The Optimality of Consols in Government Debt Management*

Davide Debortoli† Ricardo Nunes‡ Pierre Yared§

September 12, 2017

Abstract

We consider optimal government debt maturity in a deterministic economy in which the government can issue any arbitrary maturity structure and in which bond prices are a function of the government’s current primary surplus and the path of future primary surpluses. The government sequentially chooses policy, taking into account how the current primary surplus and the structure of maturity issuance—which impacts future policy—feed back into current bond prices. We establish conditions under which, in the long-run, the unique stable debt maturity structure is flat, with the government owing the same amount of resources to the private sector at all future dates. Our analysis provides a theoretical foundation for the optimal use of consols in government debt management, and we show that the short-run welfare benefits of transitioning to consols are substantive, at the expense of small long-run costs.

Keywords: Public debt, optimal taxation, fiscal policy

JEL Classification: H63, H21, E62

*We would like to thank Facundo Piguillem for comments.
†Universitat Pompeu Fabra and Barcelona GSE: davide.debortoli@upf.edu.
‡University of Surrey and CIMS: ricardo.nunes@surrey.ac.uk.
§Columbia University and NBER: pyared@columbia.edu.
1 Introduction

How should government debt maturity be structured? In this paper, we study this question in a deterministic environment in which bond prices are a function of the government’s current primary surplus and the path of future primary surpluses. The government sequentially chooses taxes and any arbitrary debt issuance strategy taking into account the impact of current policy on bond prices. Our main finding is that the unique stable distribution of government debt is flat, with the government owing the same amount to the private sector at all future dates. We also establish sufficient conditions under which government debt maturity converges to this flat structure in the long-run.

We establish these results in the dynamic fiscal policy model of Lucas and Stokey (1983). This is an economy with exogenous public spending and no capital in which the government chooses linear taxes on labor and issues public debt to finance government spending. In this environment, if the government could commit to policy at the beginning of time, then the choice of government debt maturity would be indeterminate. This is because any debt maturity structure would satisfy the present value constraints of the government at a given point in time.

We do not assume that the government commits ex-ante to policy, but we instead consider the sequentially optimal policy. More specifically, we characterize the Markov Perfect Competitive Equilibrium (MPCE) in which, at every date, a government—which needs to honor the inherited debt repayments—chooses current taxes and an issued portfolio of maturities. In doing so, the government considers how current taxes and its financing strategy affect the price of bonds through expectations of future policy.

We focus on characterizing the entire set of MPCE’s, including those with potentially discontinuous policy functions.1 In addition, we allow for any arbitrary structure of maturity issuance. This means that the payoff relevant state—the government’s portfolio of inherited maturities—is an infinite-dimensional and potentially complicated object. In this context, a stable maturity distribution is defined as one in which the inherited portfolio of maturities equals the issued portfolio, so that the distribution of maturities is time-invariant. Such a stable distribution emerges if fiscal policy and short-term interest rates are both constant over time.

Our main result is that the only stable debt maturity structure in an MPCE is flat, with the government owing the same amount of resources to the private sector at all future dates. Under such a flat maturity structure, the government sequentially chooses a

1In this regard, our approach is similar in spirit to that of Cao and Werning (2017) in their analysis of Markov equilibria in the hyperbolic consumption model.
stable fiscal policy which coincides with the optimum under full commitment. If the debt maturity were not flat, it would be optimal for the government to pursue an unstable fiscal policy which reduces (increases) the market value of outstanding (newly-issued) government liabilities. A flat maturity structure is thus the unique structure to guarantee a stable fiscal policy.

To provide some intuition for this result, suppose that the government enters the period with more long-term liabilities relative to short-term liabilities. To relax its budget constraint, the government should pursue a policy which increases short-term interest rates. Such a policy reduces the market value of its outstanding long-term liabilities and can make the government better off relative to a stable policy. The opposite is true if the government enters the period with more short-term liabilities relative to long-term liabilities. In this case, the government should relax its budget constraint by pursuing a policy which reduces short-term interest rates since this benefits the government by increasing the market value of newly issued liabilities.

In addition to this main result, we also construct examples in which optimal government debt maturity converges to a stable, flat distribution in the long-run under the unique MPCE. In these examples, the initial debt maturity structure is weighted to the short end and nearly flat on the long end, in the sense that maturities beyond a certain horizon are all the same size. The transition to a flat maturity follows from backward induction, taking into account that a flat maturity is the unique stable MPCE. More specifically, consider a government today with an inherited flat maturity structure. Because a flat maturity is stable, this government knows that if it issues a flat maturity today, future governments will do the same. Taking this into account, today’s government thus optimally issues a flat maturity, since this achieves the full commitment solution. By analogous logic, consider a government with an inherited nearly flat maturity structure. Such a government anticipates that future governments with a nearly flat maturity structure will gradually flatten out the maturity structure by issuing more consols. Anticipating this future behavior, the government today also chooses to gradually flatten out the maturity structure by issuing more consols, and is able to achieve the commitment solution. Following this logic, the unique MPCE entails the gradual flattening of the maturity structure over time.

We also consider the practical implications of our theory by analyzing an extension in which the economy experiences stable productivity growth and inflation over time, and where the government is constrained to issuing only nominal debt. In this environment, all of our theoretical results continue to hold, with a flat government debt maturity characterized by a consol with coupon payments which grow at the stable rate of nominal output. To examine the welfare benefit of transitioning to consols, we consider the optimal
sequential policy starting from a debt maturity with the same structure as that in the U.S., where the maturity distribution is weighted to the short end and the longest maturity is thirty years. Starting from this structure, the optimal policy takes the form of a gradual transition to a flat maturity where all financing is done with consols. We compare welfare under this optimal policy to welfare under a naive policy of preserving the same initial maturity structure in all future dates. Such a policy is associated with substantial welfare benefits in the short-run, equal to 0.34 percent of consumption in the first-year, at the expense of small long-run costs, equal to 0.02 percent of consumption in 30 years. These benefits come in the form of temporary tax breaks financed by the sequential issuance of consols which the government can sell along the transition path at a temporarily high price. This simple exercise suggests that gradually switching to consols is a cheap means of financing temporary tax breaks which boost welfare.

Our analysis thus provides a theoretical argument for the use of consols in debt management based on the sequential optimization of fiscal policy by the government. The use of consols has been pursued historically, most notably by the British government in the Industrial Revolution, when consols were the largest component of the British government’s debt (see Mokyr, 2011). Moreover, the introduction of consols has been discussed as a potential option in the management of U.S. government debt (e.g. Cochrane, 2015).

This paper relates to several literatures. The main contribution of this paper is to characterize the set of MPCE’s in the deterministic case of the Lucas and Stokey (1983) model. In their work, Lucas and Stokey (1983) briefly discuss the role of government debt maturity in the absence of government commitment. They argue that, if faced with the appropriately constructed inherited maturity structure, a government with full commitment from tomorrow onward will choose the same policy as a government with full commitment today. We depart from this work in two ways. First, we analyze the MPCE, which means that we allow the government to reoptimize at all future dates in an infinite horizon—not only the next date—with bond prices today responding to anticipation of policy in the future. Second, and more importantly, we analyze the entire set of MPCE’s, not only the ones which coincide with the optimal ex-ante policy under full commitment. This allows us to establish that a flat maturity structure is the unique stable structure in the entire space of MPCE’s and to also establish conditions under which convergence to a stable structure is the unique transition path in the MPCE.²

Our work also contributes to a literature on the optimal government debt maturity in

²Note that this construction requires concavity in the government’s problem along the transition path, a requirement which is needed, but not made explicit in the discussion in Lucas and Stokey (1983) of government debt maturity in the absence of commitment.
the absence of government commitment. We depart from this literature in two ways. First, we consider the optimal maturity without imposing arbitrary constraints on maturities available to the government.\textsuperscript{3} Second, given our focus, our model is most applicable to economies in which the risks of default and surprise in inflation are not salient, but the government is still not committed to a path of primary deficits and debt maturity issuance.\textsuperscript{4} In this regard, our paper is complementary to the quantitative analysis of Debortoli et al. (2017). In contrast to this work, we consider a deterministic economy and ignore the presence of shocks.\textsuperscript{5} This allows us to achieve theoretical characterization in an infinite horizon economy without confining the set of maturities available to the government. Our theoretical result that the optimal maturity structure is exactly flat in the long-run is consistent with their quantitative result that the optimal maturity structure is nearly flat in the presence of shocks.

Our paper proceeds as follows. In Section 2, we describe the model. In Section 3, we define the equilibrium. In Section 4, we provide the main results of the paper and in Section 5, we consider the quantitative implications of the model. Section 6 concludes. The Appendix provides all of the proofs and additional results not included in the text.

2 Model

We consider an economy identical to a deterministic version of Lucas and Stokey (1983), with the exception that we assume that the government chooses fiscal policy sequentially. There are discrete time periods \( t = \{0, 1, \ldots, \infty\} \). The resource constraint of the economy is

\[ c_t + g = n_t, \quad (1) \]

where \( c_t \) is consumption, \( n_t \) is labor, and \( g > 0 \) is government spending, which is exogenous and constant over time.

\textsuperscript{3}Krusell et al. (2006) and Debortoli and Nunes (2013) consider a similar environment to ours in the absence of commitment, but with only one-period bonds, for example.

\textsuperscript{4}Other work considers optimal government debt maturity in the presence of default risk, for example, Aguiar et al. (2017), Arellano and Ramanarayanan (2012), Dovis (2017), and Fernandez and Martin (2015), among others. Bocola and Dovis (2016) additionally consider the presence of liquidity risk. Bigio et al. (2017) consider debt maturity in the presence of transactions costs. Alvarez et al. (2004), Persson et al. (2006), and Arellano et al. (2013) consider lack of commitment when surprise inflation is possible.

