Dynamic OCE Choice: Time Consistency and the Separation of Time and Risk

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Abstract

No existing dynamic preference model can simultaneously satisfy time consistency, the full separation of time and risk preferences, and temporal resolution of risk indifference. In the context of consumption-saving and consumption-portfolio optimization problems, we derive necessary and sufficient conditions such that all three of these properties are satisfied by the dynamic ordinal certainty equivalent (DOCE) preference structure axiomatized in Selden and Stux (1978). These conditions ensure that DOCE resolute, naive and sophisticated consumption and asset demands are (i) identical and (ii) the same as the demands generated by Kreps and Porteus (1978) (KP) preferences. When the conditions are violated, the elasticity of intertemporal substitution can play a key role in determining whether axiomatic differences between the DOCE and KP preference models imply significantly different demand behavior.

KEYWORDS. Kreps-Porteus-Selden preferences, time consistency, separation of time and risk, temporal resolution indifference, consumption-portfolio problem

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1 Introduction

For economic models where consumers solve dynamic optimization problems under risk, assumptions on preferences play a key role in the resulting solutions and their comparative statics. At the level of preferences, the following three properties are often mentioned as being desirable: (i) time consistency (TC), (ii) the ability to fully separate time and risk preferences (SEP) and (iii) the ability to accommodate temporal resolution of risk indifference (TRI). Currently, no single dynamic preference model can simultaneously accommodate all three properties. This paper makes three contributions. First, it provides special conditions under which the three properties can be satisfied. Second, these results are shown to have important implications for the use of the preference models of SS (Selden and Stux 1978) and KP (Kreps and Porteus 1978) in dynamic consumption-saving and consumption-portfolio applications. Third when the conditions do not hold, we show that optimal consumption, saving and asset demand behavior can still be surprisingly similar for the two preference models so long as the consumer’s time preferences exhibit sufficient aversion to intertemporal substitution.

Although KP (1978) motivate the introduction of their recursive preference structure on the basis of being able to accommodate a preference for early or late resolution of uncertainty, Epstein, Farhi and Strzalecki (2014) among others argue that while early resolution can be beneficial in decision making, it may not be desirable to require this property at the pure preference level. In fact in the EZ (Epstein and Zin 1989) homothetic version of KP utility, two parameters govern the seemingly distinct time preferences, risk preferences and a preference for the resolution of uncertainty. However, when setting the parameters at different values to achieve SEP, an analyst loses her ability to control temporal resolution preferences. This limitation has been recognized from the start in Epstein and Zin (1989) and in part motivates us to explore in this paper the prospect of using DOCE (dynamic ordinal certainty equivalent) preferences. As a natural generalization of Selden (1978), DOCE preferences are based on independent risk and time preference building blocks which, respectively, are used to replace risky consumption in each period by certainty equivalent consumption and evaluate the resulting vector of certain and certainty equivalent consumption. Thus, DOCE preferences exhibit SEP. By assumption, they also exhibit TRI. In contrast to KP preferences, the attitude toward the resolution of risk is independent of the form of time and risk preferences as well as their interrelationship. However as suggested by Johnsen and Donaldson (1985), DOCE preferences in general violate time consistency. It should be noted that although the KP and DOCE utilities
in general differ, they become ordinally equivalent in a two period setting where the first period is certain. The common representation is typically referred to as the KPS (Kreps-Porteus-Selden) utility.

Given that in general, neither KP nor DOCE preferences can simultaneously satisfy TC, SEP and TRI, are there any special circumstances under which either model can satisfy the three properties? We show that in a consumption-saving setting if the distribution of asset returns is independent over time, then the consumer’s demands will be time consistent if and only if her underlying building block representations of time and risk preferences both exhibit homotheticity. In this case, the utilities take the CRRA (constant relative risk aversion) form. As a result, DOCE preferences which are defined over the subset of dynamic consumption trees where consumption along branches exhibits a special proportionality will satisfy TC as well as SEP and TRI. While the restriction that asset returns be independent over time is clearly a special case, the stronger assumption that asset returns are i.i.d. (identically and independently distributed), has been made in a number of important papers such as Levhari and Srinivasan (1969), Samuelson (1969), Weil (1993), Campbell and Cochrane (1999) and Barro (2009). Moreover, the assumption that the representations of time and risk preferences are homothetic has also been widely used for instance in the EZ special case of KP preferences. The intuition for our result is that the combination of independent returns and homotheticity permits the transformation of the choice over a multi-date-event-branch consumption tree into the choice over an equivalent single branch tree analogous to what the consumer confronts in a pure certainty time consistent setting.

It would clearly be desirable to weaken the restriction that time and risk preferences must be homothetic. In fact, it is possible to extend our result to the full class of HARA (hyperbolic absolute risk aversion)\(^1\) time and risk preferences in a consumption-portfolio setting if one adds to independent asset returns the assumptions that one of the available assets is risk free. The quasihomothetic members of the HARA class include the translated CRRA utility used for instance in the external habit model of Campbell and Cochrane (1999) and the familiar CARA (constant absolute risk aversion) form. Both of these can be viewed as being homothetic to translated origins (see Pollak 1971). The risk free asset assumption in our result is crucial in dealing with the translations. If either the homotheticity or HARA conditions is satisfied, DOCE preferences will exhibit TC, SEP and TRI on a restricted domain corresponding to the specific choice problem.\(^2\)

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\(^1\)See Gollier (2001) for a characterization of HARA preferences and their properties.

\(^2\)As discussed in Subsection 3.3, on this restricted domain KP preferences cannot distinguish
Given these results, it is natural to wonder how the time consistent DOCE and KP demands relate to one another assuming the corresponding dynamic preferences are based on the same time and risk preference building blocks and asset returns are independent over time. Since the utilities are not ordinally equivalent and both sets of demands are time consistent, one would expect that the demands would differ due to the corresponding preferences differing in terms of the properties SEP and TRI. However the DOCE and KP demand functions are identical. As a result, under the assumptions outlined above a number of key consumption saving and asset demand properties present in two period KPS applications extend to the dynamic setting. For instance, the classic two period portfolio result that the ratio of risk free to risky asset demands depends only on risk preferences and not time preferences extends to the HARA versions of DOCE and KP preferences. Giovannini and Weil (1989) prove that if asset returns are i.i.d., then the EZ special case of KP preferences result in the same consumption and asset demand behavior as generated by single period EU (expected utility) preferences or the comparable two period KPS (Kreps-Porteus-Selden) preferences. In addition to extending their result to DOCE preferences, we also extend it to the KP case based on the full class of HARA utilities. We weaken the restriction on asset returns from being identical over time and show that the conditions are necessary as well as sufficient.

Given that there is significant empirical evidence suggesting that asset returns deviate from being independent over time, how does this affect the relationship between DOCE and KP demands? Our conclusions extend to the case of infinitesimal perturbations in the independent returns assumption. However once finite perturbations of this assumption occur, DOCE preferences become time inconsistent and it becomes necessary to consider the standard (Strotz-Pollak) resolute, naive and sophisticated solution techniques for consumption-saving and consumption-portfolio problems. It is interesting to note that despite considerable discussion of time inconsistent savings, particularly in the context of quasi-hyperbolic preferences (see, for example, Laibson 1997), relatively little has been written on the saving and asset demand implications of the alternative solution techniques. early and late resolution.

3 We show (in subsection 3.5) that DOCE and KP demands continue to be identical when both preference models are based on the same time inconsistent quasi-hyperbolic time preferences. In this case, for each model resolute, naive and sophisticated choice diverge. However, respectively the DOCE and KP resolute, naive and sophisticated demands are the same.

4 See, for example, Selden (1979), Barsky (1989) and Kimball and Weil (2009).

5 A related argument is made by Kocherlakota (1990).
In order to analyze the case where asset returns are not independent over time, we generally assume a three period setting and the DOCE and KP preferences share the same CES (constant elasticity of substitution) time and CRRA risk preference building blocks. We focus on differences in demand for the resolute, naive and sophisticated DOCE and KP cases, often based on numerical simulations. Our analysis suggests that two quite different sets of conclusions can be obtained depending on the value of the EIS (elasticity of intertemporal substitution). First, when the EIS is in the range of roughly 0.20 to 0.40, as estimated in a number of certainty empirical studies, we find that the KP and DOCE resolute, naive and sophisticated period 1 consumption and asset demands exhibit the same qualitative properties and can be surprisingly close in absolute value. This suggests that axiomatic differences in the two models may not be critical. Second, if the EIS is considerably larger in the range of 1.5 to 2.0, as suggested by calibrations of some finance and macro models, then the DOCE and KP demands can differ quite substantially. In general resolute and KP demands continue to be similar, but DOCE sophisticated and KP demands can diverge significantly in terms of qualitative behavior. This difference is surprising since in a three period setting, the two models follow backward induction and the period 2 conditional saving decisions are the same. In particular as the EIS approaches infinity, the models differ in whether (i) period 1 consumption increases or decreases and (ii) the risk free to risky asset demand ratio dramatically increases or remains more or less unchanged. Both of these differences can be explained by the impact of a strong preference for intertemporal substitution. This effect is clear even in the very simple two period certainty consumption-saving problem. A substitution oriented consumer when confronting returns larger than unity tends to substantially reduce current consumption and greatly increase saving to maximize period 2 consumption. Our results suggest the critical importance of developing experimental laboratory tests to determine whether in risky consumption-saving and consumption-portfolio problems, consumer behavior can best explained by EIS values less than 0.40 or greater than 1.5.

The rest of the paper is organized as follows. In the next section, we introduce notation, definitions and the optimization problems. In Section 3, we provide our main theorems for DOCE preferences to be time consistent and provide results relating DOCE and KP demands. Section 4 provides comparisons of consumption and asset demands for the DOCE resolute, naive, sophisticated and KP cases when asset returns deviate from being independent over time. Section 5 contains

\footnote{For a review of the literature on the size of the EIS, see for instance Attanasio and Weber (2010), Havranek (2015) and Thimme (2017) and the references cited in these papers.}
concluding comments. Proofs are given in Appendix A and supporting materials are provided in Supplemental Appendix B.

2 Preliminaries

2.1 Notation and Definitions

Assume time is indexed by $t = 1, \ldots, T$. Exogenous shocks $s_t$ realize in a finite set $S$. A history of shocks up to some date $t$ is denoted by $s^t = (s_1, s_2, \ldots, s_t)$ and called a date event. Since each chance node in a tree can be reached only through one historical path, we also use $s^t$ to denote a chance node. The notation $s^{t+1} \succ s^t$ refers to the node $s^{t+1}$ following node $s^t$. Let $S$ denote the set of all nodes, $s^t$, of the tree. We consider an agent’s choices over $T$ periods, $t = 1, \ldots, T$.

For simplicity, we often focus on the $T = 3$ case where we use a different notation and denote nodes at $t = 2$ by $(21)$, $(22)$, $\ldots$ and at $t = 3$ by $(31)$, $(32)$, $\ldots$.

We next briefly describe the DOCE utility axiomatized in Selden and Stux (1978). (Their paper, although unpublished, is available on the website of Larry Selden, Columbia University Graduate Business School.) Assume a $T$ period setting, where consumption in period $t = 1$ is certain and risky in periods $t = 2, \ldots, T$. In period $t$, the consumer’s certainty time preferences over degenerate consumption streams $c = (c_t, \ldots, c_T) \ (t \in \{1, \ldots, T\})$ are represented by the the following additively separable utility

$$U_t(c) = u(c_t) + \sum_{i=t+1}^{T} \beta^{i-t} u(c_i),$$

where $0 < \beta < 1$ is the standard discount function. The consumer’s risk preferences in each period $t \in \{2, \ldots, T\}$ are identical and represented by the EU representation

$$\sum_{s^t} \pi(s^t) V(c(s^t)),$$

where $\pi(s^t)$ is the probability of the date-event (node) $s^t$ and $V$ is the NM index. DOCE preferences are assumed to be independent of when risk is resolved. This preference axiom, referred to as TRI, is one important difference from KP preferences described below. The stationary time preference $u$ and NM index $V$ will generally be assumed to satisfy $u' > 0$, $u'' < 0$, $V' > 0$ and $V'' < 0$. In what follows, we use preferences over current and future consumption conditional on the current date-event node being $s^T$.

The period $t$ certainty equivalent evaluated at node $s^T$ is defined by

$$(\tilde{c}_t|s^T) = V^{-1} \left( \sum_{s^t \succ s^T} \pi(s^t|s^T) V(c(s^t)) \right),$$
where \( \pi(s^t|s^\tau) \) is the probability of date-event \( s^t \) conditional on being at node \( s^\tau \). Thus, for a given \( s^\tau \), the DOCE representation is given by

\[
\mathcal{U}(c|s^\tau) = u(c(s^\tau)) + \sum_{t=\tau+1}^{T} \beta^{t-\tau} u(\hat{c}_t|s^\tau).
\]

In period 1, the utility is given by

\[
\mathcal{U}(c) = u(c_1) + \sum_{t=2}^{T} \beta^{t-1} u(\hat{c}_t|s_1).
\]

For the DOCE preference model, (i) risk preferences are constant over time, (ii) there is a complete separation of time and risk preferences corresponding to \( u \) and \( V \) and (iii) EU preferences are a special case where \( u \) and \( V \) are affinely equivalent. EU preferences exhibit the TRI axiom.

Kreps and Porteus (1978) derived the recursive representation

\[
\mathcal{U}(c|s^\tau) = \mathcal{U}(c, \sum_{s^{\tau+1} > s^\tau} \pi(s^{\tau+1}|s^\tau) \mathcal{U}(c|s^{\tau+1})),
\]

where \( \mathcal{U} \) is continuous and strictly increasing.\(^7\) Note that if \( \mathcal{U} \) is linear in the second argument, the KP representation converges to the EU special case. The EZ representation is a special case of the KP utility,\(^8\) where

\[
U(c_t, x) = -\left(\frac{c_t^{-\delta_1} + \beta(-\delta_2 x)^{\frac{\delta_1}{\delta_2}}}{\delta_2}\right)^{\frac{\delta_2}{\delta_1}} \quad \text{and} \quad V_T(x) = -\frac{x^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1).
\]

If \( \delta_1 = \delta_2 = \delta \), the EZ representation converges to the EU function

\[
\mathcal{U}(c|s^\tau) = -\frac{(c(s^\tau))^{-\delta}}{\delta} - E \left[ \sum_{t=\tau+1}^{T} \beta^{t-\tau} \left(\frac{c_t(s^\tau))^{-\delta}}{\delta}\right) \right].
\]

Both the KP and EZ recursive preference structures can accommodate a preference for early or late resolution of risk. However as was mentioned in the prior section, this temporal resolution preference cannot be varied independently from time and risk preferences.

The following special pairs of utilities will be used extensively in our analysis

\[
u(c_t) = -\frac{1}{\delta_1} (c_t - b)^{-\delta_1} \quad \text{and} \quad V(c_t) = -\frac{1}{\delta_2} (c_t - b)^{-\delta_2} \quad (b \geq 0, \delta_1, \delta_2 \geq -1), \quad (2)
\]

\[
u(c_t) = -\frac{\exp(-\kappa_1 c_t)}{\kappa_1} \quad \text{and} \quad V(c_t) = -\frac{\exp(-\kappa_2 c_t)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0) \quad (3)
\]

\(^7\)Unlike the DOCE and EZ cases, the KP preference building blocks are \( U \) and \( \mathcal{U} \). An EU index \( V \) can be induced from the KP utility for the final time period \( T \).

\(^8\)Weil (1990) derived an alternative specialization of the KP preference model.
and
\[ u(c_t) = \frac{1}{\delta_1} (b - c_t)^{-\delta_1} \quad \text{and} \quad V(c_t) = \frac{1}{\delta_2} (b - c_t)^{-\delta_2} \quad (\delta_1, \delta_2 < -1), \]

where for (2) \( c_t > \max(0, b) \) and for (4) \( b > c_t > 0 \). For the NM indices in (2)-(4), respectively, the risk preferences exhibit decreasing, constant and increasing absolute risk aversion. This collection of NM indices is typically referred to as the HARA class. The corresponding certainty utilities are frequently referred to as the Modified Bergson family.\(^9\) One important special case of (2) is the following pair of CES time and CRRA risk preference utilities used in the EZ special case of KP preferences
\[ u(c_t) = \frac{1}{\delta_1} c_t^{-\delta_1} \quad \text{and} \quad V(c_t) = \frac{1}{\delta_2} c_t^{-\delta_2} \quad (\delta_1, \delta_2 > -1), \]

where the \( EIS \) (elasticity of intertemporal substitution) and Arrow-Pratt relative risk aversion measures are given by, respectively,
\[ EIS = \frac{1}{1 + \delta_1} \quad \text{and} \quad -c_t \frac{V''(c_t)}{V'(c_t)} = 1 + \delta_2. \]

For the popular DARA (decreasing absolute risk aversion) case (2), it is standard to interpret \( b > 0 \) as a certain subsistence requirement.\(^10\) For the EU representation incorporating the DARA case where \( \delta_1 = \delta_2 \), Campbell and Cochrane (1999) interpret \( b > 0 \) as an external habit parameter.

### 2.2 Optimization Problems

In this subsection, we formally define the consumption-saving and consumption-portfolio problems and describe the three solution techniques of resolute, naive and sophisticated choice that are typically considered when preferences fail to be time consistent.\(^11\)

At the beginning of each period \( t = 1, \ldots, T - 1 \) there are \( J \) assets available for trade with returns \( R(s^{t+1}) = (R_j(s^{t+1}))_{j=1}^J \geq 0 \) being realized at node \( s^{t+1} \).

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\(^9\)See Pollak (1971) for a description of the Modified Bergson class.

\(^10\)For the DARA case we can have \( b < 0 \), but then the subsistence interpretation does not make sense (see Pollak 1970, p. 748). For the IARA (increasing absolute risk aversion) case (4), \( b \) can be interpreted as a bliss point.

\(^11\)In contrast to assuming resolute, naïve or sophisticated choice, Phelps and Pollak (1968) and Peleg and Yaari (1973) argue that one should think of the time inconsistent choice problem as being equivalent to a game between divergent individuals – myself today and my selves in future periods. In this paper, we do not consider the game theoretic approach. It should be noted that Caplin and Leahy (2006) argue that the sophisticated and game theoretic approaches result in equivalent solutions.
We assume that asset returns preclude arbitrage in that there exist \( \rho(s^t) > 0 \) for all \( s^t \) such that
\[
\sum_{s^{t+1} \succ s^t} \rho(s^{t+1}) R_j(s^{t+1}) = \rho(s^t) \quad \forall s^t, j.
\] (7)

The complete market special case, where the number of assets is the same as the number of states, will be assumed in some but not all of the results derived in this paper. More precisely, complete markets holds if the following assumption is satisfied.

**Assumption [CM]** At each \( s^t, t < T \), the matrix \((R(s^{t+1}))_{s^{t+1} > s^t}\) has rank \( S \).

It should be noted that if Assumption [CM] holds, then \( c(s^{t+1}) \) in eqn. (7) can be interpreted as the contingent claim price for \( c(s^{t+1}) \).

A much weaker assumption which plays a prominent role in our analysis is that there exists a one period risk free asset at each date-event.

**Assumption [RF]** For each \( s^t, t = 1, \ldots, T - 1 \), there exists an \( \omega(s^t) \), where \( \omega = \omega_1, \ldots, \omega_J \), such that
\[
\sum_j \omega_j (s^t) R_j (s^{t+1}) = 1 \quad \forall s^{t+1} \succ s^t.
\]

Note that this assumption is automatically satisfied when markets are complete.

In the next section, we focus on the case where the probability distribution of returns is independent over time. Formally, this is stated as follows.

**Assumption [IR]** Assume that for each \( s^t \), \( R(s^t) = \overline{R}(s_t) \) for some function \( \overline{R}(s_t) \) that might vary with time \( t \), but only depends on the shock, \( s_t \) and does not depend on history. Furthermore, the probabilities satisfy \( \pi(s^t|s^{t-1}) = \overline{\pi}_t(s_t) \) for some function \( \overline{\pi}_t(s_t) \).

