Habits and Leverage

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Abstract

Many stylized facts of leverage, trading, and asset prices obtain in a frictionless general equilibrium model that features agents' heterogeneity in endowments and habit preferences. Our model predicts that aggregate debt increases in expansions when asset prices are high, volatility is low, and levered agents enjoy a "consumption boom." Our model is consistent with poorer agents borrowing more and with intermediaries' leverage being a priced factor. In crises, levered agents strongly deleverage by "fire selling" their risky assets as asset prices drop. Yet, consistently with the data, their debt-to-wealth ratios increase as higher discount rates make their wealth decline faster.

1. Introduction

The financial crisis has elicited much research into the understanding of the dynamics of aggregate leverage and its impact on asset prices and economic growth. Recent empirical and theoretical research has produced a variety of results that, as argued by many, should inform a reconsideration of existing frictionless models. Amongst these we have (i) the evidence that excessive credit supply may lead to financial crises;[1] (ii) the growth in household debt and the causal relation between the deleveraging of levered households and their low future consumption growth;[2] (iii) the idea that active leveraging and deleveraging of households and financial institutions directly contributes to the rise and fall of asset prices;[3] (iv) the evidence that the aggregate leverage ratio of financial institutions is a risk factor in asset pricing;[4] (v) the view that balance sheet recessions are critical components of business cycle fluctuations;[5] and many others. Most of these explanations rely on some form of market friction, behavioral bias or both, and propose a causal effect for the effects of leverage on aggregate economic and financial phenomena. In this paper we put forward a simple frictionless general equilibrium model with endogenous leverage that offers a coherent explanation of most of these relations between agents' leverage, their consumption, and asset prices.

We posit an economy populated with agents whose preferences feature external habits. Specifically, we introduce a novel and analytically tractable "habit in utility" specification. Under these preferences, each agent's utility increases in his own consumption but decreases in the happiness of his neighbors, rather than their consumption as in traditional habit models. The weight on the external habit, moreover, differs across agents and it increases during economic downturns. That is, habits matter more in bad times than in good times. These assumptions make agents differentially more risk averse in recessions, which in turn introduces motives for risk sharing and asset trading. In addition to tractability, our preferences' specification has numerous predictions on agents' behavior that are consistent with the existing empirical literature, as further discussed below. Agents also differ in their level of endowment, which is also an important determinant of their risk bearing capacity. The model aggregates nicely to standard external habit models such as Campbell and Cochrane (1999) and Menzly, Santos and Veronesi (2004) and thus inherits the asset pricing properties of these models and in particular the dynamics of risk and return that were their original motivation.

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[1] See for instance Jordà, Schularick and Taylor (2011).
[2] See Justiniano, Primiceri and Tambalotti (2013) and Mian and Sufi (2015).
[3] See e.g. Shleifer and Vishny (2011), Geanakoplos (2010).
[4] See He and Krishnamurthy (2013) and Adrian, Etula and Muir (2014).
[5] See Huo and Ríos-Rull (2013) and Mian, Rao and Sufi (2013).

External habit models feature strong discount effects, which, as shown by Hansen and Jagannathan (1991), are required to explain the Sharpe ratios observed in financial markets. We argue that these strong discount effects are also important to understand the dynamics of risk sharing. Standard risk sharing arguments require that agents with large risk bearing capacity insure those with low risk bearing capacity. In models where, for instance, agents have CRRA preferences, such as Dumas (1989) and Longstaff and Wang (2012), this means the agents who provide the insurance consume a large share of aggregate consumption when this is large and a low share when instead aggregate consumption is low. This is obviously also the case in our framework, but in addition the share of consumption also depends on whatever state variable drives discount effects, which introduces additional sources of non-linearities in the efficient risk sharing arrangement. The reason is that in our model risk aversion changes depending on the actual realization of the aggregate endowment and thus so do the efficiency gains associated with risk sharing.

We decentralize the efficient allocation by allowing agents to trade in a claim to the aggregate endowment process and debt that is in zero net supply and provide a full characterization of the corresponding competitive equilibrium. We show that agents with higher initial endowment and/or weaker habit preferences have higher risk tolerance and thus provide insurance by issuing risk-free debt to agents with lower endowment and/or stronger habit preferences. The latter agents are more risk averse and hence want to hold risk-free debt to insure against fluctuations in their marginal utility of consumption.

A striking property of the competitive equilibrium is that the aggregate debt in the economy, scaled by output, is procyclical, an intuitive result but one that does not obtain in standard models.[6] The reason hinges on the decrease in aggregate risk aversion in good times, which makes agents with high risk tolerance willing to take on a larger fraction of the aggregate risk by issuing more risk-free debt to agents with lower risk tolerance. Thus, procyclical aggregate debt emerges naturally as the result of the optimal trading of utility maximizing agents in an equilibrium that implements an optimal risk sharing allocation.

Besides a procyclical debt-to-output ratio, our model has several additional predictions that are consistent with numerous stylized facts. First, habit heterogeneity induces agents with low endowment to leverage in equilibrium. That is, unlike most of the previous literature, our model is consistent with the empirical evidence in that poorer agents borrow more than richer agents to increase consumption. Intuitively, habit heterogeneity allows for

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[6]Most models with heterogeneous agents feature only two types of agents. Thus, leverage is necessarily inverse-U shaped in the wealth share, as it must be zero when wealth is mostly in the hands of one or the other agent. Moreover, in such models, lower aggregate risk – typical in good times – tend to reduce leverage due to lower risk-sharing needs.

a large number of low risk averse agents among those with low endowments.

Second, higher aggregate debt, scaled by aggregate output, should correlate with (i) higher valuation ratios, (ii) lower return volatility, (iii) lower future excess returns, and (iv) a "consumption boom" of those agents who lever up, who then should experience a consumption slump relative to others, on average. The reason is that as explained above, in good times aggregate debt increases as aggregate risk aversion declines. Lower risk aversion implies high valuation ratios and lower stock return volatility, as well as lower future excess returns, explaining (i) through (iii). In addition, levered agents who took up levered positions do especially well when stock market increases, implying higher consumption in good times. Mean reversion, however, implies that these same agents should also expect a relatively lower future consumption growth after their consumption binge, explaining (iv).

Our model also implies active trading. For instance, a series of negative aggregate shocks induces deleveraging of levered agents through the active sales of their positions in risky stocks. It follows that stock price declines occur exactly at the time when levered agents actively sell their risky positions to reduce leverage. This commonality of asset sales and stock price declines give the impression of a "selling pressure" affecting asset prices, when in fact equilibrium prices and quantities comove due to the variation in aggregate risk aversion, but there is no causal relation between trading and price movements. Indeed, in our model the representative agent is *independent* of agents' heterogeneity and thus the same asset pricing implications result even with identical agents and hence no trading.

While our model implies that during bad times aggregate debt declines relative to output, levered agents' debt-to-wealth ratios increase, as wealth declines faster than debt due to severe discount-rate effects. Hence, while the aggregate debt is pro-cyclical, the debt-to-wealth ratio of levered agents is countercyclical, which is broadly consistent with the empirical evidence. For instance, during 2007 - 2009 crisis the debt-to-wealth ratio of levered households increased considerably due to the decline in the value of their assets, especially housing.

Our model's predictions about leverage dynamics also sheds some light on recent empirical results in the intermediary asset pricing literature. High net-worth agents lever up to invest in risky securities, as intermediaries do in much of this literature. Because the leverage of these agents correlates with the aggregate economy risk aversion, our model implies that leverage is a priced risk factor in cross-sectional regressions. However, the sign of the price of risk depends on whether we measure leverage using market prices (e.g. debt-to-wealth ratios) or not (e.g. debt-to-output ratio), which is consistent with recent empirical evidence (Adrian, Etula, and Muir (2014) and Kelly, He, and Manela (2016)).

Finally, our model has predictions about the source of the variation in wealth inequality. Heterogeneity in endowments makes wealth inequality increase in good times, as agents with large endowment borrow and thus enjoy capital gains in those times. In contrast, heterogeneity in habits make poor agents borrow, who then enjoy an increase in their wealth in good times and lead to a lower dispersion in wealth shares. These two different sources of heterogeneity thus imply a complex dynamics of wealth dispersion over the business cycle. Once again, the model emphasizes that while asset prices affect wealth inequality, the converse does not hold, as asset prices are identical with homogeneous agents, and hence in the same model without wealth dispersion.

Our model has the considerable advantage of simplicity: All formulas for asset prices, portfolio allocation, and leverage are in closed form, no numerical solutions are required, and intuition follows from basic economic principles. Moreover, because our model aggregates to the representative agent of Menzly, Santos, and Veronesi (2004), except that we allow for time varying aggregate uncertainty, we can calibrate its parameters to match the properties of aggregate return dynamics. Our model thus, unlike most of the literature, has clears quantitative implications, not just qualitative ones.

Clearly many explanations have been put forth to explain the growth of leverage and of household debt in particular during the run up to the crisis. For instance, Bernanke (2005) argues that the global savings glut, the excess savings of East Asian nations in particular, is to blame for the ample liquidity in the years leading up to the Great Recession, which reduced rates and facilitated the remarkable rise in household leverage; Shin (2012) shows how regulatory changes, the adoption of Basel II, led European banks to increase lending in the US; Pinto (2010), Wallison (2011) and Calomiris and Haber (2014) argue that the Community Reinvestment Act played a pivotal role in the expansion of mortgage lending to risky households (but see Bhutta and Ringo (2015)); Mendoza and Quadrini (2009) show how world financial integration leads to an increase in net credit. The list goes on.

When the crisis came, the crash in prices and the rapid deleveraging of households and financial intermediaries was interpreted appealing to classic inefficient runs arguments a la Diamond and Dybvig (1983) as in Gorton and Metrick (2010) or contagion. He and Krishnamurthy (2008) connect the fall in asset prices to the shortage of capital in the intermediation sector. Finally, much research has focused on the impact that the crisis had on the consumption of households. For instance Mian and Sufi (2014) argue that debt overhang is to blame for the drop in consumption in counties where households were greatly levered.

Our point here is not to claim that these frictions are not important but simply to offer

an alternative explanation that is consistent with complete markets and that matches what we know from the asset pricing literature. We highlight that leverage is an endogenous quantity and thus cannot be used as an independent variable to explain other facts. For instance, when debt overhang is put forth as an explanation for low consumption patterns amongst levered households the alternative hypothesis of efficient risk sharing cannot be dismissed outright. Both explanations operate in the same direction and thus assessing the quantitative plausibility of one requires controlling for the other.

This paper is obviously connected to the literature on optimal risk sharing, starting with Borch (1962). Much of this literature is concerned with assessing to what extent consumers are effectively insured against idiosyncratic shocks to income and wealth.[7] Our model does not feature idiosyncratic income shocks but there is still a motive for risk sharing that is linked to different sensitivities of habits to aggregate shocks. Our paper is more closely related to Dumas (1989), Wang (1996), Bolton and Harris (2013), Longstaff and Wang (2012), and Bhamra and Uppal (2014). These papers consider two groups of agents with constant risk aversion, and trading and asset prices are generated by aggregate shocks through the variation in the wealth distribution. While similar in spirit, our model generates several novel results that do not follow from this previous work, such as procyclical debt-to-output ratio, countercyclical debt-to-wealth ratios, higher leverage amongst poorer agents, procyclical wealth dispersion, consistency with asset pricing facts, and so on. Our model is more closely related to Chan and Kogan (2002), who also consider a continuum of agents with habit preferences and heterogeneous risk aversion. In their setting, however, the risk aversions of individual agents are constant, while in our setting they are time varying in response to business cycle variation, a crucial ingredient in our model. Moreover, Chan and Kogan (2002) do not investigate the leverage dynamics implied by their model, which is instead our focus.

Our model is related to Campbell, Grossman and Wang (1993), which explores the implications for trading volume and asset prices in a model where the motivation for trade is driven by shocks to agents' risk tolerance. More recently Alvarez and Atkenson (2017) consider a model where agents' risk tolerance is subject to uninsurable idiosyncratic shocks. In our paper instead variation in risk tolerance is driven by exposure to a business cycle factor, and the source of heterogeneity, in addition to initial endowment, is the degree of exposure to that factor. Neither Campbell, Grossman and Wang (1993) or Alvarez and Atkenson (2017) analyzes the dynamics of leverage and the distribution of leverage in the population.

Finally, a recent literature (Barro and Mollerus (2014) and Caballero and Fahri (2014))

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[7]See for instance Dynarski and Sheffrin (1987), Cochrane (1991), Mace (1991) and Townsend (1994).

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studies the determinants of the supply of safe assets and its connection to aggregate activity. In our model all debt is indeed risk free and the supply of safe securities is determined by the risk bearing capacity of the wealthiest and/or least risk averse agents in the population. Importantly we are interested in the dynamics of the supply of safe assets and how it fluctuates with aggregate economic conditions.

The paper is structured as follows. The next section presents the model. Section 3 characterizes the optimal risk sharing arrangement. Decentralization of the efficient allocation and characterization of the competitive equilibrium are covered in Section 4. Section 5 evaluates the model quantitatively and Section 6 concludes. All proofs are in the Appendix.

2.   The model

Preferences. There is a continuum of agents endowed with log utility preferences defined over consumption C_{it} in excess of agent-specific external habit indices X_{it}:

$$u\left(C_{i,t}, X_{i,t}, t\right) = e^{-\rho t} \log\left(C_{it} - X_{it}\right)$$

Agents are heterogeneous in the habit indices X_{it}, which are given by:[8]

$$X_{it} = g_{it}\left(D_t - \int X_{jt} dj\right) \tag{1}$$

That is, the habit level X_{it} of agent i is proportional to the difference between aggregate output D_t and the average habit $\int X_{jt} dj$, which we call the *excess output* henceforth. A higher excess output decreases agent i's utility, an effect that captures a notion of "Envy the Joneses." As we shall show, the excess output $\left(D_t - \int X_{jt} dj\right)$ is in fact increasing in the utility of the representative agent and thus is an index of the "happiness" of the Joneses, a fact that makes agent i less happy as it pushes up his habit level X_{it} and thus reduces his utility. Our model is thus an external habit model defined on utility – as opposed to consumption – in that other people happiness impacts agent i's utility negatively.

The sensitivity of agent i's habit X_{it} to aggregate excess output $\left(D_t - \int X_{jt} dj\right)$ depends on the agent-specific proportionality factor g_{it}, which is heterogeneous across agents and depends linearly on a state variable, to be described shortly, Y_t:

$$g_{it} = a_i Y_t + b_i \tag{2}$$

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[8]Throughout, the notation $\int x_j dj$ denotes the integral of x_j over its density $f_x(x_j)$, which we do not specify for now to avoid notational clutter.

where $a_i > 0$ and b_i are heterogeneous across agents and such that $\int a_i di = 1$.

Endowment. Aggregate endowment – which we also refer to as dividends or output – follows the process

$$\frac{dD_t}{D_t} = \mu_D \ dt \ + \ \sigma_D(Y_t) \ dZ_t \tag{3}$$

where the drift rate μ_D is constant.[9] The volatility $\sigma_D(Y_t)$ of aggregate endowment – which we refer to as *economic uncertainty* – depends on the state variable Y_t, which follows

$$dY_t = k \ (\overline{Y} - Y_t) \ dt - v \ Y_t \ \left[\frac{dD_t}{D_t} - \mu_D dt\right] \tag{4}$$

That is, Y_t increases after bad aggregate shocks, $\frac{dD_t}{D_t} < \mu_D dt$, and it hovers around its central tendency $\overline{Y}$. It is useful to interpret Y_t as a *recession indicator*: During good times Y_t is low and during bad times Y_t is high. We assume throughout that Y_t is bounded below by a constant $\lambda \geq 1$. This technical restriction is motivated by our preference specification above and it can be achieved by assuming that $\sigma_D(Y_t) \to 0$ as $Y_t \to \lambda$ (under some technical conditions). We otherwise leave the diffusion terms $\sigma_D(Y_t)$ in (3) unspecified for now.

At time 0 each agent is endowed with a fraction w_i of the aggregate endowment process D_t. The fractions w_i satisfy $\int w_i di = 1$, and the technical condition

$$w_i > \frac{a_i(\overline{Y} - \lambda) + \lambda - 1}{\overline{Y}} \tag{A1}$$

which ensures that each agent has sufficient wealth to ensure positive consumption over habit in equilibrium, and hence well defined preferences. A1 is assumed throughout.

Finally, we set $b_i = \lambda(1 - a_i) - 1$ in (2), which ensures $g_{it} > 0$ for every i and for every t, and allows for a simple aggregation below. This assumption does not affect the results.

Discussion. Our preference specification differs from the standard external habit model of Campbell and Cochrane (1999) and Menzly, Santos and Veronesi (2004, MSV henceforth). In particular, our model is one *without* consumption externalities as habit levels depend only on exogenous processes and not on consumption choices. This modeling choice allows the application of standard aggregation results which considerably simplifies the analysis.

Second, our model features two relevant sources of variation across agents: Initial endowments, as summarized by the distribution of ω_i, and the sensitivity of individual habits X_{it} to excess output, as summarized by g_{it}, which results in differences in attitudes towards risk. These two dimensions seem a natural starting point to investigate optimal risk sharing as

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[9]The main results of the paper carry through with a richer specification of the drift μ_D.

well as portfolio decisions.[10] The case of homogeneous preferences ($a_i = 1$ for all i) and/or homogeneous endowments ($w_i = 1$ for all i) are of course special cases, as is the case in which habits are constant ($v = 0$ in (4)). We investigate these special cases as well below.

