Asset-based Contagion Models for Systemic Risk*

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Abstract

We develop a structural model for the analysis of systemic risk in financial markets based on asset price contagion. Specifically, we describe a mechanism of contagion where exogenous random shocks to individual agents in an economy force portfolio rebalancing and endogenously impact asset prices. This, in turn, creates a chain reaction as downstream agents trade in reaction to price changes. In our setting, this contagion is modulated through a bipartite financial holding network, which describes the relationships between agents and a universe of tradable assets through their portfolio holdings. Out approach quantifies the robustness of different financial holding networks to shocks propagated by asset price contagion by measuring the sensitivity of asset prices to exogenous shocks. We illustrate that *leverage* plays a critical role in asset price contagion. In particular, in low-leverage economies, "mutual fund" networks (where agents hold a common, diversified portfolio of risky assets) are desirable to mitigate contagion. On the hand, in high-leverage economies, "isolated" networks (where agents hold maximally diverse, unrelated portfolios) become beneficial.

1. Introduction

Traditional risk analysis takes the outcomes for an agent across possible states of nature as an exogenous given. In a systemic setting, however, negative outcomes for one agent can have an impact throughout the system. For example, they can cascade or spillover to other agents, this is an endogenous phenomena commonly known as *contagion*. Contagion is often associated with and results from an underlying network structure, which characterizes the structural mechanisms through which outcomes propagate between agents.

We consider systemic risk and contagion in financial markets. Here, there has been a growing literature on structural models for contagion. Much of this work, however, focuses on contagion that arises through direct counterparty relationships between agents. Here, the underlying network structure corresponds to bilateral trading relationships. Early work includes that of Allen and Gale

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(2000), who consider contagion arising from interbank cross holdings of deposits. The pioneering paper of Eisenberg and Noe (2001) provided a structural framework for contagion in interbank lending. Their model illustrated how shocks to individual agents can be propagated through interbank lending networks. Subsequently, there has been a substantial body of work analyzing and generalizing the Eisenberg and Noe (2001) framework. For example, Acemoglu et al. (2013) focus on the optimal structure for interbank lending networks. Cifuentes et al. (2005); Gai and Kapadia (2010); Rogers and Veraart (2013); Amini et al. (2013); Chen et al. (2014), among others, consider extensions where endowment shocks get amplified both the through counterparty network and a lack of asset liquidity. While all of this work emphasizes role of inter-institution borrowing and lending networks, in some cases, there is a single illiquid asset that can be liquidated to cover debt obligations.

However, there is reason to believe that interbank lending networks may not be significant drivers of contagion. Glasserman and Young (2014) demonstrate theoretically and empirically that expected losses due to from network effects due to interbank lending are likely to be small, unless they are magnified by mechanisms beyond simple spillover effects. If we extend the class of counterparty relationships under consideration beyond interbank lending to include derivatives (e.g., credit default swaps), it could be that counterparty network effects may be more important. However, another important source of contagion in the 2008 financial crisis were commonly held so-called "toxic assets". Indeed, commenting on how the impact of the default of Lehman in 2008 propagated across the financial sector, one observer states that[1]

> *[T]here's no evidence that any of the financial institutions that were rescued — AIG, Citigroup, Wachovia, Washington Mutual, Merrill Lynch — were weakened by their exposure to Lehman. Since they weren't, the whole idea of interconnections ... is called into question ... None of these firms was weakened by its exposure to Lehman or anyone else. They were weakened by the fact that virtually all of them held — or were suspected of holding — large amounts of what the media came to call "toxic assets."*

In this paper, we wish to understand the role of common asset holdings in the propagation of shocks in the financial system. Here, the underlying network structure is not derived from bilateral counterparty relationships, but instead takes the form of a bipartite financial holding network, linking agents to assets by their underlying portfolio holdings. When an agent experiences a shock, this forces portfolio rebalancing. This, in turn, affects asset prices. Subsequently, other agents holding common assets are impacted by the price shifts, which causes further rebalancing and downstream effects. The relationships between particular firms in our model is implicit since two firms that hold the same asset in their portfolio are connected via the asset-firm network.

Our main contributions to analytical modeling of asset-firm interaction are as follows:

- **We provide a structural model for contagion in financial network consisting of a set of firms holding common assets.**

[1] P. J. Wallison. Dodd-Frank and the Myth of 'Interconnectedness'. *The Wall Street Journal*, February 9, 2012.

We allow the portfolio choice of the firms to be any function of the asset prices, the state of the economy, and the mark-to-market wealth of firms. The asset prices are endogenously set by a market clearing condition when the economy is in equilibrium.

- **We provide analysis of how shocks are propagated by the financial network.**

  We allow a variety of exogenous shocks, including, for example, shocks to the endowments of firms, shocks to their beliefs regarding asset returns, and regulatory shocks (e.g., changes in leverage limits). Our analysis characterizes the contagion induced by the network. In particular, we identify two key components of a financial network's response to exogenous shocks: the *direct effect* is immediate response of network to the shock, and *network effect* refers to the subsequent amplification of that shocks that are mediated through changes in the prices of assets. The net impact of the shocks can be viewed as the aggregate effect along all paths through the bipartite network of firm holdings. The intensity of the effect of each path is proportional to the amount of asset held in each firm's portfolio.

- **We characterize the key features of financial holding networks that minimize systemic risk, as a function of overall leverage.**

  Given primitives, such as the asset prices, the wealth and the leverage of firms, we solve the optimization problem of designing the optimal holding network that minimizes the shock amplification amplifier under given shocks. We identify two distinct economic regimes as a function of the overall systemic leverage of the financial system, where, surprisingly, desirable asset-firm network topologies vary dramatically.

  For a low-leverage economy, a "mutual fund" holding network, where every firm holds a common market portfolio is desirable and mitigates contagion. On the other hand, for a high-leverage economy, in order to mitigate contagion, "isolated" holding networks are desirable, wherein each asset is held by a subset of firms, and each firm tends to invest all of its leveraged wealth in a single asset. Consequently, the interaction between assets via the portfolio holdings of the firms is kept at a minimum.

  In other words, networks that minimize contagion in a low-leverage economy favors diversification of investment across assets; on the other hand, the such networks in a high-leverage economy favors diversity of investment across firms. The systemic leverage becomes a defining quantity that distinguishes these two economic regimes.

The trade-off between between diversification and diversity has been explored previously. Wagner (2011) explores this trade-off in the context of an equilibrium model with an explicit model for the joint liquidation cost in a very specific portfolio choice model. We accommodate general portfolio choices; however, we focus on managing systemic risk from the perspective of a regulator. Elliott et al. (2013) investigate the trade-off between diversification, i.e., more counterparties, and integration, i.e., deeper relationship with each counterparty, in a credit default setting. A key difference between the work of Elliott et al. (2013) and this work is that the their "assets" are restricted to be claims on other firms in the network, whereas we allow for more general tradable assets and explicitly model the portfolio holding relationship between firms and tradable assets. The work of Greenwood et al. (2012) models asset-based contagion by assuming leverage targeting and a reduced-form price impact; we model this as the endogenous outcome of portfolio rebalancing. Capooni and Larsson (2014) also establish the procyclic characteristics of leverage targeting.

This paper is organized as follows. In Section 2, we describe our model. In Section 3, we introduce the price sensitivity analysis that identifies the key components in the asset-based contagion. In Section 4, we discuss the *maximal network amplifier* as a metric for risk contagion, and in Section 5 we investigate how the network configuration affects the network amplifier. We define low systemic leverage and high systemic leverage regimes, and identify the desirable network topologies for the two regimes. We conclude in Section 6.

2. Model Formulation

In this section, we propose a model for a financial network of firms and their asset holdings, where the assets prices are endogenously set through a market clearing condition. Let $\mathcal{A}$ denote the set of assets, and $\mathcal{F}$ denote the set of financial firms. The initial endowments for all of the firms are denoted by (θ^0, Θ), where $\theta^0 \in \mathbb{R}^{|\mathcal{F}|}$ are the endowments in the risk-free asset for all of the firms, and $\Theta \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}$ are the endowments in the risky assets, i.e., Θ_{ih} is the number of shares of asset i initially held by firm h. For any posted asset prices $q \in \mathbb{R}^{|\mathcal{A}|}$, the wealth of firms is given by the vector $w = \theta^0 + \Theta^\top q \in \mathbb{R}^{|\mathcal{F}|}$. Let $D = \operatorname{diag}(d) \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ denote a diagonal matrix where $d_i = \sum_{h\in\mathcal{F}} \Theta_{ih}$ is the total number of outstanding shares for asset i.

The portfolio held by the firms is denoted by the matrix $\Pi \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}$. Specifically, Π_{ih} is the percentage of the wealth of firm h that is invested in asset i. For each $h \in \mathcal{F}$, the h-th column of the matrix Π is denoted by a vector $\Pi_h \in \mathbb{R}^{|\mathcal{A}|}$, which represents the portfolio of firm h. In our model, we allow the portfolio choice Π_h to depend on:

- the vector of posted prices q,
- the wealth w_h of firm h, and
- an exogenous risk factor x.

In other words, Π_h takes the functional form $\Pi_h(q, w_h, x)$. Depending on the context, the exogenous risk factor x can denote economic factors, beliefs of investors, regulatory parameters (e.g., leverage limits), etc. Thus, the portfolio allocation rules $\Pi_h(q, w_h, x)$ captures a wide variety of portfolio choice mechanisms or preset re-balancing rules. In what follows, the exogenous variable x will be subject to a shock, and whose downstream effects we wish to understand. We also allow for the possibility that initial cash endowment θ_0 depends on x, i.e., θ_0 is a function of x. In the following example, we present a concrete form of Π as a function of asset prices q and other factors.

Example 1 (CRRA Investors). *Suppose firm h has the belief that the random payoff $p_h \in \mathbb{R}^{|\mathcal{A}|}$ of the risky assets is distributed according to a jointly log-normal distribution, i.e., $\log(p_{ih}) \sim N(\mu_h, \Sigma_h)$. Therefore, given current asset price q, the vector of log-returns $r_{ih} = \log \frac{p_{ih}}{q_i} = \log(p_{ih}) - \log(q_i)$; thus, $r_h \sim N(\hat{\mu}_h, \hat{\Sigma}_h)$ where $\hat{\mu}_{ih} = \mu_{ih} - \log(q_i)$ and $\Sigma_h = \hat{\Sigma}_h$.*

Suppose the firm h chooses its portfolio $\Pi_h \in \mathbb{R}^{|\mathcal{A}|}$ in the risky assets and $\pi_h \in \mathbb{R}$ in the risk-free asset, to maximize the expected utility $\mathsf{E}\big[U(\tilde{w}\big]$ of the random terminal wealth, and the utility function U_h has a constant relative risk aversion (CRRA) characterized by β_h, i.e., the vector (Π_h, π_h) is a solution of the optimization problem

$$\underset{\mathbf{1}^\top \Pi_h + \pi_h^0 = 1}{\text{maximize}} \ \mathsf{E}\left[\frac{\tilde{w}_h^{1-\beta_h}}{1-\beta_h}\right], \tag{1}$$

where $\tilde{w}_h = w_h^0 \sum_{i \in \mathcal{A}} \Pi_{ih} e^{r_{ih}}$ is the random wealth of firm h. Campbell and Viceira (2002) show that an approximately optimal solution to (1) is given by

$$\Pi_h = \frac{1}{\beta_h} \Sigma_h^{-1} \left(\mu_h - \log(q) - r_f \mathbf{1} + \frac{1}{2} \operatorname{diag}\{\Sigma_h\} \right). \tag{2}$$

In this example, the portfolio rule Π_h is a function of assets prices q, and exogenous parameters $x = (\beta, \mu_h, \Sigma_h)$; however, it is not a function of the current wealth w.

The portfolio rule in Example 1 is one of many portfolio choice rules that are admissible in our model. In the example, the portfolio rule Π_h is the solution of a utility maximization problem. In other cases, the rules Π can possibly be a set of predefined re-balancing rules. The model allows for very general portfolio choices that depend on asset prices and other factors.

2.1. Market Equilibrium

In this section, we will introduce our concept of market equilibrium that implicitly defines the asset prices q as a function of the initial endowment (Θ, θ^0), the portfolio allocation rules Π and the value of the exogenous risk factor x. Later in the section, we show that our equilibrium concept is equivalent to a simpler market clearing condition.