An individual is assumed to choose consumption and assets in periods \( t = 1, \ldots, T - 1 \) so as to maximize utility. We assume throughout that the individual has rational expectations in that she knows future asset returns contingent on the nodes.

In period \( t \in \{1, \ldots, T - 1\} \), at the node \( s^t \), denote the demand for asset \( j \in \{1, \ldots, J\} \) by \( n_j(s^t) \) and the vector of asset holdings by \( n = (n(s^1), \ldots, n(s^t)), \ldots, (n(s^T)) \),

where \( n(s^t) = (n_1(s^t), \ldots, n_j(s^t)) \).

Let \( I \) and \( I(s^t) \) denote, respectively, initial income and the income received from investment in period \( t - 1 \) at the beginning of period \( t > 1 \) at the node \( s^t \) and \( I(s^t) = I \) when \( t = 1 \).

The period 1 consumption-portfolio problem is defined as follows
\[
\max_{c,n} U(c) \quad S.T.
\] (8)
\[ c(s^t) = I - \sum_j n_j(s^t), \quad t = 1, \tag{9} \]
\[ c(s^t) = n(s^{t-1}) \cdot R(s^t) - \sum_j n_j(s^t), \quad 2 < t < T, \tag{10} \]
\[ c(s^t) = n(s^{t-1}) \cdot R(s^t), \quad t = T. \tag{11} \]

A special case results when there is a single asset, \( J = 1 \). In this case, the problem can be rewritten as

\[
\max_c U(c) \quad S.T. \tag{12}
\]
\[ I(s_1) = I, \tag{13} \]
\[ I(s^{t+1}) = R(s^{t+1}) (I(s^t) - c(s^t)) \quad (t = 1, \ldots, T - 1, s^{t+1} \succ s^t), \tag{14} \]
\[ c(s^T) = I(s^T). \tag{15} \]

For the consumption-saving and consumption-portfolio problems, it is assumed that in any period \( t \) the consumer can only purchase assets with maturity of one time period. To see these distinctions more clearly, see Examples 1 and 3 below.

To simplify notation, we use \((c^o, n^o)\), \((c^*, n^*)\) and \((c^{**}, n^{**})\) to denote resolute, naive and sophisticated demands, respectively. To facilitate the comparison with KP preferences below, we will use \((c^{KP}, n^{KP})\) to denote the optimal demands corresponding to KP preferences. Consistent with the certainty analysis of Strotz (1956) and Pollak (1968), DOCE demands are said to be time consistent if and only if \((c^o, n^o) = (c^*, n^*) = (c^{**}, n^{**})\), for all prices. Formally, we have the following definitions.\(^{12}\)

**Definition 1** The consumption-portfolio problem (8)-(11) is said to be solved via resolute choice if and only if the agent makes all choices at \( t = 1 \) and these choices are not revised over time as new choices become optimal. Given returns and initial income, we define resolute choice as

\[
(c^o(s^t), n^o(s^t))_{s^t \in S} \left((R(s^t))_{s^t \in S}, I\right) = \arg \max_{c(s^t), n(s^t)} U(c|s^1) \quad S.T. \]
\[ c(s^t) = I - \sum_j n_j(s^t), \quad t = 1, \]
\[ c(s^t) = n(s^{t-1}) \cdot R(s^t) - \sum_j n_j(s^t), \quad 2 < t < T, \]
and
\[ c(s^t) = n(s^{t-1}) \cdot R(s^t), \quad t = T. \]

\(^{12}\)For a more basic discussion in a certainty setting, see Selden and Wei (2016, p. 1916).
Definition 2 The consumption-portfolio problem (8)-(11) is said to be solved via naive choice if and only if the agent reoptimizes and revises her choices every period based on her current period preferences. Naive choice is defined sequentially for \( \tau = 1, 2, \ldots, T \) as

\[
(c^*(s^\tau), n^*(s^\tau)) = (c^*(s^\tau), n^*(s^\tau)) \left( (R(s^t))_{s^t \in S}, I \right)
\]

where 

\[
(c^*(s^t), n^*(s^t))_{s^t \geq s^\tau} \left( (R(s^t))_{s^t \in S}, I \right) = \arg \max_{c(s^t), n(s^t)} U(c|s^\tau) \quad S.T.
\]

\[
c(s^t) = I - \sum_j n_j(s^t), \quad t = \tau,
\]

\[
c(s^t) = n(s^{t-1}) \cdot R(s^t) - \sum_j n_j(s^t), \quad \tau < t < T,
\]

and

\[
c(s^T) = n(s^{t-1}) \cdot R(s^t), \quad t = T.
\]

Definition 3 The consumption-portfolio problem (8)-(11) is said to be solved via sophisticated choice if and only if the agent takes into account her future period preferences when making her choices in earlier periods. The sophisticated choice can be defined recursively for \( \tau = T, T-1, \ldots \) as

\[
(c^{**}(s^\tau), n^{**}(s^\tau)) (I (s^\tau)) = \arg \max_{c(s^\tau), n(s^\tau)} u(c(s^\tau)) + \sum_{t=\tau+1}^T \beta^{t-\tau} u(\hat{c}_t|s^\tau) \quad S.T.
\]

\[
c(s^t) = I(s^t) - \sum_j n_j(s^t), \quad t = \tau,
\]

and

\[
(\hat{c}_t|s^\tau) = V^{-1} \left( \sum_{s^\tau \geq s^t} \pi(s^t|s^\tau) V(c^{**}(s^t)) (n(s^{t-1}) \cdot R(s^t)) \right).
\]

One can also define time consistency at the preference as opposed to the demand level. Denote the continuation of a consumption tree starting from node \( s^t \) by \( c(s \geq s^t) \) which includes consumption at \( s^t \). Then following Epstein and Zin (1989), TC can be defined as follows.

\[\text{\footnote{It should be noted that in general a unique sophisticated choice may not exist in the recursive solution process. However for the utility functions we consider in this paper, a unique solution always exists since (quasi)homotheticity ensures concavity of the corresponding utility functions. Also, note that we have written } U(c|s^\tau) \text{ as a separable form in order to highlight the role of } (\hat{c}_t|s^\tau).}\]
Definition 4 The consumer’s preferences satisfy TC if and only if at time $t$ with some payoff history $s^t$, 
\[ c(s \succ s^{t+1}) \succeq c'(s \succ s^{t+1}) \quad (\forall s^{t+1} \succ s^t) \Rightarrow c(s \succ s^t) \succeq \ c'(s \succ s^t), \]
where $c(s^t) = c'(s^t)$.

In the certainty case, Blackorby, et al. (1973) prove that time consistency holds if and only if each period $t + 1$ utility can be embedded into the period $t$ utility for all $t \in \{1, \ldots, T - 1\}$ utilities. Johnsen and Donaldson (1985) extend this notion to the risky case, where time consistency holds if and only if the future utility function in each state can be embedded into prior periods’ utility functions. Following Blackorby et al. (1973), in the next section we link the demand and preference definitions of time consistency in our setting.

3 Time Consistent DOCE Demand

In this section, we first discuss time consistency when DOCE preferences are homothetic. Then our analysis is extended to the quasihomothetic case to allow for the more general HARA risk preferences. We show that when DOCE demands are time consistent, they can also be rationalized by time consistent KP preferences based on the same assumed building blocks utilities $(u, V)$. This equivalence of demands extends even if the DOCE and KP preferences have the same time inconsistent quasi-hyperbolic time preference $U$. Finally, when DOCE demands are time consistent, a number of properties relating to saving behavior derived for two period KPS preferences also hold for $T$-period DOCE and KP preferences.

3.1 Homothetic Preferences

In this subsection, first necessary and sufficient conditions are given such that DOCE preferences generate time consistent consumption and asset demands. Second, intuition is given for this surprising result.

3.1.1 Main Result

It is easy to see that DOCE preferences will be homothetic if and only if the building block time and risk preference representations take the CES time and CRRA risk preference forms in (5). Then, we have the following result.
Theorem 1  Suppose Assumption [IR] holds and the consumer solves the consumption-portfolio problem (8)-(11). Then DOCE demands are time consistent if and only if

\[ u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1 > -1, \delta_2 \geq -1, \delta_1, \delta_2 \neq 0), \]

\[ u(c) = \ln c \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_2 \geq -1, \delta_2 \neq 0), \]

\[ u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \ln c \quad (\delta_1 > -1, \delta_1 \neq 0), \]

and

\[ u(c) = \ln c \quad \text{and} \quad V(c) = \ln c. \quad (16) \]

It should be noted that eqn. (16) in Theorem 1 corresponds to a time consistent EU special case of DOCE preferences.

At first glance, the theorem seems very surprising: How can DOCE preferences be generally time inconsistent, but still generate time consistent demands when asset returns are independent over time? While our detailed proof of Theorem 1 gives a formal answer to this puzzle, it is useful to understand the issue in the simplest possible case. We argue next that while DOCE preferences are generally time inconsistent, one can find a well defined restricted domain on which the preferences (and corresponding demands) are time consistent.

### 3.1.2 Time Consistent Preferences over Restricted Domains

Assume the three time period tree structure in Figure 1, and denote the nodes by the following sequence of numbers 1, 21, 22, 31 and 32 corresponding naturally to the subscripts for consumption at each node.

Given the fixed tree structure in Figure 1 and a set of probabilities, a given consumption tree is fully characterized by the consumption vector

\[ c = (c_1, c_{21}, c_{22}, c_{31}, c_{32}) \in \mathbb{R}_+^5. \quad (17) \]

The vectors \((c_{21}, c_{31})\), \((c_{22}, c_{32})\) \(\in \mathbb{R}_+^2\), respectively, characterize in a natural way the upper and lower subtrees. Preferences over the full tree, upper subtree and lower subtree consumption vectors are denoted respectively by \(\succeq_1\), \(\succeq_{21}\) and \(\succeq_{22}\). Without loss of generality, it will turn out to be useful to view \(\succeq_{21}\) and \(\succeq_{22}\) as preferences over \(\mathbb{R}_+^5\) with the requirement that \(\forall c, c' \in \mathbb{R}_+^5\) and \(d, d' \in \mathbb{R}_+^2\), if \(c \succeq_{21} c'\) then

\[ d \succeq_{21}' d' \text{ whenever } (c_{21}, c_{31}) = (d_{21}, d_{31}), \ (c'_{21}, c'_{31}) = (d'_{21}, d'_{31}) \]
and if $c \succeq_{22} c'$ then

$$d \succeq'_{22} d' \text{ whenever } (c_{22}, c_{32}) = (d_{22}, d_{32}), \ (c'_{22}, c'_{32}) = (d'_{22}, d'_{32}).$$

In the current setting, the TC Definition 4 specializes to the following. The preference relations $\succeq_1, \succeq_{21}$ and $\succeq_{22}$ are said to satisfy TC over a given domain $I \subset \mathbb{R}_+^5$ if and only if whenever for all $c, c' \in I$ with $c_1 = c'_1$

$$c \succeq_{2s} c' \text{ for } s = 1, 2 \Rightarrow c \succeq_1 c',$$

with $c \succ_1 c'$ if at least one of $\succeq_{21}, \succeq_{22}$ holds strictly.

As noted earlier for $\mathbb{R}_+^5$, DOCE preferences do not satisfy time consistency. As illustrated in Examples 1 and 3 below, there can be significant differences between sophisticated and resolute choice when Assumption [IR] does not hold. However, it turns out that one can restrict the domain of preference so that they become time consistent over the restricted domain. For example, it is easy to see that for any $\bar{c} \in \mathbb{R}_+^5$, DOCE preferences are time consistent over $\{c \in \mathbb{R}_+^5 : c = \alpha \bar{c}, \alpha > 0\}$. It turns out to be more interesting and relevant to consider the set

$$I_0 = \{c \in \mathbb{R}_+^5 : c = (c_1, c_{21}, c_{22}, \alpha c_{21}, \alpha c_{22}), \alpha \in \mathbb{R}_+\}.$$

With homothetic preferences, optimal intertemporal choices will lie in this set if asset returns are independent over time. Assuming homotheticity, the period 1
utility function for DOCE preferences can be written as

\[ U(c) = u(c_1) + \beta u \circ V^{-1} \left( \sum_s \pi_s V(c_{2s}) \right) + \beta^2 u \circ V^{-1} \left( \sum_s \pi_s V(\alpha c_{2s}) \right) \]


\[ = u(c_1) + \beta u \circ V^{-1} \left( \sum_s \pi_s V(c_{2s}) \right) + \beta^2 \left( u \circ V^{-1} \left( \sum_s \pi_s V(c_{2s}) \right) u \circ V^{-1} (V(\alpha)) \right) \]


\[ = u(c_1) + \beta u \circ V^{-1} \left( \sum_s \pi_s V(c_{2s}) \right) (1 + \beta u(\alpha)) \]  

(18)

and depending on which state \( s = 1, 2 \) is realized,

\[ U(c_2|s) = u(c_{2s})(1 + \beta u(\alpha)) \]

But then it is easy to see that for any \( c_{21}, c_{22}, \alpha \) and \( c'_{21}, c'_{22}, \alpha' \), since preferences are homothetic

\[ u(c_{2s})(1 + \beta u(\alpha)) \geq u(c'_{2s})(1 + \beta u(\alpha')) \]

for \( s = 1, 2 \)

if

\[ \beta u \circ V^{-1} \left( \sum_s \pi_s V(c_{2s}) \right) (1 + \beta u(\alpha)) \geq \beta u \circ V^{-1} \left( \sum_s \pi_s V(c'_{2s}) \right) (1 + \beta u(\alpha')) . \]

Thus, we have the following proposition.

**Proposition 1** Assume the set of consumption trees corresponding to Figure 1. Homothetic DOCE preferences are TC in terms of Definition 4 over the domain \( \mathcal{I}_0 \).

Connecting Proposition 1 back to Theorem 1, one can observe that \( \mathcal{I}_0 \) corresponds to the optimal demands derived from Theorem 1. Once it is established that optimal demands lie in \( \mathcal{I}_0 \), this implies that homothetic preferences are TC. It should be noted that although we assume [IR] for asset returns, this does not imply that the consumption distribution is also independent over time. For the two state case, since second period residual income is different for the upper and lower branches, the consumption distribution will be different for the two branches as reflected in the construction of \( \mathcal{I}_0 \).
3.2 HARA Preferences

We next derive necessary and sufficient conditions for demands in the consumption-portfolio problem to be time consistent.

Theorem 2 Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8) - (11). Then DOCE demands are time consistent if and only if

(i) \[ u(c) = -(c-b)^{-\delta_1} \quad \text{and} \quad V(c) = -(c-b)^{-\delta_2} \]

\[ (\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b)) \]

or

\[ u(c) = \ln(c-b) \quad \text{and} \quad V(c) = -(c-b)^{-\delta_2} \]

\[ (\delta_2 > -1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b)) \]

or

\[ u(c) = -(c-b)^{-\delta_1} \quad \text{and} \quad V(c) = \ln(c-b) \]

\[ (\delta_1 > -1, \delta_1 \neq 0, b \in \mathbb{R}, c > \max(0, b)) \]

or

(ii) \[ u(c) = -\frac{\exp(-\kappa_1 c_1)}{\kappa_1} \quad \text{and} \quad V(c) = \frac{\exp(-\kappa_2 c_2)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0); \text{ or} \]

or

(iii) \[ u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 < -1, b > c > 0). \]

Remark 1 It should be noted that (i) for both Theorems 1 and 2, the additive time preference \( U \), eqn. (1), can have an arbitrary period 1 utility \( u_1(c_1) \) which satisfies \( u_1' > 0 \) and \( u_1'' < 0 \) and (ii) for Theorem 2, the cases covered for risk preferences include the full HARA class.

\[ ^{14} \text{Note that for the consumption-portfolio problem, we do not include the} \ \delta_2 = -1 \ \text{case since it results in corner optimal solutions.} \]
What is the intuition in Theorem 2 for why time independent returns and quasihomothetic preferences result in time consistent demands and what role is played by the assumed presence of a risk free asset? First note that, as discussed above in the consumption-portfolio case, we have time consistency for the special tree structure with the homothetic preferences (the CES time and CRRA risk utilities (5)). To see the intuition for the quasihomothetic consumption-portfolio case, assume the specific form of utility in Theorem 2(i). Then note that the risk free asset is used to fund subsistence consumption and a portfolio of assets funds saving and supernumerary consumption. Thus the presence of the risk free asset essentially translates the quasihomothetic case into the homothetic case. For the consumption-saving setting since there is no risk free asset, we have time consistent demands only for homothetic preferences.

3.3 Another Time Consistent Rationalization

We have shown that when appropriate restrictions are imposed on asset markets and DOCE time and risk preferences, demands are time consistent. Suppose that KP preferences are constructed from the same time and risk preference building block utilities (5) as in the time consistent DOCE case and one assumes that asset returns satisfy [IR]. Quite surprisingly, we next show that the two preference relations which are not ordinally equivalent over the full choice space, nevertheless result in the same demands.

**Proposition 2** Suppose Assumption [IR] holds and the consumer solves the consumption-portfolio problem (8)-(11). For DOCE preferences, further assume that

\[ u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0). \]

Then the optimal demands can also be rationalized by KP preferences, where

\[ U(c_t, x) = \left( c_t^{-\delta_1} + \beta(-\delta_2 x)^{\delta_1/2} \right)^{\delta_2/\delta_1} \quad \text{and} \quad V_T(x) = -x^{-\delta_2}. \]

We next show that the DOCE and KP demands are the same for the Theorem 2 case of HARA preferences, assuming independent returns over time and the presence of a risk free asset.

**Proposition 3** Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11). For DOCE preferences,
(i) if we assume that
\[ u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \] \( (\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0, b > c > \max(0,b)) \),
then the optimal demands can also be rationalized by KP preferences, where
\[ U(c_t, x) = -\left( (c_t - b)^{-\delta_1} + \beta (\delta_2 x)^{\frac{\delta_1}{\delta_2}} \right)^{\frac{\delta_2}{\delta_1}} \quad \text{and} \quad V_T(x) = -\frac{(x-b)^{-\delta_2}}{\delta_2}; \]

(ii) if we assume that
\[ u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \] \( (\kappa_1, \kappa_2 > 0) \),
then the optimal demands can also be rationalized by KP preferences, where
\[ U(c_t, x) = -\left( \frac{\exp(-\kappa_1 c_t) + \beta (\kappa_2 x)^{\frac{\kappa_1}{\kappa_2}}}{\kappa_2} \right)^{\frac{\kappa_2}{\kappa_1}} \quad \text{and} \quad V_T(x) = -\frac{\exp(-\kappa_2 x)}{\kappa_2}; \]

(iii) if we assume that
\[ u(c) = (b-c)^{-\delta_1} \quad \text{and} \quad V(c) = (b-c)^{-\delta_2} \] \( (\delta_1, \delta_2 < -1, b > c > 0) \),
then the optimal demands can also be rationalized by KP preferences, where
\[ U(c_t, x) = \left( (b - c_t)^{-\delta_1} + \beta (\delta_2 x)^{\frac{\delta_1}{\delta_2}} \right)^{\frac{\delta_2}{\delta_1}} \quad \text{and} \quad V_T(x) = \frac{(b-x)^{-\delta_2}}{\delta_2}. \]

The intuition for Propositions 2 and 3 is that since Assumption [IR] holds, effectively we do not receive any new information with the passage of the time. Thus the preference for early or late resolution for KP preferences cannot be distinguished from temporal resolution indifference for DOCE preferences. In fact, [IR] rules out the canonical early resolution consumption tree corresponding to the case in Figure 1 where \( c_{21} = c_{22} \)\(^{15}\). Moreover, as proved in Proposition 1, over

\(^{15}\)Suppose a consumer prefers this early resolution tree to a second late resolution consumption tree which has the same \( c_1 \) and \( c_2 \), but risk is resolved at the end of period 2 rather than at the end of period 1 in Figure 1. Then following Kreps and Porteus (1978), she is said to have a preference for early resolution. The assumption that asset returns are independent over time implies that no matter how much is saved in period 1, period 2 income will be the same on the upper and lower branches. Since preferences are also the same on the upper and lower branches, optimal \( c_2 \) and \( c_3 \) will also be the same on the two branches. Thus the restricted domain will necessarily exclude early resolution consumption trees with different \( c_3 \)-values.
the domain $I_0$, DOCE and KP preferences are indistinguishable in terms of time consistency. It is clear that property SEP holds for KP preferences. Moreover, it can be verified that assuming homothetic preferences and Assumption [IR], the KP utility takes the same form as the DOCE utility over the domain $I_0$. A similar argument can be made for quasihomothetic preferences.