Notice though that our model features no idiosyncratic shocks to individual endowment as agents simply receive a constant fraction w_i of the aggregate endowment process. In our model risk sharing motives arise exclusively because agents are exposed differently to business cycle fluctuations through their sensitivity to habits, which affects their risk tolerance. Indeed how sensitive agents are to shocks in excess output depend on the state variable Y_t. Economically, assumption (2) implies that in bad times (after negative output shocks) the habit loadings g_{it} increase, making habit preferences become more important on average. However, different sensitivities a_i imply that changes in Y_t differentially impact the external habit index as g_{it} increase more for agents with high a_i than for those with low a_i.

We highlight that our preference specification has several intuitive features that conform well with the existing evidence on household behavior. First, households' preferences are nonhomothetic in endowment. As we shall see, agents with larger endowments are less risk averse, and as a result increase the share of wealth invested in the risky asset, an empirical pattern documented in surveys of household finances even when restricted to those who participate in the stock market (Wachter and Yogo, 2010).

Second, as habit X_{it} fluctuates so do agents' attitudes towards risk. In particular the risk tolerance of all agents decreases in bad times and it does so more for some agents than for others (see expression (13) below). This behavior is consistent with the evidence in Guiso, Sapienza and Zingales (forthcoming). Using a large sample of clients of an Italian bank they show that a measure of risk aversion increased after the 2008 financial crisis even when their wealth or consumption did not decline. While in our one-factor model all variables are perfectly correlated, independent variation in X_{it} would generate such behavior. Differences in attitudes towards risk is a critical channel in our model as it is the reason why agents trade to share aggregate risk. If during bad times the risk aversion of agent i increases more than the one of agent j, for instance, then agent i wants to sell the risky asset to agent j.[11]

Finally, as discussed below in more detail, the portfolio allocation predictions of our model are consistent with the empirical evidence of Calvet and Sodini (2014), who document the importance of habit formation to explain the allocation strategies of Swedish households.

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[10]For instance, two recent theoretical contributions that consider these two sources of cross sectional variation are Longstaff and Wang (2012) and Bolton and Harris (2013). Empirically these sources of variation have been investigated by, for example, Chiappori and Paeilla (2011) and Calvet and Sodini (2014).

[11]There is substantial evidence of cross sectional dispersion in attitudes towards risk in the population. See, for instance, Barsky, Juster, Kimball and Shapiro (1997).

3. Optimal risk sharing

As already mentioned, markets are complete and therefore standard aggregation results imply that a representative agent exists, a planner, that solves the program

$$U\left(D_t, \{X_{it}\}, t\right) = \max_{C_{it}} \int \phi_i u\left(C_{it}, X_{it}, t\right) di \qquad \text{subject to} \qquad \int C_{it} di = D_t \qquad (5)$$

where all Pareto weights $\phi_i > 0$ are set at time zero, renormalized such that $\int \phi_i di = 1$ and are consistent with the initial distribution of endowments in a way to be described shortly. The first order condition implies that

$$u_C(C_{it}, X_{it}, t) = \frac{\phi_i e^{-\rho t}}{C_{it} - X_{it}} = M_t \qquad \text{for all } i, \qquad (6)$$

where M_t is the Lagrange multiplier associated with the resource constraint in (5).[12] Straightforward calculations[13] show that

$$M_t = \frac{e^{-\rho t}}{D_t - \int X_{jt} dj} \qquad \text{and} \qquad C_{it} = (g_{it} + \phi_i)\left(D_t - \int X_{jt} dj\right). \qquad (7)$$

The optimal consumption of agent i increases if the excess output, $D_t - \int X_{jt} dj$, increases or if the habit loading g_{it} increases. This is intuitive, as such agents place relatively more weight on excess output and thus want to consume relatively more. In addition, agents with a higher Pareto weight ϕ_i also consume more as they are favored by the social planner.

We finally aggregate total optimal consumption and impose market clearing to obtain

$$D_t = \int C_{it} di = \left[\int (g_{it} + \phi_i)\, di\right]\left(D_t - \int X_{it} di\right). \qquad (8)$$

Using $\int \phi_i di = 1$, we can solve for the equilibrium excess output as

$$D_t - \int X_{it} di = \frac{D_t}{\int g_{it} di + 1} > 0. \qquad (9)$$

This intermediate result also shows that individual excess consumption $C_{it} - X_{it}$ is positive for all i, which ensures all agents' utility functions are well defined.[14] Notice also an important implication of (9) and that is that preferences can be expressed as

$$u\left(C_{i,t}, X_{i,t}, t\right) = e^{-\rho t} \log\left(C_{it} - \psi_{it} D_t\right) \qquad \text{with} \qquad \psi_{it} \equiv \frac{g_{it}}{\int g_{it} di + 1}. \qquad (10)$$

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[12] This result was first derived by Borch (1962, equation (1) p. 427).

[13] It is enough to solve for C_{it} in (6), integrate across agents (recall $\int \phi_i di = 1$), and use the resource constraint to yield M_t. Plugging this expression in (6) yields C_{it}.

[14] To see this, substitute the excess output into (7) and use (1). Given g_{it} in (2), we have $\int g_{it} di + 1 = Y_t$.

Individual agents compare their own consumption to aggregate endowment properly scaled by ψ_{it}, which is agent specific and dependent on Y_t. Roughly agents care about their relative standing in society, which is subject to fluctuations. It is these fluctuations what introduces motives for risk sharing. The next proposition solves for the Pareto weights and the share of the aggregate endowment that each agent commands.

Proposition 1 *(Efficient allocation). Let the economy be at its stochastic steady state at time 0, $Y_0 = \overline{Y}$, and normalize $D_0 = \rho$. Then (a) the Pareto weights are*

$$\phi_i = a_i \lambda + (w_i - a_i) \overline{Y} + 1 - \lambda \tag{11}$$

(b) The share of the aggregate endowment accruing to agent i is given by

$$C_{it} = \left[a_i + (w_i - a_i) \frac{\overline{Y}}{Y_t} \right] D_t \qquad or \qquad s_{it} \equiv \frac{C_{it}}{D_t} = a_i + (w_i - a_i) \frac{\overline{Y}}{Y_t} \tag{12}$$

Pareto weights (11) are increasing in the fraction of the initial aggregate endowment w_i and decreasing in habit sensitivity a_i. The first result is standard. To understand the second, given optimal consumption (7), agents with higher sensitivity a_i have a higher habit loading $g_{it} = a_i(Y_t - \lambda) + \lambda - 1$ and thus would like to consume more. Given (7), for given initial endowment w_i, the Pareto weight ϕ_i must then decline to ensure that such consumption can be financed by the optimal trading strategy.

Equation (12) captures the essential properties of the optimal risk sharing rule, that is, agents with high endowment w_i or low habit sensitivity a_i enjoy a high consumption share $s_{it} = C_{it}/D_t$ during good times, that is, when the recession indicator Y_t is low, and vice versa. To grasp the intuition consider first the curvature of the utility function of an individual agent, which we refer to as "risk aversion" for simplicity:

$$Curv_{it} = -\frac{C_{it}u_{cc}(C_{it}, X_{it}, t)}{u_c(C_{it}, X_{it}, t)} = 1 + \frac{a_i(Y_t - \lambda) + \lambda - 1}{w_i \overline{Y} - a_i(\overline{Y} - \lambda) - \lambda + 1}. \tag{13}$$

Expression (13) shows that agents with higher endowment w_i or lower habit sensitivity a_i have lower risk aversion. Moreover, an increase in recession indicator Y_t increases the curvature of every agent, but more so for agents with a high habit sensitivity a_i or low endowment w_i. These variations in curvature generates the need for risk sharing as embedded in the sharing rule (12).

Preference heterogeneity and business cycle variation combine to determine the planner's transfer scheme needed to support the optimal allocation. Let $\tau_{it} > 0$ be the transfer received

by agent i at time t above her endowment $w_i D_t$; if instead the agent consumes below her endowment then $\tau_{it} < 0$. Trivial computations prove the next corollary.[15]

Corollary 2 *The transfers that implement the efficient allocation are given by*

$$\tau_{it} = - (w_i - a_i) \left(1 - \frac{\overline{Y}}{Y_t} \right) D_t. \tag{14}$$

Notice that agents for whom $w_i - a_i > 0$ receive transfers, $\tau_{it} > 0$, when $Y_t < \overline{Y}$, that is in good times, and pay $\tau_{it} < 0$ in bad times, when $Y_t > \overline{Y}$. The opposite is the case for the agents for whom $w_i - a_i < 0$. In effect, optimal risk sharing requires agents with $w_i - a_i > 0$ to insure agents with $w_i - a_i < 0$.

We emphasize an important attribute of our model and that is that habits are key to deliver all the results in our paper. Indeed, assume that $Y_t = \overline{Y}$ for all t (i.e. $v = 0$ in (4)). In this case our model collapses to an economy populated with agents with log preferences, the share of consumption of each agent is simply $s_{it} = w_i$ and, as it will be shown below, no trading occurs amongst agents. Thus, our model does not deliver risk sharing motives beyond what is induced by the habit features of our preference specification.

4. Competitive equilibrium

4.1. Decentralization

Financial markets. Having characterized the optimal allocation of risk across agents in different states of nature we turn next to the competitive equilibrium that supports it. Clearly we can introduce a complete set of Arrow-Debreu markets at the initial date, let agents trade and after that simply accept delivery and make payments. It was Arrow's (1964) original insight to show that decentralization can be achieved with a sparser financial market structure. There are obviously many ways of introducing this sparser financial market structure but here we follow many others and simply introduce a stock market and a market for borrowing and lending. Specifically we assume that each of the agents i is endowed with an initial fraction w_i of a claim to the aggregate endowment D_t. We normalize the aggregate number of shares to one and denote by P_t the price of the share to the aggregate endowment process, which is competitively traded. Second, we introduce a market for borrowing and lending between agents. Specifically we assume that there is an asset in zero net supply, a

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[15]Simply subtract from the optimal consumption allocation (12) the consumption under autarchy, $w_i D_t$.

bond, with a price B_t, yielding an instantaneous rate of return of r_t, so that $B_t = e^{\int_0^t r_u du}$. Both P_t and r_t are determined in equilibrium. Because all quantities depend on one Brownian motion (dZ_t), markets are dynamically complete.

The portfolio problem. Armed with this we can introduce the agents' problem. Indeed, given prices $\{P_t, r_t\}$ agents choose consumption C_{it} and portfolio allocations in stocks N_{it} and bonds N_{it}^0 to maximize their expected utilities

$$\max_{\{C_{it}, N_{it}, N_{it}^0\}} E_0 \left[\int_0^\infty e^{-\rho t} \log (C_{it} - X_{it}) \, dt \right]$$

subject to the budget constraint equation

$$dW_{it} = N_{it}(dP_t + D_t dt) + N_{it}^0 B_t r_t dt - C_{it} dt$$

with initial condition $W_{i,0} = w_i P_0$.

Definition of a competitive equilibrium. A competitive equilibrium is a series of stochastic processes for prices $\{P_t, r_t\}$ and allocations $\{C_{it}, N_{it}, N_{it}^0\}_{i \in \mathcal{I}}$ such that agents maximize their intertemporal utilities and markets clear $\int C_{it} di = D_t$, $\int N_{it} di = 1$, and $\int N_{it}^0 di = 0$. The economy starts at time 0 in its stochastic steady state $Y_0 = \overline{Y}$. Without loss of generality, we normalize the initial output $D_0 = \rho$ for notational convenience.

The competitive and the decentralization of the efficient allocation. We are now ready to describe the competitive equilibrium and show that it indeed supports the efficient allocation. We leave the characterization of the equilibrium for the next section.

Proposition 3 *(Competitive equilibrium). Define the surplus consumption ratio as in Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004) as*

$$S_t = \frac{D_t - \int X_{it} di}{D_t} = \frac{1}{Y_t}, \tag{15}$$

where the last equality stems from (9). Denote with some mild abuse of notation $\sigma_D(Y_t) = \sigma_D(S_t)$. Then the following price processes and allocations support the efficient allocation (12) as a competitive equilibrium outcome:

1. *Stock prices and interest rates*

$$P_t = \left(\frac{\rho + k\overline{Y} S_t}{\rho (\rho + k)} \right) D_t \tag{16}$$

$$r_t = \rho + \mu_D - (1 + v)\sigma_D^2(S_t) + k \left(1 - \overline{Y} S_t \right) \tag{17}$$

2. *The position in bonds $N_{it}^0 B_t$ and stocks N_{it} of agent i at time t are, respectively,*

$$N_{it}^0 B_t = -v (w_i - a_i) H (S_t) D_t \tag{18}$$

$$N_{it} = a_i + (\rho + k)(1 + v) (w_i - a_i) H (S_t) \tag{19}$$

where

$$H (S_t) = \frac{\overline{Y} S_t}{\rho + k(1 + v)\overline{Y} S_t} > 0 \tag{20}$$

4.2. Asset prices

The stock price in Proposition 3 is identical to the one found in MSV, which obtained in a representative consumer model. The reason is that our model does indeed aggregate to yield a representative consumer similar to the one in that paper. Indeed, having solved for the Pareto weights (11) and the individual consumption allocations we can substitute back in the objective function in (5) and obtain the equilibrium state price density.

Proposition 4 *(Representative agent and stochastic discount factor). The representative agent has utility function (up to a constant):*

$$U (D_t, \{X_{it}\}, t) = e^{-\rho t} \log \left(D_t - \int X_{jt} dj \right) \tag{21}$$

The equilibrium state price density is

$$M_t = e^{-\rho t} D_t^{-1} S_t^{-1}. \tag{22}$$

Given the risk-free rate r_t in (17), the stochastic discount factor follows

$$\frac{dM_t}{M_t} = -r_t dt - \sigma_{M,t} dZ_t \qquad with \qquad \sigma_{M,t} = (1 + v)\sigma_D(S_t), \tag{23}$$

The representative agent utility function (21) is increasing in excess output $\left(D_t - \int X_{jt} dj\right)$. This result provides the theoretical foundation to the "habit in utility" interpretation of agents' preference specification in equation (1). Each agent i's habit X_{it} increases in the representative agent's utility, which becomes the meter of comparison of his well being.

The state price density in (22) is similar to the one in Campbell and Cochrane (1999) and MSV. Equation (15) shows that the recession indicator Y_t is the inverse surplus consumption ratio of MSV. Indeed, as in this earlier work, Y_t can be shown to be linearly related to the aggregate risk aversion of the representative agent (see footnote 4 in MSV).

We are now ready to discuss the asset prices in Proposition 3. Start, briefly, with the risk free rate r_t. The terms $\rho + \mu_D - \sigma_D^2(S_t)$ in (17) are the standard log-utility terms, namely, time discount, expected aggregate consumption growth, and precautionary savings. The additional two terms, $k(1 - \overline{Y}S_t)$ and $v\ \sigma_D(S_t)$, are additional intertemporal substitution and precautionary savings terms, respectively, associated with the external habit features of the model (see MSV for details).

As for the stock price (16), the intuition for this expression is by now standard (Campbell and Cochrane (1999) and MSV). A negative aggregate shock $dZ_t < 0$ decreases the price directly through its impact on D_t, but it also increases the risk aversion Y_t and hence reduces $S_t = 1/Y_t$, which pushes down the stock price P_t further. External habit persistence models thus generate variation in prices that are driven not only by cash-flow shocks but also discount effects. Indeed, we show in the Appendix the volatility of stock returns is

$$\sigma_P(S_t) = \sigma_D(S_t)\left(1 + \frac{vk\overline{Y}S_t}{\rho + k\overline{Y}S_t}\right). \tag{24}$$

In addition, as shown in (23), the market price of risk also is time varying, not only because of the variation in aggregate consumption volatility ($\sigma_D(S_t)$) but also because of the variation in the volatility of aggregate risk aversion, given by $v\sigma_D(S_t)$. In MSV, a lower surplus consumption ratio S_t increases the average market price of risk and makes it time varying. This generates the predictability of stock returns. Indeed, denoting the total stock return as $dR_P = (dP_t + D_t dt)/dt$, the risk premium

$$E_t\left[dR_P - r_t dt\right] = \sigma_M(S_t)\sigma_P(S_t)dt \tag{25}$$

increases compared to the case with log utility both because the aggregate amount of risk $\sigma_P(S_t)$ increases and because the market price of risk $\sigma_M(S_t)$ increases.

An important property of asset prices (P_t and r_t) in our model is the following:

Corollary 5 *Asset prices are independent of the endowment distribution across agents as well as the distribution of preferences. In particular the model has identical asset pricing implications even if all agents are identical, i.e. $a_i = 1$ and $w_i = 1$ for all i.*

The asset pricing implications of our model are thus "orthogonal" to its cross sectional implications: P_t in equation (16) and r_t in (17) are independent of the distribution of either current consumption or wealth in the population. This property distinguishes our model from the existing literature such as Longstaff and Wang (2012) or Chan and Kogan (2002).

Importantly, in this earlier literature the variation in risk premia is driven by endogenous changes in the cross-sectional distribution of wealth. Roughly more risk-tolerant agents hold a higher proportion of their wealth in stocks. A drop in stock prices reduces the fraction of aggregate wealth controlled by such agents and hence their contribution to the aggregate risk aversion. The conditional properties of returns thus rely on strong fluctuations in the cross sectional distribution of wealth.

In contrast, in the present paper agents' risk aversions change, which in turn induces additional variation in premia and puts less pressure on the changes in the distribution of wealth to produce quantitatively plausible conditional properties for returns. Indeed, Corollary 5 asserts exactly that the asset pricing implications are identical even when agents are homogeneous and thus there is no variation in cross-sectional distribution of wealth. Corollary 5 thus allows us to separate cleanly the asset pricing implications of our model from its implications for trading, leverage and risk sharing, which we further discuss below. In particular, the corollary clarifies that equilibrium prices and quantities do not need to be causally related to each other, but rather comove with each other because of fundamental state variables, such as S_t in our model.

