consider again the real estate development example. Suppose we have a sample containing both undiversified individual developers and publicly traded REITs. Suppose that both individual entrepreneurs and REITs specialize in development of and not management of the properties. That is, we can take the sales value of the property upon completion of construction as given. Then, Models I and II predict that the individual entrepreneurs are more likely to develop earlier than the publicly traded REITs, because idiosyncratic risk lowers the implied option value of waiting for individual developers. However, if they also manage the properties after completion of development, then Models III and IV (flow payoff cases) suggest that the preceding prediction could be reversed because the properties are also less valuable to the undiversified individual developers. Moreover, developing real estate does not allow the entrepreneur to diversify away the uninsurable idiosyncratic risk.

5. Robustness

Our main results include the following: (i) risk aversion erodes option value and speeds up investment in the lump-sum payoff case of Model I; and (ii) risk aversion has no first-order effect on investment policy and the second-order effect is to delay investment, in the flow payoff case of Model III. These results are based on the assumptions of CARA utility and arithmetic Brownian motion payoff processes. We now relax these assumptions.[13]

First, we derive the optimal investment thresholds in Models I and III using geometric Brownian motion (GBM) payoff processes instead of arithmetic Brownian motion process (details available upon request). Consider Model I in which the lump-sum payoff process is replaced by

$$\mathrm{d}X_t = \alpha_x X_t\,\mathrm{d}t + \sigma_x X_t\,\mathrm{d}Z_t, \tag{60}$$

where $\alpha_x, \sigma_x > 0$. We assume $r > \alpha_x$ for convergence. Then we can show a result similar to Proposition 1; that is, the investment threshold $\bar{x}$ and the implied option value $G(x)$ are determined by the following free-boundary ODE:

$$rG(x) = \alpha_x x G'(x) + \frac{\sigma_x^2 x^2}{2} G''(x) - \frac{\gamma r \sigma_x^2 x^2}{2} G'(x)^2, \tag{61}$$

subject to the no-bubble condition $\lim_{x\to 0} G(x) = 0$, $\lim_{x\to\infty} G(x)/x < \infty$, and the (value-matching and smooth-pasting) boundary condition:

$$G(\bar{x}) = \bar{x} - I, \tag{62}$$

$$G'(\bar{x}) = 1. \tag{63}$$

The functional forms for the consumption rules and the value functions before investment and after investment remain the same as the ones reported in Proposition 1, but evaluated with the new option value $G(x)$, the solution to the above free-boundary problem.

As in Section 2.4, we use the asymptotic expansion method to solve for the approximate investment threshold. We can show that to the first-order approximation around σ_x^2 the

[13] We are grateful to the anonymous referee for suggesting this analysis.

investment threshold is given by

$$\bar{x}_1 = \bar{x}_0\left[1 + \left(\frac{1}{2\alpha_x} - \frac{\gamma}{2}\frac{r\bar{x}_0}{r-\alpha_x}\right)\sigma_x^2\right], \tag{64}$$

where $\bar{x}_0 = rI/(r-\alpha_x)$ is the investment threshold in the deterministic case $\sigma_x = 0$. Eq. (64) resembles (16) and implies that risk aversion speeds up investment. The intuition is that risk aversion lowers the implied option value $G(x)$ as shown in the last term in Eq. (61), but has no effect on the lump-sum project payoff X. This intuition is in accord with that in Section 2.4.

Now turn to Model III in which the flow payoff process is replaced by

$$\mathrm{d}Y_t = \alpha_y Y_t\,\mathrm{d}t + \sigma_y Y_t\,\mathrm{d}Z_t, \tag{65}$$

where $\alpha_y, \sigma_y > 0$. We assume $r > \alpha_y$ for convergence. Then we can show a result similar to Proposition 3; that is, the investment threshold $\bar{y}$ and the implied option value $g(y)$ are determined by the following free-boundary ODE:

$$rg(y) = \alpha_y y g'(y) + \frac{\sigma_y^2 y^2}{2} g''(y) - \frac{\gamma r \sigma_y^2 y^2}{2} g'(y)^2, \tag{66}$$

