

## Long Run Risk, the Wealth-Consumption Ratio, and the Temporal Pricing of Risk

By RALPH S.J. KOIJEN, HANNO LUSTIG, STIJN VAN NIEUWERBURGH,  
AND ADRIEN VERDELHAN\*

Representative agent consumption based asset pricing models have made great strides in accounting for many important features of asset returns. The long run risk (LRR) models of Ravi Bansal and Amir Yaron (2004) are a prime example of this progress. Yet, several other representative agent models, such as the external habit model of John Y. Campbell and John H. Cochrane (1999) and the variable rare disasters model of Xavier Gabaix (2008), seem to be able to match a similar set of asset pricing moments. Additional moments would be useful to help distinguish between these models. Hanno Lustig, Stijn Van Nieuwerburgh, and Adrien Verdelhan (2009) argue that the wealth-consumption ratio is such a moment. A comparison of the wealth-consumption ratio in the LRR model and in the data is favorable to the LRR model. This is no small feat because the wealth-consumption ratio is not a target in the usual calibrations of the model, and the LRR is—so far—the sole model able to reproduce both the equity premium and the wealth-consumption ratio. The LRR model matches the properties of the wealth-consumption ratio despite the fact that it implies a negative real bond risk premium. This is because it generates quite a bit of consumption cash flow risk to offset the negative discount rate risk.

This can be seen in long horizon variance ratios for consumption. So relative to the data, the consumption cash flow risk is too high and the discount rate (which is close to the long horizon real bond risk premium) seems too low.

Because of a lack of data, it is hard to assess directly whether a negative real bond risk premium is counterfactual. Yet, we know that the bond risk premium at long horizons contains crucial information about the properties of the pricing kernel. In particular, Fernando Alvarez and Urban Jermann (2005) show that the ratio of the infinite bond risk premium to the maximum risk premium is linked to the fraction of the variance of the pricing kernel that arises from the martingale component. This decomposition of the pricing kernel is model free. Like the Lars P. Hansen and Ravi Jagannathan (1991) bound, this moment directly describes a property of the pricing kernel and links it to observable asset return characteristics. The low (nominal) bond risk premium and high equity risk premium in the data suggest that most of the shocks to the pricing kernel are shocks to the martingale component.

Since the bond market data are nominal in nature, we augment the LRR model with an inflation process and study the properties of the long horizon nominal bond risk premium. We show that the long run risk model, which is successful at matching the wealth-consumption ratio, high equity risk premium, and the nominal yields at short maturities, implies too little (much) variation in the martingale component of the nominal (real) pricing kernel. This is because the nominal bond risk premium at infinite horizon is too high, or in other words because the real bond risk premium at infinite horizon is too low and thus the inflation risk premium too high. We conclude that the wealth-consumption ratio, the equity risk premium, and the long horizon bond risk premium impose tight restrictions on dynamic asset pricing models.

\*Kojien: The University of Chicago Booth School of Business, 5807 South Woodlawn Avenue, Chicago, IL 60637 (e-mail: [Ralph.Kojien@chicagobooth.edu](mailto:Ralph.Kojien@chicagobooth.edu)); Lustig: Anderson School of Management, University of California at Los Angeles, Box 951477, Los Angeles, CA 90095 (e-mail: [hlustig@anderson.ucla.edu](mailto:hlustig@anderson.ucla.edu)); Van Nieuwerburgh: Department of Finance, Stern School of Business, New York University, 44 W. 4th Street, New York, NY 10012 (e-mail: [svnieuwe@stern.nyu.edu](mailto:svnieuwe@stern.nyu.edu)); Verdelhan: Department of Finance, MIT Sloan School of Management, and Bank of France, E52–436, 50 Memorial Drive, Cambridge, MA 02142 (e-mail: [adrienv@mit.edu](mailto:adrienv@mit.edu)). We thank Rustom Irani for excellent research assistance and Ravi Bansal, Jaroslav Borovicka, Mike Chernov, Itamar Drechsler, John Heaton, Lars Hansen, Jose Scheinkman, Ivan Shaliastovich and George Tauchen for comments.