4.3. Leverage and risk sharing

We turn next to the characterization of the portfolio strategies in Proposition 3.

Corollary 6 *(Individual leverage).* *(a) The position in bonds is $N_{it}^0 B_t < 0$ if and only if $w_i - a_i > 0$. That is, agents with $w_i > a_i$ take on leverage.*

(b) The investment in stock of agent i in proportion to wealth is

$$\frac{N_{it}P_t}{W_{it}} = \frac{1 + v\left(1 - \frac{\rho}{\rho + \overline{Y}[k + (\rho+k)(w_i - a_i)/a_i]S_t}\right)}{1 + v\left(1 - \frac{\rho}{\rho + \overline{Y}kS_t}\right)} > 1 \quad \text{if and only if} \quad w_i - a_i > 0. \qquad (26)$$

Recall that, as shown in equation (14), optimal risk sharing requires transfers from agents with $w_i - a_i > 0$ to those with $w_i - a_i < 0$ when Y_t is high (or S_t is low) and the opposite when Y_t is low (or S_t is high). Equations (18) and (19) show the portfolios of stocks and bonds needed to implement the efficient allocation. This is achieved by having the agents with large risk bearing capacity, agents with $w_i - a_i > 0$, issue debt in order to insure those agents with lower risk bearing capacity, $w_i - a_i < 0$. Part (b) of Corollary 6 shows that

indeed agents with $w_i - a_i > 0$ lever up to achieve a position in stocks that is higher than 100% of their wealth.

Expression (26) shows that for given level of habit sensitivity a_i, agents with higher wealth w_i invest comparatively more in stocks. Indeed, habits render the agents' utilities nonhomothetic in wealth which results in a positive relation between wealth and the portfolio share in the risky asset. As shown, for instance, by Wachter and Yogo (2010, section 2.2) this is a result with strong empirical support.

Obviously, nonhomotheticity can obtain in a variety of settings.[16] But expression (26) has some specific implications that have been taken to the data by Calvet and Sodini (2014). Indeed we show in the appendix that expression (26) can be written as

$$\frac{N_{i,t}P_t}{W_{i,t}} = \frac{\mathsf{SR}\left(S_t\right)}{\sigma_P\left(S_t\right)}\left(1 - \frac{\theta_i D_t}{W_{i,t}}\right),\tag{27}$$

where $\mathsf{SR}\left(S_t\right) = \left(1 + v\right)\sigma_P\left(S_t\right)$ is the Sharpe ratio of the risky asset and θ_i is a household specific constant. Equation (27) is a version of equation (2) in Calvet and Sodini (2014, page 876).[17] These authors test a variety of implications of (27) in a large panel of Swedish twins (which serves to control for differences in risk preferences) and find strong support for them.

Expressions (18) and (19) show that the amount of leverage and asset allocation depend on the function $H(S_t)$, which is time varying as the recession indicator $Y_t = S_t^{-1}$ moves over time. We discuss the dynamics of leverage in the next section.

4.4.   The supply of safe assets: Leverage dynamics

A particular feature of our model is that that the risk attitudes of the agents in the economy fluctuate with the recession indicator Y_t (see equation (13)). As Y_t increases, for instance, the risk bearing capacity of the agents for whom $w_i - a_i > 0$ decreases precisely when the demand for insurance by the agents with $w_i - a_i < 0$ increases. The supply of safe assets, to use the term that has become standard in the recent literature, may decrease precisely when

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[16]Wachter and Yogo (2010) for instance write a model in which nonhomotheticity obtains because the agents have non-separable preferences over two kinds of goods, a basic good and a luxury one.

[17]Equation (2) in Calvet and Sodini (2014) is $\phi_{it} = \frac{\mathsf{SR}}{\gamma\sigma_P}\left(1 - \theta_i X_{it}/W_{it}\right)$, where $X_{i,t}$ is a subsistence or habit level in consumption. This equation obtains in a variety of habit setups (see Section II of the internet appendix of Calvet and Sodini (2014)). When we map our habit formulation to the standard one, as in expression (10), aggregate output, D_t, takes the place of "habit" in traditional models.

it is most needed,[18] an issue explored by some recent papers.[19] In this section we focus on the dynamics of the aggregate debt-to-output ratio:

$$L(S_t) \equiv \frac{-\int_{i:N_{it}^0 < 0} N_{it}^0 B_t di}{D_t}$$

where the negative sign is to make this number positive. Given (18) it is easy to see that

$$L(S_t) = v K_1 H(S_t) \quad \text{where} \quad K_1 \equiv \int_{i:(w_i - a_i) > 0} (w_i - a_i) \, di > 0, \tag{28}$$

where $H(S_t)$ was given in (20). Given that $H(S_t)$ is strictly increasing in S_t, the following corollary obtains:

Corollary 7 (*Procyclical aggregate debt*). *The aggregate debt-to-output ratio $L(S_t)$ is procyclical, increasing in good times (high S_t) and decreasing in bad times (low S_t).*

Risk sharing and leverage are in our model two related but distinct concepts. Efficient risk sharing requires marginal utilities scaled by the Pareto weights to be equated across households (see equation (6)). How the competitive equilibrium implements the efficient allocation described in Proposition 1 depends on the specific financial market structure assumed and thus so do the leverage implications of our model. With this in mind, it is useful to consider how the portfolio allocations in Proposition 3 implement the efficient allocation described in Proposition 1 through a standard replication argument. Let $P_{i,t}$ be the value of the contingent claim that at each point in time and state delivers as a dividend the consumption of agent i, $C_{i,t}$, associated with the efficient allocation (see equation (12)). The value of such contingent claim, if it existed, would be (see Appendix, expression (62)):

$$P_{it} = E_t \left[\int_t^\infty \frac{M_\tau}{M_t} C_{i\tau} d\tau \right] = \frac{\rho a_i + (\rho(w_i - a_i) + kw_i)\overline{Y} S_t}{\rho(\rho + k)} D_t. \tag{29}$$

Clearly a financial structure that features these contingent claims can equally implement the efficient allocation: Each agent would buy his corresponding contingent claim at date 0 and consume the dividends $C_{i,t}$ throughout. Following Cox and Huang (1989) the portfolio policy in Proposition 3 simply replicates the cash-flows of this contingent claim

$$N_{it} P_t + N_{it}^0 B_t = P_{it}. \tag{30}$$

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[18] In our model the debt issued by the agents with the largest risk bearing capacity is safe because they delever as negative shocks accumulate in order to maintain their marginal utility bounded away from infinity.

[19] See for instance Barro and Mollerus (2014), who propose a model based on Epstein-Zin preferences to offer predictions about the ratio of safe assets to output in the economy. Gorton, Lewellen and Mettrick (2012) and Krishnamurthy and Vissing-Jorgensen (2012) provide empirical evidence regarding the demand for safe assets. In all these papers the presence of "outside debt" in the form of government debt plays a critical role in driving the variation of the supply of safe assets by the private sector, a mechanism that is absent in this paper.

For this to be satisfied for every t (and pay C_{it} as dividend), it must be the case that the portfolio and the security have the same sensitivity to shocks dZ_t. Denoting by $\sigma_{Pi}(S_t)$ the volatility of P_{it}, the portfolio allocation N_{it} and N_{it}^0 must then satisfy

$$N_{it} = \frac{P_{it} \, \sigma_{Pi}(S_t)}{P_t \, \sigma_P(S_t)} \qquad \text{and} \qquad N_{it}^0 B_t = P_{it} - N_{it} P_t = P_{it} \left(1 - \frac{\sigma_{Pi}(S_t)}{\sigma_P(S_t)} \right). \tag{31}$$

The bond position, $N_{it}^0 B_t$, depends on the ratio of volatilities $\frac{\sigma_{Pi}(S_t)}{\sigma_P(S_t)}$: If this ratio is greater than one, the agent is leveraging his investment in the stock market. The volatility of the contingent claim is

$$\sigma_{Pi}(S_t) = \sigma_D(S_t) \left(1 + \frac{v \, (k + (\rho + k)(w_i - a_i)/a_i) \, \overline{Y} S_t}{\rho + (k + (\rho + k)(w_i - a_i)/a_i) \, \overline{Y} S_t} \right). \tag{32}$$

Comparing this expression with $\sigma_P(S_t)$ in (24), we see that $\sigma_{Pi}(S_t) > \sigma_P(S_t)$ if and only if $w_i - a_i > 0$. That is, agents with $w_i - a_i > 0$ leverage their portfolio. Intuitively, from the optimal risk sharing rule (12), agents with a high $w_i - a_i > 0$ have a high consumption share in good times, when S_t is high, and a low consumption share in bad times, when S_t is low. This particular consumption profile implies that the value of the contingent claim P_{it} is more sensitive to discount rate shocks than the stock price P_t. As a result the "replicating" portfolio requires some leverage to match such sensitivity.

Equation (31) also highlights the reason why the aggregate debt-to-output ratio, $L(S_t)$, increases in good times (high S_t). This is due to a "level effect": from (32) and (24) the ratio of volatilities actually declines as S_t increases. This is intuitive as the hypothetical contingent claim pays out more in good times and hence becomes less sensitive to discount rate shocks then. However, from (29) the value of the hypothetical contingent claim P_{it} increases in good times because the discount rate declines and more than overcomes the decline in the ratio of volatilities. As a result, aggregate debt increases in good times.

While a procyclical aggregate debt-to-output ratio may seem intuitive, it is not normally implied by, for instance, standard CRRA models with differences in risk aversion. In such models, less risk averse agents borrow from more risk averse agents, who want to hold riskless bonds rather than risky assets. As aggregate wealth becomes more concentrated in the hands of less risk-averse agents, the need of borrowing and lending declines, which in turn decreases aggregate debt. Moreover, a decline in aggregate uncertainty – which normally occur in good times – actually decreases leverage in such models, as it reduces the risk-sharing motives of trade. In our model, in contrast, the decrease in aggregate risk aversion in good times make agents with high-risk bearing capacity even more willing to take on risk and hence increase their supply of risk-free assets to those who have a lower risk bearing capacity.

Finally notice that good times, periods when S_t is high, also periods when expected excess returns are low as the market price of risk $\sigma_M(S_t)$ and so is typically aggregate uncertainty $\sigma_D(S_t)$.[20] Thus high the aggregate debt-to-output ratio $L(S_t)$ should predict low future excess returns.

4.5. Individual leverage and consumption

The following corollary follows immediately from Proposition 1 and Corollary 6.

Corollary 8 *Agents with higher debt enjoy higher consumption share during good times.*

After a sequence of good economic shocks aggregate risk aversion declines. Thus, agents with positive $(w_i - a_i)$ increase their debt and experience a consumption "boom". The two effects are not directly related, however. The increase in consumption is due to the higher investment in stocks that have higher payoffs in good times. Good times mean lower aggregate risk aversion and thus these same agents take on more leverage. Hence, our model predicts a positive comovement of leverage and consumption at the household level. An implication of this result is that agents who took on higher leverage during good times are also those that suffer a bigger drop in consumption growth as S_t mean reverts. In particular, we have the following corollary:

Corollary 9 *Agent i's consumption growth satisfies*

$$\frac{dC_{it}}{C_{it}} = \mu_{C,i,t}dt + \sigma_{C,i,t}dZ_t \tag{33}$$

where

$$\mu_{C,i,t} = \mu_D + \frac{(w_i - a_i)\overline{Y}S_t}{a_i + (w_i - a_i)\overline{Y}S_t}F(S_t) \tag{34}$$

$$\sigma_{C,i,t} = \left(1 + \frac{v(w_i - a_i)\overline{Y}S_t}{a_i + (w_i - a_i)\overline{Y}S_t}\right)\sigma_D(S_t) \tag{35}$$

with

$$F(S_t) = k(1 - \overline{Y}S_t) + (1 + v)v\ \sigma_D^2(S_t) \tag{36}$$

If $\sigma_D(S_t)$ is decreasing in S_t with $\sigma_D(\lambda^{-1}) = 0$, then the function $F(S)$ has $F'(S) < 0$ and $F(0) > 0$ and $F(\lambda^{-1}) = k(1 - \lambda^{-1}\overline{Y}) < 0$. Thus, there exists a unique solution S^* to

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[20]Note that we have not made any assumptions yet on $\sigma_D(S_t)$, except that it vanishes for $S_t \to \lambda^{-1}$.

$F(S^*) = 0$ *such that for all* i *and* j *with* $w^i - a^i > 0$ *and* $w^j - a^j < 0$ *we have*

$$E\left[\frac{dC_{it}}{C_{it}}\right] < \mu_D < E\left[\frac{dC_{jt}}{C_{jt}}\right] \qquad for \qquad S_t > S^* \tag{37}$$

$$E\left[\frac{dC_{it}}{C_{it}}\right] > \mu_D > E\left[\frac{dC_{jt}}{C_{jt}}\right] \qquad for \qquad S_t < S^* \tag{38}$$

This corollary shows that cross-sectionally agents with high $w_i - a_i > 0$ have a lower expected growth rate of consumption when S_t is high. We know that these are also times when such agents are heavily in debt. It follows then that agents who are heavily leveraged enjoy both a high consumption boom in good times, but a lower *future* expected consumption growth.[21] These agents also expect a higher consumption growth when S_t is low. Therefore, Corollaries 7 and 9 imply the following:

Corollary 10 *Periods with high aggregate debt-to-output ratio* $L(S_t)$ *forecast lower consumption growth for highly levered agents compared to those with lower leverage.*

That is, according to Corollary 10, periods of very high aggregate debt should follow on average by periods in which levered agents "retrench" and experience consumption growth that is comparatively lower than those agents who did not take on leverage. Essentially, agents with high $w_i - a_i$ are less risk averse and do not mind consumption fluctuations: they borrow and consume more in good times knowing that they will do the opposite in bad times. The opposite for agents with low $w_i - a_i$, who prefer a more stable consumption path.

This implication of our model speaks to some of the recent debates regarding the low consumption growth experienced by levered households following the Great Recession. Some have argued that the observed drop in consumption growth was purely due to a wealth effect, as levered households tend to live in counties that experienced big drops in housing values, whereas others have emphasized the critical role of debt in explaining this drop.[22] Clearly these effects are important but our contribution is to show that high leverage followed by low consumption growth is precisely what arises from standard risk sharing arguments in models that can address the observed conditional properties of asset returns, as external habit models do.

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[21]Parker and Vissing-Jørgensen (2009) use the Consumer Expenditure (CEX) Survey to show that the consumption growth of high-consumption households is significantly more exposed to aggregate fluctuations than that of the typical household.

[22]See for instance Mian and Sufi (2014, in particular pages 39-45) for a clear exposition of this debate.

4.6. Active trading in stocks and bonds

Corollary 11 *(Active trading) (a) Agents with positive leverage (i.e. with $w_i - a_i > 0$) increase their stock position in good times (S_t high) and decrease it in bad times (S_t low.) Agents with negative leverage (i.e. with $w_i - a_i < 0$) do the opposite.*

(b) Agents with higher absolute difference $|w_i - a_i|$ trade more in response to changes to the aggregate surplus consumption ratio S_t.

Corollary 11-a says that agents with positive leverage increase the number of units of stocks purchased in good times, and decrease them in bad times. That is, these agents *actively trade* in stocks. In a model with passive investors, an agent who is long stocks may mechanically find himself with a higher allocation in stocks during good times because the stock yields good returns in good times. Corollary 11-a instead says that an agent who is leveraged ($w_i - a_i > 0$) actively *increases leverage* in good times to buy more shares of stocks in such times. Such agent acts as a trend chaser, as he increases his stock positions after good market news. Conversely, agents with $w_i - a_i < 0$ do the opposite and hence act as contrarian investors. Corollary 11-b in addition predicts that there is heterogeneity in trading in that some agents react more to shocks in discount rates as proxied by S_t.

Corollary 11-a also implies that levered traders *actively deleverage* as times are getting worse (S_t declines) by actively *selling* the risky assets. In fact, deleveraging is specially pronounced during "crisis" times, as shown next:

Corollary 12 *("Panic deleveraging") The function $H(S_t)$ in (20) is increasing and concave in S_t. Therefore, both leverage and asset holdings of levered agents decrease by an increasing larger amount as time get worse, i.e. as S_t declines.*

The concavity of $H(S_t)$ has an important economic implication: during good times (S_t high) we should observe higher aggregate debt and higher asset holdings of levered agents, but less variation of both compared to bad times (S_t low). This implies that as S_t declines, levered agents decrease their amount of debt by an increasingly larger amount, giving the impression of a "panic deleveraging" during bad times.

Because deleveraging occurs as both the stock price plunges and the wealth of levered investors drops, an observer may be tempted to conclude that the "selling pressure" of deleveraging agents is the cause of the drop in the stock price. While in reality such effects may occur, in our model the joint dynamics of deleveraging and price drop happens for the

simple reason that during bad times aggregate risk aversion increases. Indeed, as shown in Corollary 5, the same asset pricing implications obtain even without heterogeneity and hence no trade. Our model then should caution against the excessive reliance on the simple intuition of price declines due to the "price pressure" of some agents in the economy.

