Dynamic Information Regimes in Financial Markets

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Abstract

We develop a dynamic model of information and asset prices in which investor information choices influence the level of available public and private information about fundamentals. We study two types of feedback. In the first, as more investors become informed, more information about fundamentals becomes available. As a consequence, two regimes emerge, one with higher prices and lower volatility, and one with lower prices and higher volatility. Information dynamics move the market between regimes, creating large market drops and rallies, with no change in fundamentals, but large changes in discount rates reflecting changes in information asymmetry between informed and uninformed investors. In the second type of feedback we study, an increase in the number of informed investors leads to greater public information through leakage or disclosures; this mechanism has a stabilizing effect. When calibrated to market data, the positive feedback model suggests a role for information dynamics in financial crises; the negative feedback model helps explain empirical findings in the literature on the market reaction to the loss of analyst coverage.

1 Introduction

We develop a dynamic model of information and asset prices in which investor information choices influence the level of available public and private information about fundamentals. We study two types of feedback in particular. In the first, as more investors become informed, more information about fundamentals becomes available, magnifying the asymmetry between informed and uninformed investors: this mechanism tends to increase price volatility and can amplify small shocks into large price drops. In the second case, an increase in the number of informed investors leads to greater public information through leakage or disclosures; this mechanism has a stabilizing effect.

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A few vignettes should help motivate our investigation:

- In 2009, the Greek government revised its estimated budget deficit. This revision triggered a large increase in investor demands for information about Greek debt, as reflected for example in media attention and internet searches. Figure 1 shows the large and persistent increase in the number of Bloomberg articles mentioning Greece starting in 2009. This increased demand for information was met with further revisions of official statistics, revelations about falsified data, stories of investment banks complicit in masking true conditions, research reports by industry analysts and non-governmental organizations, and a downgrade to junk by Standard & Poor’s, all of which arguably made more information available to investors. The price of Greece’s debt dropped sharply as the volatility of its sovereign credit default swap spreads rose.

- In June 2007, Bear Stearns disclosed that two of its hedge funds were on the brink of failure, fueling investor demand for information about the type of subprime mortgages in which the funds had invested. Indeed, Gorton and Ordoñez (2014) and Dang, Gorton, and Holmström (2012) have argued that the demand for information about “safe” collateral triggered the ensuing crisis. In our narrative, as more investors chose to incur the cost of becoming informed, more information became available — through revised credit ratings, academic and industry research, and regulatory reports. Less informed investors, fearing an informational disadvantage, fled to safer assets. Abstracting from the specifics of this setting, we calibrate our model to stock market data and find that the dynamics of
information precision alone, even without negative news about fundamentals, can produce crisis-like effects.

- An extensive literature has studied the information value of analysts’ research. Kelly and Ljungqvist (2012) use brokerage closures as an exogenous shock to the level of coverage. Following this method, Balakrishnan et al. (2014) find that firms respond with greater information disclosure, and Chen et al. (2018) find that hedge funds respond with greater information acquisition; these are feedback effects between shocks to information availability and information choices. We calibrate our model to this setting and find that it can reproduce the key empirical findings in Chen et al. (2018).

- The accounting literature documents many related associations which suggest feedback between the information environment and investors. Botosan (1997) finds that greater disclosure leads to a lower cost of capital when analyst coverage is low, suggesting that a loss of coverage should be followed by greater disclosure. Healy, Hutton, and Palepu (1999) find that greater disclosure increases institutional ownership, which can be seen as a proxy for the number of informed investors in a stock. Chen, Matsumoto, and Rajgopal (2011) find that lower institutional ownership leads to lower public disclosures in the form of earnings guidance. In Bushee, Core, Guay, and Hamm (2010), greater press coverage around announcements increases price informativeness, suggesting positive feedback from the demand for information to the level of information available. Although we do not investigate these specific effects, they point to the importance of feedback effects and thus a dynamic information environment.

Motivated by these types of examples, our model combines exogenous shocks to the quality of available information, an endogenous response by investors who may choose to become informed at a cost, and feedback from investor information choices to the information environment. Information about fundamentals falls in three categories: publicly known, privately knowable at a cost, and completely unknowable. In the interest of clarity, we treat separately the cases in which the share of public information and fraction of knowable information vary. These two settings lead to two types of feedback, with the number of informed investors influencing the precision of either public or private information.

In more detail, we develop an overlapping generations (OLG) model with a single risky asset, which pays a dividend each period, and a riskless asset. In each period, a new generation of

\[1\] Treating both together requires working with a two-dimensional state space but does not present any conceptual or theoretical difficulties.
investors observes the information environment, decides whether to become informed at a cost, sets optimal demands, and trades to clear an exogenous net supply of shares. Market clearing determines the price. At the end of the period, these investors receive their dividend and sell their shares at the new price. The notion of “generations” should not be taken literally in our setting; the OLG framework simply provides a tractable dynamic setting to model changes in information, and it ensures that investors care about future prices as well as the next dividend.

Crucially, in making their information choices at the start of the period, investors take into account the distribution of shocks to information precision and the feedback from information choices in the current period to future precision. The future precision will affect the end-of-period asset price and thus investors’ capital gains. In the positive feedback version of our model, information shocks will lead to large drops in prices and increases in volatility, and these effects are particularly noteworthy given that our investors have rational expectations.

The interplay between information and asset prices is often studied through single-period models of the type in Grossman and Stiglitz (1980), Hellwig (1980), Admati (1985), and a large subsequent literature. But there are several important features available in a dynamic model that are inaccessible in a single-period model, and these merit discussion. The first two important features we have already highlighted: feedback from investor information choices to information precision, and exogenous shocks to precision. Our emphasis is on feedback effects, and these cannot be captured in a single-period model. Persistent exogenous shocks are also important — they include the loss of analyst coverage discussed above, and also changes in accounting standards, like the introduction of mark-to-market accounting, and changes in regulatory policies on disclosures. In a single-period model, exogenous shocks are often approximated by changes in model parameters, but such changes are necessarily outside the model and, in particular, not contemplated by the agents in the model.