### I. Stock and Bond Risk Premia in the Long Run Risk Model

The long run risk literature works off the class of preferences due to David M. Kreps and Evan L. Porteus (1978) and Larry G. Epstein and Stan E. Zin (1989). Let  $U_t(C_t)$  denote the utility derived from consuming  $C_t$ . The value function of the representative agent takes the following recursive form:

$$U_t(C_t) = \left[ (1 - \delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}.$$

The time discount factor is  $\delta$ , the risk aversion parameter is  $\gamma \geq 0$ , and the intertemporal elasticity of substitution (IES) is  $\psi \geq 0$ . The parameter  $\theta$  is defined by  $\theta \equiv (1 - \gamma)/(1 - (1/\psi))$ . When  $\psi > 1$  and  $\gamma > 1$ , then  $\theta < 0$ , and agents prefer early resolution of uncertainty.

On the technology side, we adopt the specification of Bansal and Ivan Shaliastovich (2008) for consumption growth, dividend growth, and inflation:

$$\begin{aligned} \Delta c_{t+1} &= \mu_g + x_t + \sigma_{gt} \eta_{t+1} \\ x_{t+1} &= \rho x_t + \sigma_{xt} e_{t+1} \\ \sigma_{g,t+1}^2 &= \sigma_g^2 + \nu_g(\sigma_{gt}^2 - \sigma_g^2) + \sigma_{gw} W_{g,t+1} \\ \sigma_{x,t+1}^2 &= \sigma_x^2 + \nu_x(\sigma_{xt}^2 - \sigma_x^2) + \sigma_{xw} W_{x,t+1} \\ \Delta d_{t+1} &= \mu_d + \phi_x x_t + \varphi_d \sigma_{gt} \eta_{d,t+1} \\ \pi_{t+1} &= \bar{\pi}_t + \varphi_{\pi g} \sigma_{gt} \eta_{t+1} + \varphi_{\pi x} \sigma_{xt} e_{t+1} \\ &\quad + \sigma_{\pi} \xi_{t+1} \\ \bar{\pi}_{t+1} &= \mu_{\pi} + \alpha_{\pi}(\bar{\pi}_t - \mu_{\pi}) + \alpha_x x_t \\ &\quad + \varphi_{zg} \sigma_{gt} \eta_{t+1} + \varphi_{zx} \sigma_{xt} e_{t+1} \\ &\quad + \sigma_z \xi_{t+1}. \end{aligned}$$

All shocks are independent and identically distributed standard normal, except  $\text{Corr}(\eta_{t+1}, \eta_{d,t+1}) \equiv \tau_{gd}$ . This specification builds on Bansal and Yaron (2004) and delivers empirically plausible stock and nominal bond prices. Tim Bollerslev, George Tauchen, and Hao Zhou (2009) show that heteroskedasticity is key to reproduce asset pricing moments in the LRR framework. Real consumption growth

contains a persistent long run expected growth component  $x_t$ . Shocks to (short run) consumption growth have a stochastic volatility  $\sigma_{g,t+1}^2$ . As in Bansal and Shaliastovich (2007), this volatility differs from the conditional variance of the long run component  $x_t$ , which is denoted  $\sigma_{xt}^2$ . The inflation process is similar to that in Jessica Wachter (2006) and Monika Piazzesi and Martin Schneider (2006).

For our numerical results, we use the calibration of Bansal and Shaliastovich (2007), repeated in Table A1 in the online Appendix.<sup>1</sup> Table A2 summarizes the model loadings on state variables. The model matches several key features of aggregate consumption and dividend growth, as well as inflation.

A central object in the LRR model is the log wealth-consumption ratio,  $wc_t \equiv w_t - c_t$ . It is the price-dividend ratio of a claim on aggregate consumption. It is affine in the state variables  $x_t$ ,  $\sigma_{gt}^2$  and  $\sigma_{xt}^2$ :

$$\begin{aligned} wc_t &= \mu_{wc} + W_x x_t + W_{gs} (\sigma_{gt}^2 - \sigma_g^2) \\ &\quad + W_{xs} (\sigma_{xt}^2 - \sigma_x^2). \end{aligned}$$

The Appendix derives the coefficients  $W_x$ ,  $W_{gs}$  and  $W_{xs}$  as functions of the structural parameters. When the IES exceeds one, an increase in expected consumption growth and a decrease in short run or long run consumption volatility increase the wealth-consumption ratio. The log real stochastic discount factor (SDF) can now be written as a function of log consumption growth and the change in the log wealth-consumption ratio:

$$\begin{aligned} sdf_{t+1} &= [\theta \log \delta + (\theta - 1) \kappa_0^c] - \gamma \Delta c_{t+1} \\ &\quad + (\theta - 1)(wc_{t+1} - \kappa_1^c wc_t), \end{aligned}$$

where  $\kappa_0^c$  and  $\kappa_1^c$  are linearization constants, which are a function of the long run average log wealth-consumption ratio  $\mu_{wc}$ . Note that when  $\theta = 1$  ( $\gamma = (1/\psi)$ ), the above recursive preferences collapse to the standard power utility preferences, and changes in the wealth-consumption ratio are not priced. The only priced shocks

<sup>1</sup> We assume that the continuation values exist. See Hansen (2008) and Jaroslav Borovicka et al. (2009) for more on this question.