subject to the boundary conditions $\lim_{y\to 0} g(y) = 0$, and

$$g(\bar{y}) = f(\bar{y}) - I, \tag{67}$$

$$g'(\bar{y}) = f'(\bar{y}), \tag{68}$$

where the implied project value $f(y)$ after investment satisfies the ODE

$$rf(y) = y + \alpha_y y f'(y) + \frac{\sigma_y^2 y^2}{2} f''(y) - \frac{\gamma r \sigma_y^2 y^2}{2} f'(y)^2, \tag{69}$$

subject to the no-bubble condition $\lim_{y\to\infty} f(y)/y < \infty$. As in the robustness check for the lump-sum payoff case, the functional forms for the consumption rules and the value functions before investment and after investment are the same as the ones reported in Proposition 3, but evaluated with the new implied project value $f(y)$ and the new option value $g(y)$, characterized by (66)–(69). Note that unlike Proposition 3, there is no closed-form solution for $f(y)$ nor for the consumption rule after investment.

As in Section 4.1, we can show that to the first-order approximation around σ_y^2 the investment threshold is given by

$$\bar{y}_1 = \bar{y}_0\left(1 + \frac{1}{2\alpha_y}\sigma_y^2\right), \tag{70}$$

where $\bar{y}_0 = rI$ is the deterministic investment threshold. To ensure convergence for the first-order approximation, we assume $r > 2\alpha_y$. The preceding equation resembles Eq. (45) and implies that risk aversion has no first-order effect on the investment threshold in the flow payoff case. The intuition is the same as that discussed in Section 4.1. Specifically, both the implied option value $g(y)$ and the implied project value $f(y)$ are lowered by risk aversion as seen in (66) and (69). The two effects offset each other to the first-order approximation.

We can also demonstrate that, to the second-order approximation around σ_y^2, the investment threshold is given by

$$\bar{y}_2 = \bar{y}_1 + \frac{\bar{y}_0}{4(r-\alpha_y)\alpha_y^3}\left[2Ir^2\alpha_y\gamma - (r-\alpha_y)^2\right]\sigma_y^4. \tag{71}$$

This equation resembles Eq. (46) and implies that risk aversion delays investment. This result is due to the fact that risk aversion has a bigger (negative) impact on the implied project value $f(y)$ than on the implied option value $g(y)$ to the second order, which is consistent with the result and the intuition reported for the arithmetic Brownian motion case in Section 4.1.

Our preceding analysis demonstrates that our main results on the investment timing are robust to the GBM specification of the payoff process. This finding is not surprising since our intuition on the interplay between the precautionary savings effect and the option effect does not hinge upon the specifications of the payoff process.

We now discuss the specification of the utility function. CARA utility allows us to exploit its lack of wealth effect feature so that we can reduce the two-dimensional free-boundary problem to a simpler one with the payoff as the only state variable. Another widely adopted specification is the constant relative risk aversion (CRRA) utility function. Although this utility function has the homogeneity property, the budget constraints in our problem are nonlinear due to the jump of wealth at the time of investment. Thus, our problem does not have a homogeneity property so that we cannot reduce our two-dimensional problem to one with a single state variable. In this case, the investment threshold is a function of wealth and differs from that in our CARA utility models. Therefore, both our asymptotic expansion and numerical methods are not directly applicable to the CRRA utility specification. Since an agent with CRRA utility also has precautionary motives, we suspect that our key intuition, primarily building on the interaction between precautionary savings and option value, is likely to survive. Nonetheless, we should emphasize that with CRRA utility, the agent's liquid wealth will have effects on investment policy that could generate new insights. A thorough analysis is beyond the scope of the present paper.

6. Conclusions

Entrepreneurs' business investment opportunities are often nontradable and their payoffs cannot be spanned by existing traded assets for reasons such as incentives and informational asymmetries. These features invalidate the standard real options approach to investment. Extending this approach, we develop a utility-based real options model to analyze an agent's interdependent real investment, consumption, and portfolio choice decisions. We derive semi-closed-form solutions and analyze the solutions using both asymptotic expansion and numerical methods.