3.4 Extension of Two Period KPS Demand Properties

In this subsection, we show that two key demand properties which hold for two period KPS preferences extend to the DOCE setting if the conditions in Theorems 1 and 2 are satisfied. The first relates to precautionary saving which in recent years has received considerable attention in finance and macroeconomics. Gollier (2001, chapter 19) analyzes in a two period setting the properties of excess saving

$$\theta = s_{1}^{\text{risky}} - s_{1}^{\text{certain}},$$

where $s_{1}^{\text{risky}}$ and $s_{1}^{\text{certain}}$ denote, respectively, optimal period 1 saving when the return on the investment asset is risky and certain. The certain return equals the mean of the risky return. Selden and Wei (2018) prove that for KPS preferences corresponding to the CES and CRRA utilities in eqn. (5), the existence of a positive $\theta$ depends on a comparison of the $EIS$ and unity and is independent of the risk aversion parameter $\delta_2$.

**Proposition 4** Suppose Assumption [IR] holds and the consumer solves the consumption-saving problem (12)-(15). Further assume

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1).$$

Then for KP and DOCE preferences, using (6) we have

$$\theta \geq 0 \iff EIS \leq 1,$$

which is the same as the two period case. (The proof of this proposition is provided in Supplemental Appendix B.1.)

One attractive feature of the complete separation of time and risk preferences implicit in the KPS utility corresponding to (5) is that in the classic consumption-portfolio problem, optimal asset ratios are determined by risk preferences and are independent of time preferences. This result extends to the dynamic setting if the conditions in Theorem 2 are satisfied.

**Proposition 5** Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11). In each period $t \in \{1, ..., T - 1\}$,
given the node $s^t$, denote the return on the risk free asset on the branch starting from node $s^t$ by $R_f(s^t)$, \(^{16}\) the demands for risky and risk free assets by $n_f(s^t)$ and $n_f(s^t)$, respectively. If we further assume

(i) 
$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, c > \max(0, b)),$$

then in each period $t \in \{1, \ldots, T-1\}$, \(^{17}\)
$$\frac{n_f(s^t) - \frac{b}{R_f(s^t)}}{n_j(s^t)} = \eta_j(s^t)$$
are the same for KP and DOCE preferences and independent of $\delta_1$ and $\beta$;

(ii) 
$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),$$

then in each period $t \in \{1, \ldots, T-1\}$,
$$n_j(s^t) = \eta_j(s^t)$$
are the same for KP and DOCE preferences and independent of $\kappa_1$ and $\beta$; or

(iii) 
$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, b > c > 0),$$

then in each period $t \in \{1, \ldots, T-1\}$,
$$\frac{\frac{b}{R_f(s^t)} - n_f(s^t)}{n_j(s^t)} = \eta_j(s^t)$$
are the same for KP and DOCE preferences and independent of $\delta_1$ and $\beta$.

**Remark 2** There is a direct connection between Proposition 5 and a widely referenced result in Giovannini and Weil (1989, section 2.5). They prove that corresponding to the EZ special case of KP preferences associated with eqn. (5), the portfolio optimization is identical to that of a single period EU optimization

\[^{16}\text{We use the notation } R_f(s^t) \text{ instead of } R_f(s^{t+1}) \text{ since the risk free rate only depends on the starting node } s^t \text{ and is the same for each } s^{t+1} \succ s^t.\]

\[^{17}\text{If } \delta_i = 0, \text{ one can use } u(c) = \ln(c) \text{ instead of power utility as in Theorem 2. This statement also applies to subsequent results unless indicated otherwise.}\]
and hence is independent of the consumer’s time preference parameters $\delta_1$ and $\beta$. Proposition 5 extends this result to a more general set of KP preferences and establishes the connection to DOCE preferences. Also, Proposition 5 when combined with Theorem 2 can be viewed as providing necessary as well as sufficient conditions for asset ratios to be independent of time preferences since DOCE preferences are time consistent only under the indicated conditions.

### 3.5 Quasi-hyperbolic Time Preferences

So far, we have assumed that the consumer’s time and risk preferences corresponding to a given $(U, V)$-pair do not change over time. Time preferences in future periods are represented by the continuation of the current $U$ and each of the NM indices in periods $2, ..., T$ are equivalent up to a positive affine transformation. The time inconsistency inherent in DOCE preferences is attributable to asset returns failing to be independent over time. Although a bit of a digression, in this subsection we consider the case where time preferences change over time. In each period $t$, $U_t$ takes the quasi-hyperbolic discounted utility form first introduced by Phelps and Pollak (1968)\(^{18}\)

$$U_t(c_t, ..., c_T) = u(c_t) + \gamma \sum_{i=t+1}^{T} \beta^{t-i} u_t(c_i). \quad (19)$$

When $\gamma = 1$, the above utility converges to the discounted utility (1) used outside this subsection. When $\gamma \neq 1$, the period 2 continuation of (19), corresponding to $U_2$, cannot be nested in the period 1 utility $U_1$. This implies that time preferences exhibit changing tastes and in applications such as the certainty consumption-saving problem, resolute, naive and sophisticated demands will diverge.\(^{19}\) The three solution techniques also yield different demands in the consumption-portfolio problem with risky asset returns for both the DOCE and KP cases. We next show quite surprisingly, at least for us, that the equivalence of the DOCE and KP demands established in Proposition 3 extends to the case of quasi-hyperbolic time preferences. That is, respectively the resolute, naive and sophisticated DOCE and KP demand functions are the same.

**Proposition 6** Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11), where $U_t$ takes the quasi-hyperbolic discounted utility form first introduced by Phelps and Pollak (1968)\(^{18}\)

\(^{18}\)In order to be consistent with the use of $\beta$ as the discount function in the rest of this paper, we have interchanged the normal roles of $\beta$ and $\gamma$ typically used in the quasi-hyperbolic discounting literature.

\(^{19}\)The economic implications of the quasi-hyperbolic discounted form have been studied extensively (e.g., Laibson 1997 and Diamond and Koszegi 2003)
form (19). The consumer employs the resolute, naive and sophisticated solution techniques (Definitions 1 - 3) for both the DOCE and KP cases. For DOCE preferences,

(i) if we assume that

\[
 u(c) = -\frac{(c - b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c - b)^{-\delta_2}}{\delta_2}
\]

\((\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0, b \geq 0, c > \max(0, b))\),

then the optimal resolute, naive and sophisticated demands can also be rationalized by KP preferences, where

\[
 U_1(c_1, x) = -\left(\frac{(c_1 - b)^{-\delta_1} + \gamma \beta (-\delta_2 x)^{\frac{\delta_2}{\delta_1}}}{\delta_2}\right)
\]

and

\[
 U_t(c_t, x) = -\left(\frac{(c_t - b)^{-\delta_1} + \gamma \beta (-\delta_2 x)^{\frac{\delta_2}{\delta_1}}}{\delta_2}\right) \quad (t \geq 2) \quad \text{and} \quad V_T(x) = -\frac{(x - b)^{-\delta_2}}{\delta_2};
\]

(ii) if we assume that

\[
 u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),
\]

then the optimal resolute, naive and sophisticated demands can also be rationalized by KP preferences, where

\[
 U_1(c_1, x) = -\left(\frac{\exp(-\kappa_1 c_1) + \gamma \beta (-\kappa_2 x)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2}\right)
\]

and

\[
 U_t(c_t, x) = -\left(\frac{\exp(-\kappa_1 c_t) + \beta (-\kappa_2 x)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2}\right) \quad (t \geq 2) \quad \text{and} \quad V_T(x) = -\frac{\exp(-\kappa_2 x)}{\kappa_2};
\]

(iii) if we assume that

\[
 u(c) = \frac{(b - c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b - c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 < -1, b > c > 0).
\]

then the optimal resolute, naive and sophisticated demands can also be rationalized by KP preferences, where

\[
 U_1(c_1, x) = -\left(\frac{(b - c_1)^{-\delta_1} + \gamma \beta (\delta_2 x)^{\frac{\delta_2}{\delta_1}}}{\delta_2}\right)
\]
\[
U_t(c_t, x) = \left( (b - c_t)^{\delta_1} + \beta (\delta_2 x)^{\delta_2} \right)^{\frac{\delta_2}{\delta_1}} (t \geq 2) \quad \text{and} \quad V_T(x) = \frac{(b - x)^{-\delta_2}}{\delta_2}.
\]

We next show that despite the presence of time inconsistent quasi-hyperbolic time preferences corresponding to (19), the common optimal asset allocation for DOCE and KP preferences is independent of the time preference parameters \((\delta_1, \kappa_1, \gamma, \beta)\) if Assumptions [IR] and [RF] hold. Thus, Proposition 5 extends to the case of quasi-hyperbolic time preferences.\textsuperscript{20}

**Proposition 7** Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11), where \(U_t\) takes the quasi-hyperbolic form (19). In each period \(t \in \{1, \ldots, T - 1\}\), given the node \(s_t\), denote the return on the risk free asset on the branch starting from node \(s_t\) by \(R_f(s_t)\) and the demands for risky and risk free assets by \(n_j(s_t)\) and \(n_f(s_t)\), respectively. If we further assume

(i)

\[
\begin{align*}
\quad u(c) &= \frac{- (c - b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{- (c - b)^{-\delta_2}}{\delta_2} \\
(\delta_1, \delta_2) &= -1, b \geq 0, c > \max(0, b),
\end{align*}
\]

(20)

then in each period \(t \in \{1, \ldots, T - 1\}\),

\[
\frac{n_f(s_t) - \frac{b}{R_f(s_t)}}{n_j(s_t)} = \eta_j(s_t)
\]

are the same for KP and DOCE resolute, naive and sophisticated choice and are independent of \(\delta_1, \gamma\) and \(\beta\);

(ii)

\[
\begin{align*}
\quad u(c) &= -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),
\end{align*}
\]

then in each period \(t \in \{1, \ldots, T - 1\}\)

\[
n_j(s_t) = \eta_j(s_t)
\]

are the same for KP and DOCE resolute, naive and sophisticated choice and are independent of \(\kappa_1, \gamma\) and \(\beta\); or

\textsuperscript{20}It should be noted that versions of this result are shown by Palacios-Huerta and Pérez-Kakabadse (2017) for EU preferences and by Love and Phelan (2015) for the EZ special case of KP preferences. Both papers assume different settings, only investigate sophisticated choice and never consider DOCE preferences.
\begin{align*}
    u(c) &= \frac{(b - c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b - c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, b > c > 0),

\end{align*}

then in each period \( t \)

\[
\frac{b}{n_f(s^t) - n_f(s^t)} = \eta_j(s^t)
\]

are the same for KP and DOCE resolute, naive and sophisticated choice and are independent of \( \delta_1, \gamma \) and \( \beta \).

\section{Departures from Independent Asset Returns}

In this section, we first consider the effect of an infinitesimal change in asset returns from independence on DOCE resolute and sophisticated and KP demands. Second, for finite departures from independence we show that for a strong preference for intertemporal substitution, optimal period 1 consumption (and saving) and portfolio composition (as reflected by the ratio \( n_f/n_1 \))\textsuperscript{21} can exhibit very different behavior for the DOCE sophisticated and KP cases. This is true despite both preference models having exactly the same time and risk preference building blocks \((u, V)\).

\subsection{First Order Change}

We begin by proving that when asset returns depart from independence over time, DOCE demands are still time consistent and agree with KP demands up to a first order. To characterize a departure from independent returns, consider a time \( t \) choice node in a given consumption tree and the asset returns realized in period \( t \). Independence of asset returns over time implies that the returns on each branch coming out of the time \( t \) node are identical. Assume that the asset return distribution on one of the branches is subjected to an infinitesimal change.

The following two propositions are immediate consequences of applying the implicit function theorem to the first order conditions.

**Proposition 8** The consumer solves the consumption-saving problem (12) - (15) and preferences take one of the forms in Theorem 1. Then

\[
    c(s^t)^{KP} |_{R(s^t) = R(s^t)} = c(s^t)^{**} |_{R(s^t) = R(s^t)} = c(s^t)^{*.t} |_{R(s^t) = R(s^t)}
\]

\textsuperscript{21}Since there is one node in period 1, we simplify the notation by denoting the risky and risk free asset holdings respectively by \( n_1 \) and \( n_{f1} \) instead of \( n(s^1) \) and \( n_f(s^1) \).
Proposition 9 Assume Assumption [RF] holds and the consumer solves the consumption-portfolio problem (8) - (11) with preferences taking one of the forms in Theorem 2. Then

\[
\frac{\partial c(s^t)^c}{\partial R_j(s^t)} \bigg|_{R(s^t)=R_\ell(s_t)} = \frac{\partial c(s^t)^{**}}{\partial R_j(s^t)} \bigg|_{R(s^t)=R_\ell(s_t)} = \frac{\partial c(s^t)^{KP}}{\partial R_j(s^t)} \bigg|_{R(s^t)=R_\ell(s_t)}.
\]

In the next two subsections, we give conditions under which for non-infinitesimal variations from Assumption [IR], DOCE resolute and sophisticated demands and KP demands can all be surprisingly close or diverge significantly depending on the assumed time and risk preference building blocks.

### 4.2 Disentangling the Effects of Time and Risk on Saving

In this subsection, we explore the implications of finite departures from [IR] for optimal consumption and saving. Consider the special case portrayed in Figure 2. In period 1, there is a risk free asset with return \( R_{f2} \). In period 2 if the upper state is realized (with probability \( \pi_1 \)), there exists a risk free asset with return \( R_{f31} \). If the lower state is realized (with probability \( \pi_2 \)), the risk free return is \( R_{f32} \). Assume the CES time and CRRA risk preference utilities in (5). Employing these same utility building blocks, we derive optimal DOCE sophisticated and KP demands based on backward induction.\(^\text{22}\) Figure 2 facilitates a particularly clear comparison of the two sets of demands. It should be noted that for the consumption-saving setup in Figure 2, there is neither exogenous income nor capital risk. However, the KP and DOCE solution processes create risky consumption in periods 2 and 3 when viewed from the perspective of period 1. The saving in period 2 conditional on being on the up or down branch is certain. The income

\(^{22}\text{Supplemental Appendix B.2 provides supporting calculations for this subsection. We investigate the special case of risk neutral risk preferences defined by } \delta_j = -1. \text{ We show that risk neutral DOCE resolute choice results in boundary solutions. We also derive closed form analytic expressions for optimal period 1 consumption for the cases of DOCE sophisticated and KP preferences and discuss the differences.}
realized from period 1 saving is \((I - c_1)R_{f2}\), which is the same at both period 2 nodes. However, since \(R_{f31} \neq R_{f32}\) optimal \(c_{21}\) and \(c_{22}\) will in general differ as will \(c_{31}\) and \(c_{32}\).

Example 1 below illustrates that in general for the consumption-saving problem, when \([IR]\) does not hold, DOCE resolute, naive and sophisticated demands and KP demands all differ unless the EU special case holds where \(\delta_1 = \delta_2\). However, for the special "log" time preference case where \(\delta_1 = 0\), we have the following result.

**Result 1** Assume the CES and CRRA utilities in (5) and the consumer faces the consumption-saving problem associated with the Figure 2 setting. \(R_{f31} \neq R_{f32}\) implies that Assumption [IR] does not hold. When \(\delta_1 = 0 \neq \delta_2\),

(i) \(c_1^* = c_1^{**} = c_1^{KP}\),

(ii) \(c_{2i}^* \neq c_{2i}^{**} = c_{2i}^{KP} \) \((i = 1, 2)\).

We next illustrate important differences in optimal period 1 consumption particularly for the sophisticated DOCE and KP cases when Assumption [IR] does not hold and provide intuition for why \(c_1^{KP}\) can be significantly lower than \(c_1^{**}\).

\[23\] It can also be verified that \(c_1^* = c_1^{**} = c_1^{KP}\) is independent of the risk aversion parameter \(\delta_2\). The fact that \(c_1^{KP}\) is independent of \(\delta_2\) is consistent with eqn. (15) in Giovannini and Weil (1989).

---

**Figure 2:**
Example 1 Assume the consumption-saving setting associated with Figure 2 and that the time and risk preference utilities take the form in (5). Based on the following parameters

\[ R_{f2} = 1.1, R_{f31} = 1.2, R_{f32} = 1, \pi_1 = 0.5, \beta = 0.97, I = 10, \]

we performed numerical simulations of optimal \( c_1 \) as functions of \( \delta_1 \) and \( \delta_2 \) which are summarized respectively in Figures 3(a) and (b). Based on the definitions of resolute and naive choice (Definitions 1 and 2), \( c_1^* = c_1^0 \) always holds. In both Figures 3(a) and (b), \( c_1^0, c_1^{**} \) and \( c_1^{KP} \) are generally close in value as \( \delta_1 \) and \( \delta_2 \) are varied (with respectively \( \delta_2 = 5 \) and \( \delta_1 = -0.5 \) being fixed).\(^{24}\) The three curves intersect at the EU special cases in Figures 3(a) and (b) where respectively \( \delta_1 = \delta_2 = 5 \) and \( \delta_1 = \delta_2 = -0.5.\(^{25}\) In addition to \( c_1^{**} \) and \( c_1^{KP} \) differing in their monotonicity with respect to \( \delta_1 \) in Figure 3(a), they also diverge significantly in value as \( \delta_1 \) goes to \(-1 \). To provide intuition for why these differences arise, first consider sophisticated choice. Following the recursive solution process, the consumer faces a two period certainty consumption-saving problem. As \( \delta_1 \to -1 \), on the upper branch in Figure 2 where \( R_{f31} > 1 \), the consumer substitutes \( c_{31} \) for \( c_{21} \) and saves all of her period 2 income \((I - c_1)R_{f2}\) resulting in \( c_{21} \) going to zero. Similarly, on the lower branch since \( R_{f32} = 1 \) and \( \beta < 1 \), the consumer

\(^{24}\)It should be noted that the scale of the vertical axes in Figures 3(a) and (b) are different.

\(^{25}\)The three curves also intersect at \( \delta_1 = 0 \) in Figure 3(a), consistent with Result 1(i).
consumes all of her income in period 2 resulting in $c_{32}$ going to zero. Assuming $\delta_2 \geq 0$ as well as $\delta_1 \to -1$, $\hat{c}_2$, $\hat{c}_3 \to 0$ and the highly substitute oriented consumer maximizes her three period certainty utility by setting $c_1^{**} = I$. For the KP case, note that the period 1 utility is given by

$$U(c) = \left( c_1^{-\delta_1} + \beta \left( \pi_1 U_{21}^{-\delta_2} + \pi_2 U_{22}^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}},$$

where

$$U_{21} = \left( c_2^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{-\frac{1}{\delta_1}}$$

and

$$U_{22} = \left( c_2^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{-\frac{1}{\delta_1}}.$$

Applying the same argument for KP utility as was done for sophisticated choice, we have $c_{21} = 0$ and $c_{32} = 0$ when $\delta_1 \to -1$. Substituting into the above formula for the KP utility yields

$$\left( \pi_1 (c_{21} + \beta c_{31})^{-\delta_2} + \pi_2 (c_{22} + \beta c_{32})^{-\delta_2} \right)^{-\frac{1}{\delta_2}}$$

$$= \left( \pi_1 (\beta c_{31})^{-\delta_2} + \pi_2 c_{22}^{-\delta_2} \right)^{-\frac{1}{\delta_2}}$$

$$= \left( \pi_1 (\beta Rf_2 (I - c_1))^{-\delta_2} + \pi_2 (Rf_2 (I - c_1))^{-\delta_2} \right)^{-\frac{1}{\delta_2}}$$

$$= \left( \pi_1 \beta^{-\delta_2} + \pi_2 \right)^{-\frac{1}{\delta_2}} Rf_2 (I - c_1).$$

Since $\left( \pi_1 \beta^{-\delta_2} + \pi_2 \right)^{-\frac{1}{\delta_2}} Rf_2 > 1$, it follows from maximizing the KP period one utility function that $c_1 \to 0$ when $\delta_1 \to -1$. For resolute choice, where unlike the KP case one does not use backward induction, as $\delta_1 \to -1$ the boundary solution $c_1 \to 0$ is realized.