4.7. Intermediary asset pricing and the leverage risk price

Our model also sheds light on recent empirical findings in the "intermediary asset pricing" literature (Adrian, Etula and Muir (2014) and He, Kelly and Manela (2016)), which is in turn inspired by some recent theoretical advances (He and Krishnamurthy, 2013). This literature emphasizes that households access markets for risky securities largely through financial intermediaries. Intermediary capital is needed to facilitate this access and capital ratios are priced risk factors. Importantly, intermediaries lever up, issuing the safe securities that households (and other agents) use to substitute intertemporally as well as manage their risk exposures. Because households are not allowed to directly invest in the risky asset, intermediaries therefore effectively transform the safe assets held by households into investments in the risky asset and effectively price the risky asset.

This is also the case here. Indeed in our model, agents who take on leverage to purchase the risky assets also supply risk-free assets to those agents who want to limit their risk exposure (see the discussion in Section 4.4.) and thus they are akin to financial intermediaries. The only difference with the intermediary asset pricing literature is that *all* agents can invest in the risky asset themselves and therefore the marginal valuation of the risky asset is the same for both leveraged and unleveraged agents.

The intermediary asset pricing literature finds that the capital equity ratio of financial intermediaries predicts returns in the cross section. A debate in this literature is whether there is a *negative* or *positive* price of risk associated with shocks to the capital ratio of the financial intermediaries (see Adrian, Etula and Muir (2014) and He, Kelly and Manela (2016), respectively). Our model sheds light on this debate by showing first that agents' leverage is a priced factor and, second, that the leverage risk price has a different sign depending on whether we measure intermediaries' leverage using market prices or not.

Formally, in our one-factor model the conditional CAPM holds. If we could easily measure S_t in the data, we could compute expected returns off the conditional CAPM. However, suppose, reasonably, that the surplus S_t is not observable, but a monotonic transformation is, $\ell_t = Q(S_t)$. Let $d\ell_t = \mu_{\ell,t}dt + \sigma_{\ell,t}dZ_t$ where $\mu_{\ell,t}$ and $\sigma_{\ell,t}$ can be derived from Ito's lemma.

In this case, the state price density can be expressed as

$$M_t = e^{-\rho t} D_t^{-1} S_t^{-1} = e^{-\rho t} D_t^{-1} q(\ell_t)^{-1}$$

where $S_t = q(\ell_t) = Q^{-1}(\ell_t)$. The volatility of the SDF is thus $\sigma_{M,t} = \sigma_{D,t} + \frac{q'(\ell_t)}{q(\ell_t)}\sigma_{\ell,t}$ and therefore the risk premium for *any* asset with return $dR_{i,t}$ can be written as

$$E_t[dR_{it} - r_t dt] = Cov_t\left(\frac{dD_t}{D_t}, dR_{it}\right) + \frac{q'(\ell_t)}{q(\ell_t)} Cov_t\left(d\ell_t, dR_{it}\right) \qquad (39)$$

The first term corresponds to the usual log-utility, consumption-CAPM term, while the second term corresponds to the additional risk premium due to shocks to ℓ_t.[23]

Consider now a highly leveraged agent i in our economy, i.e. one with $w_i > a_i$. These agent issues risk-free bonds to other agents and use the proceeds to purchase risky securities. We can consider such agent an intermediary. Consider now the leverage of such agent. We have two potential measures, namely, its debt-to-output ratio,

$$\ell_{it} = Q_{it}^{D/O}(S_t) = -\frac{N_{it}^0 B_t}{D_t} = v\left(w_i - a_i\right) H\left(S_t\right);$$

or its debt-to-wealth ratio

$$\ell_{it} = Q_{it}^{D/W}(S_t) = -\frac{N_{it}^0 B_t}{W_{it}} = \frac{\sigma_{Wi}(S_t)}{\sigma_P(S)} - 1.$$

These two measures of leverage have different properties. In particular, $Q_{it}^{D/O}(S_t)$ is monotonically increasing in S_t while $Q_{it}^{D/W}(S_t)$ is monotonically decreasing in S_t. We then obtain the following corollary:

Corollary 13 *(price of leverage risk) (a) The price of leverage risk is positive,* $\lambda_t^{D/O} = \frac{q^{D/O\,'}(\ell_{it})}{q^{D/O}(\ell_{it})} > 0$, *when leverage is measured as the debt-to-output ratio ("book leverage").*

(b) The price of leverage risk is negative, $\lambda_t^{D/W} = \frac{q^{D/W\,'}(\ell_{it})}{q^{D/W}(\ell_{it})} < 0$, *when leverage is measured as the debt-to-wealth ratio ("market leverage").*

The economics behind this corollary are important: Our model generates strong discount effects that affect the valuation of securities. While intuitively in our model deleveraging occurs in bad times – which coincide with high marginal utility – the strong increase in discount rates pushes market prices even lower, which in turn increase leverage ratios computed off

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[23]This decomposition is for illustrative purposes only. All shocks are perfectly correlated in our model and so there is only one priced of risk factor.

market prices. The sign of leverage risk prices therefore depends on the definition of leverage employed, and specially on whether market prices are used or not in the computation.[24]

To link these results to the empirical evidence in Adrian, Etula and Muir (2014) and He, Kelly and Manela (2016), one could equate the levered agent's debt-to-output ratio to the "book leverage" of financial intermediaries, as it measures the agent's *amount* of debt; this leverage measure does not use market prices, and it is in fact procyclical. In contrast, a levered agent's debt-to-wealth ratio is akin to a measure of "market leverage" for financial intermediaries, as wealth is computed from market prices, which are affected by discount effects and is in fact countercyclical. These two different measures imply prices of "leverage risk" of opposite signs. Finally, we also note that $q^{D/O}(\ell_t)$ and $q^{D/W}(\ell_t)$ are non-linearly related with each other, and therefore the results of cross-sectional tests would not be the exact opposite, as found in the literature (see He, Kelly, and Manela (2016) and Section 5.3.)

4.8. The dynamics of wealth dispersion

Our model has implications for the dynamics of the cross sectional distribution of wealth.[25] Many factors of course matter for the distribution of wealth. For starters in our framework all wealth is financial and other forms of wealth, such as human capital, are not considered. Our model though is to our knowledge the first to explicitly consider the effect of discount rate shocks on the dynamics of the wealth distribution.

The next proposition characterizes the cross sectional dispersion of wealth, whether scaled by output or aggregate wealth, and its dependence on the surplus-consumption ratio S_t.

Proposition 14 *Let $Var^{CS}(a_i)$, $Var^{CS}(w_i)$, and $Cov^{CS}(a_i, w_i)$ denote the cross-sectional variance of preference characteristics a_i and in share w_i of aggregate endowment, and their covariance, respectively. Then, the cross-sectional variance of wealth-to-output ratio is*

$$
Var_t^{CS}\left(\frac{W_{it}}{D_t}\right) = Var^{CS}(a_i)\left(\frac{1 - \overline{Y}S_t}{\rho + k}\right)^2 + Var^{CS}(w_i)\left(\frac{\overline{Y}S_t}{\rho}\right)^2
$$
$$
+ 2Cov^{CS}(a_i, w_i)\frac{(1 - \overline{Y}S_t)(\overline{Y}S_t)}{(\rho + k)\rho} \tag{40}
$$

---

[24]Clearly, the loadings also have opposite signs for the two cases. Because $\sigma_{\ell,t} = Q'(S_t)S_t v \sigma_D(S_t)$, then $\sigma_{\ell,t} > 0$ if leverage is the debt-to-output ratio and $\sigma_{\ell,t} < 0$ when it is the debt-to-wealth ratio. Thus, $Cov_t(d\ell_t, dR_{it}) > 0$ in the former case and $Cov_t(d\ell_t, dR_{it}) < 0$ in the latter case.

[25]For a recent piece on the dynamics of the wealth distribution in the USA, see Saez and Zuckman (2016).

and the cross-sectional variance of wealth shares $W_{it}/\int_j W_j dj$ is

$$Var_t^{CS}\left(\frac{W_{it}}{\int_j W_j dj}\right) = Var_t^{CS}\left(\frac{W_{it}}{D_t}\right)\left(\frac{\rho(\rho+\kappa)}{\rho+k\overline{Y}S_t}\right)^2 \tag{41}$$

To understand the intuition behind (40), recall first that when $\overline{Y}S_t = 1$, the economy is at its stochastic steady state, which is the initial condition at time 0 when agents' wealth is $W_{i0} = w_i$, their initial endowment. Thus, (40) shows that when the system is at its stochastic steady state, the wealth dispersion is given by the dispersion in endowments w_i.

Consider now the case in which the cross-sectional covariance between endowment and preferences is zero, $Cov^{CS}(a_i, w_i) = 0$. During good times the surplus consumption ratio S_t increases. Whether this variation brings about an increase or decrease in wealth distribution, however, depends on the importance of the heterogeneity in preferences $Var^{CS}(a_i)$ relative to the dispersion in shares of aggregate endowment across the population. For instance, if $Var^{CS}(a_i) = 0$, then during good times (high S_t) the dispersion in wealth increases, while it decreases during bad times. Intuitively, when $Var^{CS}(a_i) = 0$, all agents differ from each other only in shares of aggregate endowment. Thus, agents with higher endowments, who are less risk averse, take on a more leveraged position and their wealth increase during good times, and so does the dispersion of wealth.

However, if $Var^{CS}(w_i) = 0$, then there is no dispersion in wealth at the aggregate stochastic steady state $\overline{Y}S_t = 1$, but it otherwise increases, both in good or in bad times, due to heterogeneous preferences. Agents with low a_i are less risk averse and thus take on leverage. As a consequence, they fare better than more risk averse agents in good times (S_t high) but worse in bad times (S_t low). The dispersion in wealth is thus U-shaped when heterogeneity is in only in preferences, and not in endowments.

The dispersion of wealth share in (41) is proportional to the dispersion of wealth-to-output ratio in (40), except that the proportionality factor decreases in good times. This is due to the increase in aggregate wealth $\int W_{jt} dj = P_t$ in good times. Thus, even if the dispersion of wealth-to-output ratios increases as S_t increase, the wealth share may still decline if discount rate effects are strong enough.

5. Quantitative implications

We now provide a quantitative assessment of the effects discussed in previous sections. The results in previous sections do not depend on the functional form of $\sigma_D(Y_t)$ but obviously to

simulate the model we need to specify one. We opt for a simple expression that bounds how high the volatility can get:

$$\sigma_D(Y_t) = \sigma^{max}\left(1 - \lambda Y_t^{-1}\right) \tag{42}$$

This assumption implies that dividend volatility increases when the recession index increases, but it is also bounded between $[0, \sigma^{max}]$.[26] This assumption about output volatility is consistent with existing evidence that aggregate uncertainty increases in bad times (see e.g. Jurado, Ludvigson, and Ng (2015)), it satisfies the technical condition $\sigma_D(Y) \to 0$ as $Y_t \to \lambda$, and it also allows us to compare our results with previous literature, as we obtain

$$dY_t = k(\overline{Y} - Y_t)dt - (Y_t - \lambda)\overline{v}dZ_t$$

with $\overline{v} = v\sigma^{max}$ which is similar to the one in MSV.[27]

For the calibration we use the same parameters as in MSV Table 1 to model the dynamics of Y_t. These are are reported in Panel A of Table 1. The only additional parameter is σ^{max}, which we choose to match the average consumption volatility $E[\sigma_D(S_t)] = std[\Delta \log(C_t^{data})]$, where the expectation can be computed from the stationary density of Y_t.[28]

Figure 1 reports the conditional moments implied by the model as a function of the surplus-consumption ratio S_t. As in MSV Figure 1, Panel A reports the stationary distribution of the surplus-consumption ratio S_t and shows that most of the probability mass is around $\overline{S} = 0.0294$, although S_t drops considerably below occasionally. The price-dividend ratio is increasing in S_t (panel B), while volatility, risk premium and interest rates decline with S_t (panel C). Finally, the Sharpe ratio is also strongly time varying, and it is higher in bad times (low S_t) and lower in good times (high S_t). This figure is virtually identical to Figure 1 in MSV, which highlights that our mild calibration of consumption volatility (with a maximum of only 6.4%) is such to have a minor on impact on the level of asset prices.

Given the parameters in Panel A of Table 1, we simulate 10,000 years of quarterly data and report the aggregate moments in Panel B. As in MSV, Table 1, the model fits well the asset pricing data, though both the volatilities of stock returns and of the risk free rate are higher than their empirical counterparts.[29] Still, the model yields a respectable Sharpe

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[26]The alternative of assuming e.g. $\sigma_D(Y)$ as linear in Y_t would result in $\sigma_D(Y)$ potentially diverging to infinity as Y_t increases.

[27]Technically, we also impose $\sigma_D(S_t)$ converges to zero for $S_t \leq \epsilon$ for some small but strictly positive $\epsilon > 0$ to ensure integrability of stochastic integrals. This faster convergence to zero for a strictly positive number can be achieved through a killing function, as in Cheriditto and Gabaix (2008). We do not specify such functions explicitly here, for notational convenience.

[28]See the Appendix in MSV. In addition, note that in MSV, $\alpha = \overline{v}/\sigma$ and therefore we compute $\overline{v} = \alpha\sigma$.

[29]The volatility of the risk free rate can be substantially reduced by making the natural assumption that expected dividend growth μ_D decreases in bad times, i.e. when the recession indicator Y_t is high. Indeed,

Table 1: Parameters and Moments. Panel A reports the calibrated parameters of Menzly, Santos, and Veronesi (2004), except for the new parameter σ^{max} which is chosen to match the average volatility of consumption. Panel B reports a set of moments for the aggregate stock market and interest rates, as well as consumption growth, and compares with the same moments in artificial data obtained from a 10,000-year Monte Carlo simulation of the model. Panel C similarly reports the R^2 of predictability regressions in the model and in the data, using the price-dividend ratio as predictor.

Panel A. Parameter Estimates

ρ	k	$\overline{Y}$	λ	$\overline{v}$	μ	σ^{max}
0.0416	0.1567	34	20	1.1194	0.0218	0.0641

Panel B. Moments (1952 – 2014)

	$E[R]$	$Std(R)$	$E[r_f]$	$Std(r_f)$	$E[P/D]$	$Std[P/D]$	SR	$E[\sigma_t]$	$Std(\sigma_t)$
Data	7.13%	16.55%	1.00%	1.00%	38	15	43%	1.41%	0.52%
Model	8.19%	25.08%	0.54%	3.77 %	30.30	5.80	32.64%	1.43%	1.18%

Panel C. P/D Predictability R^2

	1 year	2 year	3 year	4 year	5 year
Data	5.12%	8.25%	9.22%	9.59%	12.45%
Model	14.18%	23.67%	30.01%	33.81%	35.92

ratio of 32.64%. Finally, the simulated model generates an average consumption volatility of 1.43% with a standard deviation of 1.18%. This latter variation is a bit higher than the variation of consumption volatility in the data (0.52%), where the latter is computed fitting a GARCH(1,1) model to quarterly consumption data, and then taking the standard deviation of the annualized GARCH volatility. Our calibrated number is however lower than the standard deviation of dividend growth' volatility, which is instead around 1.50%.

The calibrated model also generates a strong predictability of stock returns (Panel C), with R^2 ranging between 14.18% at one year to 35.92% at 5 year. This predictability is stronger than the one generated in MSV and also the one in the data. This is due to the combined effect of time varying economic uncertainty (*i.e.* the quantity of risk) and time varying risk aversion (*i.e.* the market price of risk), which move in the same direction.

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in the extreme, by assuming $\mu_D(Y_t) = \overline{\mu}_D + (1 - v)\sigma_D(Y)^2 - k(1 - \overline{Y}Y_t^{-1})$, which is decreasing in Y_t, we would obtain a constant interest rates $r = \rho + \overline{\mu}_D$. No other result in the paper depend on $\mu_D(Y_t)$ and thus all the other results would remain unaltered by the change.
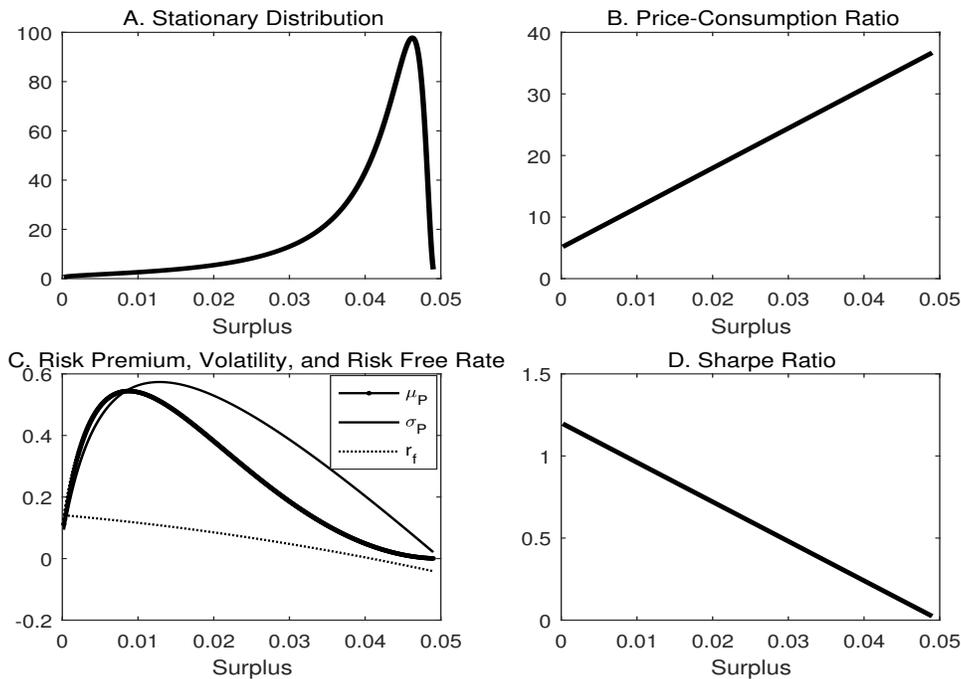
Figure 1: Conditional Moments. Panel A shows the stationary probability density function of the surplus consumption ratio S_t. Panel B shows the P/D ratio as a function of S_t. Panel C plot the expected excess return $E_t[dR_P - r_t dt]$, the return volatility $\sigma_P(S_t)$ and the interest rate $r(S_t)$ as functions of S_t. Finally, Panel D shows the Sharpe ratio $E_t[dR_P - r_t dt]/\sigma_P(S_t)$ against S_t.