We show that project volatility has both a positive and a negative effect on the implied option value. The negative effect is induced by the precautionary savings motive. For the lump-sum payoff case, risk aversion accelerates investment. Unlike in standard real options analysis, an increase in project volatility can accelerate investment if the agent has a sufficiently strong precautionary motive. We further extend our model to allow for the opportunity to hedge. We show that hedging reduces the agent's exposure to idiosyncratic risk, and hence raises the option value. In addition, hedging allows the decomposition of total project volatility into systematic volatility and idiosyncratic volatility. Idiosyncratic volatility generates an idiosyncratic risk premium. Finally, we analyze settings where investment payoffs are given in flow terms over time. Unlike in standard real options analysis, here the lump-sum and flow payoff cases have different implications. Because the precautionary savings effect matters both before and after investment in the flow payoff case, many predictions in this case differ from and can even be opposite to those in the lump-sum payoff case.

In order to analyze the effect of uninsurable idiosyncratic risk on investment in the simplest possible setting, we have intentionally ignored the wealth effect by adopting CARA utility. However, the wealth effect can potentially play an important role for quantitative assessment in settings such as entrepreneurship. We leave the incorporation of wealth effects to future research. Finally, when entrepreneurs invest in nontradable projects, they often need to make financing decisions jointly. For the real estate example, the construction and operating expenses are often largely financed by mortgages. We analyze the interaction between investment and financing decisions in Miao and Wang (2005).

Appendix A. Proofs

Proof of Proposition 1. From the first-order condition $U'(c) = V_w(w, x)$, we can derive the consumption policy before the option exercise given in (11). Substituting it into the HJB equation (5), we can show that $G(x)$ satisfies the ODE (12). Given the functional forms of the value functions, we can also show that the no-bubble, value-matching, and smooth-pasting conditions become the boundary conditions in Proposition 1. By a standard dynamic programming argument, one can show that V satisfies

$$V(w,x) = \max_{(\tau,C)} E\left[\int_0^\tau e^{-rt}U(C_t)\,\mathrm{d}t + e^{-r\tau}V^0(W_\tau + X_\tau - I)\middle|(W_0, X_0) = (w,x)\right]. \tag{A.1}$$

Consider $x < x'$. For $X_0 = x'$, let τ' be the optimal investment time and $\{C'_t : 0 \leqslant t \leqslant \tau'\}$ be the optimal consumption process before investment. Since V^0 is an increasing function and, given any sample path,

$$X_{\tau'} \equiv x + \alpha_x\tau' + \sigma_x W_{\tau'} < X'_{\tau'} \equiv x' + \alpha_x\tau' + \sigma_x W_{\tau'},$$

we have

$$\int_0^{\tau'} e^{-rt}U(C'_t)\,\mathrm{d}t + e^{-r\tau'}V^0(W_{\tau'} + X_{\tau'} - I) < \int_0^{\tau'} e^{-rt}U(C'_t)\,\mathrm{d}t + e^{-r\tau'}V^0(W_{\tau'} + X'_{\tau'} - I).$$

Taking conditional expectations yields

$$\begin{aligned}&\mathrm{E}\left[\int_0^{\tau'} e^{-rt}U(C'_t)\,\mathrm{d}t + e^{-r\tau'}V^0(W_{\tau'} + X_{\tau'} - I)\middle|(W_0, X_0) = (w,x)\right]\\ &\quad< \mathrm{E}\left[\int_0^{\tau'} e^{-rt}U(C'_t)\,\mathrm{d}t + e^{-r\tau'}V^0(W_{\tau'} + X'_{\tau'} - I)\middle|(W_0, X_0) = (w,x')\right] = V(w,x').\end{aligned}$$