The question of how excess saving $\theta$ differs for the DOCE resolute and sophisticated and KP models is of interest since, as shown in Example 1, they can exhibit very similar behavior in terms of optimal first period consumption. They also share the same time preference utility and hence the same $c_1^{\text{certain}}$. The following shows that the conclusions of Proposition 4 relating to the sign of excess saving can still hold when Assumption [IR] is violated.

**Example 2** Assume the same setting as in Example 1. To define an appropriate $c_1^{\text{certain}}$ for comparison purposes, assume that the certainty period 3 risk free return $Rf_3 = \pi_1 Rf_{31} + \pi_2 Rf_{32} = Rf_2$. Figure 4(a) shows that the value of the certainty $c_1$ is surprisingly close to the corresponding values for the DOCE sophisticated, resolute (=naive) and KP cases as long as $\delta_1$ is not too close to $-1$.26 Figure 4(b)

26Not surprisingly, the distance between the certainty $c_1$ curve and the other curves in Figure 4(a) increases if we consider more dispersed distributions of risk free returns such as $Rf_{31} = 1.5$ and $Rf_{32} = 0.7$. 28
confirms that $\epsilon^c_1$ is larger (smaller) than resolute, sophisticated and KP period 1 consumption if $\delta_1 > (<)0$. This in turn implies that $\theta > (<)0$ if $\delta_1 > (<)0$ for the DOCE and KP models. However unlike Proposition 4, in the current case the conclusion on the sign of $\theta$ can depend on the value of $\delta_2$. When $\delta_2 = 5$ all of the curves in Figure 4(b) intersect at $\delta_1 = 0$ and using (6), we have

$$\theta \gtrless 0 \iff \delta_1 \lesssim 0 \iff EIS \lesssim 1$$

for the DOCE and KP models. However, as in Figure 5, when $\delta_2$ takes on low values such as $-0.7$ positive excess saving can occur for the DOCE and KP models when $\delta_1 < 0$. (See Supplemental Appendix B.3 for supporting calculations.)

4.3 Disentangling the Effects of Time and Risk on Asset Demand

In this subsection we consider the effect of relaxing Assumption [IR] on optimal demands in the consumption-portfolio problem. In contrast to the prior subsection where the focus was on optimal period 1 consumption and saving, here we concentrate on asset demands and in particular the portfolio composition. Once asset returns are allowed to be dependent over time, the portfolio composition in general depends on both time and risk preferences. For some of our analysis, we assume a specific three period setting in which it is possible to clearly distinguish the separate effects of time and risk preferences on portfolio composition.

Before turning our focus to optimal asset demand behavior, we show that the DOCE resolute solution to the consumption-portfolio problem has a relatively
simple characterization which in practice is not complicated to compute. If the
time and risk preference utilities \( u \) and \( V \) take one of the HARA forms in Theorem
2 and Assumption [CM] holds, the period 1 DOCE utility used for resolute choice
or can be expressed in the form of a simple discounted utility. Importantly for
this case, even though Assumption [IR] is not satisfied, the optimal holdings of
contingent claims in each period depend only on risk preferences and not on time
preferences. One key assumption in this analysis of resolute choice is that the
consumer is strictly risk averse. This is done to rule out the risk neutral case
in which resolute choice in general fails to have an interior optimal solution (see
Proposition 11 in Supplemental Appendix B.2).

**Proposition 10** Consider the consumption-portfolio problem (8)-(11), where As-
sumption [CM] holds.\(^{27}\) The consumer has DOCE preferences taking one of the
forms (i)-(iii) in Theorem 2. Then for DOCE resolute choice, the period 1 utility
function can be written as

\[
U(c) = u(c_1) + \sum_{t=2}^{T} \beta^{t-1} u(c_t) = u(c_1) + \sum_{t=2}^{T} \nu_t \beta^{t-1} u(c(s^t)),
\]

where for Theorem 2 cases (i) and (iii),

\[
\nu_t = \left( \sum_{s^{t}} \pi(s^{t}) \left( \frac{\pi(s^{t}) \rho(s^{t})}{\pi(s^{t}) \rho(s^{t})} \right)^{\frac{\delta_1}{\delta_2}} \right)^{\frac{\delta_2}{\delta_1}}
\]

\(^{27}\)Assumption [RF] is not required since it is implied by Assumption [CM].
and for case (ii)
\[
\nu_t = \left( \sum_{s^t} \pi^t(s^t) \rho(s^t) \right) \frac{\kappa_1}{\kappa_2},
\]
where \(\pi^t\) denotes a given state in period \(t\) and \(\rho(\cdot)\) is defined by eqn. (7).\(^{28}\)

The intuition for Proposition 10 is that since in each period \(t\), preferences are quasihomothetic and we consider resolute choice, it follows from the first order conditions that the \(c(s^t)\) can be expressed as linear functions of \(c(\pi^t)\). Therefore, \(\hat{c}_t\) can be also expressed as a linear function of \(c(\pi^t)\). This is similar to the argument in Subsection 3.2 following Theorem 2 where optimal consumption in all other branches can be derived from the solutions along a reduced single branch tree.

**Remark 3** It is interesting to note that the risk preference parameters \(\delta_2\) and \(\kappa_2\) enter into the period 1 utility (21) via \(\nu_t\). For case (i) \(\forall t\), if one chooses, without loss of generality, \(c(s^t) = \min_{s^t} c(s^t)\), then
\[
\frac{\pi(s^t) \rho(s^t)}{\pi(\pi^t) \rho(s^t)} > 1.
\]

Defining
\[
k(s^t) = \left( \frac{\pi(s^t) \rho(s^t)}{\pi(\pi^t) \rho(s^t)} \right)^{\frac{1}{1+\delta_2}},
\]
we have
\[
\frac{\partial k(s^t)}{\partial \delta_2} < 0.
\]
Moreover,
\[
\frac{\partial \left( \sum_{s^t} \pi(s^t) k(s^t)^{-\delta_2} \right)^{\frac{1}{\delta_2}}}{\partial \delta_2} < 0.
\]
Therefore,
\[
\frac{\partial \nu_t}{\partial \delta_2} \leq 0 \Leftrightarrow \delta_1 \geq 0.
\]

There is a clear analogy between \(\nu_t\) and the standard certainty discount function \(\beta\). Increasing \(\beta\) increases the relative importance of the future discounted utility terms and results in decreased optimal \(c_1\). Analogously if \(\delta_1 > 0\), \(\nu_t\) increases with \(\delta_2\), which increases the importance of the future discounted utility terms and results in decreased optimal period 1 consumption as can be seen from
\[
\frac{\partial c_1}{\partial \nu_t} \geq 0 \Leftrightarrow \frac{\partial \nu_t}{\partial \delta_2} \geq 0 \Leftrightarrow \delta_1 \leq 0.
\]

\(^{28}\)It should be noted that cases (i) and (iii) can be written in the same general form since the shift parameter \(b\) is embedded in \(u\).
In this sense, certainty discounting is similar to changes in risk aversion when considering optimal period one consumption.

Remark 4 Gollier and Kihlstrom (2016) consider a representative agent economy, in which the agent is assumed to possess KP preferences or DOCE preferences following resolute choice. They consider an infinite number of periods and derive and compare the equilibrium term structure of interest rates for the two models. However, since they assume incomplete asset markets, the results in Proposition 10 cannot be applied to their setup.

It follows from the proof of Proposition 10 that we have for case (i)
\[
\frac{c(s^t) - b}{c(s^t) - b} = \left( \frac{\pi(s^t) \rho(s^t)}{\pi(s^t) \rho(s^t)} \right)^{-\frac{1}{1+\kappa_2}};
\]
for case (ii)
\[
c(s^t) - c(s^t) = -\frac{1}{\kappa_2} \ln \left( \frac{\pi(s^t) \rho(s^t)}{\pi(s^t) \rho(s^t)} \right);
\]
and for case (iii)
\[
\frac{b - c(s^t)}{b - c(s^t)} = \left( \frac{\pi(s^t) \rho(s^t)}{\pi(s^t) \rho(s^t)} \right)^{-\frac{1}{1+\kappa_2}}.
\]
In each case, the contingent claim ratios or differences for any period \( t \) are independent of the time preference parameters \( \delta_1, \kappa_1 \) and \( \beta \). However importantly, this conclusion does not apply to the asset demands since the consumer can only buy short term assets and the investment in the short term assets must finance contingent claim demands in all future periods. Therefore, the asset demands are derived from the set of equations corresponding to contingent claim demands in all periods, where the latter depend on both the time and risk preferences. We next consider a particularly simple and transparent setting in which it is possible to explicitly solve for asset demands. We then investigate the dependence of asset demands and portfolio composition on time and risk preference parameters.

Assume the CES time and CRRA risk preference utilities in (5) and the tree structure in Figure 1. In period 1, the consumer can buy short term risk free and risky assets, which pay off in period 2. For simplicity, consider the two state case. The short term risk free asset has the return \( R_{f2} \). The short term risky asset has the return \( R_{2s} \) with the probability \( \pi_s \) (\( s = 1, 2 \)). In period 2, depending on which state is realized, there exists a risk free asset with return \( R_{f31} \) or \( R_{f32} \). With slight abuse of our general notation, period 1 asset holdings will be denoted by \( n_1 \) and \( n_{f1} \). Then period 2 income for the two branches is given by
\[
I_{2s} = R_{2s}n_1 + R_{f2}n_{f1} \quad (s = 1, 2).
\]
For DOCE resolute choice, the period 1 utility function is

\[ U(c) = \left( c_1^{-\delta_1} + \beta \left( \pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2} \right)^{\frac{\delta_2}{\delta_2}} + \beta^2 \left( \pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2} \right)^{\frac{\delta_2}{\delta_2}} \right)^{-\frac{1}{\gamma_1}} \]

and the budget constraints are

\[ c_{31} = R_{f31} (R_{21} n_1 + R_{f2} n_1 - c_{21}), \quad c_{32} = R_{f32} (R_{22} n_1 + R_{f2} n_1 - c_{22}) \]

and

\[ I = c_1 + n_1 + n_{f1}. \]

Straightforward, although tedious calculations result in

\[
n_{f1}^0 \frac{n^0}{n_1} = \frac{k_{R21}}{R_{f32}} \left( 1 + \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right) \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right) + R_{f1} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right) \frac{1}{\gamma_1} + R_{f2} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right) \frac{1}{\gamma_1},
\]

where

\[
k = \left( \begin{array}{c}
\beta \left( \frac{R_{f2} - R_{22}}{(R_{21} - R_{f2}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{f2}) R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{\frac{1}{\gamma_1}} + \frac{\pi_1 + \pi_2}{\pi_1} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{\gamma_1}} \right) \\
\left( \frac{R_{f1} - R_{22}}{(R_{21} - R_{f2}) R_{f1}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{f2}) R_{f1}} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{\gamma_1}} + \frac{\pi_1 + \pi_2}{\pi_1} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{\gamma_1}} \right) \\
\left( \frac{R_{f1} - R_{22}}{(R_{21} - R_{f2}) R_{f1}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{f2}) R_{f1}} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{\gamma_1}} + \frac{\pi_1 + \pi_2}{\pi_1} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{\gamma_1}} \right)
\end{array} \right)^{\frac{1}{\gamma_1}}.
\]

(Supporting calculations for this subsection are provided in Supplemental Appendix B.4.) One key observation is that \( \kappa \) depends on the time preference parameters \( \delta_1 \) and \( \beta \) and hence so does the optimal asset ratio \( n_{f1}^o / n_1^o \). This differs from the analogous two period KPS solution as well as the case where Assumptions [IR] holds as shown in Proposition 5. (It should be noted that it is not possible to derive a closed form expression for the optimal asset ratio for DOCE sophisticated choice. However, we demonstrate via simulations in Example 3 below that when Assumption [IR] does not hold, the sophisticated \( n_{f1}^{**} / n_1^{**} \) in general depends on the consumer’s time preference parameters.)
For the comparable KP model, the period 1 utility function is

\[
U(c) = \left( c_{1} - \delta_{1} + \beta \left( \pi_{1} \left( c_{21}^{\delta_{1}} + \beta c_{31}^{\delta_{1}} \right)^{\frac{\delta_{2}}{\delta_{1}}} + \pi_{2} \left( c_{22}^{\delta_{1}} + \beta c_{32}^{\delta_{1}} \right)^{\frac{\delta_{2}}{\delta_{1}}} \right) \right)^{-\frac{1}{\pi_{1}}}
\]

\[
= \left( c_{1}^{\delta_{1}} + \beta \left( \pi_{1} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f31}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} (R_{f1} n_{1} + R_{f2} n_{f1})^{-\delta_{2}} + \pi_{2} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f32}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} (R_{f1} n_{1} + R_{f2} n_{f1})^{-\delta_{2}} \right) \right)^{-\frac{1}{\pi_{1}}}
\]

(22)

Solving for asset demands, one obtains

\[
\frac{n_{f1}^{KP}}{n_{1}^{KP}} = \frac{R_{f1} k_{2}^{\frac{-1}{\pi_{1}}} - R_{f2}}{1 - k_{2}^{\frac{1}{\pi_{1}}} R_{f2}},
\]

where

\[
k_{2} = \frac{\pi_{2} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f32}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} (R_{f2} - R_{f2})}{\pi_{1} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f31}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} (R_{f1} - R_{f2})}.
\]

As in the DOCE resolute case, the asset ratio \( n_{f1}^{KP}/n_{1}^{KP} \) depends on \( \delta_{1} \) and \( \beta \). It should be noted that when \( R_{f31} = R_{f32} \), we have

\[
k_{2} = \frac{\pi_{2} (R_{f2} - R_{f2})}{\pi_{1} (R_{f1} - R_{f2})}
\]

and the conditional portfolio problem converges to the two period case with the asset ratio being independent of \( \delta_{1} \) and \( \beta \). Moreover, for this case, if

\[
\pi_{1} R_{f1} + \pi_{2} R_{f2} > R_{f2},
\]

we have \( k_{2} < 1 \), implying that \( n_{1}^{KP} > 0 \). However, if \( R_{f31} \neq R_{f32} \) then it is possible for \( k_{2} > 1 \) and \( n_{1}^{KP} < 0 \). The general condition for determining the sign of \( n_{1}^{KP} \) is given by the following expression

\[
n_{1}^{KP} \geq 0 \iff \pi_{2} \left( R_{f2} - R_{f2} \right) \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f31}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f32}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} \geq \pi_{1} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f31}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}} \left( 1 + \beta \frac{1}{1+\pi_{1}} R_{f32}^{\frac{-\delta_{1}}{\delta_{1}}} \right)^{(1+\delta_{1})\delta_{2}}.
\]

(23)

Note that when \( n_{1}^{KP} < 0 \), we require that period 2 income

\[
I_{2s} = R_{2s} n_{1} + R_{f2} n_{f1} > 0 (s = 1, 2)
\]

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in order for $c_{21}$, $c_{22}$, $c_{31}$ and $c_{32}$ to be positive and for the consumer’s utility function to be well-defined. The increase in the risk free asset holdings financed by the shorting of the risky asset should not be viewed as reflecting increased intraperiod risk aversion. Rather as discussed in Supplemental Appendix B.5, it should be viewed as dynamic hedging of intertemporal risk.

We conclude this section with an example illustrating the differences in the behavior of $c_1$ and $n_{f1}/n_1$ in response to variations in the time and risk preference parameters $\delta_1$ and $\delta_2$ for the KP and DOCE resolute, naive and sophisticated cases.

**Example 3** Assume a consumption-portfolio setting consistent with Figure 1. The time and risk preference building blocks are given by (5). Assume the following parameter values

$$R_{21} = 2, R_{22} = 0.8, R_{f2} = 1.1, R_{f31} = 1.25, R_{f32} = 0.95, \pi_1 = 0.5, \beta = 0.97, I = 10.$$ 

The results from numerical simulations of optimal $c_1$ as functions of $\delta_1$ and $\delta_2$ are plotted in Figures 6(a) and (b). Paralleling Example 1, period one DOCE resolute and sophisticated and KP consumption values are generally quite close in value except when $\delta_1$ is close to $-1$. Simulations of the optimal asset ratio $n_{f1}/n_1$ as functions of $\delta_1$ and $\delta_2$ are given in Figures 7(a) and (b). Based on the definitions of resolute and naive choice, $n_{f1}^*/n_1^* = n_{f1}^0/n_1^0$. The KP and DOCE resolute and sophisticated asset ratios $n_{f1}/n_1$ converge for the EU special case.
Figure 7:

where $\delta_1 = \delta_2$. In contrast to Proposition 5 where Assumption [IR] holds, in Figure 7(a), $n_{f1}/n_1$ varies with $\delta_1$. In Figure 7(b) when $\delta_2 = 5$ and $\delta_1 = -0.6$, the KP and DOCE sophisticated asset ratios equal 15.28 and 4.43, respectively.\footnote{If one assumes for the [IR] case that $R_{f3} = \pi_1 R_{f31} + \pi_2 R_{f32} = 1.10$, then the constant $n_{f1}/n_1 = 4.70$ which is the same as the two period case.} Given that the two models share the same time and risk preference building blocks, how can this divergence be explained? Because for this example asset returns are positively correlated over time, the upper and lower branches in Figure 1 are associated respectively with (good, good) and (bad, bad) asset payoffs. But the two models react quite differently to this intertemporal risk. Consider first the KP optimization, which is based on evaluating a lottery of utility values. Referring to eqn. (22), the utility levels in the two states are respectively given by

\[
U_{21} = \left(1 + \beta^{1+\delta_1} R_{f31}^{-\frac{\delta_1}{\sigma_1}}\right)^{-\frac{(1+\delta_1)}{\sigma_1}} (R_{21} n_1 + R_{f2} n_{f1})
\]

and

\[
U_{22} = \left(1 + \beta^{1+\delta_1} R_{f32}^{-\frac{\delta_1}{\sigma_1}}\right)^{-\frac{(1+\delta_1)}{\sigma_1}} (R_{22} n_1 + R_{f2} n_{f1}).
\]

Independent of whether $\delta_1 < 0$ or $\delta_1 > 0$,

\[
\left(1 + \beta^{1+\delta_1} R_{f31}^{-\frac{\delta_1}{\sigma_1}}\right)^{-\frac{(1+\delta_1)}{\sigma_1}} > \left(1 + \beta^{1+\delta_1} R_{f32}^{-\frac{\delta_1}{\sigma_1}}\right)^{-\frac{(1+\delta_1)}{\sigma_1}}. \tag{24}
\]
This inequality implies that if $R_{21} > R_{22}$ and $n_1 > 0$, the consumer faces more risk when considering the certainty equivalent of utility values

$$\left(\pi_1 U_{21}^{-\delta_2} + \pi_2 U_{22}^{-\delta_2}\right)^{-\frac{1}{\delta_2}}$$

than considering the certainty equivalent of consumption values

$$\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{-\frac{1}{\delta_2}},$$

where

$$c_{21} = R_{21} n_1 + R_{f2} n_{f1} \quad \text{and} \quad c_{22} = R_{22} n_1 + R_{f2} n_{f1}.$$ The spread in period 2 consumption values $c_{21}$ and $c_{22}$ is increased by the period 3 return factors in (24) and the consumer faces more risk than in the two period case. As a result she attempts to compensate for this greater risk by increasing her $n_{f1}/n_1$ ratio beyond that of the two period or [IR] case where there is no intertemporal risk. In contrast, why does the DOCE sophisticated consumer seemingly perceive the risk as less worrisome as evidenced by her much smaller $n_{f1}/n_1$ ratio? This difference in perception occurs even though the KP and DOCE optimization processes both proceed via backward induction and consider the conditional optimal saving decisions based on $I_{21}$ and $I_{22}$. For the sophisticated DOCE consumer, the fact that $\delta_1 < 0$ ($EIS > 1$) has two distinct, important consequences for her asset allocation. First as in Example 1, for the certainty consumption allocation along the upper and lower branches, the substitution effect dominates the income effect. Depending on whether $R_{f3} > (\leq) 1$ ($s = 1, 2$) the consumer substitutes $c_3$ for $c_2$ ($c_2$ for $c_3$). It follows that $c_{31} > c_{21}$ on the upper branch and $c_{22} > c_{32}$ on the lower branch. This results in the period 2 and 3 consumption spreads, $c_{22} - c_{21}$ and $c_{31} - c_{32}$, being respectively smaller and larger than the spread of $I_{21} - I_{22}$. It follows that $c_{31}$ and $c_{32}$ are respectively the best and worst of the four contingent claim consumption values. The assumption that $\delta_2 = 5$ suggests a relatively risk averse consumer which in turn implies that the certainty equivalents $\tilde{c}_2$ and $\tilde{c}_3$ are near their respective lowest contingent claim outcomes. The second important implication of $\delta_1 < 0$ relates to the evaluation of $\tilde{c}_2$ and $\tilde{c}_3$. Since $\tilde{c}_3 < \tilde{c}_2$ and the DOCE consumer is substitute oriented, she will overvalue period 2 versus period 3 and the asset allocation decision will largely be determined by the period 2 spread. But as argued above, the period 2 consumption spread is smaller than the $(I_{21}, I_{22})$ distribution. Hence the DOCE sophisticated choice consumer perceives her period 2 risk as not being increased by the positive correlation of asset returns and hence she does not increase the $n_{f1}/n_1$ ratio as the KP consumer does. As indicated by

\[30\] The same is true for the KP consumer.
Figure 7(a), if $\delta_1 > 0$ ($EIS < 1$) the above argument does not apply and the asset ratios for the four models are close. (For a discussion of cases where $n_1$ and the asset ratio can be negative, see Supplemental Appendix B.5. This appendix also contains supporting calculations for Example 3.)