A third important feature of a dynamic model is that it can capture two distinct aspects of an increase in available information: greater information reduces uncertainty about the next dividend but can increase volatility in future prices and thus in capital gains. The first of these effects is clear — the information we model is information about dividends. To appreciate the second effect, note that in the absence of dividend information, price volatility is driven entirely by supply volatility; but when some investors have dividend information, this information is partly reflected in the price, so volatility in the signal adds to volatility in the price. In a single-period model, the price merely determines the cost of a claim to an end-of-period dividend. With overlapping generations, investors earn the change in price over the period as well as a dividend, so the variance in this return affects their investment decisions at the beginning of
the period. The two information effects, on dividends and prices, are potentially offsetting and lead to more complex tradeoffs than can be captured in a single-period setting.

To the best of our knowledge, our model is the first to capture a stochastic information environment, endogenous investor information choices, and feedback from these choices to available information. Spiegel (1998) develops an overlapping generations model in which all investors have the same information. Watanabe (2008) extends Spiegel’s (1998) model by introducing asymmetric information. Biais, Bossaerts, and Spatt (2010) also model asymmetric information in an OLG setting. In their model, as in Watanabe’s (2008), the fraction of informed investors and the precision of their signals are fixed and exogenous. Wang (1993) develops a continuous-time model of trading among differentially informed investors with a fixed fraction of informed investors and a fixed information environment; Wang (1994) is a discrete-time version of the model that investigates trading volume. The OLG model of Farboodi and Veldkamp (2017) incorporates a changing information environment, but the change is limited to a deterministic increase in investor information processing capacity over time. Signal precision also changes deterministically over time in Brennan and Cao (1997).

In the positive-feedback version of our model, information shocks are amplified and can produce crisis-like dynamics or, less dramatically, business cycle fluctuations. Amplification in our setting arises purely through an information channel. This adds to other amplification mechanisms, such as financial frictions (as in Bernanke and Gertler 1989, Kiyotaki and Moore 1997, and Adrian and Shin 2010) or leverage (as in Lorenzoni 2008 and Bianchi 2011). We are not suggesting that other mechanisms are less important, but rather highlighting the role that information feedback alone can play.

Information revelation is at the center of the crisis explanation of Gorton and Ordoñez (2014). In their account, a crisis results when lenders choose to acquire information about borrowers’ collateral; with less information available, borrowers with poor collateral have access to credit, and the increased supply of credit sustains higher growth. We work in an entirely different framework, but one contrast is particularly noteworthy. In Gorton and Ordoñez (2014), the information revealed is bad news; following an aggregate shock, some unobservable amount of collateral becomes bad, thus inducing more information acquisition. In our setting, only the precision of information changes — a shock may bring more news or less news, but not specifically good or bad news. An increase in precision leads to a price drop when it magnifies the information asymmetry between informed and uninformed investors, leading the uninformed to reduce their demand for the risky asset. Of course, a crisis is more likely to be precipitated by bad news, and adding a directional shock would likely further amplify the effects we observe; but
our model isolates the role that the dynamics of information precision alone can play, without a negative shock to fundamentals.

To illustrate these effects, we calibrate our model to stock market data. The equilibrium dynamics of the calibrated model fluctuate between two regimes, one with low volatility and high prices, and one with high volatility and low prices. The model can spend long intervals in each regime. A transition from one to the other can be sudden and result in a price drop of 10%, with no change in fundamentals. The two regimes emerge from investor information choices; we do not impose them in setting up the model.

This pattern has important implications: in times of market stress, a policy change that makes more information available is potentially destabilizing. We emphasize “policy change” because the information environment in our model is persistent, so the effects we study go beyond a single announcement. Releasing positive information may help calm markets, but our model indicates that this effect must be weighed against the increased volatility that can accompany increased information. There is an extensive literature studying the downsides of increased disclosures, which include reducing risk-sharing opportunities, distorting incentives for managers, inducing agents to underweight private information, and crowding out incentives for the production of additional information; see Goldstein and Yang (2017) for a survey. But the impact on prices and volatility we identify in our dynamic model is new in this context. Indeed, in a single-period setting, Goldstein and Yang (2017) show greater disclosure decreases return volatility, again highlighting the difference in perspective in a multiperiod model.

In the negative feedback or “leakage” version of our model, information dynamics can have a stabilizing rather than amplifying effect. In this formulation, as the number of informed investors increases, the amount of public information also increases, either because some of the informed leak their information or because a firm responds with increased disclosure. A large literature finds an association between reduced asymmetric information, higher liquidity, and a lower cost of capital (see, for example, Balakrishnan et al. 2014, Botosan 1997, and Ng 2011), which supports a motive for firms to disclose more public information in response to an increase in the number of privately informed investors. Chen, Matsumoto, and Rajgopal (2011) find that firms with lower institutional ownership are more likely to suspend guidance on earnings; put differently, this finding is qualitatively consistent with an increase in informed (institutional) investors driving an increase in public information.

Chen et al. (2018) study the effect of an exogenous loss of analyst coverage resulting from the closures or mergers of brokerage research departments. They document the following effects: price information efficiency (as measured by post-earnings-announcement drift) falls;
hedge funds trade more aggressively in the stock around earnings; hedge funds experience better investment performance in the stock; sophisticated investors increase their information acquisition; conditional on a large hedge fund presence in the stock, the loss of coverage has a smaller effect on price efficiency. We calibrate our model to this setting and show that it reproduces these findings. We compare results with and without feedback and confirm the importance of feedback effects for empirical predictions. Working with a dynamic model also allows us to distinguish immediate and longer-term responses to the loss of analyst coverage.