\textsuperscript{5}Angeletos (2002), Bhandari et al. (2017), Brua and Nicolini (2004), Faraglia et al. (2010), Guibaud et al. (2013), and Lustig et al. (2008) also consider optimal government debt maturity in the presence of shocks, but they assume full commitment.
There is a continuum of mass 1 of identical households that derive the following utility:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \beta \in (0, 1).$$  \hspace{1cm} (2)

$u(\cdot)$ is strictly increasing in consumption and strictly decreasing in labor, globally concave, and continuously differentiable. As a benchmark, we define first best consumption and labor $\{c^{fb}, n^{fb}\}$ as the values of consumption and labor which maximize $u(c_t, n_t)$ subject to the resource constraint (1).

Household wages equal the marginal product of labor (which is 1 unit of consumption), and are taxed at a linear tax rate $\tau_t$. $b_{t,k} \geq 0$ represents government debt purchased by a representative household at $t$, which is a promise to repay 1 unit of consumption at $t + k > t$, and $q_{t,k}$ is its price at $t$. At every $t$, the household’s allocation and portfolio $\{c_t, n_t, \{b_{t,k}\}_{k=1}^{\infty}\}$ must satisfy the household’s dynamic budget constraint

$$c_t + \sum_{k=1}^{\infty} q_{t,k} (b_{t,k} - b_{t-1,k+1}) = (1 - \tau_t) n_t + b_{t-1,1}. \hspace{1cm} (3)$$

$B_{t,k} \geq 0$ represents debt issued by the government at $t$ with a promise to repay 1 unit of consumption at $t + k > t$. At every $t$, government policies $\{\tau_t, g_t, \{B_{t,k}\}_{k=1}^{\infty}\}$ must satisfy the government’s dynamic budget constraint

$$g_t + B_{t-1,1} = \tau_t n_t + \sum_{k=1}^{\infty} q_{t,k} (B_{t,k} - B_{t-1,k+1}). \hspace{1cm} (4)$$

The economy is closed which means that the bonds issued by the government equal the bonds purchased by households:

$$b_{t,k} = B_{t,k} \forall t, k. \hspace{1cm} (5)$$

Initial debt $\{B_{-1,k}\}_{k=1}^{\infty} = \{b_{-1,k}\}_{k=1}^{\infty}$ is exogenous. We assume that there exist debt limits to prevent Ponzi schemes:

$$b_{t,k} \in [b, \bar{b}] \forall t, k. \hspace{1cm} (6)$$

---

We follow the same exposition as in Angeletos (2002) in which the government rebalances its debt in every period by buying back all outstanding debt and then issuing fresh debt at all maturities. This is without loss of generality. For example, if the government at $t - k$ issues debt due at date $t$ of size $B_{t-k,k}$ which it then holds to maturity without issuing additional debt, then all future governments at date $t - k + l$ for $l = 1, \ldots, k - 1$ will choose $B_{t-k+l,k-l} = B_{t-k,k}$, implying that $B_{t-1,1} = B_{t-k,k}$.
The government is benevolent and shares the same preferences as the households in (2). We assume that the government cannot commit to policy and therefore chooses taxes and debt sequentially.

3 Markov Perfect Competitive Equilibrium

3.1 Equilibrium Definition

We consider a Markov Perfect Competitive Equilibrium (MPCE) in which the government optimally chooses its preferred policy—which consists of taxes and an issued portfolio of debt—at every date as a function of current payoff-relevant variables, which consists of the inherited portfolio of debt. The government takes into account that its choice affects future debt and thus affects the policies of future governments. Households rationally anticipate these future policies, and their expectations are in turn reflected in current bond prices. Thus, in choosing policy today, a government anticipates that it may affect current bond prices by impacting expectations about future policy.

Formally, let $B_t \equiv \{B_{t,k}\}_{k=1}^{\infty}$ and $q_t \equiv \{q_{t,k}\}_{k=1}^{\infty}$. In every period $t$, the government chooses a policy $\{\tau_t, B_t\}$ given $B_{t-1}$. Households then choose an allocation and portfolio $\{c_t, n_t, \{b_{t,k}\}_{k=1}^{\infty}\}$. An MPCE consists of: a government strategy $\rho(B_{t-1})$ which is a function of $B_{t-1}$; a household allocation and portfolio strategy $\omega(B_{t-1}, \rho_t, q_t)$ which is a function of $B_{t-1}$, the government policy $\rho_t = \rho(B_{t-1})$, and bond prices $q_t$; and a set of bond pricing functions $\{\varphi_k(B_{t-1}, \rho_t)\}_{k=1}^{\infty}$ with $q_{t,k} = \varphi_k(B_{t-1}, \rho_t) \ \forall k \geq 1$ which depend on $B_{t-1}$ and the government policy $\rho_t = \rho(B_{t-1})$. In an MPCE, these objects must satisfy the following conditions $\forall t$:

1. The government strategy $\rho(\cdot)$ maximizes (2) given $\omega(\cdot)$, $\varphi_k(\cdot) \ \forall k \geq 1$, and the government budget constraint (4);

2. The household allocation and portfolio strategy $\omega(\cdot)$ maximizes (2) given $\rho(\cdot)$, $\varphi_k(\cdot) \ \forall k \geq 1$, and the household budget constraint (3), and

3. The set of bond pricing functions $\varphi_k(\cdot) \ \forall k \geq 1$ satisfy (5) given $\rho(\cdot)$ and $\omega(\cdot)$.

3.2 Primal Approach

Any MPCE must be a competitive equilibrium. We follow Lucas and Stokey (1983) by taking the primal approach to the characterization of competitive equilibria since this
allows us to abstract away from bond prices and taxes. Let

$$\{c_t, n_t\}_{t=0}^\infty$$  \hspace{1cm} (7)

represent a sequence. We can establish necessary and sufficient conditions for (7) to constitute a competitive equilibrium. The household’s optimization problem implies the following intratemporal and intertemporal conditions, respectively:

$$1 - \tau_t = -\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)}$$ and

$$q_{t,k} = \frac{\beta^k u_c(c_{t+k}, n_{t+k})}{u_c(c_t, n_t)}$$  \hspace{1cm} (8)

Substitution of these conditions into the household’s dynamic budget constraint implies the following condition:

$$u_c(c_t, n_t) c_t + u_n(c_t, n_t) n_t + \sum_{k=1}^\infty \beta^k u_c(c_{t+k}, n_{t+k}) b_{t,k} = \sum_{k=0}^\infty \beta^k u_c(c_{t+k}, n_{t+k}) b_{t-1,k+1}. \hspace{1cm} (9)$$

Forward substitution into the above equation and taking into account the absence of Ponzi schemes implies the following implementability condition:

$$\sum_{k=0}^\infty \beta^k (u_c(c_{t+k}, n_{t+k}) c_{t+k} + u_n(c_{t+k}, n_{t+k}) n_{t+k}) = \sum_{k=0}^\infty \beta^k u_c(c_{t+k}, n_{t+k}) b_{t-1,k+1}. \hspace{1cm} (10)$$

By this reasoning, if a sequence in (7) is generated by a competitive equilibrium, then it necessarily satisfies (1) and (10). We prove in the Appendix that the converse is also true, which leads to the below proposition that is useful for the rest of our analysis.

**Lemma 1 (competitive equilibrium)** A sequence (7) is a competitive equilibrium if and only if it satisfies (1) \(\forall t\) and (10) at \(t = 0\) given \(\{b_{-1,k}\}_{k=1}^\infty\).

Note that this result rests on the fact that the satisfaction of (10) at \(t = 0\) guarantees the satisfaction of (10) for all other future dates, since bonds can be freely chosen so as to satisfy (10) at all future dates for a given sequence (7).

### 3.3 Recursive Representation

We can use the primal approach to represent an MPCE recursively. Recall that \(\rho(B_{t-1})\) is a policy which depends on \(B_{t-1}\), and that \(\omega((B_{t-1}), \rho_t, q_t)\) is a household allocation and portfolio strategy which depends on \(B_{t-1}\), government policy \(\rho_t = \rho(B_{t-1})\), and bond prices \(q_t\), where these bond prices depend on \(B_{t-1}\) and government policy. As
such, an MPCE in equilibrium is characterized by a sequence in (7) and a debt sequence
\( \{b_{t,k}\}_{k=1}^{\infty} \), where each element at date \( t \) depends on history only through \( B_{t-1} \), the payoff relevant variables. Given this observation, in an MPCE, one can define a function \( h^k(\cdot) \)

\[
h^k(B_t) = \beta^k u_c(c_{t+k}, n_{t+k}) | B_t
\]

for \( k \geq 1 \), which equals the discounted marginal utility of consumption at \( t + k \) given \( B_t \) at \( t \). This function is useful since, in choosing \( B_t \) at date \( t \), the government must take into account how it affects future expectations of policy which in turn affect current bond prices through expected future marginal utility of consumption.

Note furthermore that choosing \( \{\tau_t, B_t\} \) at date \( t \) from the perspective of the government is equivalent to choosing \( \{c_t, n_t, B_t\} \) where one can write, with some abuse of notation, \( B_t = \{b_{t,k}\}_{k=1}^{\infty} \), and this follows from the primal approach delineated in Section 3.2. Removing the time subscript and defining \( B \equiv B_{t-1} = \{b_k\}_{k=1}^{\infty} \) as the inherited portfolio of bonds, we can write the government’s problem recursively as

\[
V(B) = \max_{c, n, B'} u(c, n) + \beta V(B')
\]

s.t.
\[
c + g = n,
\]

\[
u_c(c, n) c + u_n(c, n) n - u_c(c, n) b_1 + \sum_{k=1}^{\infty} h^k(B')(b'_k - b_{k+1}) = 0,
\]

where (14) is a recursive representation of (9). Let \( f(B) \) correspond to the solution to (12) – (14) given \( V(\cdot) \) and \( h^k(\cdot) \forall k \geq 1 \). It therefore follows that the function \( f(\cdot) \) necessarily implies functions \( h^k(\cdot) \forall k \geq 1 \) which satisfy (11). An MPCE is therefore composed of functions \( V(\cdot) \), \( f(\cdot) \), and \( h^k(\cdot) \forall k \geq 1 \) which are consistent with one another and satisfy (11) – (14).