Definition 1 (Equilibrium). *Suppose we are given initial endowments (Θ, θ^0) and the risk factor x. An equilibrium is the tuple $(q, \hat{\Theta}, \hat{\theta}^0)$, where $q \in \mathbb{R}^{|\mathcal{A}|}$ are posted asset prices, $\hat{\Theta} \in \mathbb{R}^{|\mathcal{F}| \times |\mathcal{A}|}$ is the portfolio of risky assets held by firms, and $\hat{\theta}^0 \in \mathbb{R}^{|\mathcal{A}|}$ is the vector for risk-free assets held by firms. We require that $(q, \hat{\Theta}, \hat{\theta}^0)$ satisfy:*

- *Budget Balance.*

$$\sum_{i \in \mathcal{A}} \left(\hat{\Theta}_{ih} - \Theta_{ih} \right) q_i + \left(\hat{\theta}_h^0 - \theta_h^0 \right) = 0, \quad \forall\ h \in \mathcal{F}. \tag{3}$$

- *Share Balance.*

$$D_{ii} = \sum_{h \in \mathcal{F}} \Theta_{ih} = \sum_{h \in \mathcal{F}} \hat{\Theta}_{ih}, \quad \forall\, i \in \mathcal{A}. \tag{4}$$

- *Portfolio Consistency.*

$$\hat{\Theta}_{ih} q_i = \Pi_{ih}(q, \theta^0 + \Theta^\top q, x) \left(\hat{\theta}^0_h + \sum_{j \in \mathcal{A}} \hat{\Theta}_{jh} q_j \right), \quad \forall\, i \in \mathcal{A},\ h \in \mathcal{F}. \tag{5}$$

The budget balance condition (3) requires that at the posted asset prices q, the portfolio re-balancing from (Θ, θ^0) to $(\hat{\Theta}, \hat{\theta}^0)$ does not change the wealth of any firm. This condition immediately implies that wealth is invariant under re-balancing, i.e., $w = \theta^0 + \Theta^\top q = \hat{\theta}^0 + \hat{\Theta}^\top q$. The share balance condition (4) requires that at posted asset prices q, the number of shares of each asset is not changed by the portfolio re-balancing from (Θ, θ^0) to $(\hat{\Theta}, \hat{\theta}^0)$. Recall that Π_{ih} denotes the percentage of wealth of firm h invested in asset i, but Θ_{ih} represents the portfolio in terms of the number of shares that firm h holds in asset i. Thus, the portfolio consistency condition (5) states that the final position Θ is consistent with the portfolio allocation rule Π.

In the following theorem, we introduce a simpler characterization of an equilibrium. The proof of the theorem can be found in Appendix A.

Theorem 1. *Suppose we are given initial endowments (Θ, θ^0) and the risk factor x. For a vector of posted asset prices q, there exists an equilibrium with prices q if and only if q satisfies the following* ***market clearing condition****:*

$$Dq = \Pi(q, \theta^0 + \Theta^\top q, x) \left(\theta^0 + \Theta^\top q \right). \tag{6}$$

Interaction between firms and their portfolio selection rules Π_h and the asset prices q is encapsulated in the market clearing condition (6). The left-side of (6), Dq, is the vector for market capitalization, i.e., the total dollar supply for each asset, and the right-side, $\Pi(q, \theta^0 + \Theta^\top q, x) \left(\theta^0 + \Theta^\top q \right)$, is a vector whose element is the total invested wealth for each asset, i.e., the total dollar demand of investment in the market, while the market capitalization Dq is the dollar supply for investment in the market. In order to interpret Theorem 1, it is helpful to consider the following definition.

Definition 2. *Suppose we are given initial endowments (Θ, θ^0) and the risk factor x. For a vector of posted asset prices q, firms choose portfolio Π. The* ***excess dollar demand*** *is defined as follows:*

$$EDD(q) = \Pi(q, \theta^0 + \Theta^\top q, x) \left(\theta^0 + \Theta^\top q \right) - Dq. \tag{7}$$

$EDD(q)$ is the difference between market capitalization and the total dollar demand for each asset. It is clear that q satisfies the market clearing condition in equation (6) if and only if the excess dollar demand $EDD(q) = 0$.

3. Asset Price Contagion

In this section, we wish to understand how various forms of exogenous shocks propagate through the network to generate endogenous asset price changes. We define the vector of price sensitivity with respect to changes in x as the total derivative $\frac{dq}{dx} \in \mathbb{R}^{|\mathcal{A}|}$. We show that the price sensitivity can be decomposed into a two components that contribute to price movement and characterize the contagion effects.

Let $\bar{\Theta} \triangleq D^{-1}\Theta \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ denote the number of shares Θ held by the firms normalized by the total number of outstanding shares. For each asset $i \in \mathcal{A}$, define

$$G^i \triangleq \begin{bmatrix} \frac{\partial \Pi_{i1}}{\partial q_1} & \frac{\partial \Pi_{i2}}{\partial q_1} & \cdots & \frac{\partial \Pi_{im}}{\partial q_1} \\ \frac{\partial \Pi_{i1}}{\partial q_2} & \frac{\partial \Pi_{i2}}{\partial q_2} & \cdots & \frac{\partial \Pi_{im}}{\partial q_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \Pi_{i1}}{\partial q_n} & \frac{\partial \Pi_{i2}}{\partial q_n} & \cdots & \frac{\partial \Pi_{im}}{\partial q_n} \end{bmatrix} \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|} \tag{8}$$

as the sensitivity of portfolio Π_{ih} to price vector q. Define $H \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ as

$$H \triangleq \begin{bmatrix} (G^1 w)^\top \\ (G^2 w)^\top \\ \vdots \\ (G^n w)^\top \end{bmatrix}, \tag{9}$$

where $w \in \mathbb{R}^{|\mathcal{F}|}$ denotes the vector of wealth of the firms. From the definition, it follows that

$$H_{ij} = [(G^i w)^\top]_j = \sum_{h=1}^{|\mathcal{F}|} \frac{d\Pi_{ih}}{dq_j} w_h. \tag{10}$$

Thus, the matrix H can be interpreted as the wealth-weighted cross-asset portfolio sensitivity matrix. Define $\Omega \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}$ as

$$\Omega \triangleq \begin{bmatrix} \frac{\partial \Pi_{11}}{\partial w_1} w_1 & \frac{\partial \Pi_{12}}{\partial w_2} w_2 & \cdots & \frac{\partial \Pi_{1|\mathcal{F}|}}{\partial w_{|\mathcal{F}|}} w_{|\mathcal{F}|} \\ \frac{\partial \Pi_{21}}{\partial w_1} w_1 & \frac{\partial \Pi_{22}}{\partial w_2} w_2 & \cdots & \frac{\partial \Pi_{2|\mathcal{F}|}}{\partial w_{|\mathcal{F}|}} w_{|\mathcal{F}|} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \Pi_{|\mathcal{A}|1}}{\partial w_1} w_1 & \frac{\partial \Pi_{|\mathcal{A}|2}}{\partial w_2} w_2 & \cdots & \frac{\partial \Pi_{|\mathcal{A}||\mathcal{F}|}}{\partial w_{|\mathcal{F}|}} w_{|\mathcal{F}|} \end{bmatrix}. \tag{11}$$

Then Ω is the wealth-weighted firm-asset portfolio sensitivity matrix, i.e., the component Ω_{ih} is the sensitivity of the portfolio Π_h to the wealth w_h weighted by the wealth. The main result of this section is the following theorem, whose proof can be found in Appendix A.

Theorem 2. *Suppose (q, Θ, θ^0) is a given equilibrium, and the risk factor is x_0. Assume that $\Pi(q(x), w(\theta^0(x), q(x)), x)$ is a continuously differentiable function of q, w, and x, and $\theta^0(x)$ is a continuously differentiable function of x. Suppose the matrix $\left[I - (\Pi + \Omega)\bar{\Theta}^\top - HD^{-1}\right]$ is invert-*

ible at $x = x_0$. Then there exists a neighborhood $N(x_0)$ of x_0 and a unique price function $q(x)$ on $N(x_0)$ such that for all $x \in N(x_0)$, $q(x)$ satisfies the market clearing condition,

$$Dq(x) \quad = \quad \Pi(q(x), w(\theta^0(x), q(x)), x) \left(\theta^0(x) + \Theta^\top q(x)\right), \tag{12}$$

and we have that

$$D\frac{dq}{dx} = \left[I - (\Pi + \Omega)\,\bar{\Theta}^\top - HD^{-1}\right]^{-1} \left[(\Pi + \Omega)\,\frac{d\theta^0}{dx} + \frac{\partial \Pi}{\partial x} w\right]. \tag{13}$$

The main result of Theorem 2 is to characterize the price sensitivity dq/dx to the exogenous risk factor x. A detailed interpretation of this price sensitivity is provided in the following section.

3.1. Direct Effects and Network Effects

In this section, we interpret Theorem 2 to better understand price changes. First, we define two concepts.

Definition 3 (Direct Effects). *The direct effects are defined as the vector*

$$[DE] \triangleq (\Pi + \Omega)\,\frac{d\theta^0}{dx} + \frac{\partial \Pi}{\partial x} w. \tag{14}$$

Definition 4 (Network Effects). *The network effects are defined as the matrix*

$$[NE] \triangleq \left[I - \Pi\bar{\Theta}^\top - \Omega\bar{\Theta}^\top - HD^{-1}\right]^{-1}. \tag{15}$$

Given these definitions, we can re-write the price sensitivity equation (13) as follows:

$$D\frac{dq}{dx} = [NE] \cdot [DE]. \tag{16}$$

In what follows, we will see that the direct effects capture the immediate response from the financial system when facing the external shocks, while the network effects characterize the follow-on effects through the interactions within the financial network to eventually produce the endogenous price changes.

Suppose we keep the posted price q fixed when we shock the risk factor according to $x' = x + \Delta x$, where Δx is a small external shock on x. Suppose $w = \theta^0(x) + \Theta^\top q$ is the wealth vector. When the conditions in Theorem 2 are satisfied, the excess dollar demand is given by

$$\begin{aligned}
EDD(q,x') &= \Pi(q,w(\theta^0(x')),x')\left(\theta^0(x')+\Theta^\top q\right)-Dq \\
&= \left[(\Pi+\Omega)\frac{d\theta^0}{dx}+\frac{\partial\Pi}{\partial x}w\right](x'-x)+\Pi(q,w(\theta^0(x)),x)\left(\theta^0(x)+\Theta^\top q\right) \\
&\quad -Dq+o(\Delta x) \\
&= \left[(\Pi+\Omega)\frac{d\theta^0}{dx}+\frac{\partial\Pi}{\partial x}w\right]\Delta x+o(\Delta x) \\
&= [DE]\Delta x+o(\Delta x),
\end{aligned}$$

where the third equality follows from the fact that $EDD(q,x)=0$ as market clears in equilibrium corresponding to the original value x for the risk factor. Therefore, when the posted prices are held constant, the excess dollar demand in the system proportional to the vector of direct effects. Thus, the direct effects quantify the immediate change in the dollar supply and demand for the assets across the market that results in an imbalance in the market clearing condition. If there were an outside investor who could to absorb this excess dollar demand, the posted prices could indeed remain unchanged.

Next, we describe the vector of direct effects $[DE]$ for a variety of shocks.

Example 2 (Cash Shock). *Suppose firm g has a shock on its initial cash position, i.e., the risk factor takes the form $x=\theta_g^0$. In this case, $[DE]=\Pi_g$. In other words, the direct effect $[DE]$ of this shock is given by the portfolio of firms.*

Example 3 (Changes in Beliefs). *Suppose firm g has a belief that the mean future payoff of asset k is μ_{kg}. Suppose the risk factor $x=\mu_{kg}$. Then changes in the belief of firm g with the regards to the mean payoff is equivalent to a shock on x. In this case, $[DE]=\frac{\partial\Pi_g}{\partial\mu_{kg}}w_g$.*

Example 4 (Changes in Regulation). *Suppose regulatory restrictions limit the level of leverage for firms to be below a fixed parameter $L\in\mathbb{R}_+$. We can model the impact of a change in regulation by setting the risk factor $x=L$. Then a change in regulation is equivalent to a shock on x. In this case, $[DE]=\sum_{g\in\mathcal{F}}\frac{\partial\Pi_g}{\partial L}w_g$.*

The direct effects quantifies the excess dollar demand of immediately after a shock; the network effects describe the way the excess demands subsequently propagate and the asset prices converge to a new equilibrium. In other words, asset-firm interactions and contagion within the system are characterized in the formula (15) for $[NE]$. We identify three key components, $\Pi\bar{\Theta}^\top$, $\Omega\bar{\Theta}^\top$ and HD^{-1}.