Given the wealth dynamics described in Section 2.1, $\{C'_t : 0 \leqslant t \leqslant \tau'\}$ and τ' are also feasible for $X_0 = x$. Thus, the left side of the above equation is less than or equal to $V(w,x)$ by (A.1). So, $V(w,x) < V(w,x')$ and V is increasing in x. $\square$

Proof of Proposition 2. Without risk of confusion, we still use $V^0(w)$ and $V(w,x)$ to denote the value function after and before the option exercise, respectively, when the agent can trade a risky asset. By a standard argument, V^0 satisfies the following HJB equation:

$$rV^0(w) = \max_{(c,\pi)\in\mathbb{R}^2} U(c) + [rw + \pi(\mu_e - r) - c]V^0_w(w) + \frac{(\pi\sigma_e)^2}{2} V^0_{ww}(w). \tag{A.2}$$

The transversality condition $\lim_{T\to\infty} \mathrm{E}[e^{-rT}V^0(W_T)] = 0$ must also be satisfied. Given CARA utility, one can follow Merton (1969) to derive the consumption and portfolio rules in (21)–(22) and

$$V^0(w) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + \frac{\eta^2}{2\gamma r^2}\right)\right]. \tag{A.3}$$

Before the option exercise, the value function $V(w,x)$ satisfies the following HJB equation:

$$\begin{aligned} rV(w,x) = \max_{(c,\pi)\in\mathbb{R}^2} & \; U(c) + [rw + \pi(\mu_e - r) - c]V_w(w,x) + \alpha_x V_x(w,x) \\ & + \frac{\sigma_x^2}{2} V_{xx}(w,x) + \frac{(\pi\sigma_e)^2}{2} V_{ww}(w,x) + \pi\sigma_e\sigma_x\rho V_{wx}(w,x). \end{aligned} \tag{A.4}$$

We conjecture that the value function V takes the form

$$V(w,x) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + G(x) + \frac{\eta^2}{2\gamma r^2}\right)\right], \tag{A.5}$$

where $G(x)$ is a function to be determined. Using the first-order conditions,

$$U'(c) = V_w(w,x), \pi = \frac{-V_w(w,x)}{V_{ww}(w,x)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-V_{wx}(w,x)}{V_{ww}(w,x)}\frac{\rho\sigma_x}{\sigma_e}, \tag{A.6}$$

one can derive the optimal consumption and portfolio policies before exercising the option given in (23)–(24). Plugging these expressions back into the HJB equation gives (25). As in Section 2, the boundary conditions are given by the no-bubble, value-matching, and smooth-pasting conditions similar to (6)–(8). Using these boundary conditions, one can derive the boundary conditions in Proposition 2. The rest of the proof follows a similar argument to that in Proposition 1. $\square$

Proof of Proposition 3. We conjecture that the value function after the option exercise J takes the following form:

$$J(w,y) = -\frac{1}{\gamma r}\exp[-\gamma r(w + f(y))], \tag{A.7}$$

where $f(y)$ is a function to be determined. To solve for this function, we use the first-order condition $U'(c) = J_w(w,y)$ to derive the optimal consumption rule given in (37). Substitute it back into the HJB equation (32) to derive the following ODE:

$$0 = (y - rf(y)) + \alpha_y f'(y) + \frac{\sigma_y^2}{2}[f''(y) - \gamma r f'(y)^2]. \tag{A.8}$$

It can be verified that its solution is given by (38). Moreover, it is such that the value function satisfies the transversality condition $\lim_{T\to\infty} \mathrm{E}[e^{-rT} J(W_T, Y_T)] = 0$.