5.1. The cross-section of agents' behavior: Who levers?

We now make some assumptions about the dispersion of initial endowments w_i and of preferences a_i. A full micro-founded "calibration" is clearly problematic in our setting, given the types of preference specification. We resort to illustrate the model's predictions through a reasonable numerical illustration which yields sensible quantities for some observables, such as households consumption volatility and debt levels. We assume that the habit loading parameters a_i are uniformly distributed $a_i \sim U[1-a, 1+a]$, so as $\int a_i di = 1$. The uniform distribution is a reasonable starting point, as it bounds above the parameter a_i.

Endowments w_i must meet assumption A1. While distributions can be found such that a_i and w_i are independent, A1 severely restricts the dispersion of such distributions. We instead assume that Pareto weights ϕ_i are distributed independently of preferences a_i and obtain the endowments by inverting (11):

$$w_i = \frac{\phi_i + a_i(\overline{Y} - \lambda) + \lambda - 1}{\overline{Y}}. \tag{43}$$

To ensure a skewed distribution of wealth, we assume

$$\phi_i = e^{-\sigma_w \varepsilon_i - \frac{1}{2}\sigma_w^2}$$

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with $\varepsilon_i \sim N(0,1)$. Thus, $\int_i \phi_i di = E^{CS}[\phi_i] = 1$. This procedure ensures that the Pareto weights are positive and hence all the constraints are satisfied. While all agents have random Pareto weights, and therefore contribute to the representative agent in a random manner, the procedure implies that agents with higher habit sensitivity a_i also have a higher endowment, a required condition to have well defined preferences in equilibrium.

To guide our choice of parameters a and σ_ϕ – the only two parameters that affect the whole distribution of consumption – we look at moments of individual households' consumption growth, such as average household consumption growth (arithmetic or log), its mean and median total and systematic volatility, and the cross-sectional dispersion of both. One important stumbling block to estimate the total and systematic volatility of household consumption growth is the lack of reliable panel data on households consumption, which has limited the empirical work on the time-series properties of households' consumption. However, Appendix C describes the novel methodology of Santos, Suarez, and Veronesi (2017) to estimate households' total and systematic consumption volatility from cross-sectional consumption data, and its application to the Survey of Consumer Expenditure (CEX). For our estimation, we use the cleaned dataset compiled by Kocherlakota and Pistaferri (2009) which spans the period 1980–2005.

Panel A of Table 2 reports the results. The average quarterly (arithmetic) growth rate is 6%, a very large number, which is mostly driven by the large cross-sectional heterogeneity in quarterly growth rates. Indeed, the median is slightly negative and the cross-sectional standard deviation is 40%, in line with estimates by e.g. Constantinides and Ghosh (2017). The log-growth indeed shows a slightly negative mean, which is close to the median, highlighting the positive skewness of the consumption data.

The total quarterly volatility is large, at 36%, and it displays a strong positive skewness, as its median is much lower at 27%, and its dispersion (standard deviation) is at 42%. Clearly, much of this quarterly consumption volatility is due to idiosyncratic shocks and residual seasonalities, which we do not have in our paper.[30] The quarterly systematic volatility is in fact far lower than the total volatility, as we would expect: the average is 9%, and the median is just 6%. The dispersion is still large, but reasonable, at 10.4%.

Panel B of Table 2 contains the same moments as Panel A but from the simulated model. We consider various combinations of parameters a of the uniform $U[1-a, 1+a]$ and the dispersion σ_ϕ of the lognormal distribution of Pareto weights ϕ. We focus on the first set of parameters here and discuss the other combinations in the Internet Appendix.

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[30]As explained in the Appendix, for each household i we mitigate the influence of seasonality by computing the average $\widehat{\sigma}_{it}^2$ over the three quarters of available variance observations.

Table 2: Cross-Sectional Parameters and Household Consumption Moments. Panel A reports the distribution of household consumption growth and their quarterly volatility and systematic volatility estimated from the Survey of Consumption Expenditures. The estimation methodology is briefly discussed in the Appendix and fully outlined in Santos, Suarez, and Veronesi (2017). The data are from Kocherlakota and Pistaferri (2009) and span the period 1980 to 2005. Panel B reports the same quantities in artificial data. We simulate the model with 5,000 households for a period of 1,000 years. As in the data, estimates are performed on quarterly data. The parameters of the model are as in Table 1, except for the preference and Pareto weight parameters a_i and ϕ_i, which are reported in the first column. We assume $a_i \sim U[\underline{a}, \overline{a}]$ and $\phi \sim$ Log Normal$(-\frac{1}{2}\sigma_\phi^2, \sigma_\phi^2)$. Each row corresponds to a different parameter choice.

Panel A. Households Quarterly Consumption Moments. Data

	Growth Rate (%)				Volatility (%)		
	Mean	Median	Std. Dev.		Mean	Median	Std. Dev.
Arithmetic	6.04	-0.63	40.13	Total	36.53	27.10	42.35
Logarithmic	-0.59	-0.66	35.78	Systematic	8.94	6.61	10.42

Panel B. Households Quarterly Consumption Moments. Model

$U[\underline{a}, \overline{a}]$, σ_ϕ	Arithmetic Growth Rate (%)				Volatility (%)		
	Mean	Median	Std. Dev.		Mean	Median	Std. Dev.
$U[0,2]$, 3	0.73	0.52	4.31		6.58	4.08	8.37
$U[1,1]$, 3	0.52	0.52	0.67		1.41	0.91	3.38
$U[0,2]$, 0	0.73	0.52	4.23		6.58	3.91	8.48
$U[1,1]$, 0	0.52	0.52	0.00		1.19	1.19	0.00

As can be seen, assuming a uniform $U[0,2]$ with $\sigma_\phi = 3$ provides a set of moments for consumption growth and volatility that are reasonable and close to the data (i.e. systematic volatility), with the important exception that our model is not able to generate the large cross-sectional dispersion in quarterly consumption growth. This is to be expected, as the cross-sectional dispersion in the data quarter by quarter is likely due to idiosyncratic shocks, which are absent in our model. More specifically, with those parameters, the model generates a mean growth rate of 0.7%, with the median at 0.5% and a cross-sectional dispersion of 4%. There is positive skewness, but not at the levels observed in the data, as this is "systematic skewness". Indeed, the mean consumption volatility is at 6.6%, with the median at 4.1% and dispersion at 8.4%. These values are lower than the corresponding values in Panel A for systematic volatility, but in the same ballpark. Moreover, this calibration also generate positive skewness in systematic volatility, as observed in the data.

Panel A and B of Figure 2 shows the resulting distribution of preferences and endowment in a simulation of 200,000 agents. In particular, Panel B shows a markedly skewed distribution of endowments (the extreme right tail of the distribution is omitted to provide a better visual impression). Because of the restriction $\int w_i di = 1$, the distribution shows
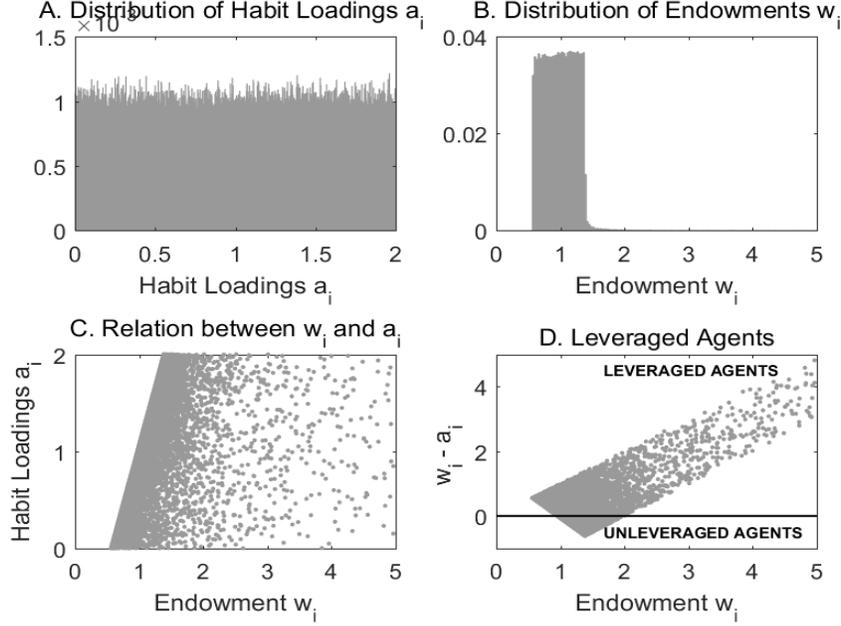
Figure 2: Preference and Endowment Distribution. Panel A plots the simulated distribution of preference parameters a_i from a uniform $[0,2]$. Panel B plots the simulated endowment distribution $w_i = \frac{\phi_i + a_i(\overline{Y} - \lambda) + \lambda - 1}{\overline{Y}}$ where $\phi_i = e^{-\sigma_w \varepsilon_i - \frac{1}{2}\sigma_w^2}$ are lognormally distributed and $\sigma_\phi = 3$. Panel C shows the relation between endowments w_i and preferences a_i. Panel D shows the relation between endowments w_i and $w_i - a_i$, where we recall that agents with $w_i - a_i > 0$ take on debt.

a large mass of agents with $w_i < 1$ to allow for some agents with a very large endowment. Panel C shows the relation between endowments on the x-axis and preference on the y-axis. The white area in the top-left corner is due to restriction A1: Agents with high habit loading a_i must have high initial endowment w_i to ensure a feasible consumption plan.

Finally, Panel D shows the relation between endowment w_i and leverage, namely, $w_i - a_i$. Indeed, recall that only agents with $w_i - a_i > 0$ lever up (see Corollary 6). Leverage is thus "U-shaped" in our calibration of the cross section in that two types of agents lever up, those with very low endowment but with also very low sensitivity to habit and those agents with very high endowment. The group with intermediate endowment, in contrast, are heterogeneous in that some leverage and some purchase the risk-free asset.

5.1.1. Households leverage ratios in good and bad times

Our assumption on the joint distribution of preferences a_i and endowments w_i in the previous section also yields a cross section of debt-to-assets that matches well its empirical counterpart. Panel A of Figure 3 plots the distribution of debt-to-assets of agents who take on debt in

simulations during three types of periods: Booms (S_t high), recessions (S_t low), and crisis (S_t very low). First, in general, agents with lower net worth (W_t) take on more debt as a fraction of assets ($N_{it}P_t$). The reason is that in the calibration above, these types of agents are less risk averse, as their a_i is on average lower. This is the effect of the constraint A1, also shown in Panels C and D of Figure 2: Agents with low endowment may have low risk aversion parameter a_i.

The second important effect of Panel A, however, is that the debt-to-asset ratio substantially increases during crises, that is, those rare times in which S_t is on the left-hand-side of its distribution (see Panel A of Figure 1). This an important channel in our model: While agents who borrow deleverage when S_t decline (Corollary 12), and hence reduce their amount of debt, the debt-to-asset ratio actually increases, because the value of assets declines by even more. That is, active deleveraging and increasing debt-to-asset ratios occur simultaneosuly when assets are valued at market values.

Panel B of Figure 3 shows that similar effects occur at the household level in the data. We use the Surveys of Consumer Finances conducted in 2007 and 2009. This last survey was conducted on the same sample of households as the 2007 survey, and thus it reflects a panel of agents. The debt-to-asset ratio of households is decreasing in their net worth. Interestingly, the debt-to-asset ratio increased substantially between 2007 and 2009 for the same agents that are ranked as of their 2009 net worth. We rank households on their net worth in 2009 to highlight how the increase in debt-to-asset ratio for these agents between 2007 and 2009 was especially due to a decline in asset value, which decreases net worth. Indeed, agents who suffered larger losses due to declining asset values will be moving to the left of the net worth distribution, and for given debt, would have a higher debt-to-asset ratio. The figure clearly indicates how the variation in assets generate an increase in debt-to-assets, in line with our model. Notice though that the model is not able to match the observed *level* of debt-to-assets amongst the poorest agents.[31]

In sum, our calibrated model is able to capture an important fact in the cross section, namely that the less wealthy lever more. This feature of our model stands in contrast with most models with heterogeneous agents, such as, for example, Dumas (1982) and Longstaff and Wang (2012). In these models less risk averse agents lever up, invest in risky stocks, and become richer as a result. These models thus imply counterfactually that leverage is more pronounced amongst richer agents. In contrast, in our model the two different sources of

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[31]The Internet Appendix documents that a similar plot obtains in the case of Spain which has a similar household survey (the "Encuesta Financiera de las Familias" or EFF). We thank Olympia Bover of the Bank of Spain for pointing out this to us.
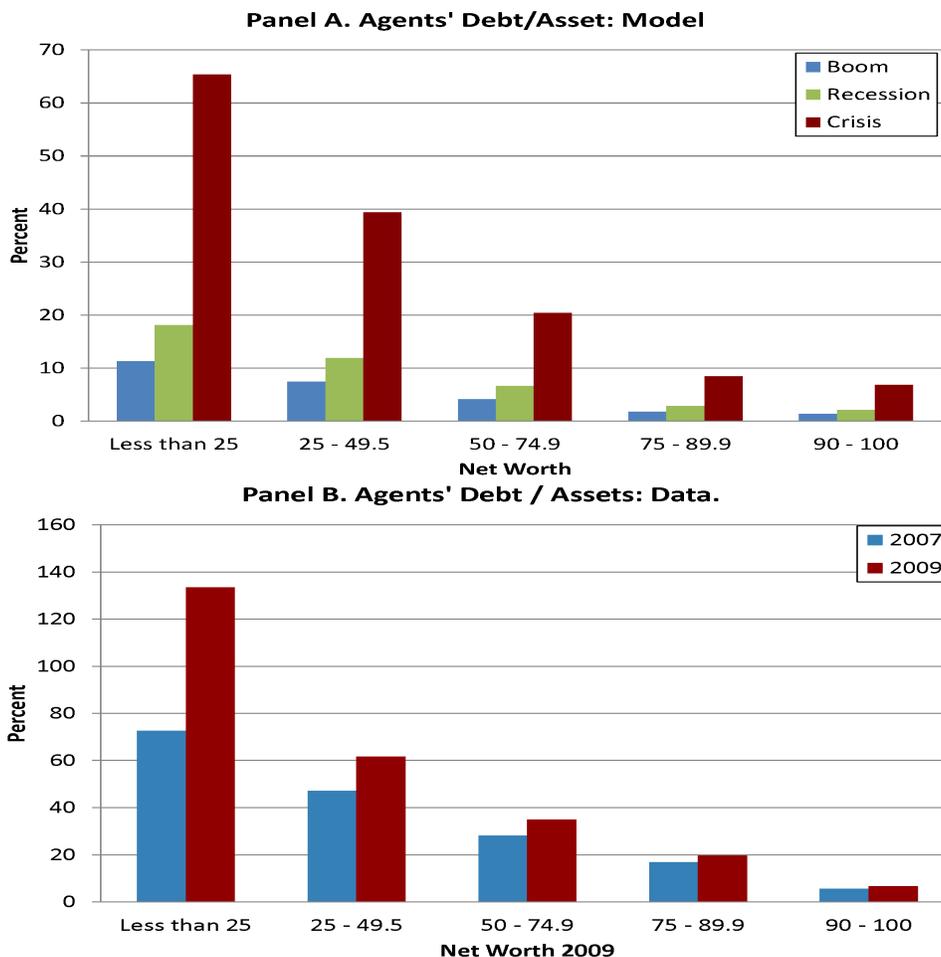
Figure 3: Debt-to-assets ratios across the wealth distribution. Model and data . Panel A plots the distribution of debt-to-asset ratios of agents who take on debt in simulations during three types of periods: Booms (S_t high), recessions (S_t low) and crisis (S_t very low). Panel B plots the distribution of debt-to-asset ratios from the Survey of Consumer Finances in 2007 and in 2009, which were conducted on the same sample of households.

heterogeneity, combined with the implicit assumption that agents with low endowment have lower habit loading a_i, imply that poor agents lever up more, consistently with the data.