- $\Pi\bar{\Theta}^\top\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ characterizes the holding-induced cross-asset interaction through Π and $\bar{\Theta}$, with Π been fixed. It indicates how the excess demand of asset i is affected by the excess demand of asset j through the fixed portfolios of all firms. In particular,

$$(\Pi\bar{\Theta}^\top)_{ij}=\sum_{h\in\mathcal{F}}\Pi_{ih}\bar{\Theta}_{jh}.$$

This component emphasizes the effect of portfolio re-balancing.

- $\Omega\bar{\Theta}^\top \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ characterizes how the excess demand of asset i is affected by the excess demand of asset j through the portfolio sensitivity with respect to wealth. In particular,

$$(\Omega\bar{\Theta}^\top)_{ij} = \sum_{h\in\mathcal{F}} \frac{\partial \Pi_{ih}}{\partial w_h}\bar{\Theta}_{jh} w_h.$$

  This component emphasizes the impact of a change in wealth.

- $HD^{-1} \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ is the normalized wealth-weighted cross-asset portfolio elasticity. It characterizes how the excess demand of asset i is affected by the excess demand of asset j through the portfolio sensitivity with respect to price. In particular,

$$(HD^{-1})_{ij} = \sum_{h\in\mathcal{F}} \frac{\partial \Pi_{ih}}{\partial q_j}\frac{w_h}{D_{jj}}.$$

In the following sections, we further examine the network effects as flows in bipartite networks, quantify the effect, and show how to design a network to minimize the network effects.

3.2. Bipartite Structure of Financial Networks

Firms often set portfolio targets Π_h and re-balance to the target portfolio when the proportions of wealth invested in the assets changes because of changes in prices. Thus, when a firm targets are certain portfolio the partial derivatives $\frac{\partial \Pi_{ih}}{\partial q_j}$ and $\frac{\partial \Pi_{ih}}{\partial w_h}$ are identically equal to zero. Therefore, $\Omega = \mathbf{0}$ and $H = \mathbf{0}$, and the network effects matrix takes the form

$$[NE] = [I - \Pi\bar{\Theta}^\top]^{-1}.$$

In the rest of this paper we will assume that all the firms in the economy employ portfolio tracking strategies.

When the spectral radius $\rho(\Pi\bar{\Theta}^\top)$ of the matrix $\Pi\bar{\Theta}^\top$ is strictly less than 1, the network effects $[NE]$ can be expanded into a infinite power series.

$$[NE] = \sum_{t=0}^{\infty}\left(\Pi\bar{\Theta}^\top\right)^t = I + \Pi\bar{\Theta}^\top + \left(\Pi\bar{\Theta}^\top\right)^2 + \dots. \tag{17}$$

This power series can be interpreted as follows.

Definition 5. *The interaction between asset prices and the firms' portfolios can be modeled by a weighted directed bipartite graph with vertex set $\mathcal{V} = \mathcal{A}\cup\mathcal{F}$ and edge set $\mathcal{E} = \mathcal{E}_1\cup\mathcal{E}_2$, where $\mathcal{E}_1$ denotes set of asset-to-firm edges, and $\mathcal{E}_2$ denotes the set of firm-to-asset edges. The edge weights are given by*

- $w_{jh} = \bar{\Theta}_{jh}$ *for edge $(j,g)\in\mathcal{E}_1$ that links asset j to firm h,*

- *$w_{hi} = \Pi_{ih}$ for edge $(h, i) \in \mathcal{E}_2$ that links firm h to asset i,*

We will call this graph the contagion graph for the network.

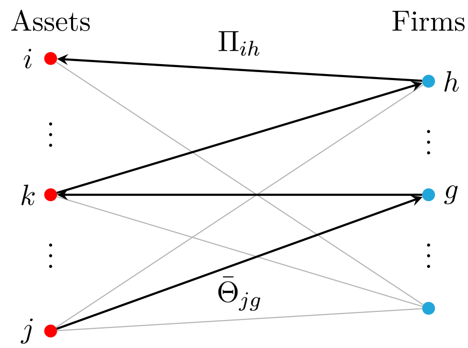

Figure 1: The Contagion Graph

A directed path from asset j to asset i in the contagion graph is a sequence of edges that connect a sequence of nodes, with the starting node being asset j and the ending node being asset i. The length of a directed path is the number edges in the path. A path between two assets in the contagion graph represents a pathway of cascading flow, through which shocks can be propagated. The contribution of a particular path to overall contagion is the product of the weights of all edges along the path. The terms of the power series in equation (17) can be interpreted as follows:

- I is the identity matrix, which captures the direct effects of the shock.
- $\Pi\bar{\Theta}^\top$ is the first-level cross-asset impact matrix. Each element $(\Pi\bar{\Theta}^\top)_{ij}$ is the impact that the excess demand of asset i has on asset j via all firms whose initial endowments consist of asset i and who invest in asset j.
- $\left(\Pi\bar{\Theta}^\top\right)^2$ is the second-level cross-asset impact matrix. Each element $\left(\Pi\bar{\Theta}^\top\right)^2_{ij}$ characterizes the impact that the excess demand of asset i has on asset j via all possible intermediate assets. Now the contagion has a second-level impact.
- $\left(\Pi\bar{\Theta}^\top\right)^t$ is the t-th-level cross-asset impact matrix for any $t = 3, 4, \ldots$. Each element $\left(\Pi\bar{\Theta}^\top\right)^t_{ij}$ characterizes the impact that the excess demand of asset i has on asset j via all possible paths with length $2t$. Now the contagion has a t-th-level impact.

To summarize, for any two assets in the financial network, the contagion effect from one to the asset can be characterized by the aggregate impact of all possible paths with all possible lengths in the network between these two assets. The intensity of the contagion is the sum of all the impact matrices of all levels according to equation (17).

In the literature, there are many articles that investigate related financial networks. For example, Eisenberg and Noe (2001) discuss the interbank lending networks. Elliott et al. (2013) provide a model for cross-holdings among institutions. While most of the existing literature emphasizes cross-firm interactions through debt obligation or institution-wise cross-holdings, the financial networks discussed in this paper characterize the bipartite connection between firms and tradable assets via holding portfolios of assets by firms and portfolios re-balancing.

For the rest of this paper, we will continue to focus on the contagion graph introduced in this section. Recall that this contagion graph is is consequence of the fact that all firms are assumed to execute a portfolio tracking strategy.

4. Network Amplifier, Holding Network, and Leverage

In this section, we introduce the network amplifier to quantify the scale of the contagion effects.

Definition 6 (Maximal Network Amplifier). *Suppose (q, Θ, θ^0) is a given equilibrium, and the risk factor is x_0. Assume that $\Pi(x)$ is a continuously differentiable function of x, and the matrix $\left[I - \Pi\bar{\Theta}^\top\right]$ is invertible at $x = x_0$. The network amplifier is defined by*

$$\mathsf{MNA} \triangleq \rho([NE]) = \rho([I - \Pi\bar{\Theta}^\top]^{-1}), \tag{18}$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix.

In what follows, we will see that under mild technical conditions, $[NE]$ has real eigenvalues. From straightforward linear algebra, it follows that

$$\mathsf{MNA} = \max_{\|[DE]\|_2 \leq 1} \|[NE] \cdot [DE]\|_2.$$

Therefore, $\|[NE] \cdot [DE]\|_2 \leq \mathsf{MNA}\|[DE]\|_2$, i.e., the network amplifier MNA is a worst-case bound on the extent to which the network amplifies the direct effects.

4.1. Holding Network and Leverage

In order to understand the key drivers of contagion, we will decompose the structure of an economy into two components. One is the leverage of firms, and the other is the configuration of the interactions of firms and assets. The goal is to separately describe the impact of network configuration and leverage on the overall contagion effects.

Let b_h denote proportion of net wealth invested by firm h in risky assets. Then

$$b_h = \sum_{i \in \mathcal{A}} \Pi_{ih}, \quad \forall\ h \in \mathcal{F}.$$

We require that $b_h > 0$ so that firm h has a positive position in risky assets. When $b_h > 1$, firm h is borrowing cash to invest in risky asset. Thus, firm h is leveraged with a b_h-to-1 ratio. The leverage vector across firms is denoted by $b \in \mathbb{R}^{|\mathcal{F}|}$.

Define $X \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{F}|}$ by setting

$$X_{ih} = \frac{\Pi_{ih}}{b_h}, \quad \forall\ i \in \mathcal{A},\ h \in \mathcal{F}.$$

Each element X_{ih} is the ratio of the wealth invested by firm h in asset i to the wealth invested by the firm in all risky assets. Therefore, $\sum_{i \in \mathcal{A}} X_{ih} = 1$ for any h, and

$$\Pi = X \operatorname{diag}(b). \tag{19}$$

We refer to X as the *holding network*. Thus, we have two ways to represent the portfolio choices of firms: one is through Π matrix, and the other is via the holding network X and the leverage vector b. In this setting, we will simplify the expression for the maximal network amplifier MNA. Specifically, we assume that the initial endowment (θ_0, Θ) is in equilibrium, i.e., we assume that the firms have traded in the past, and are now holding portfolios that they do not wish to alter. Since Π and Θ_{ih} describe the same portfolio at prices q, we have that

$$\Theta_{ih} q_i = \Pi_{ih} w_h, \quad \forall\ i\ \forall\ h, \tag{20}$$

where $w \triangleq \theta^0 + \Theta^\top q$. The proof of the following result is in Appendix A.

Theorem 3. *Suppose (q, Θ, θ^0) is a given equilibrium, with the risk factor $x = x_0$. Assume that the conditions of Theorem 2 hold, and $H = \mathbf{0}$ and $\Omega = \mathbf{0}$. Suppose the initial endowment is in equilibrium and satisfies equation (20), then we have the following results.*

(i)

$$\Pi \bar{\Theta}^\top = X \operatorname{diag}(b) \operatorname{diag}(w) \operatorname{diag}(b) X^\top M^{-1} \triangleq Y(X),$$

where $M = D \operatorname{diag}(q)$ is a diagonal matrix, where the diagonal entries are the total market capitalization of each asset.

(ii) The eigenvalues of $Y(X)$ are real and non-negative.

(iii) The maximal network amplifier is given by

$$\mathsf{MNA} = \max_{1 \le i \le |\mathcal{A}|} |1 - \lambda_i(Y(X))|^{-1}, \tag{21}$$

where each λ_i is the i-th eigenvalue of the matrix.

Theorem 3 links the magnitude of the network effects to the eigenvalues of a matrix related to the holding network X, the leverage b and the wealth w. In the following section, we will investigate how these two components contribute to contagion through the analysis of the maximal network amplifier.

5. Optimal Network Design

In this section we investigate the structure of the holding network that minimizes risk contagion for a given economy consisting of a set of firms and a set of assets. Concretely, we characterize the holding networks that minimize upper bounds on MNA, among all networks that are consistent with some aggregate economic parameters of the network. There are two holding networks with extreme structures that are of interest. One is the mutual fund economy where all firms invest in the same market portfolio, and the firms benefit from diversification across the assets that provides a buffer for shocks. The other extreme network is an isolated economy where firms hold diverse portfolios. This structure limits contagion by isolating firms. In what follows, we show that when the economy has low systemic leverage, the mutual fund economy is desirable holding network, and for an economy with high systemic leverage, the isolated structure is desirable.

5.1. Economies and Feasible Holding Networks

In our model, an economy is characterized by the following four aggregate quantities:

Definition 7 (Economy). *Suppose $D \in \mathbb{R}^{\mathcal{A}\times\mathcal{A}}$ is a diagonal matrix where diagonal entries are the numbers of shares for assets, $q \in \mathbb{R}^{\mathcal{A}}$ denotes asset prices, $w \in \mathbb{R}^{\mathcal{F}}$ denotes wealth of firms, and $b \in \mathbb{R}^{\mathcal{A}}$ denotes leverage of firms. The quantity (D, q, w, b) defines an economy if*

$$\mathbf{1}_{\mathcal{A}}^{\top} D q = w^{\top} b. \tag{22}$$

The condition (22) equates the total market capitalization $\mathbf{1}_{\mathcal{A}}^{\top} D q$ of risky assets with the total leveraged wealth $w^{\top} b$ of all the firms that is invested in risky assets. Note that Definition 7 does not specify the holding network X between the assets and firms in the financial system. The following definition characterizes the feasible holding networks for a given economy.