We present our model in Section 2. Section 3 solves the model and states our main theoretical results. Section 4 studies changes in the level of knowable information described by equation (4), and shows that positive feedback can lead to two regimes using parameters calibrated to market data. Section 5 applies the model in equation (5) to study an exogenous loss of information, focusing on the application to the loss of analyst coverage. The appendices provide proofs of our theoretical results and details of the calibration and numerical solution of the model.

2 Model

2.1 Dividends and Information Dynamics

A single infinitely-lived security pays a dividend in each period. The dividend paid at the end of period $t$ is given by

$$D_{t+1} = \bar{D} + \rho(D_t - \bar{D}) + M_{t+1} = (1 - \rho)\bar{D} + \rho D_t + M_{t+1}. \quad (1)$$

The innovation $M_{t+1}$ decomposes as

$$M_{t+1} = m_t + \theta_t + \epsilon_{t+1},$$

with the following interpretation: $m_t$ is known to informed investors; $\theta_t$ is public information; $\tilde{m}_t$ is the knowable portion of the innovation; and $\epsilon_{t+1}$ is unknowable at the beginning of period $t$. These are mean zero, normally distributed random variables, independent across time with variances given by

$$\text{var}(\tilde{m}_t) = f_t \text{var}(M) \quad \text{and} \quad \text{var}(\epsilon_{t+1}) = (1 - f_t)\text{var}(M). \quad (2)$$

$^{2}$More precisely, they are conditionally independent given all $(f_t, \phi_t)$. 

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and
\[ \text{var}(m_t) = \phi_t \text{var}(\tilde{m}_t) \quad \text{and} \quad \text{var}(\theta_t) = (1 - \phi_t) \text{var}(\tilde{m}_t). \] (3)

Thus,
\[ f_t = \text{fraction of dividend innovation that is knowable}; \]
\[ 1 - \phi_t = \text{fraction of knowable part of dividend innovation that is public}. \]

Before making investment decisions in period \( t \), all agents observe \( \theta_t \) and \( D_t \), and the time-\( t \) informed agents observe \( m_t \). A fraction \( \lambda_t \in [0,1] \) of agents are informed at time \( t \). The time-\( t \) uninformed agents, representing \( 1 - \lambda_t \) of the population, in addition to observing \( \theta_t \) and \( D_t \), also observe the market clearing price \( P_t \). Since the market-clearing price contains information about \( m_t \) through the demands of the informed traders, the uninformed also make rational inferences from the price about the innovation \( m_t \). The price is not fully revealing about \( m_t \) because of the presence of unobservable supply shocks. In this respect, for a given \( f_t \) and \( \phi_t \), our information environment is the same as in Grossman and Stiglitz (1980).

The innovation of our paper is to allow the information environment to evolve over time in response to exogenous shocks, and also in response to information decisions made by past generations of investors. We model two different information environments:

(Level of knowable information.) We set \( \phi_t \equiv \phi_0 \) and let \( f_t \) evolve according to
\[ f_{t+1} = \Pi_D (a_f + b_f \lambda_t + \kappa_f (f_t - a_f) + \epsilon_{f,t+1}). \] (4)

(Public-private mix.) We fix \( f_t \equiv f_0 \) and let \( \phi_t \) evolve according to
\[ \phi_{t+1} = \Pi_D (\phi_t + b_\phi \lambda_t + \epsilon_{\phi,t+1}). \] (5)

In these specifications, \( a_f, b_f, \kappa_f \), and \( b_\phi \) are constants, \( \epsilon_{f,t+1} \) are i.i.d. shocks, \( \epsilon_{\phi,t+1} \) are i.i.d. shocks, and \( \Pi_D \) maps the real line to a set \( \mathcal{D} \subseteq [0,1] \). Taking \( b_f > 0 \) in (4) implies positive feedback from the fraction informed \( \lambda_t \) to the precision of knowable information. Taking \( b_\phi < 0 \) in (5) implies leakage of information (or increased disclosure) as the fraction informed increases.

In equilibrium, the state of the economy will be either \( f_t \) or \( \phi_t \), depending on which version of the model we consider. We could model the joint evolution of \( f_t \) and \( \phi_t \), but we keep them separate to emphasize different consequences of the two settings. To cover both versions of

3In the simplest case, \( \Pi_D(x) = \min(1, \max(0,x)) \) projects \( x \) to \([0,1]\). For some of our theoretical results in Section 3 and for our numerical results, we will discretize \( f_t \) and \( \phi_t \) to finite subsets of the unit interval, but for now we keep the discussion general.
the model, we will sometimes refer to \((f_t, \phi_t)\) as the information state, though one of the two variables is always fixed.

Similarly, we will sometimes write model quantities (such as price coefficients) as functions of \((f_t, \phi_t)\). In particular, we restrict the fraction informed \(\lambda_t\) to be a function \(\lambda(f_t, \phi_t)\) of the information state. This means that \(\lambda_t = \lambda(f_t)\) in (4) and \(\lambda_t = \lambda(\phi_t)\) in (5). The specific form of \(\lambda(\cdot)\) will ultimately be determined endogenously as investors make their information choices.

Figure 2 summarizes the timing of the model. In each period, investors first observe the information state and choose whether to become informed. Investors then observe public information and set their demands as functions of the price, which determines the price through market clearing. At the end of the period, investors sell their shares at the new price, and the process repeats.