4 Characterization of Government Debt Maturity

We now characterize government debt maturity in the long run and along the transition path.

4.1 Preliminaries

Before proceeding with our analysis, we establish a preliminary assumption which we utilize in deriving our results. Let us define \( W(\{b_k\}_{k=1}^{\infty}) \) as the welfare of the government
under full commitment given an initial starting debt position \( \{b_k\}_{k=1}^{\infty} \):

\[
W (\{b_k\}_{k=1}^{\infty}) = \max_{\{c_k, n_k\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \beta^k u (c_k, n_k)
\]

s.t.

\[
c_k + g = n_k
\]

\[
\sum_{k=0}^{\infty} \beta^k (u_c (c_k, n_k) c_k + u_n (c_k, n_k) n_k) = \sum_{k=0}^{\infty} \beta^k u_c (c_k, n_k) b_{k+1}
\]

The program in (15)–(17) corresponds to that of a government under full commitment with \( b_{-1,k} = b_k \). We now make an assumption regarding the solution to this program under a flat maturity structure, meaning a maturity structure in which \( b_k \) is the same for all \( k \).

**Assumption 1.** Consider the solution to (15)–(17) with \( b_k = b \forall k \). \( \forall b \in [b, \bar{b}] \), the solution to this program is unique and admits \( \{c_k, n_k\} = \{c^*(b), n^*(b)\} \forall k \), where

\[
u_c (c^*(b), n^*(b)) c^*(b) + u_n (c^*(b), n^*(b)) n^*(b) = u_c (c^*(b), n^*(b)) b
\]

and

\[
c^*(b) + g = n^*(b).
\]

This assumption states that if a government under full commitment is faced with a flat maturity structure, then there is a unique optimum in which the government chooses a constant allocation of consumption and labor in the future. This assumption is intuitive. Under a flat maturity structure, every time period in the program in (15)–(17) is identical in the objective function and in the constraint set, which suggests that the optimal solution is a time-invariant allocation. A sufficient condition for Assumption 1 is that the function \( u_c (n - g, n) (n - g - b) + u_n (n - g, n) n \) is concave in \( n \) for all \( b \), which is the case for example if \( b = 0 \) so that debt is non-negative, and if the utility function satisfies

\[
u (c, n) = \log c - \eta n^\gamma / \gamma,
\]

for \( \eta > 0 \) and \( \gamma > 1 \), which corresponds to a utility function analyzed in Werning (2007).

### 4.2 Stable Government Debt Maturity

We begin by characterizing the stable level of government debt maturity. More specifically, we evaluate the characteristics of an economy in which \( b_{t+1,k} = b_{t,k}, \forall t, k \), so that government debt maturity is time-invariant. We first establish that there exists an MPCE
with a flat debt maturity which is stable.

**Lemma 2** Suppose that B satisfies \( b_k = b \ \forall k \) for some \( b \in [\underline{b}, \overline{b}] \). Then,

1. In all solutions to (12) − (14), \( c = c^*(b) \) and \( n = n^*(b) \), and
2. There exists a solution to (12) − (14) which admits \( b'_k = b \ \forall k \).

The first part of the lemma states that in any MPCE, if the government inherits a flat maturity with \( b_k = b \ \forall k \), then the unique optimal response of the government is to choose consumption and labor which coincide with the commitment optimum. This implies that in any MPCE for which B is a flat government debt maturity, it is necessary that

\[
V(B) = W(B) \quad (21)
\]

so that there is no welfare loss for the present government due to lack of commitment by future governments. Moreover, the second part of the lemma implies that one optimal—but not necessarily uniquely optimal—strategy for the government in this scenario is to choose \( b'_k = b \ \forall k \) so that the issued debt maturity structure is unchanged and continues to be flat. As such, there exists a stable MPCE with a flat government debt maturity.

Note that this lemma applies to any MPCE which is constructed, and this lemma does not rely on making assumptions regarding the structure of future government strategies. Moreover, it is critical to emphasize that this lemma establishes the existence, but not the uniqueness of this stable MPCE. We now turn to the possibility that another stable MPCE can exist.

**Lemma 3** Suppose that given \( B \), there exists a solution to (12) − (14) with a stable debt maturity structure \( b'_k = b_k \ \forall k \). Then there exists another solution to (12) − (14) with \( b'_k = \hat{b} \ \forall k \) where

\[
\hat{b} = \sum_{k=1}^{\infty} \beta^{k-1} (1 - \beta) b_k. \quad (22)
\]

This lemma states that under any MPCE with a stable distribution of debt, the government can choose the same current tax policy and deviate to a flat issuance of debt maturity and achieve the same welfare. More precisely, the government can issue a flat maturity with the same market value, as determined by (22). Moreover, Lemma 2 characterizes future welfare and future allocations following the issuance of a flat maturity today, which means that bond prices are not affected by the deviation.
This lemma implies that if there is a stable distribution of debt which is not flat, then the corresponding welfare is equal to that achieved under a flat maturity distribution with the same market value. Moreover, from (21), welfare under this MPCE equals that under commitment associated with a flat maturity distribution with the same market value:

\[ V(B) = W(\{b_k\}_{k=1}^{\infty} \mid b_k = \hat{b}, \forall k) = \frac{u(c(\hat{b}), n(\hat{b}))}{1 - \beta}. \]  

(23)

With these results in mind, we now develop an induction argument to show that the unique stable distribution of debt is flat. The argument rests on showing that if a distribution of debt is not flat, the government can deviate from stable fiscal policy in order to frontload consumption or backload consumption so as to change the value of its inherited or newly-issued government debt portfolio.

**Lemma 4** Suppose that given \( B \), there exists a solution to (12) – (14) with a stable debt maturity structure \( b'_k = b_k \forall k \) and for which \( \{c, n\} \neq \{c^{fb}, n^{fb}\} \). Then, \( B \) must satisfy \( b_1 = \hat{b} \) for \( \hat{b} \) defined in (22).

This lemma states that in any stable distribution of debt maturity in which taxes are non-zero (so that consumption and labor do not equal the first best), it must be that the primary surplus is used to pay the short-term debt \( b_1 \). If the primary surplus is in excess of, or below, this short-term debt then the government can pursue a deviation from its smooth consumption strategy to boost welfare.

For example, if the primary surplus is in excess of what the government immediately owes, this means that in equilibrium the government buys back some of its long-term debt. In this circumstance, the government can deviate to tilt the path of consumption so as to increase short-term interest rates and reduce the value of the long-term debt which it buys back.

If instead the primary surplus is below what the government owes, this means that in equilibrium the government issues fresh debt in order to repay current short-term debt. In this circumstance, the government can deviate to tilt the path of consumption so as to reduce short-term interest rates and increase the value of newly issued debt. Thus, if the primary surplus equals the amount of short-term debt that is due, the government will not engage in such deviations.

Note that in constructing these deviations, we utilize the result in Lemma 2 which allows us to characterize the continuation equilibrium if the government issues a flat government debt maturity today as part of its deviation. As such, we can explicitly show that these deviations increase welfare by relaxing the government’s budget constraint. The
reason why our argument does not hold under a stable distribution of debt maturities with zero taxes is that in this case, it is not possible to relax the government budget constraint further.

We now expand this lemma to consider longer maturities.

**Lemma 5** Suppose that given $B$, there exists a solution to (12) – (14) with a stable debt maturity structure $b'_k = b_k \forall k$ and for which $\{c, n\} \neq \{c^{lb}, n^{lb}\}$. If $b_l = \hat{b} \forall l \leq m$, then $B$ must satisfy $b_{m+1} = \hat{b}$ for $\hat{b}$ defined in (22).

This lemma considers the stable distribution of government debt maturity when all maturities below $m$ have the property that the amount owed equals the primary surplus of the government. The lemma states that if this is true, then the bond of maturity $m+1$ must also equal the primary surplus of the government.

The argument, which relies on a proof by contradiction, starts from the fact that under a stable maturity, government welfare satisfies (23), and if the amount owed at date $m + 1$ does not also equal the primary surplus, then a feasible deviation exists for the government which can increase welfare above (23). More specifically, if $b_l = \hat{b} \forall l \leq m$ but $b_{m+1} \neq \hat{b}$, a feasible strategy for the government today is to continue to choose the same consumption and labor allocation today $\{c(\hat{b}), n(\hat{b})\}$ but to deviate by not retriading the inherited maturity structure (i.e., letting the bonds mature to next period). Such a deviation is feasible whatever the expectations of future policy and their impact on current bond prices since the government is not rebalancing its portfolio.

Without specifying the exact form of the continuation equilibrium, we can show that this deviation must necessarily increase welfare. The argument rests on putting a lower bound on the welfare of future governments based on the feasible policies at their disposal. More specifically, note that after this initial deviation, future governments also have the opportunity to pursue the same strategy of choosing consumption and labor equal to $\{c(\hat{b}), n(\hat{b})\}$ and not rebalancing the portfolio of maturities. This is true up until some future date $m$ periods in the future. Therefore, the welfare of the government today from pursuing the deviation must weakly exceed

$$\sum_{l=0}^{m-1} \beta^l u(c(\hat{b}), n(\hat{b})) + \beta^m V(\hat{B}(m))$$  \hspace{1cm} (24)

where $\hat{B}(m)$ satisfies $\hat{b}_k(m) = b_{k+m} \forall k \geq 1$.