Definition 8 (Feasible Holding Networks). *Suppose we are given an economy (D, q, w, b). We denote the set of feasible holding networks by $\mathcal{X}$, defined as*

$$\mathcal{X} = \left\{ X \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|} \colon Dq = X \operatorname{diag}(b) w,\ \mathbf{1}_{\mathcal{A}}^{\top} X = \mathbf{1}_{\mathcal{F}}^{\top}, X \geq 0 \right\}.$$

Note that the condition $Dq = X \operatorname{diag}(b) w$ is just a re-formulation of the market clearing condition that ensures that dollar supply and demand for each of the assets is balanced. Also, $\mathbf{1}_{\mathcal{A}}^{\top} D q = w^{\top} b$ if and only if $\mathcal{X}$ is non-empty. To see this, suppose $\mathcal{X}$ is non-empty, and let $X \in \mathcal{X}$. Then $\mathbf{1}_{\mathcal{A}}^{\top} D q = \mathbf{1}_{\mathcal{A}}^{\top} X \operatorname{diag}(b) w = \mathbf{1}_{\mathcal{F}}^{\top} \operatorname{diag}(b) w = w^{\top} b$. Conversely, define $\bar{X} = \frac{1}{v} m \mathbf{1}_{\mathcal{F}}^{\top}$ where $m = Dq$ and $v = 1_{\mathcal{A}}^{\top} D q = w^{\top} b$. Then $\mathbf{1}_{\mathcal{A}}^{\top} D q = w^{\top} b$, implies that $\bar{X} \in \mathcal{X}$.

5.2. Optimal Holding Networks

Suppose we are given an economy. We are interested in a holding network that keeps a minimum level of systemic risk and contagion. More concretely, we want to solve

$$\min_{X \in \mathcal{X}} \mathsf{MNA}(X) = \min_{X \in \mathcal{X}} \max_{i \in \mathcal{A}} |1 - \lambda_i(Y(X))|^{-1}, \tag{23}$$

where the second formulation follows from Theorem 3. For the purpose of solving (23), we introduce two quantities:

$$\underline{\lambda_{max}} \triangleq \min_{X \in \mathcal{X}} \lambda_{max}(Y(X)), \tag{24}$$

$$\overline{\lambda_{min}} \triangleq \max_{X \in \mathcal{X}} \lambda_{min}(Y(X)). \tag{25}$$

The following theorem relates these quantities:

Theorem 4.

$$\overline{\lambda_{min}} \leq \underline{\lambda_{max}}. \tag{26}$$

The proof can be found in the Appendix A. Based on $\underline{\lambda_{max}}$ and $\overline{\lambda_{min}}$, we define two economic regimes that we show present different features of contagion.

Definition 9 (Leverage Regimes). *Suppose we are given an economy (D, q, w, b). We call the economy a low-leverage economy if $\underline{\lambda_{max}} < 1$. We will call it a a high-leverage economy if $\overline{\lambda_{min}} > 1$.*

Theorem 4 guarantees that the set of low-leverage economies is disjoint from the set of high-leverage economies. The reason why these two economic regimes are named low-leverage and high-leverage respectively will become clear after we introduce a parameter that we call systemic leverage. We show that the definitions are closely related to the scale of leverage of the economy.

Definition 10 (Systemic Leverage). *Suppose we are given an economy (D, q, w, b). We define the systemic leverage γ of the economy as*

$$\gamma = \frac{\sum_{h \in \mathcal{F}} w_h b_h^2}{\sum_{h \in \mathcal{F}} w_h b_h}.$$

Define $\bar{w}_h = \frac{w_h}{\sum_{h \in \mathcal{F}} w_h}$. Then $\bar{w}$ can be interpreted as a probability distribution, γ can be rewritten as

$$\gamma = \frac{\mathsf{E}_{\bar{w}}[b_h^2]}{\mathsf{E}_{\bar{w}}[b_h]} = \mathsf{E}_{\bar{w}}[b_h] \left(1 + \frac{\mathrm{Var}_{\bar{w}}[b_h]}{\mathsf{E}_{\bar{w}}[b_h]^2}\right),$$

where $\mathsf{E}_{\bar{w}}[b_h]$ is the mean leverage, and $\frac{\mathrm{Var}_{\bar{w}}[b_h]}{\mathsf{E}_{\bar{w}}[b_h]^2}$ is the squared coefficient of variation of the leverage under the probability mass function $\bar{w}$. This relation implies that the systemic leverage γ is larger than the average wealth-weighted leverage by a factor that depends on the variability of leverage across firms. In other words, γ is large if the average wealth-weighted leverage is high. And for a fixed average leverage is fixed, γ is larger when the coefficient of variation of the wealth-weighted leverage is higher. The following result proved in Appendix A links $\underline{\lambda_{max}}$ with γ.

Theorem 5. *In any economy,*

$$\underline{\lambda_{max}} = \gamma. \tag{27}$$

Theorem 5 implies that an economy is low-leverage whenever the systemic leverage $\gamma < 1$. In what follows, we will demonstrate analytical results for low-leverage and high-leverage economies. However, there exist economies that are neither low-leverage nor high-leverage. Establishing optimal holding networks for economies with intermediate leverage requires one to overcome technical challenges that our current model is not able to accommodate. The main purpose of the model in this paper is not to provide analytical results that cover the entire spectrum of possible economic states; rather to establish the counter-intuitive fact that the optimal structures of financial networks present completely different features for the low-leverage and high-leverage economies. In the following sections, we solve(23) for the low-leverage and the high-leverage economies.

5.3. Low-Leverage Economy

The following theorem establishes a bound for maximal network amplifier for low-leverage economy, and characterizes the network that achieves the bound. The proof is in the Appendix A.

Theorem 6. *Suppose we are given a low-leverage economy. Then*

$$\min_{X \in \mathcal{X}} \mathsf{MNA}(X) \leq \frac{1}{1 - \underline{\lambda_{max}}}, \tag{28}$$

and the upper bound is achieved by the mutual-fund *economy*

$$X^* \triangleq \frac{Dq}{\mathbf{1}_{\mathcal{A}}^\top Dq} \mathbf{1}_{\mathcal{F}}^\top,$$

where each firm invests in the market portfolio $\frac{Dq}{\mathbf{1}_{\mathcal{A}}^\top Dq}$ *where assets are held in proportion to the market capitalization of assets.*

Therefore, a desirable holding network for the low leverage economy is one where each firm invests into a common market portfolio. In this case, the network drives all the firms to diversify their investment across assets proportional to the market capitalization of assets.

Now we consider another special economy that presents interesting features. When firms have uniform leverage levels, the following theorem provides an optimal holding network. The proof is in the Appendix A.

Theorem 7. *Suppose in a low-leverage economy, every firm has a constant leverage level* $0 < \bar{b} < 1$*, and* $b = \bar{b}\mathbf{1}_{\mathcal{F}}$*. Then the maximal network amplifier* $\mathsf{MNA}(X)$ *does not depend on the configuration of the holding network* X*.*

Note that in Theorem 7 refers to the $\mathsf{MNA}(X)$ and not the bound established in Theorem 6.

5.4. High-Leverage Economy

The following result establishes a bound on the minimum achievable MNA for a high-leverage economy. The proof can be found in the Appendix A.

Theorem 8. *Suppose we are given a high-leverage economy. The minimum value of the maximal network amplifier is bounded as follows*

$$\min_{X \in \mathcal{X}} \mathsf{MNA}(X) \leq \frac{1}{\overline{\lambda_{min}} - 1}. \tag{29}$$

In the rest of this section, we characterize the networks that achieve the upper bound $(\overline{\lambda_{\min}} - 1)^{-1}$ in some special cases. In these special cases we show that the bound is achieved by holding networks where firms are isolated in the sense that they invest in a diverse set of assets.

5.4.1. Completely Symmetric Case

In this section, we examine a special case where firms are symmetric and assets are also symmetric. Again, our objective is to solve for the optimal network X that minimizes the upper bound in (29). The following theorem solves the optimization problem. The proof can be found in the Appendix A.

Theorem 9. *Suppose the leverage is* $b = \bar{b}\mathbf{1}_{\mathcal{F}}$ *where* $\bar{b} > 1$*, wealth is* $w = \bar{w}\mathbf{1}_{\mathcal{F}}$*, and the market capitalization is* $m = \bar{m}\mathbf{1}_{\mathcal{A}}$*. We also assume that* $|\mathcal{F}| = n|\mathcal{A}|$*, where* $n \in N$*. Then the upper bound on* (29) *is achieved by the holding network*

$$X^* = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}; \tag{30}$$

specifically,

$$X^*_{ih} = \begin{cases} 1, & \text{if } (i-1)n + 1 \leq h \leq in, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

The holding network X^* in (31) has a special structure.In X^*, each firm only invests in one asset. So the holding network is partitioned into separate components, where each component corresponds to a group of firms that invest in a common asset. For high-leverage economy, this isolated holding network is desirable because it achieves the bound on the MNA. We refer to holding networks of the form X^* defined in (31) as **fully isolated holding networks**.

5.4.2. General Case

In the symmetric case, we were able to show that the fully isolated holding network defined in (31) achieves the bound in (29). However, for general parameters D, q, w, and b, it is usually impossible to construct a fully isolated holding network. In this section, we construct a network very similar to a fully isolated network, and show that this network achieves the upper bound in (29) in an appropriately defined asymptotic regime.

We define our candidate network by suitably extending the construction for the symmetric case. We first describe the extension and then an asymptotic regime where the construction is asymptotically feasible and asymptotically optimal for the bound in (29). Let $\mathcal{P} = (\cup_{i\in\mathcal{A}}\mathcal{F}_i) \cup \mathcal{R}$ denote any partition of the set of firms $\mathcal{F}$. We assume that for all $i = 1, \ldots, |\mathcal{A}|$, all the firms in the set $\mathcal{F}_i$ invest in asset i, whereas the firms in the set $\mathcal{R}$ are allowed to invest in *all* assets. Let $m_i = D_{ii}q_i$ denote the market capitalization of asset i for all $i \in \mathcal{A}$. Let $m'_i = m_i - \sum_{h\in\mathcal{F}_i} w_h b_h$, denote the gap between the market capitalization m_i of asset i and the part that is covered by firms in $\mathcal{F}_i$. We assume that the firms in the set $\mathcal{R}$ hold a common portfolio that covers the mismatch m'. Thus, we obtain the holding network

$$X(\mathcal{P}) = \begin{pmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & \alpha m'_1 & \alpha m'_1 & \ldots & \alpha m'_1 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & 0 & \ldots & 0 & \alpha m'_2 & \alpha m'_2 & \ldots & \alpha m'_2 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & \ldots & 1 & \alpha m'_{|\mathcal{A}|} & \alpha m'_{|\mathcal{A}|} & \ldots & \alpha m'_{|\mathcal{A}|} \end{pmatrix}, \tag{32}$$

where $\alpha = 1/\mathbf{1}_{|\mathcal{A}|}^\top m'$. Specifically,

$$X_{ih}(\mathcal{P}) = \begin{cases} 1, & \text{if } h \in \mathcal{F}_i, \\ \alpha m'_i, & \text{if } h \in \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

It can be easily verified that the construction of $X(\mathcal{P})$ satisfies that $Dq = X\,\mathrm{diag}(b)w$, and $\mathbf{1}_{\mathcal{A}}^\top X(\mathcal{P}) = \mathbf{1}_{\mathcal{F}}^\top$. But whether $X(\mathcal{P}) \geq 0$ depends on the choice of the partition $\mathcal{P}$. In general, $X(\mathcal{P})$ may not be a feasible holding network. In what follows, we show that there exists an asymptotic regime in which asymptotically $X(\mathcal{P})$ is a feasible and achieves the bound (29).

Consider a setting where the parameters that describe the economy are randomly drawn from given distributions. Suppose for each firm h, the leverage and the wealth of the firm (b_h, w_h) is an i.i.d. random variable drawn from a fixed joint distribution for (b_h, w_h) that has finite fourth order moments, and $\bar{\gamma} = \frac{E[w_h b_h^2]}{E[w_h b_h]} > 1$. The market capitalization

$$m_i = \bar{m}_i \sum_{h \in \mathcal{F}} w_h b_h, \quad \forall\ i \in \mathcal{A},$$

where $\bar{m} \in \mathbb{R}^{|\mathcal{A}|}$ is a given fixed vector with $\bar{m} \geq 0$, and $\sum_{i \in \mathcal{A}} \bar{m}_i = 1$. In the following theorem, we will show that the construction of $X(\mathcal{P})$ inspired by the isolated holding network is asymptotically feasible and asymptotically achieves the upper bound in (29). The complete proof of Theorem 10 can be found in Appendix A.