2.2 Investor Optimization Problem

At the beginning of period \(t\), a unit mass of new (young) investors enter the market, each endowed with wealth \(W_t\), known at time \(t\). For an investor who buys \(q\) shares of the risky asset at price \(P_t\) at the beginning of the period and sells the shares at the end of the period at price

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4 When \(f\) is restricted to a finite set, a function of \(f\) can be represented as a vector indexed by the values of \(f\), and similarly for \(\phi\). We use this representation in our numerical calculations.
$P_{t+1}$, terminal wealth is given by

$$W_{t+1} = R(W_t - qP_t) + q(D_{t+1} + P_{t+1})$$

$$= RW_t + q(D_{t+1} + P_{t+1} - RP_t),$$

(6)

where $R > 1$ is the gross return on a riskless asset. It will be convenient to define the per period net profit from owning a single share of the stock as

$$\pi_{t+1} \equiv D_{t+1} + P_{t+1} - RP_t,$$

(7)

in which case the budget constraint becomes $W_{t+1} = RW_t + q\pi_{t+1}$. Agents who enter at time $t$ consume their wealth at $t+1$ and leave the market. These agents set their demands for shares of the risky asset at time $t$ by solving

$$J^I_t \equiv \max_q \mathbb{E} \left[ \mathbb{E}[W_{t+1}|I^I_t, f_{t+1}, \phi_{t+1}] - \frac{\gamma}{2} \text{var}(W_{t+1}|I^I_t, f_{t+1}, \phi_{t+1}) \right] | I^I_t],$$

(8)

where $I^U_t = \{f_t, \phi_t, \lambda_t, D_t, \theta_t, P_t, W_t\}$ is the uninformed agents’ information set at time $t$, $I^I_t = I^U_t \cup \{m_t\}$ is the informed agents’ information set, and $\gamma > 0$ is a risk aversion parameter. Similar objectives are used in Peress (2010), Van Nieuwerburgh and Veldkamp (2014), and Mondria (2010), and can be interpreted as expressing a preference for early resolution of uncertainty, in the sense of Kreps and Porteus (1978). Maximizing (8) is equivalent to maximizing

$$\mathbb{E} \left[ v \left\{ \mathbb{E} \left[ -\exp(-\gamma W_{t+1})|I^I_t, f_{t+1}, \phi_{t+1} \right] \right\} | I^I_t] \right],$$

with $v(u) = -\frac{1}{\gamma} \log(-u)$, if $W_{t+1}$ is conditionally normal, as it will be in our equilibrium. We could allow investors to condition on past values of variables in their information sets in (8), but past information will be irrelevant, given our independence assumptions. We will use the notation $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot|I^I_t]$ to denote conditioning on the time $t$ common information set.

In addition to investor demands for shares of the risky asset, we need to specify the supply. As in the OLG model of Allen, Morris, and Shin (2006), we assume that the supply $X_t$ of the risky asset is independent and identically distributed from one period to the next. As explained in Allen et al. (2006), i.i.d. supply can be interpreted as the result of trading by price-insensitive noise traders who reverse their trades at the end of each period. New investors each period thus only clear a new exogenous supply shock. We assume each $X_t$ is normally distributed with mean zero and variance $\sigma_X^2$. Furthermore, we assume that there exists a positive net supply $\bar{X}$ of the risky asset, and that this fixed supply is constant over time.
2.3 Equilibrium

Given a function \( \lambda : [0, 1] \rightarrow [0, 1] \) (yielding the fraction informed \( \lambda(f_t) \) or \( \lambda(\phi_t) \)), a market equilibrium is defined by a price process \( P_t \) and demands \( q^I_t \) and \( q^U_t \), depending on the price and other time-\( t \) information \( I^I_t \) and \( I^U_t \), that clear the market,

\[
\lambda_t q^I_t + (1 - \lambda_t) q^U_t = X + X_t,
\]
and for which \( q^I_t \) solve (8), \( t \in \{I, U\} \), for all \( t \).

Market clearing and investor optimality define a market equilibrium, given a function \( \lambda \) that determines the fraction of investors who are informed. Next we define what it means for this fraction to be determined endogenously. We suppose that investors at the beginning of the period can choose to become informed at a cost \( c_I \), incurred at the beginning of the period but after observing the current information state \((f_t, \phi_t)\). Investors’ decisions to become informed or remain uninformed thus define a mapping from the information state to the fraction informed, which is precisely \( \lambda \). We will use the following:

**Definition 2.1 (Endogenous fraction informed)** In the case of the \( f_t \) model (4), we call \( \lambda \) the endogenous fraction informed if it satisfies the following conditions for each \( f \in [0, 1] \):

(i) \( \lambda(f) = 0 \) and \( E[J^I_t - R c_I | f_t = f] < E[J^U_t | f_t = f] \); or

(ii) \( 0 \leq \lambda(f) \leq 1 \) and \( E[J^I_t - R c_I | f_t = f] = E[J^U_t | f_t = f] \); or

(iii) \( \lambda(f) = 1 \) and \( E[J^I_t - R c_I | f_t = f] > E[J^U_t | f_t = f] \).

In the case of model (5), we require the corresponding conditions with \( \phi \) in place of \( f \).

In case (ii), the fraction \( \lambda(f) \) is the point at which the marginal investor is indifferent between becoming informed and remaining uninformed. Cases (i) and (iii) cover the possibility that one choice dominates the other and is therefore selected by all investors.

3 Model Solution and Variance Beliefs

In this section, we state the main results on the solution of the model. We show that for arbitrary fixed \( \lambda(\cdot) \) and what we call variance beliefs, the model admits a market equilibrium. Then, again for fixed \( \lambda(\cdot) \), we give conditions for the existence of self-consistent variance beliefs and thus a rational expectations market equilibrium. Given a market equilibrium, we give conditions for an endogenous fraction informed \( \lambda(\cdot) \). With this endogenous \( \lambda(\cdot) \), it is possible
that the initial variance beliefs are not self-consistent. Therefore, we combine results to give conditions for an *information equilibrium*, in which conditions for a rational expectations market equilibrium and an endogenous fraction informed are jointly satisfied.