At that point $m$ periods in the future, if the government pursued a stable policy from thereafter, the market value of debt would equal $\hat{b}/(1 - \beta)$ and welfare $V(\hat{B}(m))$ would
be given by (23). Were the government to choose \( \{c(\hat{b}), n(\hat{b})\} \) at that date so as to satisfy (23), the fact that \( b_{m+1} \neq \hat{b} \) means that the primary surplus would either exceed or be below the short-term debt. However, by the arguments of Lemma 4, the government could choose at this point a non-stable policy which either reduces the market value of inherited debt or increases the market value of newly-issued debt. Such a policy would provide a continuation value \( V(\hat{B}(m)) \) which strictly exceeds (23). Based on this logic, the initial deviation which provides the government at least (24) makes the government strictly better off since (24) strictly exceeds (23). This completes the argument, since it contradicts the fact that government welfare equals (23) under the MPCE.

**Proposition 1 (stability of flat maturity)** Suppose that conditional on \( B \), there exists a solution to (12) – (14) with a stable debt maturity structure \( b'_k = b_k \forall k \) and for which \( \{c, n\} \neq \{c^K, n^K\} \). Then it is necessary that \( b_k = \hat{b} \forall k \) so that the government debt maturity is flat.

This proposition is the main result of the paper. It states that if the distribution of government debt is stable and if the equilibrium does not entail first best consumption and labor, then government debt must be flat. The reasoning for the proposition follows from induction arguments which appeal to Lemmas 4 and 5. Intuitively, if government debt is not flat, then there are opportunities for the government take advantage of this fact to decrease market value of its inherited portfolio or increase the market value of its newly-issued portfolio. Note that this result holds in any MPCE and does not appeal to any assumptions regarding the behavior of future governments.

Our result relies on the stable distribution of debt not being associated with first best consumption and labor. Under such a stable distribution, taxes are zero, the market value of debt is sufficiently negative to finance the stream of government spending forever, and the marginal benefit of resources for the government is zero. For this reason, the stable distribution of government debt maturity is undetermined in this circumstance. While such stable distribution potentially exists, we can rule such a stable distribution out if there are exogenous bounds on government debt which prevent such asset accumulation for the government.

**Corollary 1** Suppose that \( b > -g \). Then if conditional on \( B \), there exists a solution to (12) – (14) with a stable debt maturity structure \( b'_k = b_k \forall k \), it is necessary that \( b_k = \hat{b} \forall k \) so that the government debt maturity is flat.

Finally, returning to Lemma 2, note that Proposition 1 also implies that starting from a flat government debt maturity, the unique continuation equilibrium involves a flat
government debt maturity. Therefore, in any MPCE, a flat government debt maturity is an absorbing state.

**Corollary 2** Suppose that $B$ satisfies $b_k = b \ \forall k$ for some $b$ and that $\{c, n\} \neq \{c^{fb}, n^{fb}\}$. Then, in all solutions to (12) – (14) $b'_k = b \ \forall k$.

Starting from a flat government debt maturity, the current government would like to guarantee a constant level of consumption and labor going forward. Choosing a tilted maturity structure cannot guarantee such a continuation equilibrium going forward, since future governments will deviate from a smooth policy in order to relax the government budget constraint. For this reason, it chooses a flat maturity structure, and a flat maturity structure is an absorbing state.

### 4.3 Transitional Dynamics

We have established conditions under which the unique stable distribution of government debt maturity in any MPCE must be flat. A natural question concerns whether one should expect an MPCE to converge to a stable distribution in the long-run. Unfortunately, it is difficult to establish such conditions in general, but one can do so for specific examples. This is because both under commitment and under lack of commitment, the Lucas and Stokey (1983) problem is not concave, which makes it difficult to predict how government policy sequentially responds to different maturity structures.

The below proposition constructs a tractable example where concavity in the government’s problem allows us to prove that government debt maturity converges to a stable and flat distribution in the long-run.

**Proposition 2** (transition to flat maturity) Suppose that preferences satisfy (20), $b = 0$ and $\bar{b} = \gamma cd$, and $B = \{b_k\}_{k=1}^{\infty}$ satisfies $b_{k+1} \leq b_k \ \forall k \leq m$ and $b_{k+1} = b_k \ \forall k > m$. Then:

1. $V(B) = W(B)$ so that welfare under any MPCE and under commitment coincide, and

2. $B' = \{b'_k\}_{k=1}^{\infty}$ is uniquely determined and satisfies $b'_{k+1} \leq b'_k \ \forall k \leq m-1$ and $b_{k+1} = b_k \ \forall k > m-1$. Therefore, government debt maturity converges to a stable and flat distribution in finite time.

The proof of this result relies on backward induction. We have already established in Proposition 1 that starting from a flat maturity structure, meaning with $b_{k+1} = b_k$
\( \forall k \geq 1 \), the government necessarily chooses the commitment solution with a smooth policy. Now consider the problem of the government starting from a nearly flat maturity structure with \( b_{k+1} = b_k \ \forall k \geq 2 \). Such a government would like to choose a smooth policy from tomorrow onward. Moreover, it knows that by issuing a flat maturity structure appropriately, tomorrow’s government will pursue a smooth policy. As such, today’s government is able to achieve the commitment optimum. Now consider the government starting from a nearly flat maturity structure with \( b_{k+1} = b_k \ \forall k \geq 3 \). Such a government would like a temporarily unstable policy in one period, followed by a stable policy from two periods onward. This can be achieved by issuing a one period bond together with a flat maturity for all future dates. Backward induction on this argument implies that the government issues some uneven maturities on the short end and a flat maturity on the long end, and as time passes, the maturity structure becomes flatter and flatter.

Note that in this example, initial debt is positive and also weighted more heavily on the short end. As such, optimal policy follows a strategy of reducing the market value of newly-issued debt by reducing short-term interest rates. Since preferences satisfy (20), this is achieved with lower taxes along the transition path relative to the long-run. Therefore, short-term interest rates are lower along the transition path relative to the long-run, and the government sells long-term debt at a temporarily high price along the transition path. Note that in constructing the equilibrium, we impose boundaries on the initial maturities of debt \([b, \bar{b}]\). Together with the assumption on preferences, these boundaries on debt together with the monotonicity of initial debt positions guarantee that the problem of the government at all dates is concave so that characterization of the MPCE is feasible.

Our arguments behind Proposition 2 are analogous to those of Lucas and Stokey (1983). They argue that, if faced with the appropriately constructed inherited maturity structure, a government with full commitment from tomorrow onward will choose the same policy as a government with full commitment today. However, in contrast to their work, we allow the government to reoptimize at all future dates in an infinite horizon—not only the next date—with bond prices today responding to anticipation of policy in the future. Moreover, we analyze the entire set of MPCE’s, not only the ones which coincide with the optimal ex-ante policy under full commitment. As such, our result in Proposition 2 is a uniqueness result which shows that an MPCE which coincides with the commitment solution not only exists, but is the unique MPCE. This uniqueness result is achieved through backward induction given a unique stable distribution of debt established in Proposition 1.
5 Quantitative Exercise

What is the welfare benefit of pursuing optimal sequential policy of gradually transitioning to a flat maturity structure? In this section, we explore this question by comparing welfare under a sequentially optimizing government which gradually flattens the maturity of debt to welfare under a naive government which rolls its debt over in every period. To pursue this objective, we begin by extending our framework to incorporate economic growth and nominal bonds, which are the most common types of bonds issued by governments in practice.\footnote{We currently exploring additional extensions of this framework which incorporate capital accumulation.}

5.1 Incorporating Growth and Nominal Bonds

To incorporate growth and nominal bonds, let preferences satisfy (20). Suppose that the production function is given by $A_t^* n_t$, where $A_t^*$ denotes productivity, which is assumed to grow at a constant rate $\phi$, so that $A_t = (1 + \phi) A_{t-1}$. Suppose further that government spending at date $t$ is exogenous and equals $A_t g$, so that it grows at the same rate as productivity. Moreover, suppose that in the household and government budget constraints (3) and (4) there are nominal bonds $b_{t,k}^{nom}$ and $B_{t,k}^{nom}$, respectively, which are isomorphic to the real bonds we have considered. Finally, the price level $p_t$ which represents the price of consumption is exogenous and grows at a constant inflation rate $\pi$ so that $p_t = (1 + \pi) p_{t-1}$.

This extended environment is mathematically equivalent to our benchmark economy where at every date $t$ the government chooses normalized consumption $c_t/A_t$, labor $n_t$, and a normalized portfolio of real government debt maturities $\{ B_{t,k}^{nom} / (A_t p_{t+k}) \}_{k=1}^{\infty}$. Given the preferences in (20), welfare under a given sequence $\{c_t/A_t, n_t\}_{t=0}^{\infty}$ is independent of the inflation rate $\pi$ and depends only on productivity through an exogenous constant. Therefore, the transition path in an MPCE for $\{c_t/A_t, n_t\}_{t=0}^{\infty}$ is independent of productivity growth $\phi$.

Since all of the results in our benchmark economy hold, it follows that the unique stable distribution of government debt involves a flat normalized portfolio of real government debt maturities, where

$$\frac{B_{t,k+1}^{nom}}{A_{t+k+1} p_{t+k+1}} = \frac{B_{t,k}^{nom}}{A_{t+k} p_{t+k}} \forall k \geq 1. \quad (25)$$

The optimal maturity structure therefore converges to one where consols are issued, and
satisfy
\[ B_{t+1,k}^{\text{nom}} = (1 + \phi) (1 + \pi) B_{t,k}^{\text{nom}} \forall k \geq 1, \]  
so that the coupon associated with these consols grows at the rate of nominal output. We now consider the welfare benefits of transitioning to such a maturity structure.