We conjecture that the value function before the option exercise, $V(w, y)$, takes the form

$$V(w, y) = -\frac{1}{\gamma r}\exp[-\gamma r(w + g(y))], \tag{A.9}$$

where $g(y)$ is a function to be determined. From the first-order condition $U'(c) = V_w(w, y)$, we can derive the consumption policy before investment given in (39). Substituting it into the HJB equation (33), we can show that $g(y)$ satisfies the ODE (40). By a standard dynamic programming argument, we can show that V satisfies

$$V(w, y) = \max_{\tau, C} \mathrm{E}\left[\int_0^\tau e^{-rt} U(C_t)\,\mathrm{d}t + e^{-r\tau} J(W_\tau - I, Y_\tau)\middle|(W_0, Y_0) = (w, y)\right]. \tag{A.10}$$

Since it follows from (A.7) that J is increasing and concave in y, we can show that V is also increasing and concave in y. The rest of the proof follows from a similar argument to that in Proposition 1. □

Proof of Proposition 4. Without risk of confusion, we still use $J(w, y)$ and $V(w, y)$ to denote the value function after and before the option exercise, respectively, when the agent can also trade a risky asset. By a standard argument, $J(w, y)$ satisfies the HJB equation

$$\begin{aligned} rJ(w, y) = \max_{(c,\pi)\in\mathbb{R}^2} & U(c) + [rw + \pi(\mu_e - r) + y - c]J_w(w, y) + \alpha_y J_y(w, y) \\ & + \frac{\sigma_y^2}{2} J_{yy}(w, y) + \frac{(\pi\sigma_e)^2}{2} J_{ww}(w, y) + \pi\sigma_e\sigma_y\rho J_{wy}(w, y). \end{aligned} \tag{A.11}$$

The transversality condition $\lim_{T\to\infty} \mathrm{E}[e^{-rT} J(W_T, Y_T)] = 0$ must also be satisfied. We conjecture that $J(w, y)$ takes the following form:

$$J(w, y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + f(y) + \frac{\eta^2}{2\gamma r^2}\right)\right], \tag{A.12}$$

where the function f is to be determined. By the first-order conditions,

$$U'(c) = J_w(w, y), \pi = \frac{-J_w(w, y)}{J_{ww}(w, y)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-J_{wy}(w, y)}{J_{ww}(w, y)}\frac{\rho\sigma_y}{\sigma_e}, \tag{A.13}$$

we can derive the optimal consumption and portfolio policies after investment given in (53)–(54). Substituting them back into the HJB equation (A.11), one can derive the solution for $f(y)$ given in (52). It can be verified that this solution satisfies the transversality condition.

The value function before the option exercise, V, satisfies the following HJB equation:

$$\begin{aligned} rV(w, y) = \max_{(c,\pi)\in\mathbb{R}^2} & U(c) + [rw + \pi(\mu_e - r) - c]V_w(w, y) + \alpha_y V_y(w, y) \\ & + \frac{\sigma_y^2}{2} V_{yy}(w, y) + \frac{(\pi\sigma_e)^2}{2} V_{ww}(w, y) + \pi\sigma_e\sigma_y\rho V_{wy}(w, y). \end{aligned} \tag{A.14}$$

We conjecture that the value function V takes the following form:

$$V(w,y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + g(y) + \frac{\eta^2}{2\gamma r^2}\right)\right], \tag{A.15}$$

where $g(y)$ is a function to be determined. Using the first-order conditions,

$$U'(c) = V_w(w,y), \pi = \frac{-V_w(w,y)}{V_{ww}(w,y)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-V_{wy}(w,y)}{V_{ww}(w,y)}\frac{\rho\sigma_y}{\sigma_e}, \tag{A.16}$$

one can derive the optimal consumption and portfolio policies before investment given in (50)–(51). Plugging these expressions into the HJB equation gives a differential equation for $g(\cdot)$. The rest of the proof follows from a similar argument to that in Propositions 1 and 3. □

Appendix B. Complete markets solution

To derive the complete-markets solution, we assume that the agent can trade an additional risky asset that spans the idiosyncratic risk generated by the Brownian motion $\widetilde{B}$. Specifically, let the return of the second risky asset be given by $\mathrm{d}S_t/S_t = r\,\mathrm{d}t + \sigma_S\,\mathrm{d}\widetilde{B}_t$, where σ_S is a positive constant. Since idiosyncratic risk is by definition independent of market risk, this risky asset yields an expected rate of return r and does not demand a risk premium by the CAPM. Therefore, the implied unique stochastic discount factor ξ is given by $-\mathrm{d}\xi_t/\xi_t = r\,\mathrm{d}t + \eta\,\mathrm{d}B_t$ with $\xi_0 = 1$, where η is the Sharpe ratio of the market portfolio.