For some of the results in this section (Propositions 3.2–3.4), and for our numerical solution of the model, we discretize the state space by restricting $D$ — the set of values that $f_t$ and $\phi_t$ can take — to be a finite subset of $[0, 1]$. This discretization allows us to represent any function of $f$ or $\phi$ as an $n$-dimensional vector. Our numerical procedure is discussed in Appendix C.

### 3.1 Market Equilibrium

Proceeding with the first of these statements, we will show that, for any choice of $\lambda$, the model admits a market equilibrium in which the price process takes the form

$$P_t = a_t + b_t m_t + g \theta_t - c_t X_t + d D_t, \quad (10)$$

where $g$, and $d$ are constants, and $a_t, b_t, c_t$ are functions of the information state $(f_t, \phi_t)$ but do not otherwise depend on $t$.

To characterize investor demands, we will initially solve a more general version of their optimization problem (8), in which we do not assume that investors know the coefficients of the price process (the meaning of this will be clear momentarily). We then show how this leads to a market equilibrium.

If prices are given by (10), we can write terminal wealth $W_{t+1}$ in (6) as

$$W_{t+1} = RW_t + q(1 + d) D_{t+1} + q(P_{t+1} - d D_{t+1} - R P_t)$$

$$= RW_t + q[(1 + d) D_{t+1} + a_{t+1} + b_{t+1} m_{t+1} + g \theta_{t+1} - c_{t+1} X_{t+1} - R P_t]. \quad (11)$$

Note that $m_{t+1}, \theta_{t+1}$ and $X_{t+1}$ are independent of $D_{t+1}$, and of any time $t$ information. With a view to solving (8), we evaluate the conditional mean of terminal wealth as

$$E[W_{t+1} | I, f_{t+1}, \phi_{t+1}] = q[(1+d)(\mu D + \rho D_t + \theta_t + E[m_t | I_t]) + a(f_{t+1}, \phi_{t+1}) - R P_t] + RW_t, \quad t \in \{I, U\}. \quad (13)$$

In the above, we write $a_t$ from (10) as $a(f, \phi)$ to make explicit its dependence on the state
variables, and discuss it further below. For the conditional variance, we use (11)–(12) to write

\[ \text{var}(W_{t+1}|\mathcal{I}_t, f_{t+1}, \phi_{t+1}) = q^2(1 + d)^2 \text{var}[D_{t+1}|\mathcal{I}_t, f_{t+1}, \phi_{t+1}] + q^2 \text{var}[P_{t+1} - dD_{t+1}|\mathcal{I}_t, f_{t+1}, \phi_{t+1}] \]

(14)

\[ = q^2(1 + d)^2 [\text{var}(m_t|\mathcal{I}_t) + (1 - f_t)\sigma_M^2] + q^2V_B(f_{t+1}, \phi_{t+1}). \]

(15)

In the second equality, we have used (1)–(2) and introduced the variance belief function \(V_B\). If prices are indeed given by (10), then the “correct” (rational expectations) belief is given by

\[ V_B(f, \phi) = b(f, \phi)^2\phi f \sigma_M^2 + g^2(1 - \phi)f \sigma_M^2 + c(f, \phi)^2\sigma_X^2 \quad \forall f, \phi, \]  

(16)

as can be seen by comparing the last term in (14) and (15). However, we initially allow investors to have an arbitrary, strictly positive variance belief \(V_B\), which is shared by all investors.

With arbitrary \(V_B\), we do not have equality in (15); instead, we posit that investors solve their optimization problems (8) as though (15) held. In other words, investors solve (8) but with the conditional variance replaced by the right side of (15). Furthermore, we show in Appendix A.2 that given \(V_B\), \(a(f, \phi)\) in (13) is fully determined by the investor’s optimization problem.

A market equilibrium with variance belief \(V_B\) is then a price process and investor demand functions that clear the market and solve (8) with this modification.

**Proposition 3.1** Under the \(f_t\) model (4) or the \(\phi_t\) model (5), for any variance belief function \(V_B(\cdot)\) bounded above and bounded away from zero, and any \(\lambda(\cdot)\), there exists a market equilibrium with a price process of the form (10) in which \(a_t\), \(b_t\), and \(c_t\) are functions of the information state and do not otherwise depend on \(t\).

The proof of this proposition in Appendix A.1. The market equilibrium of Proposition 3.1 is not in general a rational expectations equilibrium because the variance belief \(V_B\) may not coincide with the conditional variance \(\text{var}[P_{t+1} - dD_{t+1}|\mathcal{I}_t, f_{t+1}, \phi_{t+1}]\) in (14). But we can think of agents in the model as learning over time. Starting from an initial belief, investors set their demands and clear the market at a price of the form in (10). They (or the next generation) then observe the realized variance given by the right side of (16). They update their beliefs by setting \(V_B\) equal to this realized variance, and the process repeats. This in fact is how we solve our model numerically.

Given price coefficient functions \(b\) and \(c\), the belief updating equation (16) defines a new \(V_B\), and given \(V_B\), Proposition 3.1 defines new coefficients \(b\) and \(c\). (The coefficient \(a\) depends on \(V_B\) but does not enter in the update of \(V_B\).) Combining the two steps yields a mapping from
an initial pair of functions \((b, c)\) to an updated pair \((b, c)\). We have self-consistent beliefs, i.e., a *rational expectations market equilibrium*, at a fixed point of this mapping.

We prove the existence of a fixed point in the appendix under mild restrictions on model parameters. For technical reasons, in this analysis we limit the values of \(f_t\) and \(\phi_t\) to a finite (but arbitrarily large) subset \(D\) of the unit interval.

**Proposition 3.2** Under the parameter restrictions in Appendix A.3 for any fixed \(\lambda(\cdot)\), there exists a fixed point of the variance belief updating mapping. This fixed point defines a self-consistent variance belief and thus a rational expectations market equilibrium with prices of the form in (10).