5.2 Policy under a Naive Government

Measuring the welfare benefits of the optimal policy requires establishing a benchmark. To construct this benchmark, we consider policy under a naive government which chooses a constant tax rate and rolls over its bonds in every period. Operationally, this means that the law of motion for nominal bonds satisfies

\[ B_{t+1,k}^{\text{nom}} = (1 + \phi) (1 + \pi) B_{t,k}^{\text{nom}} \forall t, \forall k \geq 1 . \]  

Under such a naive policy, normalized consumption and labor are also constant since households anticipate a stable government policy.

Let us consider an economy with the following parameters:

\[ \{ \phi = .02, \pi = .02, \beta = .9808, g = .2, \eta = 1, \gamma = 2 \} . \]

Productivity growth \( \phi \) equals the U.S. average productivity growth since 1950. Inflation \( \pi \) is chosen under the assumption of a 2 percent inflation expectation. The value of \( \beta \) implies a stable real interest rate of 4 percent. We choose \( g \) so that in this benchmark economy, government spending is 20 percent of GDP, in line with U.S. federal outlays since 1950. Our value of \( \gamma \) implies a Frisch elasticity of 1.\(^8\)

We choose initial debt \( \{ B_{k=1}^{\text{nom}} \}_{k=1}^{\infty} \) so that the market value of debt under a constant real interest rate of 4 percent equals 60 percent of GDP. We choose the distribution of maturities to match the maturity structure of U.S. federal nominal treasury bonds in 2007.\(^9\) For this exercise, we match the sum of all nominal payments—coupons and principal—due at various horizons from the perspective of 2007.\(^10\) Figure 1 displays the maturity structure of marketable U.S. federal nominal treasury bonds in 2007, showing that the maturity is heavily weighted towards the short end.

\(^8\)The parameter \( \eta \) only scales the level of output and does not affect the results.

\(^9\)Similar results emerge in more recent years. We chose 2007 since it pre-dates the maturity management performed by the Federal Reserve during periods of quantitative easing.

\(^10\)We exclude TIPS since we focus on nominal payments. We obtain similar patterns if we include TIPS and adjust for expected inflation.
The dynamics of the benchmark economy are very simple since taxes are constant and the government’s primary surplus is used to pay the interest on its debt which it rolls over from period to period. Having chosen the parameters of the benchmark economy, we now describe the features of the optimal sequential policy.
5.3 Welfare Benefit of Optimal Policy

Figure 2 describes the path of the optimal sequential policy and compares it to the naive policy.\textsuperscript{11} Panel A shows that under the optimal sequential policy, initial taxes are lower and long-run taxes are higher relative to the naive policy, which involves constant taxes. Moreover, as Panel B shows, while short-term real interest rates are constant under a naive policy, the expectation of higher taxes and lower consumption in the future under the optimal policy depresses interest rates along the transition path relative to interest rates in the long-run. Panel C shows that the market value of government debt rises over time, and panel D shows that this increase in government debt comes together with a gradual flattening out of the maturity structure. Given that the longest initial maturity is thirty years, the complete transition to a flat maturity occurs after thirty years along the transition path.

Figure 3 displays the welfare implications of the optimal policy relative to the naive policy. Adopting the optimal debt maturity structure is associated with sizeable short-term gains, at the expense of some long-term losses. For instance, in the first year the optimal policy implies an increase in output of 1.38 percent and an increase in welfare equivalent to 0.34 percent of consumption relative to the naive policy. These short-term welfare gains are achieved through a temporary cut in taxes at the expense of higher long-run taxes, which in turn leads to a reduction in output of 0.08 percent and a reduction in long-term welfare equivalent to 0.02 percent of consumption at an horizon of 30 years or longer. The benefit of the temporary tax reduction outweighs the cost of the long-run tax increase because the initial maturity structure is heavily weighted towards the short-end.\textsuperscript{12} More specifically, the path of fiscal policy temporarily reduces short-term interest rates relative to the naive policy, which allows the government to reduce the cost of rolling over its debt. In this setting, the price of consols is higher along the transition path relative to the long-run, and the government takes advantage of this high price to flatten out its debt maturity and issue consols. Since the interest rates are lower along the transition path, the government takes this opportunity to increase its liabilities as seen in Panel C.

This exercise suggests that gradually switching to consols is a cheap mean of financing temporary tax breaks which boost welfare and stimulates the economy in the short-run.

\textsuperscript{11}For this exercise, we check that the problem of the government is concave at all dates. Using arguments similar to those of Proposition 2, this implies that the MPCE coincides with the ex-ante commitment solution.

\textsuperscript{12}These short-term gains and long-term costs quantitatively balance each other out, since the overall welfare gain is 0.00008 percent of consumption.
Figure 2: Equilibrium Dynamics under Alternative Policies

Notes: The figure displays the equilibrium dynamics under the benchmark (naive) policy (dashed line) and the optimal policy (solid line).
Figure 3: Gains from Optimal Maturity Management

Notes: The figure displays the gains (+) and losses (-) in terms of welfare (left panel) and the output (right panel) under the optimal policy relative to the naive policy.


6 Concluding Remarks

We have established conditions under which, in the long-run, the unique stable debt maturity structure is flat, with the government owing the same amount of resources to the private sector at all future dates. Our analysis provides a theoretical foundation for the optimal use of consols in government debt management. We also find that gradually switching to consols could be a useful way to stimulate economic activity and welfare, particularly in the short-run.

Our analysis leaves several interesting avenues for future research. We have considered a situation in which government’s objective is to minimize its financing costs. In practice, government debt management offices also pursue other objectives, such as supporting financial stability. For example, this can be achieved either by providing liquidity to segments of the market which lack it or through the bond auction process which itself may serve a purpose of aggregating financial market information. How these factors matter for the optimal maturity management of government debt is an interesting question for future research.
References


Appendix

A.1 Proofs

Proof of Lemma 1

The necessity of these conditions is proved in the text. To prove sufficiency, let the government choose the associated level of debt \( \{ \{ b_{t,k} \} \}_{t=0}^{\infty} \) which satisfies (9) and a tax sequence \( \{ \tau_t \}_{t=0}^{\infty} \) which satisfies (8). Let bond prices satisfy (8). (9) given (1) implies that (3) and (4) are satisfied. Therefore household optimality holds and all dynamic budget constraints are satisfied along with the market clearing, so the equilibrium is competitive. ■

Proof of Lemma 2

Note that if \( b_k = b \) \( \forall k \), then from Assumption 1, the solution under commitment admits \( \{ c_t, n_t \} = \{ c^*(b), n^*(b) \} \) \( \forall t \), and this solution can be implemented with \( b'_k = b \) given (18) – (19). Since the MPCE satisfies the same constraints of the problem under commitment plus additional constraints regarding sequential optimality, it follows that

\[
W(B) = \frac{u(c^*(b), n^*(b))}{1 - \beta} \geq V(B) \quad (A.1)
\]

if \( b_k = b \) \( \forall k \). Now consider optimal policy under the MPCE in (12) – (14) given \( b_k = b \) \( \forall k \). A government has the option of choosing \( c = c^*(b) \) and \( n = n^*(b) \) together with \( b'_k = b \) \( \forall k \). This satisfies the resource constraint (13) and the implementability constraint (14) since in an MPCE, \( B' = B \) implies that \( h^k(B') = \beta^k u_c (c^*(b), n^*(b)) \) \( \forall k \) for \( h^k(B') \) defined in (11). Therefore, it follows that

\[
V(B) \geq u(c^*(b), n^*(b)) + \beta V(B). \quad (A.2)
\]

Equations (A.1) and (A.2) imply that

\[
V(B) = W(B). \quad (A.3)
\]

By Assumption 1, \( W(B) \) is uniquely characterized by \( \{ c_k, n_k \} = \{ c^*(b), n^*(b) \} \) \( \forall k \). Therefore, it follows that any solution to (12) – (14) given \( b_k = b \) \( \forall k \) admits \( c = c^*(b) \) and \( n = n^*(b) \). ■
Proof of Lemma 3

Conditional on $B$, if a solution admits $b'_k = b_k$, then this means that $B$ is an absorbing state with $B = B'$ and consumption and labor are constant and equal to some $\{c, n\}$ from that period onward. Therefore, $h^k(B') = \beta^k u_c(c, n) \forall k \geq 1$ for $h^k(B')$ defined in (11). As such, (14) can be rewritten as

$$u_c(c, n) c + u_n(c, n) n - u_c(c, n) b_1 + u_c(c, n) \sum_{k=1}^{\infty} \beta^k (b'_k - b_{k+1}) = 0 \quad (A.4)$$

which combined with (22) and the fact that $b'_k = b_k$ implies that

$$u_c(c, n) c + u_n(c, n) n = u_c(c, n) \tilde{b}. \quad (A.5)$$

Now consider the solution to the following problem given $\tilde{b}$:

$$\max_{c, n} \frac{u(c, n)}{1 - \beta} \quad \text{s.t.} \quad c + g = n \quad \text{and} \quad (A.5). \quad (A.6)$$

It is necessary that $V(B)$ be weakly below the value of (A.6). This is because the solution to $V(B)$ also admits a constant consumption and labor (as in the program in (A.6)) and since the constraint set in (A.6) is slacker, since the program ignores the role of government debt in changing future policies. Note furthermore that the value of (A.6) equals $W(\{b_k\}_{k=1}^{\infty} \mid b_k = \tilde{b} \forall k)$, where this follows from Assumption 1. Therefore,

$$V(B) \leq W(\{b_k\}_{k=1}^{\infty} \mid b_k = \tilde{b} \forall k). \quad (A.7)$$