Theorem 10. *Let $A = |\mathcal{A}|$ and $F = |\mathcal{F}|$. Suppose A is fixed and we let $F \to +\infty$. For each fixed F, we assume that the parameters (b, h, m) characterizing the economy are generated by the process described above. Then there exists a sequence of partitions $\mathcal{P}^{(F)}$ of $\mathcal{F}$ such that the sequence of holding networks $X(\mathcal{P}^{(F)})$ defined by (33) are asymptotically feasible, and achieve the bound (29) in the limit, i.e., $X(\mathcal{P}^F) \in \mathcal{X}^{(F)}$, for all but finitely many F, almost surely, and $\lambda_{\min}(Y(X(\mathcal{P}^F))) - \min_{X \in \mathcal{X}^{(F)}} \lambda_{min}(Y(X)) \to_P 0$.*

Proof Sketch. To prove the theorem, we construct the sequence of partitions $\mathcal{P}^F$ of $\mathcal{F}$. This theorem has two parts to prove. The first part is to prove that the sequence of $X(\mathcal{P}^F)$ is asymptotically feasible. The second part is to prove that the sequence of $X(\mathcal{P}^F)$ is asymptotically optimal for the bound (29).

Let $\mathcal{P}^F = \{\mathcal{F}_i\}_{i \in \mathcal{A}} + \mathcal{R}$. denote the partition for $\mathcal{F}$. IOnly the cardinality $|\mathcal{F}_i|$ of $\mathcal{F}_i$ matters for the proof, therefore we first select the cardinality of $\mathcal{F}_i$ for each $i \in \mathcal{A}$.

$$|\mathcal{F}_i| = \max\{\lceil \bar{m}_i F \rceil - 1 - \lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor, 0\}. \tag{34}$$

where $\epsilon \in (0, \frac{1}{2})$ is a small number that is arbitrarily chosen and fixed. It is clear that $|\mathcal{F}_i| < \bar{m}_i F$, so $\sum_{i \in \mathcal{A}} |\mathcal{F}_i| \leq F$. For each $i \in \mathcal{A}$, we define

$$\begin{array}{rcl} \mathcal{F}_i & = & \{h \in \mathcal{F} \colon \sum_{j=1}^{i-1} |\mathcal{F}_j| + 1 \leq h \leq \sum_{j=1}^{i} |\mathcal{F}_j|\}, \\ \mathcal{R} & = & \mathcal{F} \setminus \bigcup_{i \in \mathcal{A}} \mathcal{F}_i. \end{array} \tag{35}$$

Now based on the partition $\mathcal{P}^F$, we can construct $X(\mathcal{P}^F)$ according to equation (33). With this particular construction, we show that $|\mathcal{F}_i| \to \bar{m}_i F$, and $\frac{|\mathcal{R}|}{F} \to 0$. The rest of proof follows from these two facts, and the finiteness of the moments of the joint distribution of (b_h, w_h) technical derivation, we can prove the theorem. More details can be found in the Appendix A. ■

Recall that all the firms in the set $\mathcal{F}_i$ invest all their leveraged wealth into asset i. The firms in the set $\mathcal{R}$ spread their leveraged wealth across all assets according to (35). We show that $\frac{|\mathcal{R}|}{F} \to 0$. Therefore, the network constructed in (34) becomes asymptotically a fully isolated holding network. This indicates that an asymptotically isolated holding network achieves the bound (29) for the general case.

6. Conclusion: Diversification versus Diversity

In this work we propose a model for asset based risk contagion. In this model, the risk contagion occurs via endogenous price movements – firms trade in reaction to changes to exogenous risk factors, these trades lead to an imbalance in the demand and supply that leads to change in prices, that then feed back into the decisions of the firms. This process continues until prices converge to a new equilibrium. We show that the change in price due to the external shocks to risk factors can be decomposed into direct effects that capture the immediate impact of the shock, and network effects that capture the subsequent amplification of due to network interactions. We define the network amplifier as the worst case amplification of direct effects by the network. In the rest of the paper is devoted to designing asset-firm network that attempt to minimize this network amplifier. We identify two different economies based on the systemic leverage. We show that the desirable asset-firm networks in the two cases demonstrate very different features. In low-leverage economies, the desirable network is given by a mutual fund economy where every firm invests in a common market portfolio. This structure of network agrees with the conventional diversification argument in risk management. The common market portfolio leads to risk pooling. However, for the high-leverage case, the desirable network is an isolated holding network where firms are partitioned into groups, and each group invests all of their leveraged wealth into only asset. This network attempts to reduce risk contagion by limiting the contagion to a given pool.

From the regulator's perspective, it is critical to determine whether the current state of the economy has high or low systemic leverage, since in the desirable holding networks from the perspective of managing systemic risk demonstrate completely different and even opposite features. We show that the state of the economy is determined by the systemic leverage γ that can be computed by only monitoring wealth and leverage of all the firms – the detailed knowledge portfolio positions of each firm are not needed for this computation. If the systemic leverage is high, the regulator can then proceed to incentivize a diversity of assets or a reduction in systemic leverage.

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A. Proofs

Theorem 1. *Suppose we are given initial endowments (Θ, θ^0) and the risk factor x. For a vector of posted asset prices q, there exists an equilibrium with prices q if and only if q satisfies the following* ***market clearing condition****:*

$$Dq = \Pi(q, \theta^0 + \Theta^\top q, x) \left(\theta^0 + \Theta^\top q\right). \tag{36}$$

Proof. Suppose q satisfies the market clearing condition in equation (6). We take

$$\hat{\theta}^0_h = \left(1 - \sum_{i\in\mathcal{A}} \Pi_{ih}(q, \theta^0 + \Theta^\top q, x)\right)\left(\theta^0_h + \sum_{i\in\mathcal{A}} \Theta_{ih} q_i\right), \quad \forall\ h \in \mathcal{F},$$

and

$$\hat{\Theta}_{ih} = \frac{1}{q_i}\Pi_{ih}(q, \theta^0 + \Theta^\top q, x)\left(\theta^0_h + \sum_{j\in\mathcal{A}} \Theta_{jh} q_j\right), \quad \forall\ i \in \mathcal{A}\ \forall\ h \in \mathcal{F}.$$

Now, for any $h \in \mathcal{F}$,

$$\begin{aligned}
& \sum_{i\in\mathcal{A}} \left(\hat{\Theta}_{ih} - \Theta_{ih}\right) q_i + \left(\hat{\theta}^0_h - \theta^0_h\right) \\
= \ & \sum_{i\in\mathcal{A}} \Pi_{ih}\theta^0_h + \sum_{i\in\mathcal{A}} \Pi_{ih} \sum_{j\in\mathcal{A}} \Theta_{jh} q_j - \sum_{i\in\mathcal{A}} \Theta_{ih} q_i \\
& + \left(1 - \sum_{i\in\mathcal{A}} \Pi_{ih}\right)\theta^0_h + \left(1 - \sum_{i\in\mathcal{A}} \Pi_{ih}\right)\sum_{j\in\mathcal{A}} \Theta_{jh} q_j - \theta^0_h \\
= \ & 0.
\end{aligned}$$

So the budget balance in equation (3) holds. According to the market clearing condition in equation (6), for each $i \in \mathcal{A}$ we have

$$D_{ii} = \frac{1}{q_i} \sum_{h \in \mathcal{F}} \Pi_{ih}(q, \theta^0 + \Theta^\top q, x) \left(\theta_h^0 + \sum_{j \in \mathcal{A}} \Theta_{jh} q_j \right) = \sum_{h \in \mathcal{F}} \hat{\Theta}_{ih}.$$

So the share balance in equation (4) holds. By using the proved result of budget balance in equation (3), for each $i \in \mathcal{A}$ and each $h \in \mathcal{F}$, we can show that

$$\hat{\Theta}_{ih} = \frac{1}{q_i} \Pi_{ih}(q, \theta^0 + \Theta^\top q, x) \left(\theta_h^0 + \sum_{j \in \mathcal{A}} \Theta_{jh} q_j \right) = \Pi_{ih}(q, \theta^0 + \Theta^\top q, x) \left(\hat{\theta}_h^0 + \sum_{j \in \mathcal{A}} \hat{\Theta}_{jh} q_j \right).$$

So the portfolio match condition in equation (5) holds. Therefore $(q, \hat{\Theta}, \hat{\theta}^0)$ established an equilibrium.

Now we suppose that there exists an equilibrium with prices q. Namely there exists a matrix $\hat{\Theta} \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{F}|}$ and a vector $\hat{\theta}^0 \in \mathbb{R}^{|\mathcal{F}|}$ such that $(q, \hat{\Theta}, \hat{\theta}^0)$ establishes an equilibrium. For each $i \in \mathcal{A}$,

$$\begin{aligned} D_{ii} q_i &= \sum_{h \in \mathcal{F}} \hat{\Theta}_{ih} q_i \\ &= \sum_{h \in \mathcal{F}} \Pi_{ih}(q, \theta^0 + \Theta^\top q, x) \left(\hat{\theta}_h^0 + \sum_{j \in \mathcal{A}} \hat{\Theta}_{jh} q_j \right) \\ &= \sum_{h \in \mathcal{F}} \Pi_{ih}(q, \theta^0 + \Theta^\top q, x) \left(\theta_h^0 + \sum_{j \in \mathcal{A}} \Theta_{jh} q_j \right). \end{aligned}$$

The first equality is because of the share balance condition in equation (4). The second equality is due to the portfolio match condition in equation (5). The last equality is because of the budget balance condition in equation (3). Therefore, the market clearing condition is proved. ■

Theorem 2. *Suppose (q, Θ, θ^0) is a given equilibrium, and the risk factor is x_0. Assume that $\Pi(q(x), w(\theta^0(x), q(x)), x)$ is a continuously differentiable function of q, w, and x, and $\theta^0(x)$ is a continuously differentiable function of x. If the matrix $\left[I - (\Pi + \Omega) \bar{\Theta}^\top - H D^{-1} \right]$ is invertible at $x = x_0$, then there exists a neighborhood $N(x_0)$ of x_0 and a unique price function $q(x)$ on $N(x_0)$ such that for all $x \in N(x_0)$, $q(x)$ satisfies the market clearing condition,*

$$D q(x) = \Pi(q(x), w(\theta^0(x), q(x)), x) \left(\theta^0(x) + \Theta^\top q(x) \right), \tag{37}$$

and we have that

$$D \frac{dq}{dx} = \left[I - (\Pi + \Omega) \bar{\Theta}^\top - H D^{-1} \right]^{-1} \left[(\Pi + \Omega) \frac{d\theta^0}{dx} + \frac{\partial \Pi}{\partial x} w \right]. \tag{38}$$

Proof. Equation (37) characterizes the relation between q and x. Since $\Pi(q(x), w(\theta^0(x), q(x)), x)$

is a continuously differentiable function of q, w, and x, and $\theta^0(x)$ is a continuously differentiable function of x, we can apply implicit function theorem to equation (37). By differentiating all the terms w.r.t. x, we can investigate price sensitivity with respect to shocks on x.

$$\Pi\left(\tfrac{d\theta^0}{dx}+\Theta^\top\tfrac{dq}{dx}\right)+\tfrac{d\Pi}{dx}w-D\tfrac{dq}{dx} \quad = \quad 0. \tag{39}$$

For each asset i, we can write out an element-wise equation for equation (39).

(40)

$$\left[\Pi\frac{d\theta^0}{dx}\right]_i+\left[\Pi\Theta^\top\frac{dq}{dx}\right]_i+\sum_{h\in\mathcal{F}}w_h\left(\frac{\partial\Pi_{ih}}{\partial x}+\sum_{j\in\mathcal{A}}\frac{\partial\Pi_{ih}}{\partial q_j}\frac{dq_j}{dx}+\frac{\partial\Pi_{ih}}{\partial w_h}\left(\Theta^\top\frac{dq}{dx}+\frac{d\theta^0}{dx}\right)\right) \quad = \quad \left[D\tfrac{dq}{dx}\right]_i,$$

where $\frac{\partial\Pi}{\partial x}\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}$ is the portfolio sensitivity w.r.t. to x when asset prices q and w are fixed. $\frac{\partial\Pi_{ih}}{\partial q_j}\in\mathbb{R}$ characterizes portfolio sensitivity w.r.t. prices q, and $\frac{\partial\Pi_{ih}}{\partial w_h}\in\mathbb{R}$ characterizes portfolio sensitivity w.r.t. wealth w.