5.1.2. Wealth dispersion in good and bad times

The parameter choices for the cross-section of agents also imply the wealth dispersion patterns depicted in Figure 4. Panel A plots the cross-sectional standard deviation of wealth-to-output ratios (see equation (40)). The plot shows a strongly increasing dispersion of wealth as times get better. This is a level effect: as the aggregate wealth increase, the *level* difference of wealth-to-output ratio increases due to the lower discount rate in good times.
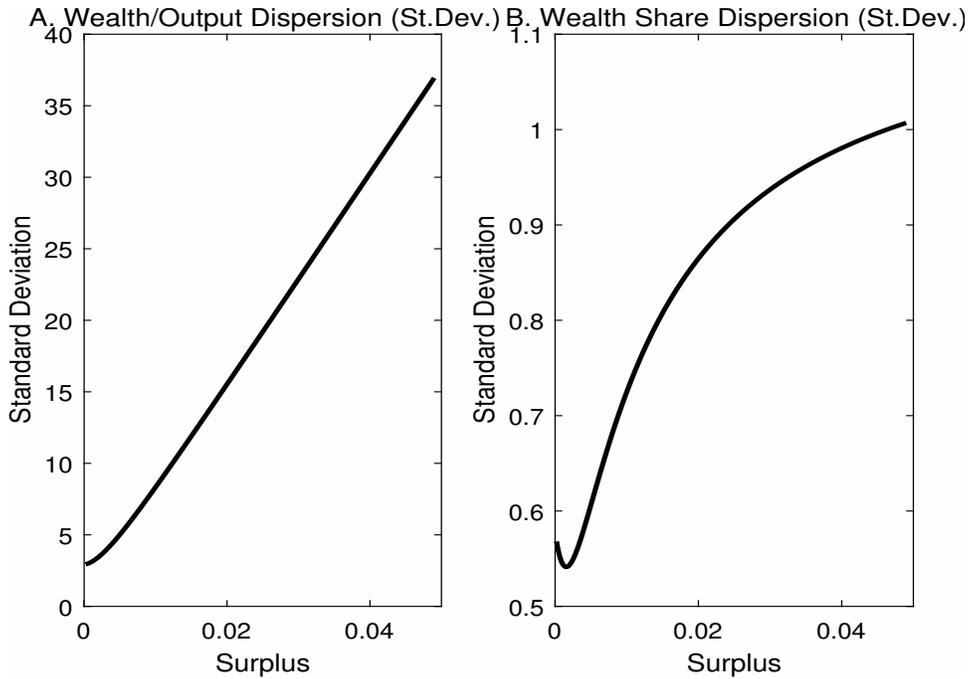
Figure 4: Wealth Dispersion. Panel A plots the cross-sectional dispersion – in standard deviation units – of the wealth to output of individual agents, plotted against the surplus consumption ratio (S_t). Panel B plots the cross-sectional dispersion – in standard deviation units – of the wealth shares, i.e. $W_{it}/\int W_{jt}dj$, against S_t. The parameters used are those in Table 1 and Figure 2.

Panel B of Figure 4 shows the cross-sectional standard deviation of wealth shares, i.e. wealth normalized by aggregate wealth (see equation (41)). For the calibrated parameters, this dispersion measure also displays a mostly increasing pattern as S_t increases, except for extremely low value of S_t. This general pattern of an increasing wealth dispersion in good times is consistent with the data, although wealth dispersion is mostly dominated by time-trends (see Saez and Zucman (2016)).

While our calibrated model implies an increasing wealth dispersion in good times, it is not able to generate a size of wealth dispersion that is comparable to the data. For instance, the top 1% of the population in our model only holds about 3.5% of aggregate wealth, against over 35% in the data in recent times. The reason is that many factors that are absent from our model, such as human wealth and returns to entrepreneurship, have first order effects on the wealth distribution. Our model instead highlights the role of discount rate shocks on the distribution of wealth and on wealth dispersion in particular. Moreover the model implies that a large increase in the cross sectional dispersion in wealth forecasts low future returns, an implication for which there is support in the data (Gomez, 2017).
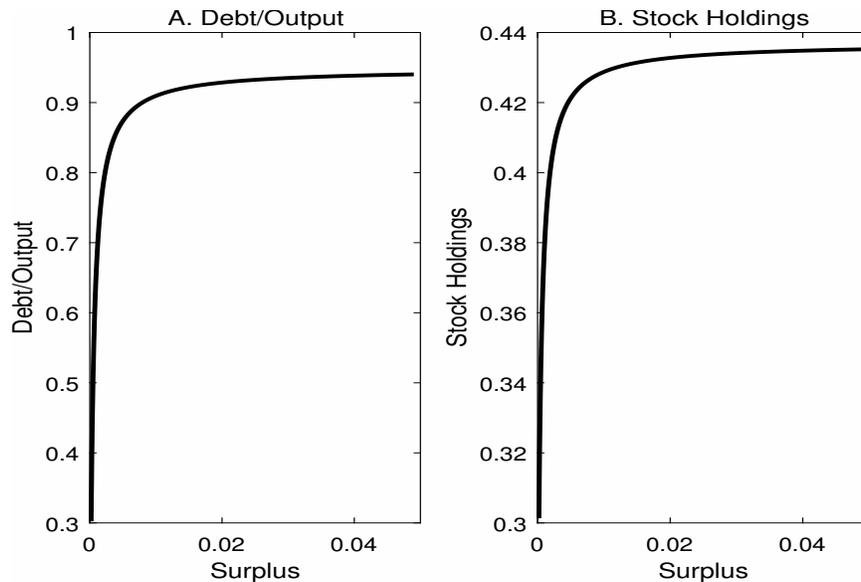
Figure 5: Aggregate Debt/Output and Stock Holdings of Levered Agents Panel A plots the aggregate debt/output in the economy as a function of the surplus consumption ratio S_t. Panel B reports the aggregate holdings of stocks from the agents who have leveraged positions (i.e. those in Panel A). The parameters used are those in Table 1 and Figure 2.

5.2. Aggregate leverage and stock holdings in good and bad times

Equation (28) shows that the aggregate debt-to-output is $L(S_t) = v \, K_1 \, H(S_t)$ where $K_1 = \int_{i:w_i-a_i>0}(w_i - a_i)di$. Similarly, from (19) the aggregate stock holding of levered agents can be written as

$$N^{Lev}(S_t) = K_0 + (\rho + k)\,(1 + v)\,K_1 H(S_t) \tag{44}$$

where $K_0 = \int_{i:w_i-a_i>0} a_i di$ and recall $H(S_t)$ is given by (20). The calibrated parameters imply $K_0 = 0.2483$ and $K_1 = 0.1526$.

As discussed in Corollaries 7 and 12, $H(S_t)$ is increasing and concave in S_t. That is, leverage and the aggregate stock exposure of levered agents are not only procyclical, but they also decline increasingly faster as times get worse, i.e. as S_t decline. Panels A and B of Figure 5 shows the patterns of $L(S_t)$ and the aggregate stock holdings of the levered agents, $N^{Lev}(S_t)$, under the parameter choices in Table 1. The concavity of $H(S_t)$ is especially strong for very low levels of S_t: Deleveraging accelerates as bad times morph into severe distress. It is important to emphasize that these results do not depend on the specific assumptions made on the functional form for $\sigma_D(S_t)$ as the function $H(S_t)$ does not depend on it.

As already mentioned, this non-linear behavior of debt and risky asset holdings of levered agents with respect to the surplus consumption ratio suggests that levered agents "fire sell"

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risky assets to decrease leverage in bad times. This is shown in the simulated path illustrated in Figure 6. Panel A shows 100 years of artificial quarterly data of the surplus consumption ratio S_t, while panel B reports the corresponding economic uncertainty $\sigma_D(S_t)$. Panel C shows the variation in the price-dividend ratio due to variation in the surplus consumption ratio, with a visible drop of the stock price from 30 to less than 10 in the middle of the simulated sample. Panel D shows the stock return volatility, which increases dramatically during bad times, as it increases to almost 60% during the "crisis".

Panel E demonstrates the impact of the variation of the surplus consumption ratio on the aggregate debt-to-output ratio and the aggregate stock holdings of levered agents. As it is apparent, the variation of both quantities is rather limited most of the time, except during extreme bad events. It is thus in these occasions, as the surplus consumption ratio drops and economic uncertainty increases, that levered agents decrease their indebtness and liquidate their positions in risky assets.

Finally, Panel F shows the debt-to-wealth ratio of the levered agents, and it highlights that the model is consistent with the observation that the efforts of all levered agents to delever simultaneously results in an increase in debt-to-wealth ratios. Indeed, while Panel E shows that aggregate debt declines during bad times, Panel F shows that the aggregate debt-to-wealth ratio actually increases, as levered agents' wealth declines faster then the decline in debt leverage.

In sum thus, as economic conditions deteriorate (a drop in S_t) prices fall but agents only delever and liquidate stock positions slowly. As bad times turn into severely distressed conditions, deleveraging and stock liquidation accelerates, creating the impression of a panic selling episode. Leverage *ratios*, debt-to-wealth, increase sharply as prices drop faster than the deleveraging. These results obtain in the absence of any contagion effects, liquidity dry ups or debt overhang considerations. They are the result of the optimal trading of utility maximizing agents in an equilibrium that implements an optimal risk sharing allocation. Our claim, again, is not that these particular frictions do not matter but rather to argue that the dynamics in quantities and prices observed in crises obtain naturally in risk sharing models that feature the strong discount effects needed to obtain reasonable asset pricing implications. Tests aimed at uncovering the aforementioned frictions have to control for the component of these dynamics that are the result of optimal risk sharing.
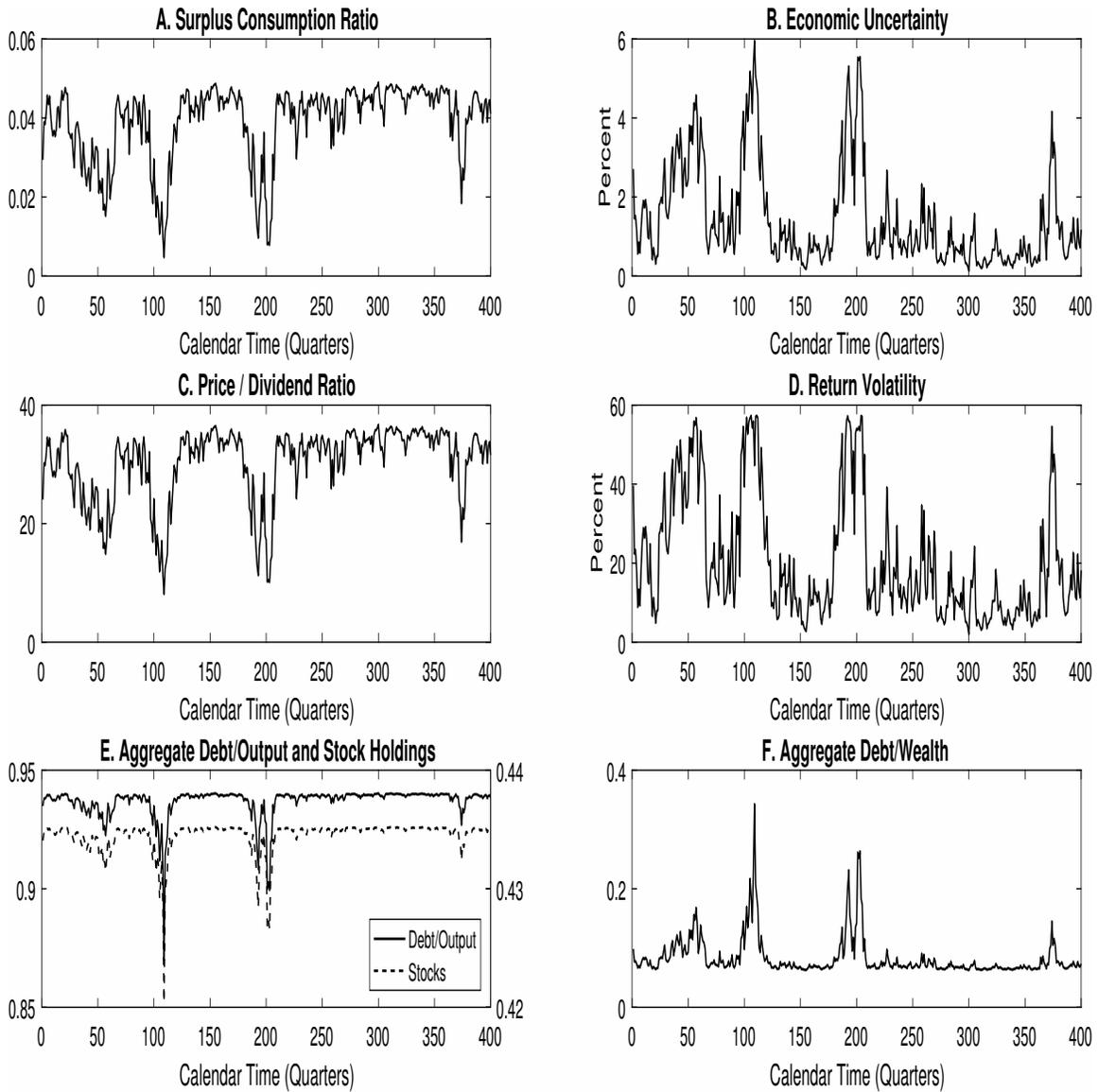
Figure 6: "Fire Sales" in a Simulation Run . This figure plots the time series of several quantities in 100 years of quarterly artificial data. Panel A reports the surplus consumption ratio S_t. Panel B reports the consumption volatility $\sigma_D(S_t)$. Panel C and D report the price-dividend ratio and the stock return volatility, respectively. Panel E reports the aggregate leverage, defined as debt-to-output ratio (solid black line, left axis), and the aggregate position in risky stock of levered agents (grey dashed line, right axis). Panel F reports the aggregate debt-to-wealth ratio of levered agents. This simulated sample was selected to highlight the effect of crises, that are quite visible in the panels.

5.3. Intermediary Asset Pricing

We showed in subsection 4.7. that in our model agents' leverage should show up as a risk factor but with different signs in the market price of risk depending on whether leverage is measured as total debt-to-output ratio ("book leverage", as it does not depend on prices) or as total debt-to-wealth ratio ("market leverage", as it depends on the market value of wealth). While this is just an interpretation of the forces that shape average returns, it is informative nonetheless to see whether in simulations the standard Fama-MacBeth cross-sectional regressions that are used in this literature would pick up our endogenous leverage ratios as risk factors and with the different signs depending on definitions.

For concreteness, we take the leverage ratios of the most leveraged agent in our simulated economy and use them as risk factors, both to predict aggregate returns in the time series and in the cross-section. As discussed in subsection 4.7., book leverage is defined as $N_{it}B_t/D_t$ and market leverage as $N_{it}B_t/W_{it}$. For convenience, we consider as test assets the securities P_{it} in expression (29) that pay the dividend C_{it} over time. To meaningfully compare the coefficients to the data, we normalize both in simulations and in the empirical data the leverage factors to have mean zero and variance one. In the data, we use the standard Fama-French 25 portfolios sorted by size and book-to-market as test assets. The leverage factors are obtained from the dataset made available by He, Kelly, and Manela (2017). In particular, we transform their capital ratio factor into a market debt/equity factor.[32] The book leverage is the one in Adrien, Etula, and Muir (2014), also reported in the He, Kelly, and Manela (2017) dataset. The sample is 1970 through 2012.

Table 3 shows the results of Fama-MacBeth cross-sectional regressions in the data (Panel A) and in the simulations (Panel B). The first column reports the CAPM regression, in which the aggregate market portfolio is the main risk factor. As is well known, the CAPM fails to price these portfolios. The R^2 is a puny 6.5%, the alpha is strongly positive, and the average market return is negative. The second column shows that market leverage is able to explain a large fraction of the variation of the portfolios. The market return becomes positive (but not statistically significant), the alpha is zero, and the market price of risk is negative, and significant. Finally, column III shows the same results for book leverage, and obtains similar results, but now with a positive market price of risk. The different signs of the market prices of risk is exactly the prediction of Corollary 13.

Panel B reports the result in the simulations. As mentioned, in our model the conditional

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[32]In He, Kelly and Manela (2017), capital ratio = Equity/(Debt + Equity) which we transform into Debt/Equity = 1/(capital ratio) − 1.

Table 3: The Market Price of Leverage Risk. Panel A reports Fama-MacBeth regressions where the set of test portfolios are the standard 25 Fama-French portfolios sorted on size and book to market. Column I reports the standard CAPM regression. Column II adds to the market, the market leverage ($N_{it}B_t/W_{it}$ in the model). Market leverage is defined as Debt/Equity = 1/(capital ratio) $-$ 1, which is transformation of the measure introduced in He, Kelly and Manela (2017), which is capital ratio= Equity/(Debt + Equity). Column III reports the same regression where instead of using market leverage we use book leverage, defined as in Adrien, Etula, and Muir (2014). The sample period is 1970-2012. t-statistics are in parenthesis. Panel B reports the same regressions but in a sample of simulated data from our model. The set of test portfolios are the contingent claims that pay the efficient allocation C_{it} for each household i (see (12)) and returns are calculated using prices P_{it} (see expression (29)).

	I	II	III
Panel A - Data			
α	3.19	0.76	1.07
	(3.05)	(0.62)	(0.97)
Market Return	-0.89	0.97	0.82
	(-0.72)	(0.69)	(0.61)
Market Leverage		-0.22	
		(-2.13)	
Book Leverage			0.63
			(3.07)
R^2 (%)	6.54	50.77	53.35

	I	II	III
Panel B - Model			
α	0.02	0.10	0.08
Market Return	2.05	1.96	1.98
Market Leverage		-0.04	
Book Leverage			0.03

CAPM holds, and indeed the first column shows a strong quarterly coefficient of 2.05, which corresponds to 8.2 risk premium, consistent with the time-series average return. We do not report cross-sectional R^2, as they are all 100% in simulations (recall that the model has only one shock, and thus all returns are perfectly correlated). Similarly, we do not report t-statistics, as they are all very large given the large number of artificial data (except for the alpha's, which are close to zero). Column II shows that the estimated market price of risk of market leverage is negative, while column III shows that the estimated market price of risk of book leverage is positive, consistently with Panel A and with the results in Corollary 13. The magnitudes though are smaller which is unsurprising as the conditional CAPM holds in our framework. Moreover, the model doesn't offer a counterpart to value- and size-sorted portfolios and hence the test asset average returns don't display as large a spread as in the

Table 4: The Predictability of Aggregate Stock Returns. Panels A and B show time series regressions of market returns on book and market leverage (see notes to Table 3) lagged one to five years. The sample period is 1970-2012. Panels C and D replicate the same regression in simulated data. $t-$statistics are in parenthesis.