TABLE 1—RISK PREMIA

	Mean	SD	AR(1)
			Data
<i>WC</i>	88.59	14.11	0.96
<i>PD</i>	27.53	7.20	0.95
<i>ERP</i>	6.90	9.54	0.92
<i>BRP</i> <sup>s</sup>	0.92	1.04	0.89
Model			
<i>WC</i>	48.97	12.59	0.99
<i>PD</i>	21.71	12.17	0.99
<i>ERP</i>	6.25	0.49	0.99
<i>BRP</i> <sup>s</sup>	2.97	0.46	0.99

*Notes:* This table reports the mean, standard deviation and autocorrelation of the annualized wealth-consumption ratio (*WC*), price-dividend ratio (*PD*), equity risk premium (*ERP*), and the five-year nominal bond risk premium (*BRP*<sup>s</sup>). The moments from the data are in the upper panel and are taken from Lustig et al. (2009). They pertain to the period 1953–2008. The lower panel reports the moments obtained from model simulations.

are short run consumption growth shocks  $\eta_{t+1}$ . Hence, the empirical failures of the power utility model and the successes of the LRR model must be attributable to their respective implications for the wealth-consumption ratio. Lustig, Van Nieuwerburgh and Verdellhan (2009) estimate the wealth-consumption ratio in the data, using a preference free no-arbitrage approach. Table 1 shows that the LRR model's implications are broadly consistent with the data. In particular, the LRR model implies that the claim to aggregate consumption is not very risky, resulting in a high mean wealth-consumption ratio of 50 and a low consumption risk premium.

Next, we turn to stock prices. Like the wealth-consumption ratio, the price-dividend ratio of the claim to aggregate dividends is affine in the same three state variables. The bulk of the risk premium is compensation for long run consumption risk, and short run, and long run consumption growth volatility risk. Table 1 shows that the model matches the properties of the price-dividend ratio and the equity risk premium well. Because the dividend claim has more exposure to long run risk ( $\phi_x > 1$ ), it ends up being much riskier than the consumption claim. This is reflected in a low price-dividend ratio of 22 and a high equity risk premium of 6.25 percent per year.

Finally, the log price of a  $n$ -period nominal bond is affine in the same three state variables, as well as in expected inflation  $\bar{\pi}_t$ . Expected

inflation (short run volatility) unambiguously increases (decreases) nominal bond yields. The effect of long run growth (long run volatility) on nominal yields is positive (negative) at short maturities but negative (positive) at long maturities. These sign reversals at long maturities do not arise for real yields; they result from a negative correlation between expected inflation and long run growth. Consistent with the findings of Bansal and Shaliastovich (2007), Table A3 shows that the LRR model matches the one-year to five-year nominal bond yields well. The yield levels are close to the average yields in the Fama-Bliss data for 1952–2008, and the five-minus-one year yield spread of 1.18 percent is reasonably close to the historical 0.56 percent spread.

However, the same table shows that nominal yields on longer horizon bonds are very high in the model. For example, the difference between the 30-year and the 5-year bond yield is 6.44 percent per year. The same spread between constant maturity Treasury yields in the 1952–2008 data is only 0.33 percent. Hypothetical 200-year nominal yields are 20 percent per year in the model. Likewise, the nominal bond risk premium increases sharply with maturity. Table 1 shows that the five-year nominal bond risk premium is 2.97 percent, which is substantially higher than the 0.92 percent premium we estimate in the data. Table A3 shows that the one-year risk premium on a 200-year bond is as high as 24.4 percent. In the next section, we connect the high nominal bond risk premium at very long maturities to one of the components of a decomposition of the SDF. The bottom panel of Table A3, which is for real yields, is informative about the origins of the high nominal yields and risk premia. It shows that the real yield curve is downward sloping. Real bond risk premia are negative at all horizons and are as low as  $-16$  percent at the 200-year maturity. Real bonds are a hedge in the LRR model because their returns are high in those states of the world where the representative agent's intertemporal marginal rate of substitution is high (long run growth is low or economic uncertainty is high). To generate an upward sloping nominal yield curve, inflation risk must more than offset this hedging effect. Current and future inflation are unexpectedly high exactly when long run growth is unexpectedly low ( $\varphi_{\pi,x} < 0$  and  $\varphi_{z,x} < 0$ ), generating a capital loss on the bond in high marginal

utility states of the world. When the inflation risk is calibrated to match nominal yield data for maturities of one through five years, it also implies a very high nominal bond risk premium at very long horizons.