By definition, the elements of H and Ω we identify that

$$H_{ij}=[(G^iw)^\top]_j=\sum_{h\in\mathcal{F}}\frac{\partial\Pi_{ih}}{\partial q_j}w_h,\quad \forall\, i,j\in\mathcal{A}. \tag{41}$$

$$\Omega_{ih}=\frac{\partial\Pi_{ih}}{\partial w_h}w_h,\quad \forall\, i\in\mathcal{A}\;\forall\, h\in\mathcal{F}. \tag{42}$$

Writing equation (40) in a matrix form and re-arranging terms, we obtain the following equations.

$$\begin{aligned}\Pi\tfrac{d\theta^0}{dx}+\Pi\Theta^\top\tfrac{dq}{dx}+\tfrac{\partial\Pi}{\partial x}w+H\tfrac{dq}{dx}+\Omega\Theta^\top\tfrac{dq}{dx}+\Omega\tfrac{d\theta^0}{dx} &\quad=\quad D\tfrac{dq}{dx}\\ \left[D-(\Pi+\Omega)\,\Theta^\top-H\right]\tfrac{dq}{dx} &\quad=\quad (\Pi+\Omega)\,\tfrac{d\theta^0}{dx}+\tfrac{\partial\Pi}{\partial x}w\\ \left[I-(\Pi+\Omega)\,\bar{\Theta}^\top-HD^{-1}\right]D\tfrac{dq}{dx} &\quad=\quad (\Pi+\Omega)\,\tfrac{d\theta^0}{dx}+\tfrac{\partial\Pi}{\partial x}w.\end{aligned} \tag{43}$$

For the last equality, we let $\bar{\Theta}=D^{-1}\Theta\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ denote the normalized initial portfolios. When $\left[I-(\Pi+\Omega)\,\bar{\Theta}^\top-HD^{-1}\right]$ is invertible, we can solve for $\frac{dq}{dx}$ in equation (43), the result of the theorem follows. ■

Theorem 3. *Suppose (q,Θ,θ^0) is a given equilibrium, with the risk factor $x=x_0$. Assume that the conditions of Theorem 2 hold, and $H=\mathbf{0}$ and $\Omega=\mathbf{0}$. Suppose the initial endowment is in equilibrium and satisfies equation (20), then we have the following results.*

(i)

$$\Pi\bar{\Theta}^\top=X\,\mathrm{diag}(b)\,\mathrm{diag}(w)\,\mathrm{diag}(b)X^\top M^{-1}\triangleq Y(X),$$

where $M=D\,\mathrm{diag}(q)$ is a diagonal matrix, where the diagonal entries are the total market capitalization of each asset.

(ii) The eigenvalues of $Y(X)$ are real and non-negative.

(iii) The maximal network amplifier is given by

$$\mathsf{MNA} = \max_{1\le i\le|\mathcal{A}|} |1-\lambda_i(Y(X))|^{-1}, \tag{44}$$

where each λ_i is the i-th eigenvalue of the matrix.

Proof. Each element of $\Pi\bar{\Theta}^\top$ is

$$(\Pi\bar{\Theta}^\top)_{ij} = \sum_{h=1}^{|\mathcal{F}|} \Pi_{ih}\frac{w_h\Pi_{jh}}{q_jD_{jj}} = \frac{1}{q_jD_{jj}}\sum_{h=1}^{|\mathcal{F}|}\Pi_{ih}\Pi_{jh}w_h. \quad \forall\ i\ \forall\ h$$

Now we can re-write $\Pi\bar{\Theta}^\top$ as

$$\Pi\bar{\Theta}^\top = \Pi\,\mathrm{diag}(w)\Pi^\top M^{-1} = X\,\mathrm{diag}(b)\,\mathrm{diag}(w)\,\mathrm{diag}(b)X^\top M^{-1} = Y(X).,\tag{45}$$

Since $N = X\,\mathrm{diag}(b)\,\mathrm{diag}(w)\,\mathrm{diag}(b)X^\top$ is a positive semidefinite matrix, so it has real non-negative eigenvalues. We note that $M^{-1} = D^{-1}\,\mathrm{diag}(q)^{-1}$ is a positive diagonal matrix, $L = M^{-\frac{1}{2}}NM^{-\frac{1}{2}}$ is also a positive semidefinite matrix, so it also has real non-negative eigenvalues. Since $\Pi\bar{\Theta}^\top = M^{\frac{1}{2}}LM^{-\frac{1}{2}}$ is similar to L, $\Pi\bar{\Theta}^\top = Y(X)$ has real non-negative eigenvalues.

Therefore, the maximal network amplifier is simplified to equation (21). ■

Theorem 4.

$$\overline{\lambda_{min}} \le \underline{\lambda_{max}}.\tag{46}$$

Proof. We define

$$\gamma = \frac{\sum_{h\in\mathcal{F}} w_hb_h^2}{\sum_{h\in\mathcal{F}} w_hb_h}.$$

To prove this theorem, we need to establish several facts.

$$\underline{\lambda_{max}} = \gamma,$$

and

$$\overline{\lambda_{min}} \le \gamma.$$

First, for $\underline{\lambda_{max}}$, we can show that the optimization program in equation (24) has an optimal solution $X^* = \frac{1}{v}Dq\mathbf{1}_{\mathcal{F}}^\top$ for equation (24) such that

$$\underline{\lambda_{max}} = \lambda_{max}(Y(X^*)) = \gamma$$

and

$$Y(X^*)Dq = \gamma Dq.$$

Namely, the corresponding eigenvector is Dq. To obtain this result, we re-write the original program (24) as

$$\begin{array}{lrcl}
\underset{X\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}}{\text{minimize}} & & t & \\
\text{subject to} & tI_{\mathcal{A}} - X\,\mathrm{diag}(b)\,\mathrm{diag}(w)\,\mathrm{diag}(b)X^\top M^{-1} & \succeq & \mathbf{0}, \\
& Dq & = & X\,\mathrm{diag}(b)w, \\
& \mathbf{1}_{\mathcal{A}}^\top X & = & \mathbf{1}_{\mathcal{F}}^\top, \\
& X & \geq & \mathbf{0}.
\end{array} \tag{47}$$

This program is equivalent to the following semidefinite program by using Schur complement.

$$\begin{array}{lrcl}
\underset{X\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}}{\text{minimize}} & & t & \\
\text{subject to} & \begin{pmatrix} \mathrm{diag}(w)^{-1} & \mathrm{diag}(b)X^\top M^{-\frac{1}{2}} \\ M^{-\frac{1}{2}}X\,\mathrm{diag}(b) & tI_{\mathcal{A}} \end{pmatrix} & \succeq & \mathbf{0}, \\
& Dq & = & X\,\mathrm{diag}(b)w, \\
& \mathbf{1}_{\mathcal{A}}^\top X & = & \mathbf{1}_{\mathcal{F}}^\top, \\
& X & \geq & \mathbf{0}.
\end{array} \tag{48}$$

Take $X = \frac{1}{v}Dq\mathbf{1}_{\mathcal{F}}^\top$ and $t = \gamma$. It is not hard to show that (X, t) is feasible in program (48). Now, we want to prove that $(X, t) = (\frac{1}{v}Dq, \gamma)$ is indeed the optimal solution for program (48). To prove it, we notice that the constraint $X \geq 0$ is not binding at $(X, t) = (\frac{1}{v}Dq, \gamma)$. We construct the following program that modifies program (48) by removing the constraint $X \geq 0$.

$$\begin{array}{lrcl}
\underset{X\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}}{\text{minimize}} & & t & \\
\text{subject to} & \begin{pmatrix} \mathrm{diag}(w)^{-1} & \mathrm{diag}(b)X^\top M^{-\frac{1}{2}} \\ M^{-\frac{1}{2}}X\,\mathrm{diag}(b) & tI_{\mathcal{A}} \end{pmatrix} & \succeq & \mathbf{0}, \\
& Dq & = & X\,\mathrm{diag}(b)w, \\
& \mathbf{1}_{\mathcal{A}}^\top X & = & \mathbf{1}_{\mathcal{F}}^\top.
\end{array} \tag{49}$$

The optimal value of the new program (49) is no greater than optimal value of the new program (48). We want to prove that $(X, t) = (\frac{1}{v}Dq, \gamma)$ is the optimal solution for program (49). The dual program for the primal program (49) is the following.

$$
\begin{array}{llrcl}
 & \underset{Z,\ \mu_A,\ \mu_F}{\text{maximize}} & \multicolumn{3}{l}{-\mu_A^\top Dq - \mu_F^\top b - tr(W^{-1}Z_{FF})} \\
 & \text{subject to} & Z = \begin{pmatrix} Z_{FF} & Z_{FA} \\ Z_{FA}^\top & Z_{AA} \end{pmatrix} & \succeq & 0, \\
(50) & & tr(Z_{AA}) & = & 1, \\
 & & 2Z_{FA}M^{-\frac{1}{2}} & = & w\mu_A^\top + \mu_F \mathbf{1}_A^\top, \\
 & & Z & \in & \mathbb{R}^{(|\mathcal{F}|+|\mathcal{A}|)\times(|\mathcal{F}|+|\mathcal{A}|)}, \\
 & & \mu_A & \in & \mathbb{R}^{|\mathcal{A}|},\ \mu_F \in \mathbb{R}^{|\mathcal{F}|}.
\end{array}
$$

Take $\mu_A = 0$, $\mu_F = -\frac{2}{w^\top b}Wb$ and $Z = uu^\top$, where $u^\top = (u_F^\top, u_A^\top)$,

$$u_F = -\frac{1}{\sqrt{w^\top b}}Wb,$$

and

$$u_A = \frac{1}{\sqrt{w^\top b}}\operatorname{diag}(M^{\frac{1}{2}}).$$

Notice that Z is a symmetric rank-one matrix. So Z is positive-semidefinite. Also,

$$Z_{FF} = u_F u_F^\top, \quad Z_{FA} = u_F u_A^\top, \quad Z_{AA} = u_A u_A^\top.$$

We can compute that

$$tr(Z_{AA}) = \frac{1}{w^\top b}d^\top q = 1,$$

and

$$2Z_{FA}M^{-\frac{1}{2}} = 2u_F u_A^\top M^{-\frac{1}{2}} = -\frac{2}{w^\top b}Wb\mathbf{1}_A^\top = w\mu_A^\top + \mu_F\mathbf{1}_A^\top.$$

So (μ_A, μ_F, Z) is a feasible solution for the dual program (50). The objective value is

$$-\mu_A^\top Dq - \mu_F^\top b - tr(W^{-1}Z_{FF}) = b^\top\frac{2}{w^\top b}Wb - b^\top\frac{1}{w^\top b}Wb = \gamma.$$

So far, we have demonstrated that there exists a dual feasible solution for the dual program (50), such that the dual objective value is also γ. $(X, t) = (\frac{1}{v}Dq, \gamma)$ is indeed the optimal solution for program (49). Therefore, it is also the optimal solution for program (48).

Now we prove the other fact, $\overline{\lambda_{min}} \le \gamma$.

Let $R = \operatorname{diag}(b)\operatorname{diag}(w)\operatorname{diag}(b)$. Take $T = M^{-1/2}XRX^\top M^{-1/2}$. Since T is similar to

$XRX^\top M^{-1}$, they have the same eigenvalues. We re-write the original program (25) as

$$
\begin{aligned}
&\underset{X\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}}{\text{maximize}} \quad \lambda_{min}\left(T\right) \\
&\text{subject to} \qquad Dq \;=\; X\,\text{diag}(b)w, \\
&\qquad\qquad\quad \mathbf{1}_{\mathcal{A}}^\top X \;=\; \mathbf{1}_{\mathcal{F}}^\top, \\
&\qquad\qquad\qquad X \;\geq\; \mathbf{0}.
\end{aligned} \tag{51}
$$

Recall $v = \mathbf{1}^\top Dq = w^\top b$. Notice that $T = \left(\frac{1}{v}M\right)^{-1/2} X\left(\frac{1}{v}R\right)X^\top\left(\frac{1}{v}M\right)^{-1/2}$. Let $e = \text{diag}(\left(\frac{1}{v}M\right)^{1/2})$. According to the constraints on X in program (51), we can show that $T \succeq \mathbf{0}$, $T \geq \mathbf{0}$, and

$$e^\top Te = \mathbf{1}_{\mathcal{A}}^\top X\left(\frac{1}{v}R\right)X^\top\mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{F}}^\top\left(\frac{1}{v}R\right)\mathbf{1}_{\mathcal{F}} = \gamma.$$

So we can construct the following program as a relaxed version of program (51) .

$$
\begin{aligned}
&\underset{X\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}}{\text{maximize}} \quad \lambda_{min}\left(T\right) \\
&\text{subject to} \qquad e^\top Te \;=\; \gamma, \\
&\qquad\qquad\qquad T \;\succeq\; \mathbf{0}, \\
&\qquad\qquad\qquad T \;\geq\; \mathbf{0}.
\end{aligned} \tag{52}
$$

Program (52) is equivalent to the following.

$$
\begin{aligned}
&\underset{X\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{F}|}}{\text{maximize}} \qquad t \\
&\text{subject to} \quad T - tI \;\succeq\; \mathbf{0}, \\
&\qquad\qquad\quad e^\top Te \;=\; \gamma, \\
&\qquad\qquad\qquad T \;\succeq\; \mathbf{0}, \\
&\qquad\qquad\qquad T \;\geq\; \mathbf{0}.
\end{aligned} \tag{53}
$$

The dual of the program (53) is the following.

$$
\begin{aligned}
&\underset{\beta\in\mathbb{R};S_1,S_2\in\mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}}{\text{minimize}} \qquad \beta\gamma \\
&\text{subject to} \quad Tr(S_1) \;=\; 1, \\
&\qquad\qquad\quad S_1 + S_2 \;\leq\; \beta e^\top e, \\
&\qquad\qquad\quad S_1,\ S_2 \;\succeq\; \mathbf{0}.
\end{aligned} \tag{54}
$$

One observation is that $\beta = 1$, $S_1 = ee^\top$, and $S_2 = 0$ is dual feasible for program (54), since $Tr(ee^\top) = \sum_{i\in\mathcal{A}}\frac{D_{ii}q_i}{v} = 1$. Then we know that the optimal value of program (54) is no greater than γ. Since program (54) is a relaxation of program (51), $\overline{\lambda_{min}} \leq \gamma$. Now the result of this theorem follows immediately.