Now consider the welfare of the government in the MPCE if, instead of choosing $b'_k = b_k \forall k$, it instead chooses $b'_k = \tilde{b} \forall k$ with $c = c^*(\tilde{b})$ and $n = n^*(\tilde{b})$. It follows from Lemma 2 that under this perturbation, $h^k(B') = \beta^k u_c(c^*(\tilde{b}), n^*(\tilde{b})) \forall k \geq 1$, which implies that the resource constraint (13) and implementability constraint (14) are satisfied under this deviation. Because the continuation value associated with this deviation is $W(\{b_k\}_{k=1}^{\infty} \mid b_k = \tilde{b} \forall k)$, it follows that for this deviation to be weakly dominated:

$$W(\{b_k\}_{k=1}^{\infty} \mid b_k = \tilde{b} \forall k) \leq V(B). \quad (A.8)$$

Given (A.7) and (A.8), it follows that $W(\{b_k\}_{k=1}^{\infty} \mid b_k = \tilde{b} \forall k) = V(B)$. Therefore, given $B$, there exists another solution to (12) $-$ (14) with $b'_k = \tilde{b} \forall k$ which achieves the same welfare.
Proof of Lemma 4

We prove this by contradiction. By Lemma 3,

\[ V(B) = W(\{b_k\}_{k=1}^\infty) |_{b_k = \tilde{b} \forall k} = \frac{u(c^*(\tilde{b}), n^*(\tilde{b}))}{1 - \beta} \]  

(A.9)

for \( \tilde{b} \) defined in (22). Now suppose that \( b_1 \neq \tilde{b} \). Given the definition of \( \tilde{b} \), this means that \( \tilde{b} \in (\underline{b}, \bar{b}) \). Now suppose that the government deviates to \( b_k' = \tilde{b} \neq \hat{b} \forall k \) so that from tomorrow onward, consumption is \( c^*(\hat{b}) \) and labor is \( n^*(\hat{b}) \), where this follows from Lemma 2. This means that \( h_k(\hat{B}) = \beta u_c(c^*(\hat{b}), n^*(\hat{b})) \) under the deviation. In order to satisfy the resource constraint and implementability condition, let the government deviate today to a consumption and labor allocation \( \{\tilde{c}, \tilde{n}\} \) which satisfies

\[ \tilde{c} + g = \tilde{n} \]  

(A.10)

and

\[ u_c(\tilde{c}, \tilde{n})\tilde{c} + u_n(\tilde{c}, \tilde{n})\tilde{n} - (u_c(\tilde{c}, \tilde{n}) - u_c(c^*(\hat{b}), n^*(\hat{b})))b_1 = u_c(c^*(\hat{b}), n^*(\hat{b}))(\tilde{b} + \frac{\beta}{1 - \beta}(\hat{b} - \tilde{b})) \]  

(A.11)

where we have appealed to the definition of \( \tilde{b} \) in (22). For such a deviation to be weakly dominated, it must be that

\[ V(B) \geq u(\tilde{c}, \tilde{n}) + \beta W(\{b_k\}_{k=1}^\infty) |_{b_k = \tilde{b} \forall k} . \]  

(A.12)

Clearly, the value of the right hand side of (A.12) equals \( V(B) \) if \( \tilde{b} = \hat{b} \). Therefore, it must be that \( \tilde{b} = \hat{b} \) with \( \{\tilde{c}, \tilde{n}\} = \{c^*(\hat{b}), n^*(\hat{b})\} \) maximizes the right hand side of (A.12) subject to (A.10), and (A.11). More specifically, we can consider the solution to the following program

\[ \max_{\tilde{c}, \tilde{n}, \hat{b}} u(\tilde{c}, \tilde{n}) + \beta W(\{b_k\}_{k=1}^\infty) |_{b_k = \tilde{b} \forall k} \text{ s.t. (A.10) and (A.11).} \]  

(A.13)

For the deviation to not strictly increase welfare, \( \tilde{b} = \hat{b} \) must be a solution to (A.13). By Assumption 1, \( W(\{b_k\}_{k=1}^\infty) |_{b_k = \tilde{b} \forall k} = u(c^*, n^*)/(1 - \beta) \) where \( \{c^*, n^*\} = \{c^*(\hat{b}), n^*(\hat{b})\} \) are the unique levels of consumption and labor which maximize welfare given \( \hat{b} \) and are defined in (18) and (19). Letting \( \mu_1 \) represent the Lagrange multiplier on the implementability condition for the program defining \( W(\{b_k\}_{k=1}^\infty) |_{b_k = \tilde{b} \forall k} \) in (15) – (17), it follows from first
order conditions that

\[
\mu_1 \left( u_c(c^*, n^*) + u_n(c^*, n^*) + u_c(c^*, n^*) + u_n(c^*, n^*) + u_{cc}(c^*, b^*) (c^* - \tilde{b}) + u_{cn}(c^*, n^*) (c^* - \tilde{b} + n^*) + u_{nn}(c^*, n^*) n^* \right) = 0
\]

(A.14)

Since \( \{c^*, n^*\} \neq \{c^{fb}, n^{fb}\} \) by the statement of the lemma, (A.14) implies that \( \mu_1 \neq 0 \). Using this observation, implicit differentiation of (18) and (19) taking (A.14) into account implies

\[
c'^*(\tilde{b}) = n'^*(\tilde{b}) = -\mu_1 \frac{u_c(c^*, n^*)}{u(c^*, n^*) + u_n(c^*, n^*)}.
\]

(A.15)

Finally, by the Envelope condition,

\[
\frac{dW(\{b_k\}_{k=1}^\infty)_{\mid b_k = \tilde{b}} \forall k}{db} = -\mu_1 \frac{u_c(c^*, n^*)}{1 - \beta}.
\]

(A.16)

Now consider the solution to (A.13). Let \( \mu_0 \) correspond to the Lagrange multiplier on (A.11). First order conditions with respect to \( \tilde{c} \) and \( \tilde{n} \) imply

\[
\mu_0 \left( u_c(\tilde{c}, \tilde{n}) + u_n(\tilde{c}, \tilde{n}) + u_c(\tilde{c}, \tilde{n}) + u_n(\tilde{c}, \tilde{n}) + u_{cc}(\tilde{c}, \tilde{n})(\tilde{c} - b_1) + u_{cn}(\tilde{c}, \tilde{n})(\tilde{c} - b_1 + \tilde{n}) + u_{nn}(\tilde{c}, \tilde{n}) \tilde{n} \right) = 0
\]

(A.17)

Since \( \{\tilde{c}, \tilde{n}\} \neq \{c^{fb}, n^{fb}\} \) by the statement of the lemma, (A.17) implies that \( \mu_0 \neq 0 \). Since the solution admits \( \tilde{b} = \hat{b} \in (b, \bar{b}) \), then we can ignore the bounds on government debt, and first order conditions with respect to \( \tilde{b} \) taking into account (A.15) and (A.16) yields

\[
\mu_0 \mu_1 \frac{u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*)}{u_c(c^*, n^*) + u_n(c^*, n^*)} \left( \hat{b} - b_1 + \frac{\beta}{1 - \beta} (\hat{b} - \tilde{b}) \right) + (\mu_0 - \mu_1) \frac{\beta}{1 - \beta} = 0
\]

(A.18)

Now consider if \( \tilde{b} = \hat{b} \) so that \( \{\tilde{c}, \tilde{n}\} = \{c^*, n^*\} \). In that case, use (A.14) and (A.17) to substitute in for \( \mu_0 \) and \( \mu_1 \) into (A.18):

\[
(u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*)) (\hat{b} - b_1) = 0.
\]

(A.19)

If it were the case that \( \hat{b} \neq b_1 \), then (A.19) would imply that \( u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*) = 0 \), which from (A.18) would imply that \( \mu_0 = \mu_1 \). However, in this case, (A.14) and (A.17) given \( \{\tilde{c}, \tilde{n}\} = \{c^*, n^*\} \) would imply that \( \hat{b} = b_1 \), yielding a contradiction. Therefore,
\( \hat{b} = b_1 \).\hfill \blacksquare

**Proof of Lemma 5**

Suppose that \( b_l = \hat{b} \forall l \leq m \). Given \( B \), let \( \hat{B}(1) \) represent the portfolio which sets \( \hat{b}_k = b_{k+1} \) so that no retraining takes place. Note that in such a portfolio, \( \hat{b}_1 = b_2 \). Define \( \hat{B}(2) \) analogously as the portfolio involving no retraining at the next date, so that \( \hat{b}_k = b_{k+2} \) under \( \hat{B}(2) \), and define \( \hat{B}(l) \forall l \leq m \) analogously. In any MPCE for which \( b_1 = \hat{b} \), a possible deviation sets \( \{c, n\} = \{c^*(\hat{b}), n^*(\hat{b})\} \) and \( b'_k = b_{k+1} \) so that no retraining takes place, where this deviation satisfies the resource constraint and implementability condition given (18) – (19). For such a deviation to be weakly dominated, it is necessary that:

\[
V(B) \geq u(c^*(\hat{b}), n^*(\hat{b})) + \beta V(\hat{B}(1)).
\]

(A.20)

Forward induction on this argument implies that

\[
V(B) \geq \beta \sum_{l=0}^{m-1} \beta^l u(c^*(\hat{b}), n^*(\hat{b})) + \beta^m V(\hat{B}(m)).
\]

(A.21)

Combining (A.9) with (A.21), we achieve

\[
V(B) \geq V(\hat{B}(m)).
\]

(A.22)

Now consider optimal policy starting from \( \hat{B}(m) \). Note that since \( b_l = \hat{b} \forall l \leq m \), then following the same arguments as in the proof of Lemma 3, a feasible strategy starting from \( \hat{B}(m) \) is to issue a flat debt maturity with all bonds equal to \( \hat{b} \). Such a strategy ensures a constant consumption and labor allocation forever equal to \( \{c^*(\hat{b}), n^*(\hat{b})\} \). As such, it follows that (A.22) holds with equality and that choosing a flat maturity structure going forward is optimal.