The results of this example suggest that when asset returns are not independent over time, if one follows much of the certainty empirical literature in assuming that the $EIS$ is in the range of 0 and 0.4 (or using eqn. (6) $\delta_1 > 1.5$), then the DOCE and KP preference models generate qualitatively quite similar consumption and asset demand behavior. Alternatively, if one accepts the long-run risk and some macro $EIS$ calibrations of 1.5 to 2.0 (or equivalently $-0.5 < \delta_1 < -0.33$), then the demands differ significantly and differences in the respective preferences and in particular their underlying properties of TC, SEP and TRI become critical.

5 Concluding Comments

In this paper, we provide conditions such that DOCE preferences exhibit TC, SEP and TRI on a restricted domain of consumption trees corresponding to consumption-saving and consumption-portfolio problems. Under these same conditions, optimal consumption and asset demands for KP preferences are the same as the common DOCE resolute, naive and sophisticated demands. When the key Assumption [IR] is relaxed, the demands for the KP and different DOCE solution techniques can be close but also can diverge significantly.

Two extensions of our work would seem interesting. The first relates to the critical role played by the value of the EIS measure for the case of KP and DOCE preferences based on the CRRA and translated CRRA preference models (5) and (2). Significant differences in both optimal consumption and asset demands can arise when the $EIS > 1$ (or $\delta_1 < 0$). Given that existing empirical research is inconclusive on whether the EIS measure is larger or smaller than unity, it would seem desirable to investigate this question particularly in the context of the simple dynamic structure in Example 3. Although a number of challenges exist in applying parametric or non-parametric tests to this setting, it would nevertheless seem to be an important area for future research.

The second extension relates to comparing equilibrium asset returns based on the KP and three DOCE dynamic solution techniques when Assumption [IR] does not hold. For instance for the CES and CRRA utilities (5) as one varies $\delta_1$ and $\delta_2$ as in Figure 7, how do the divergent behaviors of $n_1^{KP}/n_1^{KP}$ and $n_1^{*}/n_1^{*}$ get reflected in terms of the equilibrium risky and risky free returns and the equity

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Appendix

A Proofs

A.1 Proof of Theorem 1

To prove Theorems 1, it is useful to first state the following lemma that follows from generalizing the argument in Subsection 3.1.2.

Lemma 1 Suppose preferences are homothetic and for each \( \tau = 1, \ldots, T-1 \) there exist \( \alpha^*_\tau(s) \), \( \tau = \tau + 1, \ldots, T \), \( s = 1 \ldots S \) such that at each \( \tau \) naive choice satisfies

\[
c(s^t) = \alpha^*_\tau(s_t)c(s^{t-1}) \quad \forall t = \tau + 1, \ldots, T, \ s_t = 1 \ldots, S.
\]

Then choices are time consistent and naive and resolute choice are identical.\(^\ddagger\)

Proof. Generalizing eqn. (18) we obtain

\[
U(\alpha|s^\tau) = u(c(s^\tau)) + \beta u \circ V^{-1} \left( \sum_{s_{T+1}} \pi_{T+1} (s_{T+1}) V(\alpha_{T+1} c(s^\tau)) \right) + \ldots + \\
\beta^T u \circ V^{-1} \left( \sum_{s_{T+1}} \pi_{T+1} (s_{T+1}) \ldots \sum_{s_T} \pi_T (s_T) V(\alpha_T \ldots \alpha_1 c(s^\tau)) \right) \\
= u(c(s^\tau)) + \beta u \circ V^{-1} \left( \sum_{s_{T+1}} \pi_{T+1} (s_{T+1}) V(\alpha_T \ldots \alpha_1 c(s^\tau)) \right) K_{T+1},
\]

where \( K_{T+1} \) is recursively defined as

\[
K_T = 1 + \beta u \circ V^{-1} \left( \sum_{s_T} \pi_T (s_T) V(\alpha_{T-1} c(s_T)) \right)
\]

and

\[
K_t = 1 + \beta u \circ V^{-1} \left( \sum_{s_t} \pi_t (s_t) V(\alpha_{t-1} c(s_t)) \right) K_{t+1}
\]

for \( t = \tau + 1, \ldots, T - 1 \). By the same argument as in the proof of Proposition 1, it is now clear that if \( U(\alpha|s^\tau) > U(\tilde{\alpha}|s^\tau) \) and \( \tilde{\alpha}_t(s) = \alpha_t(s) \) for all \( t \leq \tau \) and all \( s \)

\(^\ddagger\)One important caveat relates to possible inconsistencies that may arise between micro-demand and equilibrium return properties as noted by Selden and Wei (2018).

\(^\ddagger\)If resolute choice and naive choice coincide, it can easily be shown that sophisticated choice is the same.
then $U(\alpha|s^{\tau-1}) > U(\tilde{\alpha}|s^{\tau-1})$. Therefore if $\alpha$ is preferred to $\tilde{\alpha}$ at $\tau$ following naive choice so must it be preferred following naive choice at $\tau - 1$ and, by induction, preferred by resolute choice. This completes the proof.

We are now in a position to prove Theorem 1.

**Proof of Theorem 1** The following first order conditions are necessary and sufficient for naive choice at $s^\tau$ under Assumption [IR]

\[
u'(c(s^\tau)) \left( V \circ u^{-1} \right)' \left( \sum_{s^\tau \succ s^\tau} \pi(s') u(c(s')) \right) = \\
\beta(V \circ u^{-1})' \left( \sum_{s^\tau+1 \succ s^\tau} \pi(s^{\tau+1}) u(c(s^{\tau+1})) \right) \sum_{s^\tau+1 \succ s^\tau} R(s_{t+1}) \pi(s^{\tau+1}) u' \left( c \left( s^{\tau+1} \right) \right),
\]

for all $s^\tau$, $t < T$. Since $u(.)$ and $V(.)$ are assumed to be homothetic, it is clear that any budget-feasible solution must satisfy

\[c(s^t) = \alpha_{t-1}(s_t) c \left( s^{t-1} \right) \quad \forall t = 1, \ldots, T, \ s_t = 1 \ldots, S\]

and sufficiency follows directly from Lemma 1.

To prove necessity consider a simple version of the model with three periods, $t = 1, 2, 3$, and an event tree as depicted in Figure 1. Suppose markets are complete and, for simplicity, there are Arrow securities available for trade. To satisfy Assumption [IR] suppose that the prices of the Arrow securities at $t = 2$ are identical and denoted by $p(2)$.

The first order conditions for optimal naive choice at $t = 2$ are

\[p(2) u'(c_{2s}) = \beta u'(c_{3s}), \quad (s = 1, 2)\]

and, at $t = 1$, planning for $t = 2$, are

\[p(2) V'(c_{2s})(u \circ V^{-1})' \left( \sum_s \pi_s V(c_{2s}) \right) = \beta(u \circ V^{-1})' \left( \sum_s \pi_s V(c_{3s}) \right) V'(c_{3s}).\]

The first equation implies

\[c_{3s} = u'^{-1} \left( \frac{p(2)}{\beta} u'(c_{2s}) \right)\]

and substituting this into the second equation we obtain

\[p(1) V'(c_{2s})(u \circ V^{-1})' \left( \sum_s \pi_s V(c_{2s}) \right) = \]

\[\beta(u \circ V^{-1})' \left( \sum_s \pi_s \left( u'^{-1} \left( \frac{p(1)}{\beta} u'(c_{2s}) \right) \right) \right) \]

\[V' \left( u'^{-1} \left( \frac{p(2)}{\beta} u'(c_{2s}) \right) \right). \quad (A.1)\]
Denote the price \( p(2) \) simply by \( p \). Then we consider variations in \( p(2) = p \) as well as first period prices \( p(1) \) that keep second period consumption fixed. Taking the derivative with respect to \( p \) on both sides and then setting \( p = \beta \) one obtains

\[
1 = \frac{(u \circ V^{-1})'' \left( \sum_s \pi_s V(c_{2s}) \right) \sum_s \pi_s (u' \circ (c_{2s}) u''(c_{2s}))}{(u \circ V^{-1})'(\sum_s \pi_s V(c_{2s}))} + \frac{V''(u' \circ (c_{2s})) u'(c_{2s})}{V'(c_{2s})}.
\]

Taking the derivatives with respect to \( c_{2s}, s = 1, 2 \), we obtain\(^{33}\)

\[
\frac{d}{dc} \frac{f'(g(c))g(c)}{f(c)} = 0,
\]

where \( f(c) = V''(c) \) and \( g(c) = u'(c) \).

Since

\[
g^{-1}(g(c))g'(c) = 1,
\]

we obtain

\[
\frac{d}{dc} \frac{f'(g(c))g(c)}{g'(c)f(c)} = 0.
\]

Consider the following ordinary differential equation

\[
\frac{d}{dc} \left( \frac{f'(g(c))g(c)}{f(c)g'(c)} \right) = 0.
\]

We have

\[
\frac{f'(g(c))g(c)}{f(c)g'(c)} = K_1,
\]

where \( K_1 \) is a constant. Therefore,

\[
\frac{f'(g(c))g(c)}{f(c)g'(c)} = \left( \ln f(c) \right)' = K_1 \frac{g'(c)}{g(c)} = K_1 \left( \ln g(c) \right)'
\]

implying that

\[
\ln f(c) = K_1 \ln g(c) + K_2,
\]

where \( K_2 \) is a constant. Thus we have

\[
f(c) = K_3 (g(c))^{K_1},
\]

where \( K_3 \) is a constant.

Assuming \( K > 0 \), we can write \( V'(c) = u'^K \) and \( V'^{-1}(x) = u'^{-1}(x^{\frac{1}{K}}) \). Substituting this into (A.1) we obtain

\[
p(u \circ V^{-1})' \left( \sum_s \pi_s V(c_{2s}) \right) = \beta(u \circ V^{-1})' \left( \sum_s \pi_s V \left( u'^{-1} \left( \frac{p}{\beta} u'(c_{2s}) \right) \right) \right) \left( \frac{p}{\beta} \right)^{\frac{1}{K}}.
\]

\(^{33}\)This is possible since we can vary the prices of both Arrow securities at \( t = 1 \) independently.
Since \( u \circ V^{-1}(x) = x^\nu \) for some \( \nu \) it follows that the above can only hold if 
\( u \left( (u')^{-1} (x) \right) \) is homothetic. In this case we can write 
\[
  u \left( (u')^{-1} (x) \right) = a x^\delta.
\]
Then we have 
\[
  (u')^{-1} (x) = u^{-1} (a x^\delta).
\]
Assuming 
\[
  (u')^{-1} (x) = y,
\]
then 
\[
  u^{-1} (a x^\delta) = y \iff x = \left( \frac{u(y)}{a} \right)^{\frac{1}{\delta}}.
\]
Therefore, we have 
\[
  u'(x) = a \left( u(x) \right)^\delta.
\]
Thus if \( \delta \neq 1 \), we have 
\[
  \frac{d \left( u(x) \right)^{1-\delta}}{dx} = a \left(1 - \delta\right) \Rightarrow u(x) = (a \left(1 - \delta\right) x + c)^{-\frac{1}{1-\delta}}.
\]
This corresponds to the DARA or IARA case of the HARA class. If \( \delta = 1 \), 
\[
  \frac{d \ln u(x)}{dx} = a \left(1 - \delta\right) \Rightarrow u(x) = \exp \left( a \left(1 - \delta\right) x + c \right).
\]
A simple numerical example shows that DARA and IARA utilities within the 
HARA class do not produce time consistent demand unless the condition of The-
orem 2 holds. This completes the proof.

**A.2 Proof of Theorem 2**

For translated CRRA preferences (2), notice that the demand is identical to de-
mend with homothetic preferences and tradable endowments.

For negative exponential preferences, the result follows from the fact (see Pollak 
1971) that 
\[
  \lim_{d \to -\infty} \left( 1 + \frac{-\beta}{d} x \right)^d = - \exp(\beta x).
\]
Thus, the result follows directly from the proof of Theorem 1.
A.3 Proof of Proposition 2

It follows from Theorem 1 that DOCE demands are time consistent. Next we prove that DOCE and KP preferences generate the same demands. First, observe that for both KP and DOCE preferences, homogeneity of the utility function, together with Assumption [IR] implies that the optimal solution must satisfy

\[ c(s^t) = \alpha_{t-1}(s_t)c(s^{t-1}), \]

where \( \alpha_{t-1}(s_t) \) are constants that depend on the shock \( s_t \) and the previous time period \( t - 1 \). This implies that we can write the choice problems alternatively by having agents choose over \( t \) as well as initial consumption (subject to budget constraints). The key insight is that DOCE and KP preferences generate identical indirect utility functions over \( t \). To see this, observe that DOCE utility can be written as follows

\[
\mathcal{U}^{DOCE}(c, \alpha|s^T) = -\frac{c(s^T)^{-\delta_1}}{\delta_1} \frac{c(s^T)^{-\delta_1}}{\delta_1} \beta \left( \sum_{s \in S^T} \pi_{\tau + 1}(s_{\tau + 1}) \alpha_{\tau}(s_{\tau + 1})^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} - \frac{c(s^T)^{-\delta_1}}{\delta_1} \beta^2 \left( \sum_{s \in S^T} \pi_{\tau + 2}(s_{\tau + 2}) \alpha_{\tau + 1}(s_{\tau + 2})^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} - \ldots - \frac{c(s^T)^{-\delta_1}}{\delta_1} \beta^{T-\tau} \left( \sum_{s \in S^T} \pi_T(s_T) \alpha_T(s_T)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}}.
\]

KP utility can be written as

\[
\mathcal{U}^{KP}(c, \alpha|s^T) = -\frac{c(s^T)^{-\delta_1}}{\delta_1} - \frac{\beta \left( \sum_{s \in S^T} \pi(s_{\tau + 1}|s^T) U(c|s^T)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}}}{\delta_1} - \frac{\beta \left( \sum_{s \in S^T} \pi_{\tau + 2}(s_{\tau + 2}) \alpha_{\tau + 1}(s_{\tau + 2})^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}}}{\delta_1} - \ldots - \frac{\beta \left( \sum_{s \in S^T} \pi_T(s_T) \alpha_T(s_T)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}}}{\delta_1} \cdot (1 + \ldots)
\]

which, when multiplied out, is identical to DOCE utility. Thus KP and DOCE preferences generate the same demands.
A.4 Proof of Proposition 3

It follows from Theorem 2 that DOCE demands are time consistent. It follows from the proof of Theorem 2 and Proposition 2 that the DOCE and KP preferences generate the same demands assuming the same time and risk preferences.

A.5 Proof of Proposition 5

It follows from Proposition 3 that DOCE demands are time consistent and the same as those for KP preferences. Therefore, it is enough to consider DOCE resolute choice. Consider case (i) with $b = 0$. As in the proof of Proposition 2, homogeneity of the utility function, together with Assumption [IR] implies that the optimal solution must satisfy

$$c(s^t) = \alpha_{t-1}(s_t)c(s^{t-1}),$$

where $\alpha_{t-1}(s_t)$ are constants that depend on the shock $s_t$ and the previous time period $t - 1$. Note that

$$\mathcal{U}^{DOCE}(c, \alpha|s^\tau) = -\frac{c(s^\tau)^{-\delta_1}}{\beta_1} - \frac{1}{\beta_1} \left( \sum_{s_{\tau+1}} \pi_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} -$$

$$\frac{1}{\beta_1} \beta_2^2 \left( \sum_{s_{\tau+2}} \pi_{\tau+2}(s_{\tau+2})c(s_{\tau+2})^{-\delta_2} \right) \left( \sum_{s_{\tau+2}} \pi_{\tau+2}(s_{\tau+2})c(s_{\tau+2})^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} - \ldots -$$

$$\frac{1}{\beta_1} \beta_2^{T-\tau} \left( \sum_{s_{\tau+1}} \pi_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}}.$$

Following resolute choice, the optimal asset ratios $n_f(s^\tau)/n_j(s^\tau)$ are determined by maximizing the EU

$$\sum_{s_{\tau+1}} \pi_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2},$$

which is independent of time preference parameters $\delta_1$ and $\beta$. Since Assumption [IR] holds, in period $\tau$, each $s^\tau$ gives the same result and hence

$$\frac{n_f(s^\tau)}{n_j(s^\tau)} = \eta_j(s^\tau).$$

For other cases, the argument is the same as the proof of Theorem 2.
A.6 Proof of Proposition 6

The proof directly follows that of Propositions 2 and 3 where exponential discounting is replaced by quasi-hyperbolic discounting.

A.7 Proof of Proposition 7

Consider case (i) with $b = 0$. For both the KP and DOCE cases, homogeneity of the utility function, together with Assumption [IR] implies that resolute choice or sophisticated choice must satisfy

$$c(s^t) = \alpha_{t-1}(s_t)c(s^{t-1}),$$

where $\alpha_{t-1}(s_t)$ are constants that depend on the shock $s_t$ and the previous time period $t - 1$. Note that

$$U^{DOCE}(c, \alpha|s^\tau) = -\frac{c(s^\tau) - \delta_1}{\delta_1} - \frac{\gamma}{\delta_1} \left( \sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \right)^{\delta_1 \delta_2} - \gamma \beta^2 \left( \sum_{s_{\tau+2}} \bar{\pi}_{\tau+2}(s_{\tau+2})c(s_{\tau+2})^{-\delta_2} \right)^{\delta_1 \delta_2} - \ldots - \gamma \beta^{T-\tau} \left( \sum_{s_{T}} \bar{\pi}_T(s_T)c(s_T)^{-\delta_2} \right)^{\delta_1 \delta_2}$$

and

$$U^{KP}(c, \alpha|s^\tau) = -\frac{c(s^\tau) - \delta_1}{\delta_1} - \frac{\gamma}{\delta_1} \left( \sum_{s_{\tau+1} \neq s^\tau} \pi(s_{\tau+1}|s^\tau) U(c|s_{\tau+1}^{-\delta_2}) \right)^{\delta_1 \delta_2}$$

Therefore, the optimal asset ratios $n_f(s^\tau)/n_j(s^\tau)$ are determined by maximizing the EU

$$\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2},$$

which is independent of time preference parameters $\delta_1$, $\beta$ and $\gamma$. Since Assumption [IR] holds, in period $\tau$, each $s^\tau$ gives the same result and hence

$$\frac{n_f(s^\tau)}{n_j(s^\tau)} = \eta_j(s^\tau).$$
For other cases, the argument is the same as the proof of Theorem 2.