## II. Decomposing the SDF

Let the SDF be the growth rate of the pricing kernel:  $SDF_{t+1} = M_{t+1}/M_t$ . Following Alvarez and Jermann (2005), Hansen, John C. Heaton, and Nan Li (2008), and Hansen and Jose A. Scheinkman (2009), we study a factorization of the SDF. Under mild regularity conditions, any pricing kernel  $M$  can be decomposed in two parts:  $M_t = M_t^p M_t^T$ . The first component,  $M_t^p$ , is a martingale  $E_t[M_{t+1}^p] = M_t^p$ , and the second component  $M_t^T$  is defined as:

$$M_t^T = \lim_{\tau \rightarrow \infty} \frac{\beta^{t+\tau}}{P_t(\tau)},$$

for some number  $\beta$ .<sup>2</sup>  $M_t^T$  is the dominant pricing component for long term bonds. We obtain expressions for both components of the SDF, as well as for their logs. We do this decomposition both for the nominal and for the real SDF, where the nominal log SDF is  $sdf_{t+1}^S = sdf_{t+1}^{S,p} - \pi_{t+1}$ . We focus on the nominal decomposition here.

We define the *conditional variance ratio*  $\omega_t$  as the ratio of the conditional variance of the martingale component of the nominal log SDF to the conditional variance of the entire nominal log SDF:

$$\begin{aligned} \omega_t &= \frac{\text{Var}_t[sdf_{t+1}^{S,p}]}{\text{Var}_t[sdf_{t+1}^S]} \\ &= \frac{1 - \mathbb{E}_t[r_{t+1}^{b,S,e}(n)] - \frac{1}{2} \text{Var}_t[r_{t+1}^{b,S,e}(n)]}{\max_i \mathbb{E}_t[r_{t+1}^{i,S,e}(n)] - \frac{1}{2} \text{Var}_t[r_{t+1}^{i,S,e}(n)]}. \end{aligned}$$

<sup>2</sup> For related work, see Bansal and Bruce N. Lehman (1997), Hansen et al. (2008), Hansen and Scheinkman (2009), Hansen (2008), Borovicka et al. (2009), and Ralph S.J. Koijen, Lustig, and Van Nieuwerburgh (2009) derive this decomposition for affine models. The Appendix applies their insights to the LRR model, which belongs to this class. Hansen and Scheinkman (2009) and Hansen (2008) give parametric examples within the affine class in which uniqueness fails. We do not study this issue here.

We show in the Appendix that  $\omega_t$  equals one minus the ratio of the log bond risk premium on a nominal infinite maturity bond (without Jensen adjustment) to the maximum nominal risk premium in the economy (without Jensen adjustment).

Alvarez and Jermann (2005) show that, in a model without the martingale component, the infinite horizon bond is the highest risk premium in the economy. Conversely, in a model with just the martingale component, bond risk premia of all maturities are zero, and the yield curve is flat. Hence, to have realistic term structure implications, the SDF cannot have only a martingale component, but the variation of  $M_t^T$  must not be too large. In the data, long horizon nominal bond risk premia are low compared to, say, equity risk premia. Hence, the data discipline  $\omega_t$  to be close to one on average. Alvarez and Jermann (2005) argue that this conclusion holds both for nominal and for real bonds. An important caveat, though, is that risk premia on bonds with infinite horizons are not precisely measured because such bonds do not exist, and actual long term bonds might offer convenience yields.

Table A4 reports moments of the SDF and its components for the benchmark LRR calibration. Not surprisingly, the martingale component of the SDF is more volatile than the dominant pricing component,  $M_t^T$ . Our key finding is that the nominal variance ratio  $\omega_t$  is very low: only 0.37 on average. The reason is that in the LRR model the long horizon nominal bond risk premium is very high, relative to the maximum nominal risk premium in the economy. Because the real bond risk premium is highly negative, the real variance ratio is much higher than one: 1.66 on average. Hence, the LRR model fails to generate a conditional variance ratio which is close to one. Inflation introduces too much volatility in the dominant pricing component of the nominal SDF.