The agent's joint consumption, investment, and asset allocation decision can then be formulated as a two-stage problem with the agent (i) choosing an investment policy to maximize the option value so that the agent's total wealth is maximized and (ii) choosing optimal consumption given this total wealth.

We first derive the solution for the lump-sum payoff case. Using the unique stochastic discount factor ξ, we can write the option value maximization problem as follows:

$$\Phi(x) = \max_{\tau} \mathrm{E}[\xi_\tau(X_\tau - I)|X_0 = x]. \tag{B.1}$$

By a standard argument, we can derive explicit expressions for the option value and the investment threshold,

$$\Phi(x) = \frac{1}{\lambda_x}e^{\lambda_x(x-x^*)}, \tag{B.2}$$

$$x^* = I + \frac{1}{\lambda_x}, \tag{B.3}$$

where $\lambda_x = -\sigma_x^{-2}(\alpha_x - \rho\eta\sigma_x) + \sqrt{\sigma_x^{-4}(\alpha_x - \rho\eta\sigma_x)^2 + 2r\sigma_x^{-2}} > 0$.

For the flow-sum payoff case, we can similarly write the market option value as

$$\Psi(y) = \max_{\tau} \mathrm{E}\left[\int_\tau^\infty \xi_t Y_t\,\mathrm{d}t - \xi_\tau I \,\middle|\, Y_0 = y\right]. \tag{B.4}$$

By a standard argument, we derive the following explicit expressions for the option value and the investment threshold:

$$\Psi(y) = \frac{1}{r\lambda_y}e^{\lambda_y(y-y^*)}, \tag{B.5}$$

$$y^* = rI - \frac{\alpha_y - \rho\eta\sigma_y}{r} + \frac{1}{\lambda_y}, \tag{B.6}$$

where $\lambda_y = -\sigma_y^{-2}(\alpha_y - \rho\eta\sigma_y) + \sqrt{\sigma_y^{-4}(\alpha_y - \rho\eta\sigma_y)^2 + 2r\sigma_y^{-2}} > 0$.

We observe that, under complete markets, the lump-sum and flow payoff formulations are mathematically equivalent, since we can discount cash flows using the unique stochastic discount factor ξ. Specifically, by defining $X_t = \xi_t^{-1}\mathrm{E}_t(\int_t^\infty \xi_s Y_s \,\mathrm{d}s) = F(Y_t)$, we can show that the problems (B.1) and (B.4) are equivalent. Thus, they deliver the same option value $\Phi(x) = \Psi(y)$ and investment timing strategy. However, this equivalence fails when the investment opportunity is not tradable and not spanned by existing traded assets.

Appendix C. Approximation method

To describe our approximation solution methodology, we sketch out the procedure for the self-insurance model with a lump-sum payoff. Essentially identical procedures can be applied to the models in Sections 3 and 4. We divide the procedure into four steps.

Step 1: Solve for the case with deterministic payoff ($\sigma_x^2 = 0$). With $\sigma_x = 0$, risk attitude (γ) does not affect the investment threshold. The implied option value $G_0(x)$ and the investment threshold $\bar{x}_0$ are both known in closed form and are given by

$$G_0(x) = \frac{\alpha_x}{r}\exp\left[\frac{r}{\alpha_x}(x - \bar{x}_0)\right], \quad x \leqslant \bar{x}_0, \tag{C.1}$$

$$\bar{x}_0 = I + \frac{\alpha_x}{r}. \tag{C.2}$$

Step 2: Consider small σ_x^2. Conjecture that the approximate option value and the investment threshold are

$$G(x) \approx G_0(x) + G_1(x)\sigma_x^2, \tag{C.3}$$

$$\bar{x}_1 = \bar{x}_0 + \delta_1\sigma^2, \tag{C.4}$$

where $G_0(x)$ and $\bar{x}_0$ are solved in Step 1, and $G_1(x)$ and δ_1 are the coefficient function and the coefficient to be determined.