Panel A. Predictability with Book Leverage. Data

	1 year	2 year	3 year	4 year	5 year
Coef ($\times 100$)	-1.78	-1.79	-2.17	-3.13	-9.89
	(-0.83)	(-0.72)	(-0.89)	(-1.03)	(-3.29)
R^2	0.01	0.01	0.01	0.01	0.07

Panel B. Predictability with Market Leverage. Data

	1 year	2 year	3 year	4 year	5 year
Coef ($\times 100$)	3.66	6.21	8.56	10.03	13.06
	(1.57)	(1.50)	(2.18)	(2.51)	(3.84)
R^2	0.04	0.07	0.10	0.12	0.19

Panel C. Predictability with Book Leverage. Model

	1 year	2 year	3 year	4 year	5 year
Coef ($\times 100$	-3.57	-7.28	-10.49	-12.62	-14.13
	(-3.45)	(-3.09)	(-3.18)	(-3.34)	(-3.52)
R^2	0.02	0.05	0.08	0.10	0.12

Panel D. Predictability with Market Leverage. Model

	1 year	2 year	3 year	4 year	5 year
Coef ($\times 100$)	5.86	10.91	14.69	17.35	19.36
	(8.08)	(7.69)	(7.55)	(7.54)	(7.74)
R^2	0.06	0.12	0.17	0.20	0.22

data.[33] Still, the simulation results highlight that endogenous leverage ratios – which only proxy for shocks to risk aversion – show up in cross-sectional regressions as risk factors and with different signs depending on their definitions as found in empirical work.

Finally, Table 4 contains the results from predictability regressions, in the data (Panels A and B) and in the model (Panels C and D). In the data, we see that market leverage (Panel B) is a better predictor of future stock returns than book leverage. The R^2 are larger and coefficients are significant starting with the three-year horizon. Book leverage, in contrast, is only significant at the 5-year horizon. Consistently with the results in Table 3, the predictability coefficient are opposite to each other.

---

[33]The spread in annual average returns across portfolios is 12% in the data while only 4.7% in the model.

Panels C and D of Table 4 show similar results in simulations. In this case, book leverage (Panel C) is always significant, but we note both a lower R^2 and $t-$statistics compared to market leverage (Panel D). That is, our model is in fact consistent with the empirical finding that market leverage should be a better predictor of future stock returns. Indeed, while book leverage and market leverage are clearly related to each other, they are not perfectly correlated. In our simulated data, market leverage and book leverage have a correlation of -83% in levels, and -75% in first differences. In the data, they have a correlation of -39% in levels and -31% in first difference. The lack of perfect correlation in simulation is due to the non-linearities implicit in the model.

In sum, under the interpretation that leveraged agents act like banks in that they provide riskless assets to other more risk averse agents to invest in, the predictions of the model are similar to those in the data. The relevant point here is that the leverage of such institutions is endogenous, and the fact that one can find empirically that leverage is a risk factor it does not necessarily mean that banks are the marginal investors. It may as well mean that leverage is proxy for aggregate risk aversion, as in this model.

6. Conclusions

We propose a general equilibrium model with heterogeneous agents, habits, and countercyclical uncertainty that is able to tie together several stylized facts related to leverage, consumption, and asset prices. The model predicts that aggregate leverage should be procyclical, it should correlate with high valuation ratios, low volatility, and with a "consumption boom" of levered agents. Agents actively trade in risky assets and delever in bad times by "fire selling" their risky positions as their wealth decline and debt-to-wealth ratios increase.

An important message of the paper is to emphasize that leverage is an *endogenous* quantity and thus that some caution must be taken when making causal statements about the impact of leverage on other economic quantities. For instance, in our model agents who increased leverage during good times suffer low consumption growth in bad times. There is nothing inefficient in this: Those agents who *decide* to take on higher leverage supply the safe assets that other agents use to hedge. Similarly, the increase in leverage in good times is the result of an optimal, efficient risk-sharing allocation, and should predict low future asset pricing returns. To reiterate, it is not high leverage that implies that future return are low (because it increases the chance of a financial crisis, for instance), but rather the fact that lower risk premia due to subsided discount rate shocks induce agents with higher risk bearing capacity to take higher leverage to achieve their optimal consumption profile.

Similarly, the leverage, or other measures of capital ratios, of financial intermediaries is also an endogenous quantity. Fluctuations in leverage may simply be driven by (unobservable) shocks to aggregate discount rates, as in our model. Thus, the fact that intermediary leverage is a powerful predictor of returns should be unsurprising: Fluctuations in leverage simply proxy for the fluctuations in the attitudes towards risk of the agents in the economy. Leverage is of course observed whereas the underlying shocks to risk aversion are not and thus the pricing success of leverage in the cross section of stock returns.

Our model is simple in that it only has one state variable, all quantities move in lock-step and thus there is an unrealistic perfect (positive or negative) correlation between leverage, prices, volatility, expected return, consumption, and so on. It is this assumption which allows for closed form solutions in quantities and prices and thus obtain a better understanding of the various economic forces at work. Future research should focus on generalizing our simple setting to obtain more realistic dynamics.

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APPENDIX

Appendix A: Proofs

Preliminary results in section 3. the Lagrangean

$$\mathcal{L}\left(C_i\right) = \int \phi_i u\left(C_{it}, X_{it}, t\right) di - M_t \left(\int C_{it} di - D_t\right)$$

implies that agents' marginal utilities satisfy

$$\phi_i u_c\left(C_{it}, X_{it}, t\right) = M_t. \tag{45}$$

Thus, consumption satisfies

$$C_{it} - X_{it} = \phi_i e^{-\rho t} M_t^{-1} \tag{46}$$

The individual excess consumption is inversely related to the Lagrange multiplier M_t. To obtain the equilibrium value of M_t, we integrate across agents

$$\int C_{it} di - \int X_{it} di = \left(\int \phi_i di\right) e^{-\rho t} M_t^{-1} = e^{-\rho t} M_t^{-1}$$

Using the market clearing condition $D_t = \int C_{it} di$ we find that the Lagrangean multiplier is

$$M_t = e^{-\rho t} \frac{1}{D_t - \int X_{it} di} \tag{47}$$

Finally, plugging this expression into (46) we obtain that agent i's consumption is given by

$$C_{it} - X_{it} = \phi_i \left(D_t - \int X_{jt} dj\right) \tag{48}$$

Each agent's excess consumption over habit is proportional to aggregate excess output. This condition also implies that in equilibrium, the ratio of any two agents' marginal utilities is constant (and equal to the ratio of Pareto weights), a standard result with complete markets. Substituting X_{it} from (1) and using (2) we obtain the optimal consumption of agent i in (7). $\qquad\square$

The proof of Propositions 1 and 3 follow after proof of Proposition 4, to which we first turn.

Proof of Proposition 4. From equation (48), we have $(C_{it} - X_{it}) = \phi_i \left(D_t - \int X_{jt} dj\right)$. Substituting into the representative agent utility (5) we obtain the utility function of the representative agent (up to a constant) as in (21).

As for the state price density, the Lagrange multiplier at time t in equation (47) provides the marginal utility of the representative agent. Using (9) we find:

$$M_t = e^{-\rho t} D_t^{-1} Y_t \tag{49}$$

The interest rate and SDF can be found by applying Ito's lemma to M_t. $\qquad\square$

Proof of Proposition 1. From market completeness, the wealth of agent i is always equal to the discounted value of his optimal consumption, which can be written as

$$C_{i,t} = (g_{it} + \phi_i)\left(D_t - \int X_{jt} dj\right) \tag{50}$$

$$= (a_i(Y_t - \lambda) + \lambda - 1 + \phi_i) S_t D_t \tag{51}$$

We then have

$$
\begin{aligned}
W_{i,t} &= E_t\left[\int_t^\infty \frac{M_\tau}{M_t}C_{i,\tau}d\tau\right] \\
&= D_tS_tE_t\left[\int_t^\infty e^{-\rho(\tau-t)}D_\tau^{-1}S_\tau^{-1}C_{i,\tau}d\tau\right] \\
&= D_tS_tE_t\left[\int_t^\infty e^{-\rho(\tau-t)}(a_i(Y_\tau-\lambda)+\lambda-1+\phi_i)d\tau\right] \\
&= D_tS_tE_t\left[\int_t^\infty e^{-\rho(\tau-t)}(a_iY_\tau-a_i\lambda++\lambda-1+\phi_i)d\tau\right] \\
&= D_tS_t\left[a_i\frac{(Y_t-\overline{Y})}{\rho+k}+\frac{a_i(\overline{Y}-\lambda)+\lambda-1+\phi_i}{\rho}\right]
\end{aligned}
\tag{52}
$$

where we used the fact that $E_t[Y_\tau]=\overline{Y}+(Y_t-\overline{Y})e^{-k(\tau-t)}$. At time 0, the economy starts at its stochastic steady state, $Y_0=\overline{Y}$, which implies $S_0=\overline{S}=1/\overline{Y}=1/Y_0$. In addition, assume $D_0=\rho$. Agent i's endowment is w_i. Therefore, we obtain that the budget constraint implies

$$
\begin{aligned}
w_i = W_{i,0} &= D_0S_0\left[a_i\frac{(Y_0-\overline{Y})}{\rho+k}+\frac{a_i(\overline{Y}-\lambda)+\lambda-1+\phi_i}{\rho}\right] \\
&= D_0S_0\left[\frac{a_i(\overline{Y}-\lambda)+\lambda-1+\phi_i}{\rho}\right] \\
&= \overline{S}\left[a_i(\overline{Y}-\lambda)+\lambda-1+\phi_i\right] \\
w_i/\overline{S} &= \left[a_i(\overline{Y}-\lambda)+\lambda-1+\phi_i\right]
\end{aligned}
$$

or

$$
\phi_i = w_i\overline{Y}-\left[a_i(\overline{Y}-\lambda)+\lambda-1\right].
\tag{53}
$$

proving part (a). The consumption/output ratio (51) can then be written as

$$
\begin{aligned}
\frac{C_{i,t}}{D_t} &= (a_i(Y_t-\lambda)+\lambda-1+\phi_i)S_t \\
&= (a_i(Y_t-\lambda)+w_i\overline{Y}-a_i(\overline{Y}-\lambda))S_t \\
&= (a_i(Y_t-\overline{Y})+w_i\overline{Y})S_t \\
&= a_i(1-\overline{Y}S_t)+w_i\overline{Y}S_t
\end{aligned}
$$

proving part (b). □

The curvature of the utility function (13) can be obtained from the definition of curvature and by substituting C_{it} and ϕ_i in the resulting expression.

Proof of Proposition 3. Part 1. The pricing function for the consumption claim is

$$
P_t = E_t\left[\int_t^\infty \frac{M_\tau}{M_t}D_\tau d\tau\right]
\tag{54}
$$

$$
= D_tY_t^{-1}E_t\left[\int_t^\infty e^{-\rho(\tau-t)}D_\tau^{-1}Y_\tau D_\tau d\tau\right]
\tag{55}
$$

$$
= D_tS_tE_t\left[\int_t^\infty e^{-\rho(\tau-t)}Y_\tau d\tau\right]
\tag{56}
$$

$$
= D_tS_t\int_t^\infty e^{-\rho(\tau-t)}E_t\left[Y_\tau\right]d\tau
\tag{57}
$$

$$
= D_tS_t\int_t^\infty e^{-\rho(\tau-t)}(\overline{Y}+(Y_t-\overline{Y})e^{-k(\tau-t)})d\tau
\tag{58}
$$

48

$$= D_t S_t \left(\frac{\overline{Y}}{\rho} + \frac{(Y_t - \overline{Y})}{\rho + k} \right) \tag{59}$$

$$= D_t S_t \left(\frac{\rho Y_t + k \overline{Y}}{\rho(\rho + k)} \right) \tag{60}$$

The interest rate follows from Ito's lemma as the drift rate of the state price density (49).

Part 2. Given the results of Propositions 1 and 4, and the standard result that the efficient allocation maximize agents' utility, the only part left to show is the optimal allocation to stocks and bonds. From Cox and Huang (1989), the dynamic budget equation can be written as the present value of future consumption discounted using the stochastic discount factor. The optimal allocation can be found by finding the "replicating" portfolio, that is, the position in stocks and bonds that satisfies the static budget equation.

We denote for simplicity

$$\sigma_Y(Y) = v \sigma_D(Y) \tag{61}$$

First, note that the process for surplus consumption ratio is

$$
\begin{aligned}
dS_t &= -Y_t^{-2} dY_t + Y_t^{-3} dY_t^2 \\
&= -Y_t^{-2} k(\overline{Y} - Y_t) dt + Y_t^{-1} \sigma_Y(Y) dZ_t + Y_t^{-1} \sigma_Y(Y)^2 dt \\
&= Y_t^{-1} k(1 - \overline{Y}/Y_t) dt + Y_t^{-1} \sigma_Y(Y) dZ_t + Y_t^{-1} \sigma_Y(Y)^2 dt \\
&= Y_t^{-1} \left(k(1 - \overline{Y}/Y_t) + \sigma_Y(Y)^2 dt \right) dt + Y_t^{-1} \sigma_Y(Y) dZ_t
\end{aligned}
$$

Consider now agents' wealth obtained in (52). Substituting ϕ_i from (53) which we can write it as

$$W_{i,t} = D_t \frac{1}{\rho} \left[a_i \frac{\rho}{\rho + k} (1 - \overline{Y} S_t) + w_i \overline{Y} S_t \right] \tag{62}$$

$$= D_t \frac{1}{\rho(\rho + k)} \left[a_i \rho + (w_i(\rho + k) - a_i \rho) \overline{Y} S_t \right] \tag{63}$$

By definition, $W_{i,t} = E_t \left[\int_t^\infty \frac{M_\tau}{M_t} C_{i,\tau} d\tau \right] = P_{i,t}$, and thus this expression also verifies formula (29).

From Ito's lemma, the diffusion of wealth process $dW_{i,t}/W_{i,t}$ is

$$\sigma_{W,i}(S_t) = \sigma_D(S_t) + \frac{(w_i(\rho + k) - a_i \rho) \overline{Y} Y_t^{-1} \sigma_Y(Y_t)}{a_i \rho + (w_i(\rho + k) - a_i \rho) \overline{Y} Y_t^{-1}} \tag{64}$$

By market completeness (Cox and Huang (1989)), agent i's wealth is always equal to his allocation to stocks and bonds

$$W_{it} = N_{i,t} P_t + N_{i,t}^0 B_t$$

From this latter expression, N_{it} must be chosen to equate the diffusion of the portfolio to the diffusion of wealth. That is, such that

$$N_{it} P_t \sigma_P(S_t) = W_{i,t} \sigma_{W,i}(S_t)$$

Solving for N_{it} gives

$$
\begin{aligned}
N_{it} &= \frac{W_{it} \sigma_{Wi}(Y)}{P_t \sigma_P(Y)} \\
&= \frac{\left(\rho a_i + (w_i(\rho + k) - \rho a_i) \overline{Y}/Y_t \right)}{\left(\rho + k \overline{Y}/Y_t \right)} \left(\frac{\sigma_D(Y) + \frac{(w_i(\rho+k) - \rho a_i) \overline{Y} Y_t^{-1} \sigma_Y(Y)}{\left(\rho a_i + (w_i(\rho+k) - \rho a_i) \overline{Y}/Y_t \right)}}{\sigma_D(Y) + \frac{k \overline{Y} Y_t^{-1} \sigma_Y(Y)}{\left(\rho + k \overline{Y}/Y_t \right)}} \right) \\
&= \frac{\left(\rho a_i + (w_i(\rho + k) - \rho a_i) \overline{Y}/Y_t \right)}{\left(\rho + k \overline{Y}/Y_t \right)} \left(\frac{\frac{\sigma_D(Y)\left(\rho a_i + (w_i(\rho+k) - \rho a_i) \overline{Y}/Y_t \right) + (w_i(\rho+k) - \rho a_i) \overline{Y} Y_t^{-1} \sigma_Y(Y)}{\left(\rho a_i + (w_i(\rho+k) - \rho a_i) \overline{Y}/Y_t \right)}}{\frac{\sigma_D(Y)\left(\rho + k \overline{Y}/Y_t \right) + k \overline{Y} Y_t^{-1} \sigma_Y(Y)}{\left(\rho + k \overline{Y}/Y_t \right)}} \right)
\end{aligned}
$$

$$
= \left(\frac{\sigma_D\left(Y\right)\left(\rho a_i + \left(w_i\left(\rho+k\right) - \rho a_i\right)\overline{Y}/Y_t\right) + \left(w_i\left(\rho+k\right) - \rho a_i\right)\overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)}{\sigma_D\left(Y\right)\left(\rho + k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)} \right)
$$

$$
= \frac{\sigma_D\left(Y\right)\rho a_i + \sigma_D\left(Y\right)\overline{Y}/Y_t\left(w_i\left(\rho+k\right) - \rho a_i\right) + w_i\left(\rho+k\right)\overline{Y}Y_t^{-1}\sigma_Y\left(Y\right) - \rho a_i\overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)}{\sigma_D\left(Y\right)\left(\rho + k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)}
$$

$$
= a_i + \left(\rho+k\right)\frac{\sigma_D\left(Y\right)\overline{Y}/Y_t + \overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)}{\sigma_D\left(Y\right)\left(\rho + k\overline{Y}/Y_t\right) + k\overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)}\left(w_i - a_i\right)
$$

$$
= a_i + \left(\rho+k\right)\frac{\sigma_D\left(Y\right)\overline{Y}/Y_t + \overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)}{\sigma_D\left(Y\right)\rho + k\left(\sigma_D\left(Y\right)\overline{Y}/Y_t + \overline{Y}Y_t^{-1}\sigma_Y\left(Y\right)\right)}\left(w_i - a_i\right)
$$

$$
= a_i + \left(\rho+k\right)\frac{\overline{Y}/Y_t\left[\sigma_D\left(Y\right) + \sigma_Y\left(Y\right)\right]}{\sigma_D\left(Y\right)\rho + k\overline{Y}/Y_t\left[\sigma_D\left(Y\right) + \sigma_Y\left(Y\right)\right]}\left(w_i - a_i\right)
$$

$$
= a_i + \left(\rho+k\right)\frac{\overline{Y}/Y_t\sigma_M\left(Y\right)}{\sigma_D\left(Y\right)\rho + k\overline{Y}/Y_t\sigma_M\left(Y\right)}\left(w_i - a_i\right)
$$

where

$$
\sigma_M(Y) = \sigma_D(Y) + \sigma_Y(D)
$$

Finally, substituting $\sigma_Y(Y) = v\sigma_D(Y)$ from definition (61) and deleting $\sigma_D(Y)$ throughout, the result follows.