This conditional variance ratio is tightly linked to the dynamics of the wealth-consumption ratio. With power utility, e.g constant relative risk aversion preferences (CRRA), the change in the log wealth-consumption ratio is no longer a priced factor in the log SDF, and the real SDF now has only the martingale component. When  $\theta = 1$ , the real bond risk premium is zero at all maturities. The nominal bond risk premium and maximum risk premium are very small. While the average variance ratio  $\omega$  is closer to the data, the power utility model

generates an equity risk premium puzzle and a nominal interest rate of 20 percent per year for the one- through five-year yields, both of which are highly counterfactual.

Our analysis raises the question of whether a change in the calibration of the LRR model may solve these issues. In the Appendix, we consider changes both on the real and on the nominal side of the economy. The variance ratio  $\omega_t$  changes noticeably with  $\rho_x$ ,  $\alpha_x$ , and  $\alpha_\pi$ . However, we find it difficult to obtain a calibration that successfully matches the ratio  $\omega_t$ , its components, and all the moments of consumption growth, inflation, and equity and bond returns.

### III. Conclusion

Matching the wealth-consumption ratio and the  $\omega$  ratio is a challenge for dynamic asset pricing models. This challenge is not unique to the LRR model, but equally applies to the habit and the rare disasters model. Future research should investigate how these models can be modified to match the variance ratio. Nonneutrality of inflation is an interesting avenue for future research.

### REFERENCES

- Alvarez, Fernando, and Urban J. Jermann.** 2005. "Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth." *Econometrica*, 73(6): 1977–2016.
- Bansal, Ravi, and Bruce N. Lehman.** 1997. "Growth Optimal Portfolio Restrictions on Asset Pricing Models." *Macroeconomic Dynamics*, 1(2): 333–54.
- Bansal, Ravi, and Ivan Shaliastovich.** 2007. "Risk and Return in Bond, Currency, and Equity Markets." Unpublished.
- Bansal, Ravi, and Amir Yaron.** 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance*, 59(4): 1481–509.
- Bollerslev, Tim, George Tauchen, and Hao Zhou.** 2009. "Expected Stock Returns and Variance Risk Premia." *Review of Financial Studies*, 22(11): 4463–92.
- Borovicka, Jaroslav, Lars Peter Hansen, Mark Hendricks, and Jose A. Scheinkman.** 2009. "Risk Price Dynamics." Unpublished.
- Campbell, John Y., and John H. Cochrane.** 1999. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy*, 107(2): 205–51.
- Epstein, Larry G., and Stanley E. Zin.** 1989. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework." *Econometrica*, 57(4): 937–69.
- Gabaix, Xavier.** 2008. "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance." Unpublished.
- Hansen, Lars Peter.** 2008. "Modeling the Long Run: Valuation in Dynamic Stochastic Economies." <http://home.uchicago.edu/~lhansen/main.pdf>.
- Hansen, Lars Peter, and Ravi Jagannathan.** 1991. "Implications of Security Market Data for Models of Dynamic Economies." *Journal of Political Economy*, 99(2): 225–62.
- Hansen, Lars Peter, and Jose A. Scheinkman.** 2009. "Long-Term Risk: An Operator Approach." *Econometrica*, 77(1): 177–234.
- Hansen, Lars Peter, John C. Heaton, and Nan Li.** 2008. "Consumption Strikes Back? Measuring Long-Run Risk." *Journal of Political Economy*, 116(2): 260–302.
- Koijen, Ralph S.J., Hanno Lustig, and Stijn Van Nieuwerburgh.** 2009. "The Cross-Section and Time-Series of Stock and Bond Returns." Unpublished.
- Kreps, David M., and Evan L. Porteus.** 1978. "Temporal Resolution of Uncertainty and Dynamic Choice Theory." *Econometrica*, 46(1): 185–200.
- Lustig, Hanno, Stijn Van Nieuwerburgh, and Adrien Verdelhan.** 2009. "The Wealth-Consumption Ratio." Unpublished.
- Piazzesi, Monika, and Martin Schneider.** 2007. "Equilibrium Yield Curves." In *NBER Macroeconomics Annual 2006*, ed. Daron Acemoglu, Kenneth Rogoff, and Michael Woodford, 389–442. Cambridge, MA: MIT Press.
- Wachter, Jessica A.** 2006. "A Consumption-Based Model of the Term Structure of Interest Rates." *Journal of Financial Economics*, 79(2): 365–99.