Step 3: Plugging the approximate (C.3) into the ODE (12) and boundary conditions (13)–(14) and keeping the terms up to σ_x^2, we have the following:

$$\alpha_x G_1'(x) + \frac{1}{2}G_0''(x) - \frac{\gamma r}{2}G_0'(x)^2 = rG_1(x), \tag{C.5}$$

subject to $G_1(\bar{x}_1) = 0$ and $G_1'(\bar{x}_1) = -r\delta_1/\alpha_x$. Note that unlike the original nonlinear ODE (12) for $G(x)$, we now have a free-boundary problem defined by a first-order ODE (C.5) for $G_1(x)$ with certain boundary conditions.

Step 4: Solving the above differential equation gives our reported solution in (16) and

$$G_1(x) = \frac{r}{2\alpha_x^2}(\bar{x}_0 - x)e^{-\frac{r}{\alpha_x}(\bar{x}_0 - x)} - \frac{\gamma}{2}\left[e^{-\frac{r}{\alpha_x}(\bar{x}_0 - x)} - e^{-\frac{2r}{\alpha_x}(\bar{x}_0 - x)}\right], \; x \leqslant \bar{x}_1.$$

Appendix D. Computation method

We describe the solution method to the free-boundary problem described in Proposition 3. The problems described in other propositions can be solved similarly. We use the projection method implemented with collocation (Judd, 1998). The traditional shooting method and finite difference method are inefficient for our nonlinear problem and extensive simulations.

We first rewrite the second-order ODE (40) as a system of first-order ODEs. Let $\Delta(y) = g'(y)$. Then (40) can be rewritten as

$$\Delta'(y) = \frac{2}{\sigma_y^2}(rg(y) - \alpha_y \Delta(y)) + \gamma r \Delta(y)^2. \tag{D.1}$$

The boundary conditions are

$$\lim_{y \to -\infty} g(y) = 0, \tag{D.2}$$

$$g(\overline{y}) = f(\overline{y}) - I, \tag{D.3}$$

$$\Delta(\overline{y}) = 1/r. \tag{D.4}$$

Note that (D.2) states that when y goes to minus infinity, the agent never exercises the investment option, and hence the implied option value is equal to zero.

The idea of the algorithm is to first ignore the smooth-pasting condition (D.4) and then to solve a two-point boundary value problem with a guessed threshold value $\overline{y}_0$. Since the boundary condition (D.2) is open ended, we pick a very small negative number $\underline{y}$ and set $g(\underline{y}) = 0$. The true value of the threshold is found by adjusting $\overline{y}_0$ so that the smooth-pasting condition (D.4) is satisfied. We then adjust $\underline{y}$ so that the solution is not sensitive to this value. The algorithm is outlined as follows.

Step 1: Start with a guess $\overline{y}_0$ and a preset order n.

Step 2: Use a Chebyshev polynomial to approximate g and Δ:

$$g(y; a) = \sum_{i=0}^{n} a_i T_i(y), \Delta(y; b) = \sum_{i=0}^{n} b_i T_i(y), \tag{D.5}$$

where $T_i(y)$ is the Chebyshev polynomial of order i, and $a = (a_0, a_1, \ldots, a_n)$ and $b = (b_0, b_1, \ldots, b_n)$ are $2n+2$ constants to be determined. Substitute the above expressions into the preceding system of ODEs and evaluate it at n roots of $T_n(y)$. Together with the two boundary conditions, we then have $2n+2$ equations for $2n+2$ unknowns $a = (a_0, a_1, \ldots, a_n)$ and $b = (b_0, b_1, \ldots, b_n)$. Let the solution be $\widehat{a}$ and $\widehat{b}$.

Step 3: Search for $\overline{y}_0$ such that the smooth-pasting condition, $\Delta(\overline{y}_0; \widehat{b}) = 1/r$, is approximately satisfied.

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