■

Theorem 5. *In any economy,*

$$\underline{\lambda_{max}} = \gamma. \tag{55}$$

Proof. This result follows immediately from the proof of Theorem 4. ■

Theorem 6. *Suppose we are given a low-leverage economy. Then*

$$\min_{X \in \mathcal{X}} \mathsf{MNA}(X) \leq \frac{1}{1 - \underline{\lambda_{max}}}, \tag{56}$$

and the upper bound is achieved by the mutual-fund *economy*

$$X^* \triangleq \frac{Dq}{\mathbf{1}_{\mathcal{A}}^\top Dq} \mathbf{1}_{\mathcal{F}}^\top,$$

where each firm invests in the market portfolio $\frac{Dq}{\mathbf{1}_{\mathcal{A}}^\top Dq}$ where assets are held in proportion to the market capitalization of assets.

Proof. First, for a low-leverage economy, $\underline{\lambda_{max}} < 1$ by definition. There exists a feasible holding network X such that $\lambda_{max}(Y(X))<1$. According to equation (23), we can establish the upper bound in the inequality (56). The result that the upper bound is achieved by the mutual fund economy follows immediately from the proof for Theorem 4. ■

Theorem 7. *Suppose in a low-leverage economy, every firm has a constant leverage level $0 < \bar{b} < 1$, and $b = \bar{b}\mathbf{1}_{\mathcal{F}}$. Then the maximal network amplifier $\mathsf{MNA}(X)$ does not depend on the configuration of the holding network X.*

Proof. When every firm has a constant leverage level $0 < \bar{b} < 1$, $\gamma = \bar{b} < 1$. So this is a low-leverage case. We observe that for any feasible holding network $X \in \mathcal{X}$,

$$\begin{aligned} \Pi\bar{\Theta}^\top Dq &= X \operatorname{diag}(b) \operatorname{diag}(w) \operatorname{diag}(b) X^\top D^{-1} \operatorname{diag}(q)^{-1} Dq \\ &= X \operatorname{diag}(b) \operatorname{diag}(w) \operatorname{diag}(b) X^\top \mathbf{1}_{\mathcal{A}} \\ &= X \operatorname{diag}(b) \operatorname{diag}(w) \operatorname{diag}(b) \mathbf{1}_{\mathcal{F}} \\ &= \bar{b} X \operatorname{diag}(b) \operatorname{diag}(w) \mathbf{1}_{\mathcal{F}} \\ &= \bar{b} X \operatorname{diag}(b) w \\ &= \bar{b} Dq. \end{aligned}$$

This proves that for any feasible holding network $X \in \mathcal{X}$, $\bar{b}$ is always an eigenvalue of the matrix $Y(X) = X \operatorname{diag}(b) \operatorname{diag}(w) \operatorname{diag}(b) X^\top M^{-1}$. The corresponding eigenvector is Dq.

Since X is a matrix with non-negative components, $Y(X)$ is a matrix with non-negative components. Also, Dq is an eigenvector with strictly positive components. By Corollary 8.1.30 in Horn and Johnson (2012), the eigenvalue associated with Dq must be the spectral radius of $Y(X)$. So we can show that for any feasible holding network $X \in \mathcal{X}$, $\bar{b}$ is always the largest eigenvalue of $Y(X)$. Since $\bar{b} < 1$, the maximal network amplifier is always

$$\mathsf{MNA}(X) = \frac{1}{1-\bar{b}}.$$

■

Theorem 8. *Suppose we are given a high-leverage economy. The minimum value of the maximal network amplifier is bounded as follows*

$$\min_{X \in \mathcal{X}} \mathsf{MNA}(X) \leq \frac{1}{\overline{\lambda_{min}} - 1}. \tag{57}$$

Proof. For a high-leverage economy, $\overline{\lambda_{min}} > 1$ by definition. There exists a feasible holding network X such that $\lambda_{min}(Y(X))>1$. According to equation (23), we can establish the upper bound in the inequality (57). ■

Theorem 9. *Suppose the leverage is $b = \bar{b}\mathbf{1}_{\mathcal{F}}$ where $\bar{b} > 1$, wealth is $w = \bar{w}\mathbf{1}_{\mathcal{F}}$, and the market capitalization is $m = \bar{m}\mathbf{1}_{\mathcal{A}}$. We also assume that $|\mathcal{F}| = n|\mathcal{A}|$, where $n \in N$. Then the upper bound on* (29) *is achieved by the holding network*

$$X^* = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix} \tag{58}$$

Specifically,

$$X^*_{ih} = \begin{cases} 1, & \textit{if } (i-1)n + 1 \leq h \leq in, \\ 0, & \textit{otherwise.} \end{cases} \tag{59}$$

Proof. First, X^* is a feasible solution for program (25). According to the definition of X^*, we obtain that

$$\begin{aligned} Y(X^*) &= X^* \operatorname{diag}(b) \operatorname{diag}(w) \operatorname{diag}(b) X^{*\top} M^{-1} \\ &= \bar{b}\tfrac{\bar{w}\bar{b}}{\bar{m}} \tfrac{|\mathcal{F}|}{|\mathcal{A}|} I_{\mathcal{A}} \\ &= \bar{b} I_{\mathcal{A}}. \end{aligned}$$

The last equality is because of the fact that the total market capitalization $\bar{m}|\mathcal{A}|$ equals the total wealth invested in risky assets $\bar{w}\bar{b}|\mathcal{F}|$. All eigenvalues of $Y(X^*)$ are $\bar{b}$.

$$\lambda_{min}(Y(X^*)) = \bar{b}.$$

In this case, $\mathsf{MNA}(X^*) = \frac{1}{\bar{b}-1}$.

We observe that $\overline{\lambda_{min}} \geq \lambda_{min}(Y(X^*)) = \bar{b} > 1$, so this is a high-leverage economy. We notice that $\gamma = \bar{b}$ for this symmetric case. According to the proof of Theorem 4, we immediately obtain that $\overline{\lambda_{min}} \leq \gamma = \bar{b}$. Therefore, $\overline{\lambda_{min}} = \bar{b}$, and the upper bound on (29) is achieved by the holding network X^*. ■

Theorem 10. *Let $A = |\mathcal{A}|$ and $F = |\mathcal{F}|$. Suppose A is fixed and we let $F \to +\infty$. For each fixed F, we assume that the parameters (b, h, m) characterizing the economy are generated by the process described above. Then there exists a sequence of partitions $\mathcal{P}^{(F)}$ of $\mathcal{F}$ such that the sequence of holding networks $X(\mathcal{P}^{(F)})$ defined by* (33) *are asymptotically feasible, and achieve the bound* (29) *in the limit, i.e., $X(\mathcal{P}^F) \in \mathcal{X}^{(F)}$, for all but finitely many F, almost surely, and $\lambda_{\min}(Y(X(\mathcal{P}^F))) - \min_{X \in \mathcal{X}^{(F)}} \lambda_{min}(Y(X)) \to_P 0$.*

Proof. To prove the theorem, we will construct the sequence of partitions $\mathcal{P}^F$ of $\mathcal{F}$. This theorem has two parts to prove. The first part is to prove that the sequence of $X(\mathcal{P}^F)$ asymptotically belongs to feasible holding networks $\mathcal{X}^{(F)}$. The second part is to prove that the sequence of $X(\mathcal{P}^F)$ asymptotically produces the optimal solution for equation (25).

For a given F, we construct a partition $\mathcal{P}^F$ of $\mathcal{F}$. The partition consists of non-intersecting subsets of $\mathcal{F}$ specified by $\mathcal{P}^F = \{\mathcal{F}_i\}_{i\in\mathcal{A}} + \mathcal{R}$. Here $\mathcal{F}_i \subseteq \mathcal{F}$ denotes firms which invest in each asset i, and $\mathcal{R} \subseteq \mathcal{F}$ denotes the remaining firms. In this construction, only the cardinality $|\mathcal{F}_i|$ of $\mathcal{F}_i$ matters for this proof. So we first decide the cardinality of $\mathcal{F}_i$ for each $i \in \mathcal{A}$.

$$|\mathcal{F}_i| = \max\{\lceil \bar{m}_i F\rceil - 1 - \lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon}\rfloor, 0\}. \tag{60}$$

where $\epsilon \in (0, \frac{1}{2})$ is a small number that is arbitrarily chosen and fixed. It is clear that $|\mathcal{F}_i| < \bar{m}_i F$, so $\sum_{i\in\mathcal{A}} |\mathcal{F}_i| \leq F$. For each $i \in \mathcal{A}$, we define

$$\mathcal{F}_i = \{h \in \mathcal{F}\colon \sum_{j=1}^{i-1} |\mathcal{F}_j| + 1 \leq h \leq \sum_{j=1}^{i} |\mathcal{F}_j|\}.$$

$$\mathcal{R} = \mathcal{F} \setminus \bigcup_{i\in\mathcal{A}} \mathcal{F}_i.$$

Now based on the partition $\mathcal{P}^F$, we can construct $X(\mathcal{P}^F)$ according to equation (32). To prove the asymptotic feasibility of $X(\mathcal{P}^F)$, we introduce some notations. For a given F, let E_F denotes the event that $X(\mathcal{P}^F) \geq 0$.

$$Pr\left(E_F\right) = 1 - Pr\left(\cup_{i\in\mathcal{A}}\{m_i' < 0\}\right) \geq 1 - \sum_{i\in\mathcal{A}} Pr\left(m_i' < 0\right).$$

In the above equation, the first equality is because the only possibility of $X(\mathcal{P}^F)$ has negative elements is from events $\{m_i' < 0\}$. The inequality is because of the relaxation of the union of events. We can bound $Pr\left(m_i' < 0\right)$. For each $i \in \mathcal{A}$, define

$$Z_i = \sum_{h \in \mathcal{F}_i} b_h w_h - \bar{m}_i \sum_{h \in \mathcal{F}} b_h w_h.$$

The mean of Z_i is

$$\mu_{Z_i} = \left(|\mathcal{F}_i| - \bar{m}_i F\right) \bar{b}\bar{w}.$$

Since $|\mathcal{F}_i| < \bar{m}_i F$, $\mu_{Z_i} < 0$. The variance of Z_i is

$$\sigma^2_{Z_i} = \left(|\mathcal{F}_i| + \bar{m}_i^2 F\right) \sigma^2_{b_h w_h}.$$

For $Pr\left(m_i' < 0\right)$ we have

$$Pr\left(m_i' < 0\right) = Pr\left(Z_i > 0\right) = Pr\left(Z_i - \mu_{Z_i} > -\mu_{Z_i}\right) \leq Pr\left(|Z_i - \mu_{Z_i}| > \frac{|\mu_{Z_i}|}{\sigma_{Z_i}} \sigma_{Z_i}\right) \leq \frac{\sigma^2_{Z_i}}{\mu^2_{Z_i}}.$$

The first inequality is because $\mu_{Z_i} < 0$ and we relax the one-side condition to a two-side condition. The second inequality is due to Chebyshev's inequality. As $F \to +\infty$,

$$\frac{\sigma^2_{Z_i}}{\mu^2_{Z_i}} = \frac{\left(|\mathcal{F}_i| + \bar{m}_i^2 F\right)}{\left(|\mathcal{F}_i| - \bar{m}_i F\right)^2} \frac{\sigma^2_{b_h w_h}}{\bar{b}^2 \bar{w}^2} = \frac{\frac{\left(|\mathcal{F}_i| + \bar{m}_i^2 F\right)}{F}}{\left(\frac{|\mathcal{F}_i| - \bar{m}_i F}{F^{1/2}}\right)^2} \frac{\sigma^2_{b_h w_h}}{\bar{b}^2 \bar{w}^2} \to 0.$$