### A.8 Proof of Result 1

Assume $\delta_1 = 0$. It is enough to consider the tree structure in Figure 2. For DOCE resolute choice, the period 1 utility is

$$
\mathcal{U}(c) = \exp \left( \frac{\ln c_1 + \beta \ln \left( \pi_1 c_{21}^{\delta_2} + \pi_2 c_{22}^{\delta_2} \right)^{-\frac{1}{\delta_2}}}{\beta^2 \ln \left( \frac{\pi_1 (R_{f31} (R_{f2} (I - c_1) - c_{21}) - \delta_2)}{\pi_2 (R_{f32} (R_{f2} (I - c_1) - c_{22}) - \delta_2)} \right)^{-\frac{1}{\delta_2}}} \right).
$$

The first order conditions are

$$
\frac{\pi_1 c_{21}^{\delta_2-1}}{\left( \pi_1 c_{21}^{\delta_2} + \pi_2 c_{22}^{\delta_2} \right)^{-\frac{1}{\delta_2}}} = \frac{\beta \pi_1 (R_{f31} (R_{f2} (I - c_1) - c_{21}))^{-\delta_2-1}}{\left( \pi_1 (R_{f31} (R_{f2} (I - c_1) - c_{21}) - \delta_2)}{\pi_2 (R_{f32} (R_{f2} (I - c_1) - c_{22}) - \delta_2)} \right)^{-\frac{1}{\delta_2}},
$$

and

$$
\frac{\pi_2 c_{22}^{\delta_2-1}}{\left( \pi_1 c_{21}^{\delta_2} + \pi_2 c_{22}^{\delta_2} \right)^{-\frac{1}{\delta_2}}} = \frac{\beta \pi_2 (R_{f32} (R_{f2} (I - c_1) - c_{22}))^{-\delta_2-1}}{\left( \pi_1 (R_{f31} (R_{f2} (I - c_1) - c_{21}) - \delta_2)}{\pi_2 (R_{f32} (R_{f2} (I - c_1) - c_{22}) - \delta_2)} \right)^{-\frac{1}{\delta_2}},
$$

implying that

$$
\left( \frac{R_{f31}}{R_{f32}} \right)^{1+\frac{1}{\delta_2}} c_{21} = \frac{R_{f31} (R_{f2} (I - c_1) - c_{21})}{R_{f32} (R_{f2} (I - c_1) - c_{22})}. \tag{A.2}
$$

For DOCE sophisticated choice, we have

$$
c_{21} = \frac{R_{f2} (I - c_1)}{1 + \beta} \quad \text{and} \quad c_{22} = \frac{R_{f2} (I - c_1)}{1 + \beta}. \tag{A.3}
$$

Substituting eqn. (A.3) into (A.2), yields

$$
\left( \frac{R_{f31}}{R_{f32}} \right)^{1+\frac{1}{\delta_2}} = \frac{R_{f31}}{R_{f32}}.
$$

This condition can hold if and only if $\delta_2 = 0$, which contradicts the assumption $\delta_2 \neq 0$. Therefore, DOCE resolute and sophisticated choices are different. For KP preferences, since the period 2 conditional demands $c_{21}^K$ and $c_{22}^K$, respectively, are the same as $c_{21}^*$ and $c_{22}^*$ and $c_1^K = c_1^*$, the unconditional demands are also the same.
A.9 Proof of Proposition 10

Since the asset prices are assumed to be 1 and Assumption [CM] holds, it follows from eqn. (7) that \( \rho(s^t) \) can be viewed as the contingent claim price of \( c(s^t) \). For case (i), the first order conditions are

\[
\frac{\pi(s^t)}{\rho(s^t)} \left( \frac{c(s^t) - b}{\rho(s^t)} \right)^{-1-\delta_2} = \rho(s^t),
\]

implying that

\[
(c(s^t) - b)^{-\delta_2} = \left( \frac{\pi(s^t)}{\rho(s^t)} \right)^{-\frac{\delta_2}{1+\delta_2}} \left( \frac{c(s^t) - b}{\rho(s^t)} \right)^{-\delta_2} \]

and

\[
\tilde{c}_t - b = \left( \sum_{s^t} \pi(s^t) \left( \frac{\pi(s^t) \rho(s^t)}{\rho(s^t)} \right)^{-\frac{\delta_2}{1+\delta_2}} \left( \frac{c(s^t) - b}{\rho(s^t)} \right)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} (c(s^t) - b).
\]

Therefore, the period 1 utility function can be rewritten as

\[
u_1 + \sum_{t=2}^{T} \beta^{t-1} u(c_t) = u(c_1) + \sum_{t=2}^{T} \nu_t \beta^{t-1} u(c(s^t)),
\]

where

\[
u_t = \left( \sum_{s^t} \pi(s^t) \left( \frac{\pi(s^t) \rho(s^t)}{\rho(s^t)} \right)^{-\frac{\delta_2}{1+\delta_2}} \left( \frac{c(s^t) - b}{\rho(s^t)} \right)^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}}.
\]

A similar argument applies to case (iii). For case (ii), the first order conditions are

\[
\frac{\pi(s^t) \exp(-\kappa_2 c(s^t))}{\pi(s^t) \exp(-\kappa_2 c(s^t))} = \frac{\rho(s^t)}{\rho(s^t)},
\]

implying that

\[
c(s^t) = c(s^t) + \frac{1}{\kappa_2} \ln \left( \frac{\pi(s^t) \rho(s^t)}{\pi(s^t) \rho(s^t)} \right) \quad \text{and} \quad \tilde{c}_t = \ln \left( \sum_{s^t} \frac{\pi(s^t) \rho(s^t)}{\rho(s^t)} \right)^{-\frac{1}{\kappa_2}} + c(s^t).
\]

Therefore, the period 1 utility function can be rewritten as

\[
u_1 + \sum_{t=2}^{T} \beta^{t-1} u(c_t) = u(c_1) + \sum_{t=2}^{T} \nu_t \beta^{t-1} u(c(s^t)),
\]

where

\[
u_t = \left( \sum_{s^t} \frac{\pi(s^t) \rho(s^t)}{\rho(s^t)} \right)^{\frac{\delta_1}{\delta_2}}.
\]
References


B Supplemental Appendix

B.1 Proof of Proposition 4

As proved in Proposition 2, KP and DOCE preferences generate the same demands. Therefore, it is enough to consider DOCE sophisticated choice. Without loss of generality, consider the three period case with identical returns in each period. It can be verified that

\[
c_1^{DOCE} = \frac{I}{1 + \left( \beta \left( 1 + \beta^{\frac{1+\delta_1}{1+\delta_1}} \hat{R}^{-\delta_1} \right) \right)^{1+\delta_1} \hat{R}^{-\delta_1}},
\]

where

\[
\hat{R} = \left( E\hat{R}^{-\delta_2} \right)^{-\frac{1}{\delta_2}}.
\]

For the certainty case, it can be verified that

\[
c_1^{certain} = \frac{I}{1 + \left( \beta \left( 1 + \beta^{\frac{1}{1+\delta_1}} R_f^{-\delta_1} \right) \right)^{1+\delta_1}},
\]

Since

\[
\theta = s_1^{DOCE} - s_1^{certain} = c_1^{certain} - c_1^{DOCE},
\]

where \( s_1 = I - c_1 \), we have

\[
\theta \gtrless 0 \iff \left( 1 + \beta^{\frac{1}{1+\delta_1}} \hat{R}^{-\delta_1} \right)^{1+\delta_1} \hat{R}^{-\delta_1} \gtrless \left( 1 + \beta^{\frac{1}{1+\delta_1}} R_f^{-\delta_1} \right)^{1+\delta_1} R_f^{-\delta_1}.
\]

Since

\[
\hat{R} < E\hat{R} = R_f,
\]

we have

\[
\left( 1 + \beta^{\frac{1}{1+\delta_1}} \hat{R}^{-\delta_1} \right)^{1+\delta_1} \hat{R}^{-\delta_1} \gtrless \left( 1 + \beta^{\frac{1}{1+\delta_1}} R_f^{-\delta_1} \right)^{1+\delta_1} R_f^{-\delta_1} \iff \delta_1 \gtrless 0,
\]

implying that

\[
\theta \gtrless 0 \iff \delta_1 \gtrless 0.
\]

This is consistent with the result in the consumption-saving setting for the two period case in Selden and Wei (2018). Next consider the four period case with identical returns in each period. In the consumption-saving setting, for the DOCE preferences, it can be verified that

\[
c_1^{DOCE} = \frac{I}{1 + \left( \beta \left( 1 + \beta^{\frac{1}{1+\delta_1}} \hat{R}^{-\delta_1} \right) \left( 1 + \beta^{\frac{1+\delta_1}{1+\delta_1}} \hat{R}^{-\delta_1} \right)^{\delta_1+1} \hat{R}^{-\delta_1} \right)^{1+\delta_1}}.
\]
For the certainty case, it can be verified that
\[
c_{1}^{\text{certain}} = \frac{I}{1 + \left( \beta \left( 1 + \beta \frac{1}{1 + \delta_{1}} R_{f}^{-\delta_{1}} \right) \left( 1 + \beta \frac{1}{1 + \delta_{1}} R_{f}^{-\delta_{1}} \right)^{\delta_{1}+1} R_{f}^{-\delta_{1}} \right)^{\frac{1}{1+\delta_{1}}}}.
\]
Therefore, we still have
\[
\theta \leq 0 \iff \delta_{1} \leq 0.
\]
Based on induction, the above result can be extended to the $T$ period case given Assumption [IR].

**B.2 Supporting Calculations for Section 4.2**

Clearly Assumption [IR] is violated for the case in Figure 2 since the risk free returns for the upper and lower branches differ. Given the assumed form of preferences, if Assumption [IR] holds and $R_{f31} = R_{f32} = R_{f3}$ it follows from Theorem 1 that the DOCE demands will be time consistent and resolute, naive and sophisticated choice will agree. It follows from Proposition 2 that the KP and DOCE demands will be the same. To provide intuition for the impact of relaxing Assumption [IR], it is first useful to consider the following certainty problem

\[
\max_{c_{1},c_{2},c_{3}} (c_{1}^{-\delta_{1}} + \beta c_{2}^{-\delta_{1}} + \beta^{2} c_{3}^{-\delta_{1}})^{-\frac{1}{\delta_{1}}} \quad \text{S.T. } I = c_{1} + \frac{c_{2}}{R_{f2}} + \frac{c_{3}}{R_{f2}R_{f3}}.
\]

From the first order conditions, optimal first period consumption is given by
\[
c_{1} = \frac{I}{1 + B_{1}^{\frac{1}{1+\delta_{1}}}},
\]
where
\[
B_{1} = \beta R_{f2}^{-\delta_{1}} \left[ \left( \frac{1}{1 + \beta \frac{1}{1+\delta_{1}} R_{f3}^{\frac{1}{1+\delta_{1}}}} \right)^{-\delta_{1}} + \beta \left( \frac{\beta \frac{1}{1+\delta_{1}} R_{f3}^{\frac{1}{1+\delta_{1}}}}{1 + \beta \frac{1}{1+\delta_{1}} R_{f3}^{\frac{1}{1+\delta_{1}}}} \right)^{-\delta_{1}} \right],
\]

or equivalently,
\[
B_{1} = \beta R_{f2}^{-\delta_{1}} \left( 1 + \beta \frac{1}{1+\delta_{1}} R_{f3}^{-\delta_{1}} \right)^{1+\delta_{1}}.
\]

To provide some intuition on how to interpret the two terms inside the bracket on the right hand side of eqn. (B.2), note first that based on the constraint in (B.1) one can introduce the following convention for defining prices for consumption in each period as follows:
\[
p_{1} = 1, \quad p_{2} = \frac{1}{R_{f2}} \quad \text{and} \quad p_{3} = \frac{1}{R_{f2}R_{f3}}.
\]
It can be verified that
\[ p_1c_1 = \frac{I}{1 + \beta \frac{1}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} + \beta \frac{2}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} R_{f_3} \frac{\delta_1}{1+\xi_1}}, \]

\[ p_2c_2 = \frac{\beta \frac{1}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} I}{1 + \beta \frac{1}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} + \beta \frac{2}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} R_{f_3} \frac{\delta_1}{1+\xi_1}}, \]

and

\[ p_3c_3 = \frac{\beta^2 \frac{2}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} R_{f_3} \frac{\delta_1}{1+\xi_1} I}{1 + \beta \frac{1}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} + \beta \frac{2}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} R_{f_3} \frac{\delta_1}{1+\xi_1}}. \]

Computing the relative marginal propensities to consume (MPCs), we obtain the following
\[ \frac{\partial (p_2c_2)}{\partial I} = \beta \frac{1}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} \quad \text{and} \quad \frac{\partial (p_3c_3)}{\partial I} = \beta^2 \frac{2}{1+\xi_1} R_{f_2} \frac{\delta_1}{1+\xi_1} R_{f_3} \frac{\delta_1}{1+\xi_1} \]

and the expression (B.3) can be rewritten in terms of the relative MPCs as
\[ B_{1}^{\frac{1}{1+\xi_1}} = \frac{\partial (p_2c_2)}{\partial (p_1c_1)} / \partial I + \frac{\partial (p_3c_3)}{\partial (p_1c_1)} / \partial I. \]

To derive the DOCE sophisticated demands when \( R_{f31} \neq R_{f32} \), we follow the solution process in Definition 3. Assuming the upper state is realized, the sophisticated DOCE consumer solves the optimization problem
\[ \max_{c_{21}} \left( c_{21}^{-\delta_1} + \beta \left( R_{f31} (R_{f2} (I-c_1) - c_{21}) \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}}. \]

The first order condition is
\[ c_{21}^{-\delta_1-1} = \frac{\beta R_{f31}}{\left( R_{f31} (R_{f2} (I-c_1) - c_{21}) \right)^{1+\delta_1}}. \]

It follows that
\[ c_{21}^{-\delta_1-1} = \frac{\beta (R_{f2} (I-c_1) - c_{21})^{-\delta_1-1}}{R_{f31}^{\delta_1}}, \]

implying that
\[ c_{21} = \frac{R_{f2} (I-c_1)}{1 + \beta \frac{1}{1+\xi_1} R_{f31} \frac{\delta_1}{1+\xi_1}}. \]

For the lower branch, similarly, one can obtain
\[ c_{22} = \frac{R_{f2} (I-c_1)}{1 + \beta \frac{1}{1+\xi_1} R_{f32} \frac{\delta_1}{1+\xi_1}}. \]
Hence

\[ c_{31} = \frac{R_{f2} \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}} (I - c_1)}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} \quad \text{and} \quad c_{32} = \frac{R_{f2} \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}} (I - c_1)}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} . \]

Therefore, the period 1 utility function becomes

\[ (c_1^{-\delta_1} + \beta (\pi_1 c_2 + \pi_2 c_2) - \delta_1 + \beta^2 (\pi_1 c_3 + \pi_2 c_3)^{1-\delta_1}) \] 

\[ = \left( \frac{c_1^{-\delta_1}}{\beta} \left( \frac{\pi_1 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} + \frac{\pi_2 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} \right) (I - c_1)^{-\delta_1} - (I - c_1)^{1-\delta_1} \] 

\[ + \beta^2 \left( \frac{\pi_1 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} + \frac{\pi_2 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} \right) (I - c_1)^{-\delta_1} . \]

The first order condition is

\[ 0 = c_1^{-\delta_1} - \beta \left( \frac{\pi_1 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} + \frac{\pi_2 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} \right) (I - c_1)^{-\delta_1} - \beta^2 \left( \frac{\pi_1 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} + \frac{\pi_2 R_{f2}^{\frac{1}{1+\gamma_1}}}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} \right) (I - c_1)^{-\delta_1} . \]

The optimal DOCE sophisticated solution is given by

\[ c_1^{s*} = \frac{I}{1 + B_1^{\frac{1}{1+\gamma_1}}} , \]  

(B.4)

where

\[ B_1 = \beta R_{f2}^{-\delta_1} \left( \frac{\pi_1}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} - \delta_2 + \frac{\pi_2}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} - \delta_2 \right) + \beta^2 R_{f2}^{-\delta_1} \left( \frac{\pi_1}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} - \delta_2 + \frac{\pi_2}{1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3}^{\frac{\delta_1}{1+\gamma_1}}} - \delta_2 \right) \]  

(B.5)

It is clear that the risk confronting the consumer in period 1 is which pair of period 2 and 3 conditional demands will hold, which in turn depends on whether \( R_{f31} \) or \( R_{f32} \) is realized. This explains why terms such as \( 1 + \beta^{\frac{1}{1+\gamma_1}} R_{f3s}^{\frac{\delta_1}{1+\gamma_1}} \) (\( s = 1, 2 \)) appear in the solution (B.4)-(B.5).
Define the certainty equivalent returns $\hat{R}_{f3}^1$ and $\hat{R}_{f3}^2$ as follows

$$
\frac{1}{1 + \hat{R}_{f3}^1 \beta^{\frac{1}{1+\pi_1}}} = \left( \frac{\pi_1}{1 + \beta^{\frac{1}{1+\pi_1}} R_{f31}^1} \right)^{-\delta_2} + \left( \frac{\pi_2}{1 + \beta^{\frac{1}{1+\pi_1}} R_{f32}^1} \right)^{-\delta_2}
$$

and

$$
\frac{2\hat{R}_{f3}^2 \beta^{\frac{1}{1+\pi_1}}}{1 + 2\hat{R}_{f3}^2 \beta^{\frac{1}{1+\pi_1}}} = \left( \frac{\pi_1}{1 + \beta^{\frac{1}{1+\pi_1}} R_{f31}^2} \right)^{-\delta_2} + \pi_2 \left( \frac{\beta^{\frac{1}{1+\pi_1}} R_{f32}^2}{1 + \beta^{\frac{1}{1+\pi_1}} R_{f32}^1} \right)^{-\delta_2}
$$

Then $B_1$ in eqn. (B.5) can be rewritten as

$$
B_1 = \beta R_{f2}^{-\delta_1} \left[ \left( \frac{1}{1 + \hat{R}_{f3}^1 \beta^{\frac{1}{1+\pi_1}}} \right)^{-\delta_1} + \beta \left( \frac{2\hat{R}_{f3}^2 \beta^{\frac{1}{1+\pi_1}}}{1 + 2\hat{R}_{f3}^2 \beta^{\frac{1}{1+\pi_1}}} \right)^{-\delta_1} \right],
$$

which is exactly analogous to the certainty eqn. (B.2). Clearly, the certainty equivalent returns $\hat{R}_{f3}^1$ and $\hat{R}_{f3}^2$ depend not just on the risk preference parameter $\delta_2$ but also on the time preference parameter $\delta_1$. Thus although at the DOCE preference level time and risk preferences can be specified independently, at the sophisticated demand level when Assumption [IR] does not hold time and risk preferences are clearly intertwined in the sense that the certainty equivalents depend on $\delta_1$ and $\beta$ as well as $\delta_2$.\(^{34}\)

Next we solve for optimal period 1 consumption assuming KP preferences based on the same CES and CRRA utilities in (5). The period 2 optimization problems are the same as for the DOCE case. The period 1 utility is given by

$$
U(c) = \frac{\pi_1}{\pi_1 - \delta_1} \left( c_1^{-\delta_1} + \beta \left( \pi_1 U_{21}^{-\delta_2} + \pi_2 U_{22}^{-\delta_2} \right) \right)^{-\frac{1}{\pi_1}},
$$

where

$$
U_{21} = \left( c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{-\frac{1}{\pi_1}} \quad \text{and} \quad U_{22} = \left( c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{-\frac{1}{\pi_1}}.
$$