We can give a simple sufficient condition for Proposition 3.2. If the risk-free return \(R \in [1, 1.2]\) and the model parameters satisfy

\[
(1 + d)\gamma\sigma_M\sigma_X \leq 0.28,
\]

where \(d = \rho/(R - \rho)\), then there exists a fixed point for the variance belief updating mapping. This simple condition is satisfied by our numerical examples in later sections. More general conditions are given in the appendix.

Two special cases of Proposition 3.2 are worth mentioning. If we fix \(f_t \equiv 0\) and \(\lambda \equiv 0\) we have an OLG model without asymmetric information, similar to the one in Spiegel (1998).

As in Spiegel’s (1998) model, the coefficients in the price function can be expressed through solutions of quadratic equations. If we fix \(f_t\) and \(\lambda\) at constant strictly positive values, we get a model similar to Watanabe’s (2008), which has asymmetric information but a fixed information environment and no feedback.

### 3.2 Information Equilibrium

Propositions 3.1 and 3.2 take \(\lambda(\cdot)\) as exogenous. We need to show that our notion of the endogenous fraction informed in Definition 2.1 is meaningful. Given a variance belief \(V_B\), for

\[
V_B^2 + \left[2(1 + d)^2\sigma_M^2 - \frac{R^2}{\gamma^2\sigma_X^2}\right] V_B + (1 + d)^4\sigma_M^4 = 0.
\]

The two roots describe two market equilibria, one with high price variance and one with low price variance. However, the high variance equilibrium is unstable under arbitrarily small parameter perturbations; only the low variance equilibrium is robust to such changes. In our numerical experiments, we find that if we start from a low value of \(V_B(\cdot)\) we converge to the low variance equilibrium.
every $\phi$ and $f$, we need to find a $\lambda$ that makes investors exactly indifferent between paying the cost $c_I$ of becoming informed or staying uninformed; if no such $\lambda$ exists, we set $\lambda$ equal to zero or one according to Definition 2.1. The details of this calculation are given in Appendix B.1.

The following proposition shows that this procedure does indeed generate an endogenous fraction informed. Because changing $\lambda$ changes the evolution of $f_t$ or $\phi_t$, we need to be a bit more explicit about how we map these variables to the finite set $D$ in (4) and (5); these details are discussed in the proof of the following proposition in Appendix B.2.

**Proposition 3.3** Suppose the shocks $\epsilon_{f,t}$, $\epsilon_{\phi,t}$ have a density. Then for any strictly positive variance belief there exists an endogenous $\lambda$ in the sense of Definition 2.1.

This result holds, in particular, at a self-consistent variance belief. That is, given an exogenous $\lambda(\cdot)$ and the associated self-consistent variance belief $V_B$, we can solve for a new endogenous $\lambda(\cdot)$. However, once we change $\lambda$, the variance belief may no longer be self-consistent. When we solve the model with endogenous beliefs numerically, we start with an arbitrary variance belief, we then solve for the endogenous fraction informed (as provided by Proposition 3.3), we then calculate the realized variance (16) using the endogenous $\lambda$, update the variance belief and repeat. We can formulate this process as starting with a pair of coefficient functions $(b,c)$, from which we calculate $\lambda$ and then a new $(b,c)$. Combining the two steps yields a mapping from an initial $(b,c,\lambda)$ to a new $(b,c,\lambda)$. A fixed-point of this mapping defines an information equilibrium, in the sense that it yields a market equilibrium in which investors do not want to deviate from their information choices.

**Existence of an Information Equilibrium**

In this section, which is technical and can be skipped on a first reading, we establish that the procedure discussed in the prior paragraph has a fixed point. Proposition 3.3 ensures the existence of an endogenous $\lambda$, but it does not guarantee uniqueness, so we need a somewhat more general formulation to establish existence of an information equilibrium. For any coefficient functions $(b,c)$, let $\Lambda_0(b,c)$ denote the set of $\lambda$ satisfying Definition 2.1 which we know from Proposition 3.3 is nonempty. Let $\Lambda(b,c)$ denote the set of all convex combinations of elements of $\Lambda_0(b,c)$. If there is just one $\lambda$ in $\Lambda_0(b,c)$, then $\Lambda(b,c) = \Lambda_0(b,c) = \{\lambda\}$. In our numerical experiments, instances of multiple $\lambda$ satisfying Definition 2.1 occur rarely, and we have never encountered multiple solutions $\lambda$ when using self-consistent variance beliefs. However, because

---

6More precisely, we start with a variance belief within the region where Proposition 4.2 ensures the existence of a fixed point.
we have not proved the uniqueness of $\lambda$, we need to work with the potentially larger set $\Lambda(b, c)$ in establishing the existence of an information equilibrium $(b, c, \lambda)$.

A convex combination $\lambda \in \Lambda(b, c)$ represents a mixed equilibrium in the following heuristic sense. For each $f \in \mathcal{D}$, we can write

$$\lambda(f) = w_f \lambda_1(f) + (1 - w_f) \lambda_2(f),$$

(18)

with $w_f \in [0, 1]$ and $\lambda_1, \lambda_2 \in \Lambda_o(b, c)$, $\lambda_2(f) > \lambda_1(f)$. Interpret this to mean that a fraction $w_f$ of investors thought equilibrium $\lambda_1(f)$ would be selected, and a fraction $1 - w_f$ thought $\lambda_2(f)$ would be selected. At the outcome $\lambda(f)$, the marginal investor is not indifferent between becoming informed or not. A fraction $w_f$ of investors, expecting an outcome of $\lambda_1(f)$, will see $\lambda(f)$ as too high, and in response a fraction $w_f (\lambda(f) - \lambda_1(f))$ of investors will switch from informed to uninform$. Similarly, a fraction $1 - w_f$, expecting $\lambda_2(f)$, will see $\lambda(f)$ as too low, resulting in a fraction $(1 - w_f)(\lambda_2(f) - \lambda(f))$ switching from uninformed to informed. But then (18) implies that these effects offset each other, leaving the fraction informed at $\lambda(f)$.