Now we prove by contradiction that \( b_{m+1} = \hat{b} \). Suppose it were the case that \( b_{m+1} \neq \hat{b} \). This means that starting from \( \hat{B}(m) \), the immediate debt which is owed by the government does not equal \( \hat{b} \). If this is the case, then the same arguments as those in the proof of Lemma 4 imply that there exists a deviation from the government’s equilibrium strategy at \( \hat{B}(m) \) which can strictly increase the government’s welfare. However, if this is the case, (A.22) which holds with equality is violated. Therefore, it must be that \( b_{m+1} = \hat{b} \).\hfill \blacksquare
Proof of Proposition 1 and Corollaries 1 and 2

The proof of Proposition 1 follows directly by induction after appealing to Lemmas 4 and 5.

To prove the first corollary, note that for the statement of Proposition 1 to be false, it is necessary that \{c, n\} = \{c^{fb}, n^{fb}\}. However, if this is the case, then (A.4) implies that

\[
c^{fb} + \frac{u_n(c^{fb}, n^{fb})}{u_c(c^{fb}, n^{fb})} n^{fb} = -g = \sum_{k=1}^{\infty} \beta^{k-1} (1 - \beta) b_k \geq b
\]  

which contradicts \(b > -g\).

To prove the second corollary, note that from Lemma 2, it is necessary that the continuation equilibrium starting from a flat government debt maturity entail consumption and labor equal to \{c^*(b), n^*(b)\} forever. The arguments in the proof of Lemmas 4 and 5 imply that if the government were to choose a non-flat maturity structure going forward, future governments would not choose \{c^*(b), n^*(b)\} forever. Therefore, all solutions to (12) – (14) admit \(b'_k = b \forall k\).

Proof of Proposition 2

We first prove two useful lemmas to establish this result.

**Lemma 6** Suppose that preferences satisfy (20) and \(B = \{b_{t,k}\}_{k=1}^{\infty}\) satisfies \(b_{t,k} \in [0, \gamma c^{fb}] \forall k \geq 1\). \(W(\{b_{t,k}\}_{k=1}^{\infty})\) is uniquely characterized by the following system of equations for some \(\mu_1 > 0\) representing the Lagrange multiplier on the implementability constraint (17):

\[
\begin{align*}
c_{t+k} + g &= n_{t+k} \forall k \geq 1, \quad (A.24) \\
\frac{1}{c_{t+k}} - \eta \gamma_k^{-1} &= \mu_1 \left( \gamma \eta \gamma_k^{-1} - \frac{b_{t,k}}{c_{t+k}} \right) \forall k \geq 1 \text{ and} \quad (A.25) \\
\sum_{k=1}^{\infty} \beta^{k-1} (1 - \eta \gamma_k) &= \sum_{k=1}^{\infty} \beta^{k-1} \frac{b_{t,k}}{c_{t+k}} \quad (A.26)
\end{align*}
\]

**Proof.** Consider the relaxed problem in (15) – (17), where (17)—which now takes into account the preference assumption in (20)—is replaced with

\[
\sum_{k=1}^{\infty} \beta^{k-1} \left( 1 - \eta \gamma_k - \frac{b_{t,k}}{c_{t+k}} \right) \geq 0, \quad (A.27)
\]

where we have taken into account that the objective is being maximized at date \(t + 1\).
starting from debt maturity \( \{b_{t,k}\}_{k=1}^{\infty} \). The objective function is strictly concave in both arguments. Substitute in for consumption in the implementability condition (A.27) given the resource constraint (16). Using (20), the second derivative of the left hand side of (A.27) with respect to \( n_{t+k} \) is

\[
-\eta \gamma (\gamma - 1) n_{t+k}^{\gamma - 2} - 2 \frac{b_{t,k}}{(n_{t+k} - g)^2} < 0
\]

which means that the constraint set in (A.27) is convex. Therefore, the relaxed problem admits a unique solution.

We can show that constraint (A.27) binds in the relaxed problem, so that the solution to the relaxed problem coincides with the solution to the constrained problem. Suppose that (A.27) does not bind in the relaxed problem. Then first order conditions would imply that \( \{c_{t+k}, n_{t+k}\} = \{c^{fb}, n^{fb}\} \forall k \geq 1 \). However, if that were the case, satisfaction of (A.27) requires \( c^{fb} - n^{fb} = -g \geq (1 - \beta) \sum_{k=1}^{\infty} b_{t,k} \) but this contradicts the fact that \( b_{t,k} \geq b = 0 \). Therefore, the solution is uniquely defined by the first order condition (A.25) and the constraints of the problem (A.24) and (A.26).

This lemma states that under commitment, the consumption and labor allocation is uniquely defined by the system of equations defining constraints and first order conditions. We now use this observation to describe the behavior of the government if it expects the government in the future to follow the commitment policy. The following preliminary result is useful.

**Lemma 7** Suppose that preferences satisfy (20), \( B = \{b_{t-1,k}\}_{k=1}^{\infty} \) satisfies \( b_{t-1,k} \in \left[0, \gamma c^{fb}\right] \forall k \geq 1 \), and \( B = \{b_{t-1,k}\}_{k=1}^{\infty} \) satisfies \( b_{t-1,k+1} \leq b_{t-1,k} \forall k \leq m \) and \( b_{t-1,k+1} = b_{t-1,k} \forall k > m \), where \( b_{t-1,k+1} < b_{t-1,k} \) for some \( k \). The solution under commitment \( \{c_{t+k}, n_{t+k}\}_{k=0}^{\infty} \) admits

\[
1 - \eta n_t^{\gamma_t} - \frac{b_{t-1,1}}{c_t} < 0, \quad (A.28)
\]

\[
c_{t+k} \leq c^{fb} \text{ and } n_{t+k} \leq n^{fb} \forall k \geq 0, \quad (A.29)
\]

\[
\eta \gamma n_{t+k}^{\gamma-1} - \frac{b_{t-1,k+1}}{c_{t+k}^2} \geq 0 \forall k \geq 0, \text{ and } \quad (A.30)
\]

\[
\eta \gamma n_{t+k}^{\gamma-1} - \frac{b_{t-1,k+1}}{c_{t+k}^2} < \eta \gamma n_{t+k+1}^{\gamma-1} - \frac{b_{t-1,k+2}}{c_{t+k+1}^2} \forall k \geq 0 \quad (A.31)
\]

31
Proof. We proceed by establishing each part separately.

**Proof of (A.28). Step 1.** Suppose by contradiction that

\[ 1 - \eta m^{\gamma} t - \frac{b_{t-1,1}}{c_t} \geq 0. \quad (A.32) \]

The contradiction assumption implies that

\[ -\eta \gamma m^{\gamma-1} t + \frac{b_{t-1,1}}{c_t^2} < 0. \quad (A.33) \]

To see why, consider the first order conditions which characterize the problem, where \( \mu \) represents the Lagrange multiplier on the implementability condition (17):

\[
\frac{1}{c_{t+k}} - \eta m^{\gamma-1}_{t+k} = \mu \left( \eta \gamma n^{\gamma-1}_{t+k} - \frac{b_{t-1,k+1}}{c_{t+k}^2} \right) \quad \forall k \geq 0.
\]

(A.34)

Note that by the arguments of the proof of Lemma 6, \( \mu > 0 \). Now if (A.33) does not hold, then (A.34) for \( k = 0 \) would imply that

\[
\frac{1}{c_t} - \eta m^{\gamma-1}_t \leq 0 \leq -\eta \gamma m^{\gamma-1}_t + \frac{b_{t-1,1}}{c_t^2}
\]

which means that

\[
(\gamma - 1) \eta m^{\gamma-1}_t \leq \frac{b_{t-1,1} - c_t}{c_t^2}
\]

which can only hold if \( b_{t-1,1} \geq c_t \). But if this is the case, (A.32) cannot hold. Therefore, (A.33) must be satisfied.

**Step 2.** If (A.33) holds, then

\[-\eta \gamma n^{\gamma-1}_{t+k} + \frac{b_{t-1,k+1}}{c_{t+k}^2} < 0 \quad \forall k \geq 0. \quad (A.35)\]

To see why, implicit differentiation of (A.34) implies that

\[
\frac{\partial n_{t+k}}{\partial b_{t-1,k+1}} \bigg|_{\mu} = \frac{\partial c_{t+k}}{\partial b_{t-1,k+1}} \bigg|_{\mu} = \mu \frac{1}{1 + \eta (\gamma - 1) c_{t+k}^2 n_{t+k}^{\gamma-2} (1 + \gamma \mu) + 2 \mu \frac{b_{t-1,k+1}}{c_{t+k}}} > 0 \quad (A.36)
\]

This means that the left hand side of (A.34) decreases in \( b_{t-1,k+1} \), holding \( \mu \) constant. Therefore, the right hand side must also decrease in \( b_{t-1,k+1} \). Since \( b_{t-1,1} \geq b_{t-1,k} \) \( \forall k \), it follows from (A.33) that (A.35) must hold.
Step 3. We now show that

\[ 1 - \eta n_{t+k}^{\gamma} - \frac{b_{t-1,k+1}}{c_{t+k}} \leq 1 - \eta n_{t+k+1}^{\gamma} - \frac{b_{t-1,k+2}}{c_{t+k+1}} \]  

(A.37)

Suppose (A.37) does not hold. This would imply that

\[ (-\eta n_{t+k}^{\gamma} - \frac{b_{t-1,k+1}}{c_{t+k}}) - (-\eta n_{t+k+1}^{\gamma} - \frac{b_{t-1,k+1}}{c_{t+k+1}}) > \frac{b_{t-1,k+1}}{c_{t+k+1}} - \frac{b_{t-1,k+2}}{c_{t+k+1}} \]  

(A.38)

The fact that \( b_{t-1,k+1} \geq b_{t-1,k+2} \) implies that the right hand side of (A.38) weakly exceeds 0. Moreover, (A.35)—and the fact that the left hand side of (A.35) is decreasing in \( c_{t+k} \) and \( n_{t+k} \)—together with (A.36) implies that the left hand side of (A.38) is weakly below 0. Therefore, (A.38) cannot hold. Moreover, this inequality is strict if \( b_{t-1,k+2} < b_{t-1,k+1} \).