The above convergence is due to the following two observations.

$$\frac{\left(|\mathcal{F}_i| + \bar{m}_i^2 F\right)}{F} \to \bar{m}_i + \bar{m}_i^2. \tag{61}$$

$$\frac{|\mathcal{F}_i| - \bar{m}_i F}{F^{\frac{1}{2}}} \to \left(\bar{m}_i\right)^{\frac{1}{2}+\epsilon} \left(F\right)^{\epsilon} \to +\infty. \tag{62}$$

To show the above results, we first notice that in equation (60), the definition of $|\mathcal{F}_i|$ provides the following facts.

$$\frac{\lceil \bar{m}_i F \rceil - 1}{\bar{m}_i F} \in [1 - \frac{1}{\bar{m}_i F}, 1) \to 1. \tag{63}$$

$$\frac{\lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor}{F} = \frac{\lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor}{(\bar{m}_i F)^{\frac{1}{2}+\epsilon}} \frac{\bar{m}_i}{(\bar{m}_i F)^{\frac{1}{2}-\epsilon}} \to 0. \tag{64}$$

$$\frac{\lceil \bar{m}_i F \rceil - 1 - \bar{m}_i F}{F^{\frac{1}{2}}} \in [-\frac{1}{F^{\frac{1}{2}}}, 0) \to 0. \tag{65}$$

$$(66)\qquad \frac{\lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor}{F^{\frac{1}{2}}} = \frac{\lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor}{(\bar{m}_i F)^{\frac{1}{2}+\epsilon}} (\bar{m}_i)^{\frac{1}{2}+\epsilon} (F)^{\epsilon} \to (\bar{m}_i)^{\frac{1}{2}+\epsilon} (F)^{\epsilon} \to +\infty.$$

By equation (63) and equation (64), we obtain that

$$\frac{\lceil \bar{m}_i F \rceil - 1 - \lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor}{F} \to \bar{m}_i.$$

Then

$$(67)\qquad \frac{|\mathcal{F}_i|}{F} \to \bar{m}_i,$$

and equation (61) immediately follows. By equation (65) and equation (66), we obtain that

$$\frac{\lceil \bar{m}_i F \rceil - 1 - \bar{m}_i F - \lfloor (\bar{m}_i F)^{\frac{1}{2}+\epsilon} \rfloor}{F^{\frac{1}{2}}} \to (\bar{m}_i)^{\frac{1}{2}+\epsilon} (F)^{\epsilon} \to +\infty.$$

Then equation (62) follows. Now we have proved equation (61) and equation (62).

Then $Pr\,(m_i' < 0) \to 0$, and hence $Pr\,(E_F) \to 1$. By using the second Borel-Cantelli lemma, we obtain that

$$Pr\left(\limsup_{F\to\infty} E_F\right) = 1.$$

Namely, $X(\mathcal{P}^F) \geq 0$ for all but finitely many F, as $F \to +\infty$. Additionally, equation (32) provides that $X(\mathcal{P}^F)$ always satisfy all other constraints apart from $X(\mathcal{P}^F) \geq 0$ in the definition (8) of the feasible holding networks $\mathcal{X}^{(F)}$. Therefore $X(\mathcal{P}^F)$ asymptotically belongs to feasible holding networks $\mathcal{X}^{(F)}$ almost surely.

Now we prove that the sequence of $X(\mathcal{P}^F)$ asymptotically produces the optimal solution for equation (25) almost surely. According to equation (32), we can calculate $Y(X(\mathcal{P}^F))$.

$$Y(X(\mathcal{P}^F)) = X(\mathcal{P}^F)\,\mathrm{diag}(b)\,\mathrm{diag}(w)\,\mathrm{diag}(b) X(\mathcal{P}^F)^{\top} M^{-1} = A(X(\mathcal{P}^F)) + B(X(\mathcal{P}^F)),$$

where

$$A(X(\mathcal{P}^F)) = \begin{pmatrix} \frac{1}{m_1}\sum_{h\in\mathcal{F}_1} w_h b_h^2 & 0 & \dots & 0 \\ 0 & \frac{1}{m_2}\sum_{h\in\mathcal{F}_2} w_h b_h^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{m_{|\mathcal{A}|}}\sum_{h\in\mathcal{F}_2} w_h b_h^2 \end{pmatrix},$$

$$B(X(\mathcal{P}^F)) = \begin{pmatrix} \frac{\alpha^2}{m_1}|\mathcal{R}|m_1^{'}m_1^{'} & \frac{\alpha^2}{m_2}|\mathcal{R}|m_1^{'}m_2^{'} & \dots & \frac{\alpha^2}{m_2}|\mathcal{R}|m_1^{'}m_{|\mathcal{A}|}^{'} \\ \frac{\alpha^2}{m_1}|\mathcal{R}|m_2^{'}m_1^{'} & \frac{\alpha^2}{m_2}|\mathcal{R}|m_2^{'}m_2^{'} & \dots & \frac{\alpha^2}{m_2}|\mathcal{R}|m_2^{'}m_{|\mathcal{A}|}^{'} \\ \dots & \dots & \dots & \dots \\ \frac{\alpha^2}{m_1}|\mathcal{R}|m_{|\mathcal{A}|}^{'}m_1^{'} & \frac{\alpha^2}{m_2}|\mathcal{R}|m_{|\mathcal{A}|}^{'}m_2^{'} & \dots & \frac{\alpha^2}{m_2}|\mathcal{R}|m_{|\mathcal{A}|}^{'}m_{|\mathcal{A}|}^{'} \end{pmatrix} = \alpha^2|\mathcal{R}|m'm'^{\top}M^{-1}.$$

In the above formulation, matrix $A(X(\mathcal{P}^F))$ characterizes the asset-to-asset impact generated by $\cup_{i\in\mathcal{A}}\mathcal{F}_i$. Meanwhile, matrix $B(X(\mathcal{P}^F))$ characterizes the asset-to-asset impact generated by the remaining firms $\mathcal{R}$. For each asset $i \in \mathcal{A}$,

$$A_{ii}(X(\mathcal{P}^F)) = \frac{\sum_{h\in\mathcal{F}_i} w_h b_h^2}{m_i} = \frac{\frac{\sum_{h\in\mathcal{F}_i} w_h b_h^2}{|\mathcal{F}_i|}}{\frac{\sum_{h\in\mathcal{F}} w_h b_h}{F}} \frac{|\mathcal{F}_i|}{\bar{m}_i F} \tag{68}$$

As $F \to +\infty$, $|\mathcal{F}_i| \to +\infty$. Recall equation (67),

$$\frac{|\mathcal{F}_i|}{\bar{m}_i F} \to 1.$$

In our construction, each b_h and w_h are in fact indexed by F implicitly. For a given F, each b_h and w_h are drawn from the given distribution for $h = 1 \dots F$. For different F, b_h and w_h are a different set of realizations of the same i.i.d. random variables. So we identify a triangular array of i.i.d. random quantities. We can apply the weak law of large numbers for triangular arrays. So the other components of equation (68) converge.

$$\frac{\sum_{h\in\mathcal{F}_i} w_h b_h^2}{|\mathcal{F}_i|} \to E[w_h b_h^2], \quad \text{in probability.}$$

$$\frac{\sum_{h\in\mathcal{F}} w_h b_h}{F} \to E[w_h b_h], \quad \text{in probability.}$$

Therefore, as $F \to +\infty$,

$$A_{ii}(X(\mathcal{P}^F)) \to \bar{\gamma}, \quad \text{in probability.} \tag{69}$$

For every asset i and j,

$$B_{ij}(X(\mathcal{P}^F)) = \frac{\alpha^2}{m_j}|\mathcal{R}|m_i^{'}m_j^{'} \le \frac{|\mathcal{R}|}{m_j} \to 0, \quad \text{in probability.} \tag{70}$$

In the above expression, the inequality is due to the fact that as $F \to +\infty$, $\alpha^2 m_i^{'}m_j^{'} = \frac{m_i^{'}}{\mathbf{1}_{|\mathcal{A}|}^{\top}m'}\frac{m_j^{'}}{\mathbf{1}_{|\mathcal{A}|}^{\top}m'} \le 1$ almost surely since $m_i^{'} \ge 0$ almost surely for any i. In addition, the statement $\frac{|\mathcal{R}|}{m_j} \to_P 0$ in equation (70) can be proved by the weak law of large numbers for triangular arrays.

$$\frac{|\mathcal{R}|}{m_j} = \frac{F - \sum_{k\in\mathcal{A}} |\mathcal{F}_k|}{F} \frac{1}{\frac{\bar{m}_j \sum_{h\in\mathcal{F}} w_h b_h}{F}} \to \left(1 - \sum_{k\in\mathcal{A}} \bar{m}_k\right) \frac{1}{\bar{m}_j E[w_h b_h]} \to 0, \quad \text{in probability.}$$

Therefore

$$B(X(\mathcal{P}^F)) \to \mathbf{0}, \quad \text{in probability.}$$

We have obtained that

$$Y(X(\mathcal{P}^F)) = A(X(\mathcal{P}^F)) + B(X(\mathcal{P}^F)) \to \bar{\gamma} I, \quad \text{in probability.}$$

We conclude that asymptotically the dominant term is $A(X(\mathcal{P}^F))$, i.e., the asset-to-asset impact generated by $\cup_{i\in\mathcal{A}}\mathcal{F}_i$. The other contributing term $B(X(\mathcal{P}^F))$ that characterizes the asset-to-asset impact generated by the remaining firms $\mathcal{R}$ vanishes as $B(X(\mathcal{P}^F)) \to \mathbf{0}$.

Since eigenvalues are continuous functions of matrix elements, according to the continuous mapping theorem, it follows that

$$\lambda_i(Y(X(\mathcal{P}^F))) \to \bar{\gamma}, \quad \text{in probability,} \quad \forall\, i.$$

We immediately know that

$$\lambda_{min}(Y(X(\mathcal{P}^F))) \to \bar{\gamma}, \quad \text{in probability.} \tag{71}$$

Also, we notice when $F \to +\infty$,

$$\gamma^F = \frac{\sum_{h\in\mathcal{F}} w_h b_h^2}{\sum_{h\in\mathcal{F}} w_h b_h} = \frac{\frac{\sum_{h\in\mathcal{F}} w_h b_h^2}{F}}{\frac{\sum_{h\in\mathcal{F}} w_h b_h}{F}} \to \bar{\gamma}, \quad \text{in probability.} \tag{72}$$

We recall from the proof of Theorem 4 that for any given F and generated wealth w and leverage b of firms, the following inequalities always hold.

$$\lambda_{min}(X(\mathcal{P}^F))\mathbf{1}_{\{X(\mathcal{P}^F)\in\mathcal{X}^{(F)}\}} \le \max_{X\in\mathcal{X}^{(F)}} \lambda_{min}(Y(X)) \le \gamma^F. \tag{73}$$

The first inequality is because when $X(\mathcal{P}^F) \in \mathcal{X}^{(F)}$, the feasible solution doesn't exceed the optimal solution, and the optimal solution is non-negative. The second inequality is from the proof of Theorem 4.

According to equation (72), the right hand side of inequality (73) converges to $\bar{\gamma}$ in probability. For the first part of this theorem, we have proved that the sequence of $X(\mathcal{P}^F)$ that we construct satisfies that

$$X(\mathcal{P}^F) \in \mathcal{X}^{(F)}, \quad \text{for all but finitely many } F\text{, almost surely.}$$

This implies that $\mathbf{1}_{\{X(\mathcal{P}^F)\in\mathcal{X}^{(F)}\}}$ converges to 1 almost surely. With the result in equation (71), we apply Slutsky's theorem to obtain that $\lambda_{min}(X(\mathcal{P}^F))\mathbf{1}_{\{X(\mathcal{P}^F)\in\mathcal{X}^{(F)}\}}$ converges to $\bar{\gamma}$ in distribution. Since $\bar{\gamma}$ is a constant, the left hand side of inequality (72) also converges to $\bar{\gamma}$ in probability.

According to the sandwich theorem for convergence, we can conclude that

$$\max_{X\in\mathcal{X}^{(F)}} \lambda_{min}(Y(X)) \to \bar{\gamma}, \quad \text{in probability.} \tag{74}$$

Therefore, $\lambda_{\min}(Y(X(\mathcal{P}^F))) - \min_{X\in\mathcal{X}^{(F)}} \lambda_{min}(Y(X)) \to_P 0$.

■