Similarly, we have that the amount in bonds is

$$
N_{it}^0 B_t = W_{it} - N_{it}P_t
$$

$$
= D_t\frac{1}{\rho}\left(\frac{\rho}{\rho+k}a_i + \left(w_i - \frac{\rho}{\rho+k}a_i\right)\overline{Y}/Y_t\right) - N_{it}D_t\frac{\left(\rho + k\overline{Y}/Y_t\right)}{\rho\left(\rho+k\right)}
$$

$$
= D_t\frac{1}{\rho\left(\rho+k\right)}\left[\left(\rho a_i + \left(w_i\left(\rho+k\right) - \rho a_i\right)\overline{Y}/Y_t\right) - N_{it}\left(\rho + k\overline{Y}/Y_t\right)\right]
$$

$$
= D_t\frac{1}{\rho\left(\rho+k\right)}\left[a_i\left(\rho + k\overline{Y}/Y_t\right) + w_i\left(\rho+k\right)\overline{Y}/Y_t - a_i\left(\rho+k\right)\overline{Y}/Y_t - N_{it}\left(\rho + k\overline{Y}/Y_t\right)\right]
$$

$$
= D_t\frac{1}{\rho\left(\rho+k\right)}\left[a_i\left(\rho + k\overline{Y}/Y_t\right) + \left(w_i - a_i\right)\left(\rho+k\right)\overline{Y}/Y_t - N_{it}\left(\rho + k\overline{Y}/Y_t\right)\right]
$$

$$
= D_t\frac{1}{\rho}\left[\overline{Y}/Y_t - \frac{\overline{Y}/Y_t\sigma_M\left(Y\right)}{\sigma_D\left(Y\right)\rho + k\overline{Y}/Y_t\sigma_M\left(Y\right)}\left(\rho + k\overline{Y}/Y_t\right)\right]\left(w_i - a_i\right)
$$

$$
= D_t\frac{1}{\rho}\left[\frac{\overline{Y}/Y_t\left[\sigma_D\left(Y\right)\rho + k\overline{Y}/Y_t\sigma_M\left(Y\right)\right] - \overline{Y}/Y_t\sigma_M\left(Y\right)\left(\rho + k\overline{Y}/Y_t\right)}{\sigma_D\left(Y\right)\rho + k\overline{Y}/Y_t\sigma_M\left(Y\right)}\right]\left(w_i - a_i\right)
$$

$$
= -D_t\left[\frac{\overline{Y}/Y_t\left(\sigma_M\left(Y\right) - \sigma_D\left(Y\right)\right)}{\sigma_D\left(Y\right)\rho + k\overline{Y}/Y_t\sigma_M\left(Y\right)}\right]\left(w_i - a_i\right)
$$

$$
= -D_t\left[\frac{\overline{Y}/Y_t\left(\sigma_M\left(Y\right)/\sigma_D\left(Y\right) - 1\right)}{\rho + k\overline{Y}/Y_t\sigma_M\left(Y\right)/\sigma_D\left(Y\right)}\right]\left(w_i - a_i\right)
$$

Finally, substituting $\sigma_Y(Y) = v\sigma_D(Y)$ from definition (61) and deleting $\sigma_D(Y)$ throughout, the result follows. $\square$

Proof of Corollary 5. Immediate from Proposition 3 and 4. The state price density and the price of stocks are independent of cross-sectional quantities. $\square$

Proof of Corollary 6. Part (a) is immediate from the expression for N_{it}^0 in Proposition 3. Part (b) can be shown as follows:

$$
\frac{N_{it}P_t}{W_{it}} = \frac{\sigma_{Wi}\left(Y\right)}{\sigma_P\left(Y\right)}
$$

50

$$= \frac{\sigma_D(Y) + \frac{\left(w_i - \frac{\rho}{\rho+k}a_i\right)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\frac{\rho}{\rho+k}a_i + \left(w_i - \frac{\rho}{\rho+k}a_i\right)\overline{Y}/Y_t\right)}}{\sigma_D(Y) + \frac{k\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho + k\overline{Y}/Y_t\right)}}$$

$$= \frac{\sigma_D(Y) + \frac{(w_i(\rho+k) - \rho a_i)\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho a_i + (w_i(\rho+k) - \rho a_i)\overline{Y}/Y_t\right)}}{\sigma_D(Y) + \frac{k\overline{Y}Y_t^{-1}\sigma_Y(Y)}{\left(\rho + k\overline{Y}/Y_t\right)}}$$

$$= \frac{\sigma_D(Y) + \sigma_Y(Y)\left(\frac{(w_i(\rho+k) - \rho a_i)\overline{Y}Y_t^{-1}}{\left(\rho a_i + (w_i(\rho+k) - \rho a_i)\overline{Y}/Y_t\right)}\right)}{\sigma_D(Y) + \sigma_Y(Y)\left(\frac{k\overline{Y}Y_t^{-1}}{\left(\rho + k\overline{Y}/Y_t\right)}\right)}$$

$$= \frac{\sigma_D(Y) + \sigma_Y(Y)\left(1 - \frac{\rho}{\rho + [k + (\rho+k)(w_i - a_i)/a_i]\overline{Y}/Y_t}\right)}{\sigma_D(Y) + \sigma_Y(Y)\left(1 - \frac{\rho}{\left(\rho + k\overline{Y}/Y_t\right)}\right)}$$

Finally, substituting $\sigma_Y(Y) = v\sigma_D(Y)$ from definition (61) and deleting $\sigma_D(Y)$ throughout, the result follows. □

Proof of Corollary 7. It is immediate from the expression of $H(S_t)$ to verify it is increasing in S_t. □

Proof of Corollary 8. Immediate from the fact $L(S_t)$ is increasing and the fact that agents with $w_i - a_i > 0$ are leveraged and have C_{it}/D_t that is increasing in S_t. □

Proof of Corollary 9. The expressions of the drift and diffusion of dC_{it}/C_{it} stem from the application of Ito's lemma to the consumption $C_{it} = D_t[a_i + (w_i - a_i)\overline{Y}S_t]$. The remaining part follows from the statement in the corollary. □

Proof of Corollary 10. Immediate from Corollary 9. □

Proof of Corollary 11. Immediate from Corollary 6 and $H(S_t)$ being increasing. □

Proof of Corollary 12. Immediate from Corollary 6 and $H(S_t)$ being concave. □

Proof of Corollary 13. Part (a) follows from the fact that $H(S_t)$ is increasing in S_t, which implies that $q^{D/O}(\ell_t)$ – the inverse function of $Q_{it}^{D/O}(S_t)$ – is also increasing in ℓ_t. Similarly, part (b) follows from the fact that the ratio $\sigma_{W,i}(S_t)/\sigma_P(S_t)$ is decreasing in S_t and thus so is $q^{D/W}(\ell_t)$, the inverse function of $Q_{it}^{D/W}(S_t)$. □

Proof of Proposition 14 . From (62) we can rewrite

$$\frac{W_{i,t}}{D_t} = a_i\left(\frac{1 - \overline{Y}S_t}{\rho + k}\right) + w_i\left(\frac{\overline{Y}S_t}{\rho}\right) \tag{65}$$

and

$$\frac{W_{i,t}}{\int W_{j,t}dj} = \frac{W_{i,t}}{D_t} \times \frac{D_t}{P_t}$$

where D_t/P_t is in (16). The results then immediately follow from the definition of cross-sectional variance. □

Appendix B: Derivation of expression (27)

Let

$$\mathsf{SR}\left(S_t\right) \equiv \frac{E_t\left[dR_t - r_t dt\right]}{\sigma_P\left(S_t\right)} = (1+v)\,\sigma_D\left(S_t\right) \qquad \text{and} \qquad \theta_i \equiv \frac{va_i}{(1+v)\,(\rho+k)}. \tag{66}$$

Finally define

$$\Omega_i\left(S_t\right) \equiv \frac{\rho}{\rho + \left[k + \frac{(\rho+k)(\omega_i - a_i)}{a_i}\right]\overline{Y}S_t}$$

The share of wealth invested in the risky security is

$$
\begin{align}
\frac{N_{i,t}P_t}{W_{i,t}} &= \left(\frac{\sigma_D\left(S_t\right)}{\sigma_P\left(S_t\right)}\right)\left[1 + v\left(1 - \Omega_i\left(S_t\right)\right)\right] \tag{67} \\
&= \left(\frac{\sigma_D\left(S_t\right)}{\sigma_P\left(S_t\right)}\right)\left[1 + v\left(1 - \Omega_i\left(S_t\right)\right)\right] \tag{68} \\
&= \left(\frac{(1+v)\,\sigma_D\left(S_t\right)}{\sigma_P\left(S_t\right)}\right)\left[1 - \frac{v}{1+v}\Omega_i\left(S_t\right)\right] \tag{69} \\
&= \frac{\mathsf{SR}\left(S_t\right)}{\sigma_P\left(S_t\right)}\left[1 - \frac{v}{1+v}\Omega_i\left(S_t\right)\right], \tag{70}
\end{align}
$$

where we have made use of (66).

The key is to show that

$$
\begin{align}
\Omega_i\left(S_t\right) &\equiv \frac{\rho}{\rho + \left[k + \frac{(\rho+k)(\omega_i - a_i)}{a_i}\right]\overline{Y}S_t} \tag{71} \\
&= \frac{\rho a_i}{\rho a_i + \left[\rho\left(\omega_i - a_i\right) + k\omega_i\right]\overline{Y}S_t} \tag{72} \\
&= \left(\frac{D_t}{\rho\left(\rho+k\right)}\right)\frac{\rho a_i}{\frac{\left[\rho a_i + \left[\rho(\omega_i - a_i) + k\omega_i\right]\overline{Y}S_t\right]D_t}{\rho(\rho+k)}} \tag{73} \\
&= \left(\frac{a_i}{\rho+k}\right)\frac{D_t}{W_{i,t}} \tag{74}
\end{align}
$$

in the text we use both $P_{i,t}$ and $W_{i,t}$ interchangeably. Define θ_i as in (66) and substitute in (70) to obtain (27). □

Appendix C: Estimating household total and systematic consumption volatility

A challenge in the literature regarding the estimation of consumption volatility – the systematic and idiosyncratic components – is the lack of reliable high frequency panel data. In this Appendix we illustrate how we can use only cross-sectional information across households and then the time series across cohorts to estimate both components. This section contains the main methodology, and we refer the reader to Santos, Suarez and Veronesi (2017) for the full methodology.

Consider the simple continuous time model, which generalizes the one derived in the model as we allow consumption to have cross-sectionally independent shocks:

$$\frac{dC_{it}}{C_{it}} = \mu_{it}dt + \sigma_{it}dZ_{it} \tag{75}$$

In this process, both μ_{it} and σ_{it} are cross-sectionally different from each other and time varying. We are interested in estimating σ_{it}. From Ito's lemma we have

$$d\log\left(C_{it}\right) = \left(\mu_{it} - \frac{1}{2}\sigma_{it}^2\right)dt + \sigma_{it}dZ_{it} \tag{76}$$

Therefore, for every i and t, we have

$$\widehat{\sigma}_{it}^2 = \frac{2}{dt}\left[\frac{dC_{it}}{C_{it}} - d\log\left(C_{it}\right)\right] \tag{77}$$

This quantity is independent of dZ_{it} (it is a dt term) and on whether shocks are correlated with each other or not. Therefore, the (rescaled) difference between arithmetic and log consumption growth isolates the consumption variance of agent i at time t. This is a (noisy) observation of variance itself, and we are going to treat it as such.

In our model all consumption processes are perfectly correlated, and there are no idiosyncratic shocks. To calibrate the model we thus assume a common shock to dZ_{it}, that is

$$dZ_{it} = \rho dZ_t + \sqrt{1-\rho^2}dZ_{it}^*$$

where dZ_{it}^* are uncorrelated across i. This assumption implies that all consumption process across every two agents have correlation ρ^2:

$$Corr\left(\frac{dC_{it}}{C_{it}}\frac{dC_{jt}}{C_{jt}}\right) = \rho^2 dt$$

Santos, Suarez, and Veronesi (2017) relax this assumption but given the scope of the current calibration, this assumption simplifies the methodology.

Consider now the cross-sectional average of consumption growth $E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right]$. This quantity follows the dynamic process

$$
\begin{aligned}
E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right] &= E_t^{CS}\left[g_{it}\right]dt + E_t^{CS}\left[\sigma_{it}dZ_{it}\right] \\
&= E_t^{CS}\left[g_{it}\right]dt + \rho E_t^{CS}\left[\sigma_{it}\right]dZ_t + \sqrt{(1-\rho^2)}E_t^{CS}\left[\sigma_{it}dZ_{it}^*\right]
\end{aligned}
$$

From the law of large numbers the idiosyncratic shocks average to zero

$$E_t^{CS}\left[\sigma_{it}dZ_{it}^*\right] = E_t^{CS}\left[\sigma_{it}\right]E^{CS}\left[dZ_{it}^*\right] = 0$$

Therefore the average arithmetic consumption growth follows

$$E_t^{CS}\left[\frac{dC_{it}}{C_{it}}\right] = E_t^{CS}\left[g_{it}\right]dt + E_t^{CS}\left[\sigma_{it}\right]\rho dZ_t$$

Hence, the squared variation of average consumption growth in continuous time has

$$\left(E_t^{CS} \left[\frac{dC_{it}}{C_{it}} \right] \right)^2 = E_t^{CS} \left[\sigma_{it} \right]^2 \rho^2 dt + o\left(dt \right)$$

That is, we can identify the average systematic volatility of consumption growth from the squared variation of the cross-sectional average of consumption growth, a result that is not surprising.

We are interested however to also identify the whole distribution of systematic volatility $\{\sigma_{it}^2 \rho^2\}_i$. Given our estimates of σ_{it}^2 obtained earlier we just need to estimate ρ^2, which can be done from the following estimator:

$$\widehat{\rho}^2 = \frac{E_t^{CS} \left[\frac{dC_{it}}{C_{it}} \right]^2}{E_t^{CS} \left[\sigma_{it} \right]^2} / dt \tag{78}$$

The systematic variance of agent i at time t is then

$$V_{it}^2 = \widehat{\sigma}_{it}^2 \widehat{\rho}^2 \tag{79}$$

To conclude this section, we note that to estimate time t quantities – idiosyncratic and total volatility components – we only need cross-sectional information. We then use time series information across cohorts of households to compute averages.

CEX Data

We exploit the dataset of Kocherlakota and Pistaferri (2009) and Toda and Welsh (2015). We refer the reader to those papers for a more detailed description of the data. In a nutshell, the data are from the survey of consumer expenditure (CEX). Households are surveyed for four consecutive quarters, in January, February, and March cycles. Thus, the growth rate can be observed at most at quarterly frequency, i.e. $dt = 0.25$. While this is a large time difference, Monte Carlo simulations indicate that the methodology above provides reliable estimates for the distribution of consumption volatility.

For each agent i, we have 3 observations of its variance as in (77). To minimize the impact of seasonalities, we then take the average of the three observations of $\widehat{\sigma}_{it}^2$ across the three quarters

$$\widehat{V}_t^i = \frac{1}{3}\widehat{\sigma}_{it}^2 + \frac{1}{3}\widehat{\sigma}_{it+.25}^2 + \frac{1}{3}\widehat{\sigma}_{it+.5}^2$$

For every year t in a given cycle (Jan, Feb, and Mar), we can then compute the distribution of consumption volatility across households. For instance, we can compute the mean, the median, and various percentiles α

$$\widehat{V}_t^{Ave} = Average\left[\widehat{V}_t^i \right]; \quad \widehat{V}_t^{Med} = Median\left[V_t^i \right]; \quad \widehat{V}_t^{\alpha} = Percentile\left[V_t^i, \alpha \right]$$

Similarly, for every t we can compute an observation for $\widehat{\rho}_t$ from estimator (78). We can thus obtain the systematic component of volatilities as above:

$$\widehat{V}_t^{Sys,Ave} = Average\left[\widehat{V}_t^i \widehat{\rho}_t^2 \right]; \quad \widehat{V}_t^{Sys,Med} = Median\left[V_t^i \widehat{\rho}_t^2 \right]; \quad \widehat{V}_t^{Sys,\alpha} = Percentile\left[V_t^i \widehat{\rho}_t^2, \alpha \right]$$

We finally take average across cohorts (Jan, Feb, and March), and finally across time. Panel A of Table 2 contains the results.