Since the conditional demands $c_{21}, c_{22}, c_{31}$ and $c_{32}$ are the same as those for DOCE preferences, we have

$$
U_{21} = \left( \frac{R_{f2} (I - c_1)}{1 + \beta^{\frac{1}{1+\pi_1}} R_{f31}^1} \right)^{-\delta_1} + \beta \left( \frac{R_{f2} \beta^{\frac{1}{1+\pi_1}} R_{f31}^1 (I - c_1)}{1 + \beta^{\frac{1}{1+\pi_1}} R_{f31}^1} \right)^{-\delta_1}
$$

\(^{34}\)Since $\hat{R}_{f3}$ and $\hat{R}_{f3}$ are in general different, eqn. (B.7) cannot be simplified to a form analogous to eqn. (B.3).
and

\[ U_{22} = \left( \frac{R_{f2} (I - c_1)}{1 + \beta^{1+\delta_1} R_{f32}^{\frac{1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left( \frac{R_{f2} \beta^{1+\delta_1} R_{f32}^{\frac{1}{1+\delta_1}} (I - c_1)}{1 + \beta^{1+\delta_1} R_{f32}^{\frac{1}{1+\delta_1}}} \right)^{-\delta_1} \left( \frac{1}{\beta} \right)^{-\frac{1}{\delta_1}}. \]

Therefore, the period 1 utility function is

\[ c_1^{-\delta_1 - 1} = \beta \left( \frac{1}{\pi_1} \left( \frac{R_{f2}(I - c_1)}{1 + \beta^{1+\delta_1} R_{f31}^{\frac{1}{1+\delta_1}}} \right)^{\delta_1} \left( 1 + \beta^{1+\delta_1} R_{f32}^{\frac{1}{1+\delta_1}} \right)^{-\delta_1} \right) - \delta_1. \]

Solving for optimal demands recursively yields

\[ c_1 = \frac{I}{1 + B_2^{1+\delta_1}}, \]

where

\[ B_2 = \beta R_{f2}^{-\delta_1} \left( \frac{1}{\pi_1} \left( 1 + \beta^{1+\delta_1} R_{f31}^{\frac{1+\delta_1}{\delta_1}} \right) + \pi_2 \left( 1 + \beta^{1+\delta_1} R_{f32}^{\frac{1+\delta_1}{\delta_1}} \right) \right)^{\delta_1}. \]

Comparing (B.9) with the corresponding expression for the certainty case (B.3), if we define the certainty equivalent \( \hat{R}_{f3} \) based on the following

\[ 1 + \beta^{1+\delta_1} \hat{R}_{f3}^{\frac{1}{1+\delta_1}} = \left( \frac{\pi_1}{\pi_1} \left( 1 + \beta^{1+\delta_1} \hat{R}_{f31}^{\frac{1+\delta_1}{\delta_1}} \right) + \pi_2 \left( 1 + \beta^{1+\delta_1} \hat{R}_{f32}^{\frac{1+\delta_1}{\delta_1}} \right) \right)^{\delta_1}, \]

then \( B_2 \) in eqn. (B.9) can be rewritten as

\[ B_2 = \beta R_{f2}^{-\delta_1} \left( 1 + \beta^{1+\delta_1} \hat{R}_{f3}^{\frac{1}{1+\delta_1}} \right)^{1+\delta_1}, \]

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which is exactly analogous to eqn. (B.3). It should be noted that in eqn. (B.10) \( \delta_1 \) enters into the intertemporal risk terms in each state

\[
1 + \beta \frac{1}{1+\gamma} R_{f31}^{\delta_1} \quad \text{and} \quad 1 + \beta \frac{1}{1+\gamma} R_{f32}^{\delta_1}
\]
as in the sophisticated DOCE expression (B.6). However, for the KP case, \( \delta_1 \) also enters into the outer exponent \( \frac{(1+\delta_1)\delta_2}{\delta_1} \). This is an important difference from the DOCE sophisticated demands.

Next we show how to apply Karush–Kuhn–Tucker (KKT) conditions to solve for the resolute choice in the risk neutral case.

**Proposition 11** Consider the consumption-saving problem (12) - (15) with the tree structure in Figure 2 and \( \delta_2 = -1 \)

\[
U(c_1, c_{21}, c_{22}) = \left( \begin{array}{c} c_1^{\delta_1} + \beta (\pi_1 c_{21} + \pi_2 c_{22})^{\delta_1} + \\
\beta^2 \left( \pi_1 R_{f31} (R_{f2} (I - c_1) - c_{21}) \right)^{\delta_1} + \\
\pi_2 R_{f32} (R_{f2} (I - c_1) - c_{22}) \end{array} \right)^{-\frac{1}{\gamma_1}}.
\]

We have the following resolute choice:

(i) if

\[
\pi_1 \geq \frac{1}{1 + \beta \frac{1}{1+\gamma} R_{f31}^{\delta_1}},
\]

then

\[
c^*_1 = \frac{\beta \frac{2}{1+\gamma} (R_{f2} R_{f31})^{\frac{\delta_1}{1+\gamma}} I}{1 + \beta \frac{1}{1+\gamma} R_{f31}^{\delta_1} + \beta^2 \frac{2}{1+\gamma} (R_{f2} R_{f31})^{\frac{\delta_1}{1+\gamma}}};
\]

(ii) if

\[
\frac{1}{1 + \beta \frac{1}{1+\gamma} R_{f31}^{\delta_1}} > \pi_1 > \frac{1}{1 + R_{f31} \beta \frac{1}{1+\gamma} R_{f32}^{\delta_1}},
\]

then

\[
c^*_1 = \frac{\left( R_{f2} (\beta - 1) (\pi_2 R_{f2})^{1-\delta_1} + \beta R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31})^{1-\delta_1} \right)^{-\frac{1}{1+\gamma_1}} I}{1 + \left( R_{f2} (\beta - 1) (\pi_2 R_{f2})^{1-\delta_1} + \beta R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31})^{1-\delta_1} \right)^{-\frac{1}{1+\gamma_1}}};
\]

(iii) if

\[
\pi_1 \leq \frac{1}{1 + R_{f31} \beta \frac{1}{1+\gamma} R_{f32}^{\delta_1}},
\]

then

\[
c^*_1 = \frac{\left( (\beta - 1) R_{f2} \pi_2 (R_{f2} - \gamma) + \beta R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31} + \pi_2 R_{f32} \gamma) \right)^{-\frac{1}{1+\gamma_1}} I}{1 + \left( (\beta - 1) R_{f2} \pi_2 (R_{f2} - \gamma) + \beta R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31} + \pi_2 R_{f32} \gamma) \right)^{-\frac{1}{1+\gamma_1}}};
\]

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Proof. An agent solves

$$\max u(c_1) + \beta u \left( \sum_{s=1}^{2} \pi_s c_{2s} \right) + \beta^2 u \left( \sum_{s=1}^{2} \pi_s c_{3s} \right) \quad S.T.$$

$$c_1 = I - n_1, \quad c_{2s} = n_1 R_{f2} - n_{2s}, c_{3s} = n_{2s} R_{f3s}, s = 1, 2,$$

$$c_1 \geq 0, c_{2s} \geq 0, c_{3s} \geq 0, s = 1, 2.$$

Assuming that $u(.)$ satisfies an Inada condition we can drop the first constraint and obtain

$$n_1 R_{f2} - n_{2s} \geq 0, \quad n_{2s} \geq 0, s = 1, 2.$$

Denote by $\mu_{1s}$ $(s = 1, 2)$ the Lagrange multiplier associated with the inequality constraint $n_1 R_{f2} - n_{2s}$ and $\mu_{2s}$ $(s = 1, 2)$ the Lagrange multiplier associated with the inequality constraint $n_{2s}$. The KKT conditions are

$$-u'(c_1) + \beta R_{f2} u'(\sum_s \pi_s c_{2s}) + \sum_s R_{f2} \mu_{1s} = 0, \quad \text{(B.11)}$$

$$-u'(\sum_s \pi_s c_{2s}) + \beta R_{f3s} u'(\sum_s \pi_s c_{3s}) - \mu_{1s} + \mu_{2s} = 0 \quad (s = 1, 2), \quad \text{(B.12)}$$

$$\mu_{ts} \geq 0 \quad (t = 1, 2, \ s = 1, 2),$$

and

$$\mu_{1s} (n_1 R_{f2} - n_{2s}) = 0 \quad \text{and} \quad \mu_{2s} n_{2s} = 0 \quad (s = 1, 2).$$

Without loss of generality, assume $R_{f31} > R_{f32}$. It is easy to see that the equations

$$-u'(\sum_s \pi_s c_{2s}) + \beta R_{f3s} u'(\sum_s \pi_s c_{3s}) - \mu_{1s} + \mu_{2s} = 0 \quad (s = 1, 2)$$

can only have a solution if $\mu_{22} > 0$ or $\mu_{11} > 0$. Complementary slackness implies that $c_{22} = 0$ or $c_{21} = 0$. As we will show, there are three cases.

1. Under the condition

$$\pi_1 \geq \frac{1}{1 + \beta^{-\frac{1}{\gamma + \gamma_1}} R_{f31}^{\frac{\gamma_1}{\gamma_1+1}}},$$

we have $\mu_{22} > 0, \mu_{11} = 0$, i.e. $c_{22} = n_1 R_{f2}, c_{32} = 0$.

2. Under the condition

$$\frac{1}{1 + \beta^{-\frac{1}{\gamma + \gamma_1}} R_{f31}^{\frac{\gamma_1}{\gamma_1+1}}} > \pi_1 > \frac{1}{1 + R_{f31} \beta^{-\frac{1}{\gamma + \gamma_1}} R_{f32}^{\frac{1}{\gamma_1+1}}},$$

we have $\mu_{22} > 0, \mu_{11} > 0$, and $c_{21} = n_1 R_{f2}, c_{31} = 0, c_{22} = 0$ and $c_{32} = n_1 R_{f2} R_{f32}$.
3. Under the condition

\[ \pi_1 \leq \frac{1}{1 + R_{f31}^\beta R_{f32}^{\frac{1}{\alpha_{f31}}}}. \]

we have \( \mu_{22} = 0, \mu_{11} > 0, \) and \( c_{22} = 0, c_{32} = n_1 R_{f2} R_{f32}. \)

We will solve the three cases, one by one.

Case 1: First solve for a solution for \( \mu_{22} > 0, \mu_{11} = 0. \) We have \( n_{22} = 0 \) and from the two first order conditions that hold with equality we obtain two linear equations

\[ I - n_1 = (\beta R_{f2})^{\frac{1}{\alpha_{f31}}} (\pi_1 (n_1 R_{f2} - n_{21}) + \pi_2 n_1 R_{f2}) \]

and

\[ n_1 R_{f2} - \pi_1 n_{21} = (R_{f31}^\beta)^{\frac{1}{\alpha_{f31}}} n_1 R_{f31}. \]

This implies

\[ n_{21} = \frac{n_1 R_{f2}}{\pi_1 (1 + \beta^{\frac{1}{\alpha_{f31}}} R_{f31}^{\frac{1}{\alpha_{f31}}})} \quad \text{and} \quad s_1 = \frac{1 + \beta^{\frac{1}{\alpha_{f31}}} R_{f31}^{\frac{1}{\alpha_{f31}}}}{1 + \beta^{\frac{1}{\alpha_{f31}}} R_{f31}^{\frac{1}{\alpha_{f31}}} + \beta^{\frac{\alpha_{f31}}{\alpha_{f31}}} (R_{f2} R_{f32})^{\frac{1}{\alpha_{f31}}}}. \]

Since we require \( c_{21} \geq 0, \) which is equivalent to \( n_1 R_{f2} - n_{21} \geq 0, \) condition (B.13) is obtained.

Case 2: In the next step we solve for \( \mu_{22} > 0, \mu_{11} > 0 \) – in this case we have

\[ n_{21} = R_{f2} n_1 \quad \text{and} \quad n_{22} = 0. \]

This is the correct solution if inequality (B.13) does not hold and if

\[ u'(\pi_2 R_{f2} n_1) > \beta R_{f32} u'(\pi_1 R_{f2} R_{f31} n_1) \]

or

\[ \pi_2 < \pi_1 (\beta R_{f32})^{\frac{1}{\alpha_{f31}}} \pi_1 R_{f31}. \]

Using \( \pi_2 = 1 - \pi_1 \) this gives eqn. (B.14). In this case we obtain from the KKT condition (B.12) at (B.14) that

\[ \mu_{11} = -u'(\pi_2 n_1 R_{f2}) + \beta R_{f31} u'(\pi_1 R_{f2} R_{f31} n_1). \]

Plugging this into the KKT condition (B.11) we obtain the following non-linear equation:

\[ -u'(I - n_1) + \beta R_{f2} u'(\pi_2 n_1 R_{f2}) + R_{f2} (-u'(\pi_2 n_1 R_{f2}) + \beta R_{f31} u'(\pi_1 R_{f2} R_{f31} n_1)) = 0. \]
This is equivalent to

\[(I - n_1)^{-\delta_1 - 1} = R_{f2}(\beta - 1)n_1^{-1-\delta_1} (\pi_2 R_{f2})^{-1-\delta_1} + n_1^{-1-\delta_1} R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31})^{-1-\delta_1},\]

which implies

\[I - n_1 = n_1 \left( R_{f2}(\beta - 1) (\pi_2 R_{f2})^{-1-\delta_1} + R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31})^{-1-\delta_1} \right)^{-\frac{1}{1+\delta_1}} \]

and

\[n_1 = \frac{I}{1 + \left( R_{f2}(\beta - 1) (\pi_2 R_{f2})^{-1-\delta_1} + R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31})^{-1-\delta_1} \right)^{-\frac{1}{1+\delta_1}}}.\]

Case 3: Finally, the third and last case is \(\mu_{11} > 0\) and \(\mu_{22} = 0\). The optimal \(n_{22}\) is determined by

\[u'(\pi_2(n_1 R_{f2} - n_{22})) = R_{f32} u'(\pi_1 n_1 R_{f2} R_{f31} + \pi_2 n_{22} R_{f32}),\]

or substituting for \(u'\) yields

\[\pi_2(n_1 R_{f2} - n_{22}) = (\beta R_{f32})^{-\frac{1}{1+\delta_1}} (\pi_1 n_1 R_{f2} R_{f31} + \pi_2 n_{22} R_{f32}),\]

which implies

\[n_{22} = \gamma n_1, \quad \gamma = \frac{\pi_2 R_{f2} - (\beta R_{f32})^{-\frac{1}{1+\delta_1}} \pi_1 R_{f2} R_{f31}}{\pi_2 + (\beta R_{f32})^{-\frac{1}{1+\delta_1}} \pi_2 R_{f32}}.\]

Similarly as in Case 2 the optimal choice is determined by

\[0 = -u'(I - n_1) + R_{f2} u'(\pi_2(n_1 R_{f2} - n_{22})) + R_{f2} u'(\pi_2(n_1 R_{f2} - n_{22})) + R_{f31} u'(\pi_1 R_{f2} R_{f31} n_1 + \pi_2 R_{f32} n_{22})).\]

Plugging in \(n_{22}\) we obtain

\[I - n_1 = n_1 ((\beta - 1) R_{f2} \pi_2 (R_{f2} - \gamma) + R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31} + \pi_2 R_{f32} \gamma))^{-\frac{1}{1+\delta_1}}\]

and hence

\[n_1 = \frac{I}{1 + ((\beta - 1) R_{f2} \pi_2 (R_{f2} - \gamma) + R_{f2} R_{f31} (\pi_1 R_{f2} R_{f31} + \pi_2 R_{f32} \gamma))^{-\frac{1}{1+\delta_1}}}.\]
B.3 Supporting Calculations for Example 2

For the certainty case, it can be easily verified that

\[ c_1 = \frac{I}{1 + \beta^{\frac{2}{n_f 2}} R_{f 2}^{-\frac{1}{n_f 2}} + \beta^{\frac{2}{n_f 3}} R_{f 2}^{-\frac{1}{n_f 3}} R_{f 3}^{-\frac{1}{n_f 3}}} \, . \]

For DOCE resolute choice, the resulting first order conditions are

\[ c_1^{-\delta_1 - 1} = \beta^2 \left( \frac{\pi_1 (R_{f 31} (R_{f 2} (I - c_1) - c_{21}))^{-\delta_2}}{1 + \pi_2 (R_{f 32} (R_{f 2} (I - c_1) - c_{22}))^{-\delta_2}} \right) \times \]

\[ \left( \frac{\pi_1 R_{f 31} R_{f 2} (R_{f 31} (R_{f 2} (I - c_1) - c_{21}))^{-\delta_2 - 1}}{1 + \pi_2 R_{f 32} R_{f 2} (R_{f 32} (R_{f 2} (I - c_1) - c_{22}))^{-\delta_2 - 1}} \right) , \]

\[ (\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2})^{\frac{\delta_1}{\delta_2} - 1} c_{21}^{-\delta_2 - 1} \]

\[ = \beta R_{f 31} \left( \frac{\pi_1 (R_{f 31} (R_{f 2} (I - c_1) - c_{21}))^{-\delta_2}}{1 + \pi_2 (R_{f 32} (R_{f 2} (I - c_1) - c_{22}))^{-\delta_2}} \right) \times (R_{f 31} (R_{f 2} (I - c_1) - c_{21}))^{-\delta_2 - 1} \]

and

\[ (\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2})^{\frac{\delta_1}{\delta_2} - 1} c_{22}^{-\delta_2 - 1} \]

\[ = \beta R_{f 32} \left( \frac{\pi_1 (R_{f 31} (R_{f 2} (I - c_1) - c_{21}))^{-\delta_2}}{1 + \pi_2 (R_{f 32} (R_{f 2} (I - c_1) - c_{22}))^{-\delta_2}} \right) \times (R_{f 32} (R_{f 2} (I - c_1) - c_{22}))^{-\delta_2 - 1} \, . \]

B.4 Supporting Calculations for Section 4.3

It follows from the constraints

\[ c_{31} = R_{f 31} (R_{21} n_1 + R_{f 2} n_{f 1} - c_{21}) \quad \text{and} \quad c_{32} = R_{f 32} (R_{22} n_1 + R_{f 2} n_{f 1} - c_{22}) \]

that

\[ n_1 = \frac{c_{31}}{R_{f 31}} + c_{21} - \frac{c_{32}}{R_{f 32}} - c_{22} \quad \text{and} \quad n_{f 1} = \frac{R_{21} \left( c_{31} - c_{22} \right) - R_{22} \left( c_{32} - c_{21} \right)}{(R_{21} - R_{22}) R_{f 2}} . \]
Therefore, the period 1 budget constraint is

\[ I = c_1 + n_1 + n_{f1} \]

\[ I = c_1 + \frac{c_{f1}}{R_{f1}} + c_{21} - \frac{c_{f2}}{R_{f2}} - c_{22} \]

\[ \frac{R_{21} (c_{f2} + c_{22}) - R_{22} (c_{f1} + c_{21})}{(R_{21} - R_{22}) R_{f2}} \]

\[ = c_1 + \frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} c_{21} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} c_{22} \]

\[ + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f3} R_{f2}} c_{31} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} c_{32}. \]

Thus the first order conditions for DOCE resolute choice are

\[ \frac{\pi_1 c_{21}^{1-\delta_2} c_{31}^{1-\delta_2}}{\pi_2 c_{22}^{1-\delta_2}} = \frac{R_{f2} - R_{22}}{R_{21} - R_{f2}} \iff c_{22} = \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{\frac{1}{1+\delta_2}} c_{21} \]

and

\[ \frac{\pi_1 c_{31}^{1-\delta_2}}{\pi_2 c_{32}^{1-\delta_2}} = \frac{(R_{f2} - R_{22}) R_{f32}}{(R_{21} - R_{f2}) R_{f31}} \iff c_{32} = \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{1+\delta_2}} c_{31}. \]