We establish existence of an information equilibrium — a joint equilibrium in $(b, c, \lambda)$ — within the broader class of information choices in $\Lambda(b, c)$. For the following, let $M(\lambda)$ be the set of market equilibrium parameters $(b, c)$ consistent with the fraction informed function $\lambda = \{\lambda(f), f \in \mathcal{D}\}$; these are the fixed points in Proposition 3.2.

**Proposition 3.4** Suppose the conditions of Propositions 3.2 and 3.3 hold. Then there exists an information equilibrium $(b, c, \lambda)$, meaning that $(b, c) \in M(\lambda)$ and $\lambda \in \Lambda(b, c)$. In other words, $(b, c)$ defines a market equilibrium given $\lambda$, and $\lambda$ defines a (possibly mixed) information equilibrium given $(b, c)$.

### 4 Price and Volatility Cycles

As discussed in the introduction, our model is motivated by the idea that as more investors become informed, more information may become available. This type of feedback can arise at the onset of market stress, when firms and governments are pressured to reveal more information in response to heightened investor attention. In this section, we will show that this dynamic can lead to periods of low and high volatility and high and low prices driven purely by changes in the information state, with no change in fundamentals. In other words, we can generate transitions similar to business cycles or even financial crises through changes in the level of information, without necessarily the release of negative information.
4.1 Model Behavior

To provide insight into the model, we turn to a numerical example. We postpone details on parameter values to our discussion of model calibration in Section 4.2. The solid line in Figure 3 shows $\lambda$ as a function of $f$ in model (4). We calculate this curve by starting from a flat variance belief function and iteratively updating the variance belief and $\lambda$ as discussed in Section 3. This iterative process converges very quickly in our numerical experiments.

At low levels of information precision $f$, the figure shows a flat section where $\lambda(f) = 0$; with little information available, no investor chooses to bear the cost of becoming informed. Once $f$ increases to just above 0.4, we have a positive fraction of investors informed, and this fraction generally increases with the precision $f$.

Figure 3: The solid line shows the fraction informed $\lambda(f_t)$ in information state $f_t$, and the dashed line shows the mapping from $\lambda$ to $f_{t+1}$ without exogenous shocks. Each circle shows a point where $f_t = f_{t+1}$ when the shocks in (4) are zero; these points are labeled with their respective $(f, \lambda)$ values.

To interpret the dashed line in Figure 3, shut off the exogenous shocks in the evolution of $f_t$ by setting $\epsilon_{f,t+1} = 0$ in (4). The dashed line shows the mapping from $\lambda$ to the next value of $f$. That is, starting from any $f_t = f$ on the horizontal axis, reading up to the solid then across

For some parameter values, at $f$ near 1 we have a small decline in $\lambda(f)$. The possibility of a decline in $\lambda(f)$ as $f$ increases reflects the dual roles of information in a multiperiod model. Becoming informed benefits an investor by reducing uncertainty about the end-of-period dividend. However, as more investors become informed, the variance of the end-of-period asset price increases, so the net effect on the variance of an investor’s end-of-period wealth is indeterminate.
to the dashed line and back down to the horizontal axis yields $f_{t+1}$. Points where the two lines cross represent equilibrium combinations of $(f, \lambda(f))$ in a model without exogenous shocks.\footnote{More precisely, the three circled points in the figure are cases where $f_t = f_{t+1}$ when $\epsilon_{t+1} = 0$.}

Consider, for example, the circled point near $f = 0.48$, $\lambda(f) = 0.071$. Starting at that $f$, the endogenous fraction informed $\lambda(f)$ is precisely the value that keeps the information state at $f$ under the evolution in (4) without endogenous shocks. The model still has feedback from $\lambda$ to $f$ (and $f$ to $\lambda$), but $f_t$ remains fixed. The same argument applies to the intersection near $f = 0.88$. In the lower left, the curves intersect throughout an interval where $\lambda(f) = 0$, and we have a fixed point at $(a_f, 0)$ because the dynamics in (4) drive $f_t$ to $a_f$ when $\lambda_t = \epsilon_{f,t+1} = 0$. But this perspective is somewhat misleading in a way that illustrates the advantage of a genuinely dynamic model over a static one. Without exogenous shocks, the three equilibria in the figure would seem to be equally valid solutions. But the middle equilibrium, near $f = 0.48$, is unstable: starting just to the right of the intersection will drive $f_t$ to the equilibrium near $f = 0.88$, whereas starting just to the left will drive $f_t$ to the interval where $\lambda(f) = 0$. The middle equilibrium is in some sense illusory, though it is a valid equilibrium without shocks.

If we reintroduce shocks in the evolution (4) and study the long-run distribution of $f_t$ using the endogenous $\lambda$ curve in the figure, we find that $f_t$ spends significant time in the region where $\lambda(f) = 0$, and it spends significant time near $f = 0.88$, but the region near $f = 0.48$ holds no particular attraction; the dynamic model with exogenous shocks sees through the illusory equilibrium. Figure 4 shows the steady-state $f_t$ distribution (indicated by the blue circles in the left panel of the figure), calculated using a Markov chain representation.\footnote{See Appendix C for details.} The distribution is bimodal, showing that the economy spends the majority of its time in the vicinity of the two stable fixed points from Figure 3, but not near the middle, unstable fixed point in Figure 3.