Step 4. Since \( b_{t-1,k+1} \geq b_{t-1,k} \) for all \( k \) and strict for some \( k \), it follows from (A.37) that

\[ 1 - \eta n_{t+k}^{\gamma} - \frac{b_{t-1,1}}{c_{t}} \leq 1 - \eta n_{t+k}^{\gamma} - \frac{b_{t-1,k+1}}{c_{t+k}} \ \forall k \geq 1 \]  

(A.39)

where this inequality must be strict for some \( k \). Substitution of (A.39) into the implementability condition (17) establishes (A.28).

Proof of (A.29) and (A.30). Suppose that (A.30) is violated so that

\[ n_{t+k}^{\gamma} - \frac{b_{t-1,k+1}}{c_{t+k}} < 0. \]  

(A.40)

From the arguments in the proof of Lemma 6 \( \mu > 0 \) in (A.34). Therefore, if (A.30) does not hold, this implies that \( c_{t+k} > c^{fb} \) and \( n_{t+k} > n^{fb} \). Substituting in the bounds on debt together with the fact that \( c_{t+k} > c^{fb} \) and \( n_{t+k} > n^{fb} \) implies that (A.40) becomes

\[ \gamma c^{fb} < \eta c_{t+k}^{\gamma} n_{t+k}^{\gamma-1} < b_{t-1,k+1} \]

However, this violates the fact that \( b_{t-1,k+1} \leq \bar{b} = \gamma c^{fb} \). This means that (A.30) must hold. From (A.34) this means that \( c_{t+k} \leq c^{fb} \) and \( n_{t+k} \leq n^{fb} \).


We proceed in three steps.

Step 1. Given \( \{b_{t-1,k}\}_{k=1}^{\infty} \) at date \( t \) which satisfies \( b_{t-1,k+1} \leq b_{t-1,k} \ \forall k \leq m \) and \( b_{t-1,k+1} = b_{t-1,k} \ \forall k > m \), suppose that from \( t+1 \) onward, the government chooses the commitment solution. Using Lemma 6 note that the objective of the government at date
is to solve the following program:

$$
\max_{c_t,n_t,b_{t,k} \in [b,\infty]} \mu_1 \left\{ \sum_{k=0}^{\infty} \beta^k \left( \log c_{t+k} - \eta n_{t+k}^{\gamma} \right) \right\}
$$

s.t.

$$
c_t + g = n_t \quad \text{and} \quad (A.24), \quad (A.42)
$$

$$
\sum_{k=0}^{\infty} \beta^k (1 - \eta n_{t+k}^{\gamma}) = \sum_{k=0}^{\infty} \beta^k b_{t-1,k+1} c_{t+k},
$$

and (A.25) − (A.26).

We can show that the solution to this problem equals \( W (\{b_{t,k}\}_{k=1}^{\infty}) \). This program is similar to the program under commitment with the exception of the additional constraints in (A.44). Therefore, if constraints (A.44) are ignored, the problem achieves the unique commitment solution. Let us consider the relaxed problem which ignores (A.44). First order conditions to the relaxed problem, letting \( \mu_0 \) represent the Lagrange multiplier on (A.43) yield

$$
\frac{1}{c_{t+k}^{\gamma-1}} - \eta n_{t+k}^{\gamma-1} = \mu_0 \left( \gamma \eta n_{t+k}^{\gamma-1} - \frac{b_{t-1,k+1}}{c_{t+k}^{2}} \right) \quad \forall k \geq 0.
$$

For conditions (A.44) to be satisfied, let \( \{b_{t,k}\}_{k=1}^{\infty} \) and \( \mu_1 \) satisfy

$$
b_{t,k} = b_{t-1,k+1} + \left( 1 - \frac{\mu_0}{\mu_1} \right) c_{t+k} \left( \eta \gamma n_{t+k}^{\gamma-1} - \frac{b_{t-1,k+1}}{c_{t+k}^{2}} \right) \quad \forall k \geq 1
$$

and

$$
\mu_1 = \frac{\sum_{k=1}^{\infty} \beta^k c_{t+k} \left( \eta \gamma n_{t+k}^{\gamma-1} - \frac{b_{t-1,k+1}}{c_{t+k}^{2}} \right)}{\sum_{k=1}^{\infty} \beta^k c_{t+k} \left( \eta \gamma n_{t+k}^{\gamma-1} - \frac{b_{t-1,k+1}}{c_{t+k}^{2}} \right) - \left( \frac{b_{t-1,1}}{c_t} - (1 - \eta n_{t}^{\gamma}) \right) \mu_0}
$$

It can be verified that (A.46) and (A.47) satisfy (A.44) given (A.43) and (A.45). Note that \( \mu_1 \neq 0 \) since this would imply by (A.25) that \( c_{t+k} = c^{fb} \) and \( n_{t+k} = n^{fb} \) \forall k, and this is not possible by the proof of Lemma 6. If the values of \( b_{t,k} \) are feasible and between 0 and \( \gamma c^{fb} \), then this establishes that welfare equals that under commitment \( W (\{b_{t-1,k+1}\}_{k=1}^{\infty}) \).

**Step 1a.** We first verify that \( b_{t,k} \leq \gamma c^{fb} \). Suppose instead that \( b_{t,k} > \gamma c^{fb} \) for some \( k \). From Lemma 7 \( c_{t+k} \leq c^{fb} \) and \( n_{t+k} \leq n^{fb} \). Therefore, \( b_{t,k}/c_{t+k} \geq \gamma \). Substituting this
into the implementability condition in (A.26), this means that
\[
\sum_{k=1}^{\infty} \beta^{k-1} (1 - \eta n_{t+k}) = \sum_{k=1}^{\infty} \beta^{k-1} \frac{b_{t+k}}{c_{t+k}} \geq \frac{\gamma}{1 - \beta},
\]
but this is a contradiction since \(1 - \eta n_{t+k} < 1 < \gamma\). Therefore, the values of \(b_{t,k}\) are feasible.

\textbf{Step 1b.} We now check that \(b_{t,k} \geq 0\). Suppose by contradiction that this is not the case. Then we can establish that \(\mu_1 > 0\). This is because the left hand side of (A.25) is weakly positive, which means that for (A.25) to hold with \(\mu_1 \leq 0\), this would require
\[
\gamma \eta n_{t+k} \leq \frac{b_{t+k}}{c_{t+k}} \leq 1 - \eta n_{t+k},
\]
\[
\gamma \eta n_{t+k} - \frac{b_{t+k}}{c_{t+k}} \leq 0,
\]
which means that \(b_{t,k} > 0\), which violates the contradiction assumption. From (A.30), the numerator in (A.47) is weakly positive. Therefore, \(\mu_1 \geq \mu_0\), and (A.46) implies that \(b_{t,k} \geq 0\).

\textbf{Step 2.} Suppose that for any \(B = \{b_k\}_{k=1}^{\infty}\) which satisfies \(b_{k+1} \leq b_k \forall k \leq m - 1\) and \(b_{k+1} = b_k \forall k > m - 1\), \(V(B) = W(B)\). Then, for any \(B = \{b_k\}_{k=1}^{\infty}\) which satisfies \(b_{k+1} \leq b_k \forall k \leq m\) and \(b_{k+1} = b_k \forall k > m\), \(V(B) = W(B)\). To see why, consider the problem of the government starting from \(B = \{b_k\}_{k=1}^{\infty}\) which satisfies \(b_{k+1} \leq b_k \forall k \leq m\) and \(b_{k+1} = b_k \forall k > m\). Suppose that the government chooses the policy and the debt maturity as in step 1. It then follows that \(V(B') = W(B')\) which implies that \(V(B) = W(B)\). We are left to check that \(b'_{k+1} \leq b'_k \forall k \leq m - 1\) and \(b'_{k+1} = b'_k \forall k > m - 1\). The fact that \(b'_{k+1} = b'_k \forall k > m - 1\) follows from \(b_{k+1} = b_k \forall k > m\) and (A.46) which defines newly issued debt maturities. To verify that \(b'_{k+1} \leq b'_k \forall k \leq m - 1\), note that this follows from \(b_{k+1} \leq b_k \forall k \leq m\) and (A.46), taking into account that \(\mu_1 \geq \mu_0 > 0\) from step 1 together with (A.30), (A.31), and (A.36).

\textbf{Step 3.} From Proposition 1, for any \(B = \{b_k\}_{k=1}^{\infty}\) which satisfies \(b_{k+1} = b_k \forall k > 0\), \(V(B) = W(B)\). By induction then using step 2, this means that \(\forall B = \{b_k\}_{k=1}^{\infty}\) which satisfies \(b_{k+1} = b_k \forall k > m\), \(V(B) = W(B)\). This establishes part 1 of the proposition. Part 2 of the proposition follows from the construction in step 2, taking into account that \(\mu_1 > 0\) is uniquely defined by (A.47).