Therefore, following Proposition 10, the period 1 DOCE utility function can be transformed into a certainty utility of the single branch \((c_1, c_{21}, c_{31})\), which is

\[
\begin{pmatrix}
  c_{1}^{-\delta_1} + \beta \left( \pi_1 + \pi_2 \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{\frac{\delta_2}{1+\delta_2}} c_{21}^{\delta_1} \right) \\
  + \beta^2 \left( \pi_1 + \pi_2 \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{\delta_2}{1+\delta_2}} c_{31}^{\delta_1} \right)
\end{pmatrix}
\]

(B.15)

and the budget constraint can be rewritten as

\[
I = c_1 + \left( \frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{\frac{1}{1+\delta_2}} \right) c_{21} + 
\]

\[
\left( \frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f3} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f3} R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{\frac{1}{1+\delta_2}} \right) c_{31}. \]

(B.16)
The first order condition is

\[
\beta \left( \frac{1}{\pi_1 + \pi_2} \left( \frac{\pi_2 (R_{f2} - R_{f2})}{\pi_1 (R_{21} - R_{f2})} \right) - \frac{\delta_1}{\pi_2} \right) \frac{\delta_1}{\pi_2} c_{21}^{-\delta_1}
\]

\[
\beta^2 \left( \frac{1}{\pi_1 + \pi_2} \left( \frac{\pi_2 (R_{f2} - R_{f2}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right) - \frac{\delta_1}{\pi_2} \right) \frac{\delta_1}{\pi_2} c_{31}^{-\delta_1}
\]

\[
= \frac{R_{f2} - R_{22}}{(R_{21} - R_{f2}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^\frac{1}{1+\delta_1}
\]

or equivalently,

\[c_{31} = \kappa c_{21},\]

where

\[
\kappa = \beta \left( \frac{R_{f2} - R_{22}}{(R_{21} - R_{f2}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^\frac{1}{1+\delta_1} \right) \frac{1}{1+\delta_1}
\]

\[
\left( \frac{1}{\pi_1 + \pi_2} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right) - \frac{\delta_1}{\pi_2} \right) \frac{\delta_1}{\pi_2}
\]

\[
\left( \frac{R_{f2} - R_{22}}{(R_{21} - R_{f2}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^\frac{1}{1+\delta_1} \right) \frac{1}{1+\delta_1}
\]

\[
\left( \frac{1}{\pi_1 + \pi_2} \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right) - \frac{\delta_1}{\pi_2} \right) \frac{\delta_1}{\pi_2}
\]

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Therefore, we have

\[
\frac{n_{f_1}}{n_1} = \frac{R_{21} \left( \frac{c_{22}}{R_{f_{32}}} + c_{22} \right) - R_{22} \left( \frac{c_{21}}{R_{f_{31}}} + c_{21} \right)}{R_f \left( \frac{c_{21}}{R_{f_{31}}} + c_{21} - \frac{c_{22}}{R_{f_{32}}} - c_{22} \right)}
\]

\[
= \frac{R_{21} \left( \frac{\pi_2 (R_{f_2 - R_{22}}) R_{f_{32}}}{\pi_1 (R_{f_2 - R_{22}}) R_{f_{31}}} \right)^{1+\sigma_2} + \left( \frac{\pi_2 (R_{f_2 - R_{22}})}{\pi_1 (R_{f_2 - R_{22}})} \right)^{1+\sigma_2}}{R_f \left( \frac{\pi_2 (R_{f_2 - R_{22}}) R_{f_{32}}}{\pi_1 (R_{f_2 - R_{22}}) R_{f_{31}}} \right)^{1+\sigma_2} - \left( \frac{\pi_2 (R_{f_2 - R_{22}})}{\pi_1 (R_{f_2 - R_{22}})} \right)^{1+\sigma_2}}.
\]

B.5 Supporting Calculations for Example 3

For DOCE sophisticated choice, the period 1 utility function is

\[
U(c) = \left( c_1^{-\delta_1} + \beta \left( \pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2} \right)^{\delta_1} + \beta^2 \left( \pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2} \right)^{\delta_1} \right)^{-\frac{1}{\delta_1}}
\]

\[
= \left( c_1^{-\delta_1} + \beta \left( \frac{R_{21} n_{1} + R_{22} n_{1}}{1 + \beta \frac{1}{R_{f_{31}}}} \right)^{-\delta_2} + \beta^2 \left( \frac{R_{22} n_{1} + R_{f_{22} n_{1}}}{1 + \beta \frac{1}{R_{f_{31}}}} \right)^{-\delta_2} \right)^{\delta_1} + \beta^2 \left( \frac{R_{21} n_{1} + R_{f_{22} n_{1}}}{1 + \beta \frac{1}{R_{f_{31}}}} \right)^{-\delta_2}
\]

The DOCE sophisticated case simulations in Example 3 can follow the first order conditions based on the above equation. For KP preferences, the period 1 utility
function is

$$U(c) = \left( c_1^{-\delta_1} + \beta \left( \pi_1 \left( c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{\delta_2 \over \delta_1} + \pi_2 \left( c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{\delta_2 \over \delta_1} \right) \right)^{-1 \over \delta_1}$$

$$= \left( c_1^{-\delta_1} + \beta \left( \pi_1 \left( 1 + \beta^{1 \over 1+\delta_1} R_{f31}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} \left( R_{21} n_1 + R_{f2} n_{f1} \right)^{-\delta_2} + \right) \right)^{-1 \over \delta_1} \pi_2 \left( 1 + \beta^{1 \over 1+\delta_1} R_{f32}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} \left( R_{22} n_1 + R_{f2} n_{f1} \right)^{-\delta_2}$$

Defining

$$k_2 = \pi_2 \left( 1 + \beta^{1 \over 1+\delta_1} R_{f32}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} \left( R_{f2} - R_{f2} \right) \pi_1 \left( 1 + \beta^{1 \over 1+\delta_1} R_{f31}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} \left( R_{21} - R_{f2} \right),$$

we have the following conditional demands

$$n_{f1} = \frac{\left( R_{21} k_2^{1+\delta_2} - R_{22} \right) (I - c_1)}{R_{f2} - R_{f2} + k_2^{1+\delta_2} \left( R_{21} - R_{22} \right)} \quad \text{and} \quad n_1 = \frac{\left( 1 - k_2^{1+\delta_2} \right) R_{f2} (I - c_1)}{R_{f2} - R_{f2} + k_2^{1+\delta_2} \left( R_{21} - R_{f2} \right)} .$$

The period 1 utility function can be rewritten as

$$U(c) = \left( c_1^{-\delta_1} + \beta \left( \pi_1 \left( c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{\delta_2 \over \delta_1} + \pi_2 \left( c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{\delta_2 \over \delta_1} \right) \right)^{-1 \over \delta_1}$$

$$= \left( c_1^{-\delta_1} + \beta \left( \pi_1 \left( 1 + \beta^{1 \over 1+\delta_1} R_{f31}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} \left( R_{21} - R_{f2} \right) \left( I - c_1 \right) \right)^{-\delta_2} + \right) \left( \pi_2 k_2^{1+\delta_2} \left( 1 + \beta^{1 \over 1+\delta_1} R_{f32}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} \left( R_{22} - R_{f2} \right) \left( I - c_1 \right) \right)^{-\delta_2}$$

The first order condition is

$$c_1^{-\delta_1 - 1} = \beta \left( \frac{\frac{\left( R_{21} - R_{f2} \right) R\_{f2}}{R_{f2} - R_{f2} + k_2^{1+\delta_2} \left( R_{21} - R_{f2} \right)}}{\pi_1 \left( 1 + \beta^{1 \over 1+\delta_1} R_{f31}^{-\delta_1 \delta_2 \over \delta_1} \right)^{1+\delta_1 \delta_2 \over \delta_1} + \right) \right)^{\delta_1 \over \delta_2} \left( I - c_1 \right)^{-1 - \delta_1} ,$$

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implying that

\[ c_{1KP} = \frac{I}{1 + \beta_1^{1 + \delta_1}} \left( \pi_1 \left( 1 + \beta_1^{1 + \delta_1} R_{f31}^{-\delta_1 \delta_2} \right) + \pi_2^{-\delta_1 \delta_2} \left( 1 + \beta_1^{1 + \delta_1} R_{f32}^{-\delta_1 \delta_2} \right) \right)^{-\delta_1 \delta_2} \]

For resolute choice, the period 1 DOCE utility function can be transformed into a certainty utility of the single branch \((c_1, c_{21}, c_{31})\) as in eqn. (B.15) and the budget constraint can be rewritten as eqn. (B.16). The first order conditions are

\[ c_{1}^{1 - \delta_1} = \beta \left( \pi_1 + \pi_2 \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{-\delta_2 \delta_2} \right) c_{21}^{ - \delta_1 - 1} \]

and

\[ c_{1}^{1 - \delta_1} = \beta^2 \left( \pi_1 + \pi_2 \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{-\delta_2 \delta_2} \right) c_{31}^{ - \delta_1 - 1} \]

Therefore

\[ c_{21} = \left( \beta \left( \pi_1 + \pi_2 \left( \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \right)^{-\delta_2 \delta_2} \right) \right)^{1 + \delta_1} \]

and

\[ c_{31} = \left( \beta^2 \left( \pi_1 + \pi_2 \left( \frac{\pi_2 (R_{f2} - R_{22}) R_{f32}}{\pi_1 (R_{21} - R_{f2}) R_{f31}} \right)^{-\delta_2 \delta_2} \right) \right)^{1 + \delta_1} \]
It follows that

\[
c_1^o = \frac{I}{1 + \left( \frac{R_{f2} - R_{f2}}{(R_{f2} - R_{f2})R_{f2}} + \frac{R_{f2} - R_{f2}}{(R_{f2} - R_{f2})R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{f2})}{\pi_1 (R_{f2} - R_{f2})} \right) \frac{1}{\delta_2} \right) + \beta \left( \frac{\pi_1 + \pi_2 (R_{f2} - R_{f2})}{\pi_1 (R_{f2} - R_{f2})} \right) \frac{1}{\delta_2}}{\frac{R_{f2} - R_{f2}}{(R_{f2} - R_{f2})R_{f2}} + \frac{R_{f2} - R_{f2}}{(R_{f2} - R_{f2})R_{f2}} \left( \frac{\pi_2 (R_{f2} - R_{f2})}{\pi_1 (R_{f2} - R_{f2})} \right) \frac{1}{\delta_2}}}
\]

The considerable variation in the \(n_{f1}/n_1\) ratio in Figure 7(b) suggests a significant difference in risk attitudes. As argued in Subsection 4.3, there are two dimensions of risk – the intraperiod portfolio risk in period 2 and the interperiod risk corresponding to the correlation pattern of the period 2 risky and period 3 risk free asset returns. The latter phenomenon can very clearly be observed if we switch the pattern of returns for \(R_{f31}\) and \(R_{f32}\) in Figure 7(b) to that shown in Figure 8. In the latter case, the period 2 risk can be viewed as being partially hedged by the period 3 risk as the intertemporal correlation has gone from positive to negative. As a result, continuing to assume that \(\delta_1 = -0.6\) and \(\delta_2 = 5\), the \(n_{f1}/n_1\) ratio for all four models drops substantially from the case in Figure 7(b), where the asset return intertemporal correlation is positive. Moreover, it is not surprising that the asset ratio for each of the models is the same when \(R_{f31} = R_{f32}\) since there is no intertemporal risk. Also, the common ratio is intermediate between the positive correlation case of Figure 7(b) and the negative correlation case of Figure 8.

The behavioral response to interperiod risk in the form of explicit intertemporal hedging can take a different form if we consider the more extreme negative and positive correlation cases, respectively,

\[
(R_{f31}, R_{f32}) = (0.8, 2.0) \quad \text{and} \quad (R_{f31}, R_{f32}) = (2.0, 0.8).
\]

See Figure 9(a) and (b) where \(\delta_1 = 1.5\) and \(\delta_2 = 5\), and the corresponding Table 1. Note that in a two period setting, if \(E\tilde{R} > R_f\) a risk averse investor will always
Figure 8:

Figure 9:
hold the risky asset in positive quantity. However, that is not the case in Figure 9(b). The KP and DOCE resolute asset ratios at first increase with \( \delta_2 \) as the consumer increases \( n_{f1} \) and reduces \( n_1 \). As the risky asset demand approaches 0, the graphs become discontinuous. As \( \delta_2 \) increases past 2.8, the consumer begins to short the risky asset (eqn. (23) provides the necessary and sufficient condition for \( n_1^{KP} < 0 \)). One can view the reduction in the positive quantity of \( n_1 \) as an attempt to decrease the period 2 portfolio intraperiod risk whereas the shift to shorting the risky asset can be thought of as a move to decrease the intertemporal risk via dynamic hedging. To see this more explicitly, refer to the row corresponding to \( n_1 \) and column "\( R_{f31} = 2 \) and \( R_{f32} = 0.8 \)" in Table 1. For both the KP and sophisticated DOCE consumer, \( n_1 < 0 \). The motivation for this short-selling of the risky asset is to bolster the lower income \( I_{22} = n_1R_{22} + n_{f1}R_{f2} \) in the down state and thereby realize a better \((c_{22}, c_{32})\)-pair. By shorting \( n \) and increasing the purchase of \( n_{f1} \), the consumer reduces her income \( I_{21} \) because \( R_{21} > R_{f2} \) and increases \( I_{22} \) since \( R_{f2} > R_{22} \). Therefore, the period 2 income is higher when the bad state is realized and the consumer reduces her "bad, bad" outcome of a low \( (R_{22}, R_{f32}) \) outcome by sacrificing gains in the "good, good" case \( (R_{21}, R_{f31}) \) associated with \( I_{21} \). It should be emphasized that this phenomenon only occurs in the positive correlation case where \( (R_{f31}, R_{f32}) = (2.0, 0.8) \). For the negative correlation case, since the risk is already partially hedged by the asset returns automatically, the consumer will hold more risky assets to increase her expected returns. This is consistent with both the KP and DOCE sophisticated values of \( n_1 \) being positive and larger than the risk free asset holdings in the negative correlation column "\( R_{f31} = 0.8 \) and \( R_{f32} = 2.0 \)" in Table 1.

To discuss the asset ratio behavior analytically, assume \( \delta_1 > -1 \) and \( \delta_2 \geq 0 \). Define\(^37\)

\[
\tilde{c}_2^* = \max_{n_1 + n_{f1} = I} V^{-1} (\pi_1 V((n_1R_{21} + n_{f1}R_{f2})R_{f31}) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2})R_{f32}))
\]

\[
(n_{i3}^*, n_{i2}^*) = \arg \max_{n_1 + n_{f1} = I} V^{-1} (\pi_1 V((n_1R_{21} + n_{f1}R_{f2})R_{f31}) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2})R_{f32})) \]

\[
\tilde{c}_2^* = \max_{n_1 + n_{f1} = I} V^{-1} (\pi_1 V((n_1R_{21} + n_{f1}R_{f2})) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2})))
\]

\(^35\)In Figure 9(b), we do not show the sophisticated choice curve for \( \delta_2 < 2 \) due to instability in the numerical simulation. Also for \( \delta_2 < 2 \), the curve increasing with \( \delta_2 \) corresponds to both the DOCE resolute and KP cases as the \( n_{f1}/n_1 \) ratios are too close to distinguish.

\(^36\)In Figure 9(a) and (b), the asset ratio is also negative when \( \delta_2 \) is close to \(-1\). However in this case, we have \( n_{f1} < 0 \) instead of \( n_1 < 0 \) and the negative ratio does not correspond to dynamic hedging. In contrast, it simply suggests that when the investor is almost risk neutral, she pursues the higher return through shorting the risk free asset and buying more risky asset.

\(^37\)Note that in the rest of this appendix, * denotes optimization instead of naive choice.
<table>
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<tr>
<th></th>
<th>( R_{f31} = 0.8 )</th>
<th>( R_{f31} = 2 )</th>
<th>( R_{f32} = 2 )</th>
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<tr>
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</table>

\( \delta_1 = 1.5, \delta_2 = 5, R_{21} = 2, R_{22} = 0.8 \)
\( R_{f2} = 1.1, \pi_1 = 0.5, \beta = 0.97, I = 10 \)

Table 1:
\((n_2^*, n_f^*) = \arg \max_{n_1+n_f = I} V^{-1}(\pi_1 V((n_1R_{21} + n_f R_{f2})) + \pi_2 V((n_1R_{22} + n_f R_{f2}))).\)

It can be verified that
\[ R_{f31} \beta > 1, \quad R_{f32} \beta < 1, \quad \beta c_3^* < \beta c_3^* \text{ and } I < \beta^2 c_3^*.\]

It is easy to compute the optimal first period portfolios in the limit. It is clear that as \(\delta_1 \to -1\) we obtain for resolute choice that
\[ n_{f1}^o \to n_f^3, \quad n_1^o \to n_1^3.\]

Note that in the period 3 optimization problem since \(R_{f31} > R_{f32}\), we effectively have larger dispersion between the up and down states and hence risk than the two period case, which is the reason why we have a larger ratio of \(n_{f1}/n_1\) for the resolute choice than the two period case. For KP preferences, we obtain
\[(n_{KP}^*, n_{f}^{KP*}) = \arg \max_{n+n_f = I} \beta \pi_1 V((n_1R_{21} + n_f R_{f2})R_{f31}) + \pi_2 V((n_1R_{22} + n_f R_{f2})).\]  

(B.19)

Comparing the optimization problems (B.18) and (B.19), since \(\beta < 1\) and \(R_{f32} < 1\), it follows that in eqn. (B.19) the consumer is essentially underweighting or discounting the good state \(n_1R_{21} + n_f R_{f2}\) and overweighting the bad state \(n_1R_{22} + n_f R_{f2}\). Because the agent is risk averse, it is easy to see that for \(\delta_1\) sufficiently close to \(-1\) we have
\[ n_{f1}^o / n_1^o < n_{f1}^{KP} / n_1^{KP}, \]
and that the difference in these ratios can be large, depending on returns. Clearly for the case of sophisticated choice we have \(n^{**} = n_{f}^{**} = 0\). Unfortunately, this says little about the limit \(n_{f1}^{**} / n_{f1}^{**}\) as \(\delta_1 \to -1\).

It is useful to understand to what degree the ratio between risky and risk-free asset holdings in period 1 is a useful measure of risk-aversion. Clearly, for recursive utility (KP or EU) we can write the period 1 maximization problem as
\[ \max_{n,n_f} u(I - n_1 - n_f) + \beta \sum_s \pi_s W_s(n_1R_{2s} + n_f R_{f2}). \]

For the case of independent returns the value functions \(W_s\) do not depend on \(s\) and it makes sense to talk about risk-aversion. Otherwise it generally makes no sense to map portfolio-choice in the first period to any kind of risk-aversion. However, for the case of homothetic utility, holding everything else fixed, it is easy to see that we can write
\[ W_s(n_1R_{2s} + n_f R_{f2}) = \Gamma_s(\delta_1) V(n_1R_{2s} + n_f R_{f2}) \]
for some $\Gamma_s$ that depends on all parameters but in particular on $\delta_1$. A change of $\delta_1$ there does not change the risk-aversion but it changes the probabilities. For our example above

$$W_s(I) = \max_c (c^{-\delta_1} + \beta((I - c)R_{f3s})^{-\delta_1})^{-\frac{1}{\delta_1}}.$$

By Pratt’s theorem it is decreasing in $\delta_1$. It is useful to consider the two cases $\delta_1 = 0$ and $\delta_1 = -1$. For the former we obtain

$$W_s(I) = \frac{1}{2} I \sqrt{\beta R_{f3s}}.$$

For $\delta_1 = -1$ we obtain

$$W_s(I) = \max (I, \beta R_{f3s}I).$$

This gives a simple condition under which the preference for the risk-free asset increases as $\delta_1$ decreases from $\delta_1 = 0$ to $\delta_1 = -1$. For $\delta_1 = 0$ the probabilities for each state $s$ are twisted by a factor $\sqrt{\beta R_{f3s}}$. For $\delta_1 = -1$ they are twisted by $\beta R_{f31}$ in state 1 and by 1 in state 2. For many parameter values this makes state 2 relatively more likely and therefore the risky asset a better asset.