The red triangles in the left panel of Figure 4 show the steady-state distribution of $f_t$ when $\lambda$ is held fixed at its equilibrium mean of 0.0731. The distribution is unimodal, indicating that $f_t$ now spends most of its time near the middle of the interval (which happens to be in the vicinity of the unstable fixed point in Figure 3). This contrast points out the crucial role played by feedback through $\lambda$ in generating two regimes. These regimes lead to price and volatility cycles in our model, as we discuss in the next section.

\subsection*{4.2 Model Calibration}

In calibrating the model to the aggregate market, we take one period in the model to represent one month. We estimate a monthly dividend process of the form (1) using daily dividend...
Figure 4: The left panel shows the equilibrium steady state distribution of $f_t$. The right panel shows the low-to-high regime transition probability $P[f_{t+i} > 0.5 | f_t = .17]$ (solid line) and the high-to-low regime transition probability $P[f_{t+i} < 0.5 | f_t = .88]$ (dashed line) as a function of $i$ (measured in months).

data for the S&P 500 index from 1998–2018, then aggregating this up to the quarterly level, and estimating an ARMA(1,1) process for the quarterly dividend. From this we back out the monthly parameters $\rho = .967$ and $\sigma_M = 0.0471$. Details are given in Appendix D.

We adopt the normalization $\bar{D} = 1$ and $\bar{X} = 1$, so dividends and share supplies are measured in units of their monthly averages. We calibrate $\sigma_X^2$ to match monthly turnover, meaning the number of shares traded per month divided by the shares outstanding. Recall from Section 2.2 that in each period $t$, investors buy the new supply $X_t$ originating from liquidity demanders, and investors from the previous period sell back $X_{t-1}$ shares to the previous period’s liquidity demanders unwinding their trades. The total trading volume in period $t$ is therefore $|X_t| + |X_{t-1}|$. Using the normality of the supply shocks, the expected volume per period becomes

$$\mathbb{E}[|X_t| + |X_{t-1}|] = 2\sigma_X \sqrt{\frac{2}{\pi}} \approx 1.596\sigma_X. \quad (19)$$

Panel B of Figure 13 shows that the average weekly turnover of the Dow Industrials index is 0.065.\footnote{Lo and Wang (2000, Table 3) show that from 1987-1996, weekly turnover on a value-weighted index of LYSE and AMEX common shares was 1.25%. Therefore the monthly turnover on this index was $52/12 \times 1.25% = 5.42\%$.} To model a period of stress, we assume that turnover, or the turnover expectation of market participants, is four times higher than normal, so $\sigma_X = 4 \times 0.065 \times 1/2 \times \sqrt{\pi/2} = 0.1629$.

We use a monthly gross risk-free return of $R = 1.0015$ and set the risk-aversion parameter
Table 1: Calibrated parameters for model (4)

<table>
<thead>
<tr>
<th>$\tilde{D}$</th>
<th>$\tilde{X}$</th>
<th>$R$</th>
<th>$\sigma_X$</th>
<th>$\rho$</th>
<th>$\sigma_M$</th>
<th>$\gamma$</th>
<th>$c_I$</th>
<th>$\phi_0$</th>
<th>$P[\epsilon_f \neq 0]$</th>
<th>$\epsilon_f$</th>
<th>$a_f$</th>
<th>$b_f$</th>
<th>$\kappa_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.0015</td>
<td>0.1629</td>
<td>0.967</td>
<td>0.0471</td>
<td>0.46</td>
<td>0.2627</td>
<td>0.35</td>
<td>0.03 \times 2</td>
<td>0.135</td>
<td>0.175</td>
<td>0.384</td>
<td>0.91</td>
</tr>
</tbody>
</table>

at $\gamma = 0.46$. This yields an annualized excess return (see discussion in Section 4.3) of roughly 15%, which is not unreasonable for periods of stress. We choose a per month cost of being informed of $c_I = 0.2627$, which should be compared with a monthly aggregate average dividend of 1 (since $\tilde{X} = \tilde{D} = 1$). This high cost of information is comparable to the 2/20 fee structure of many hedge funds, and leads to an equilibrium number of informed of under 18% of the overall population.

For the dynamics of the $f_t$ process in (4), we set $a_f = 0.175$, $\kappa_f = 0.91$ and $b_f = 0.384$. A value of $\kappa_f$ close to 1 makes the information state persistent, and a positive $b_f$ produces positive feedback from the fraction informed to the level of accessible information. When $f_t$ is low, $\lambda(f_t) = 0$, and $f_t$ goes to a steady state level of $a_f = 0.175^{11}$. We fix $\phi = 0.35$, which introduces a high degree of information asymmetry between informed and uninformed investors, and leads to a large drop in price in the high-information regime; see Section 4.3.

Beyond these qualitative considerations, these specific parameters were chosen to produce plausible model dynamics. Finally, for the shocks $\epsilon_{f,t+1}$, we use a three-point distribution taking values $\{-0.135, 0, 0.135\}$ with probabilities $\{0.03, 0.94, 0.03\}$, so shocks are rare.\(^{12}\)

Model parameters are summarized in Table 1.

### 4.3 Price Drops and Volatility Spikes

Figure 5 shows model quantities calculated using the parameters in Table 1. The first three panels show the price coefficient functions $a$, $b$, and $c$ from (10). The lower right-hand panel show the expected net profit from owning one share of the stock. We discuss this shortly. When no investors are informed, no dividend information is reflected in the price, and $b = 0$. As $f$ increases to the point where some investors become informed, $b$ and $c$ both increase, which will drive up price variance. The increase in $c$ is due to the growing informational disadvantage of the uninformed relative to the informed, as $f$, and therefore $\lambda$, increases. As informed enter

\(^{11}\)Fama and French (2000) show the $R^2$’s of year-ahead firm-level earnings forecasts to be between 5% and 20%.

\(^{12}\)Parts of our theoretical analysis assume the shocks have a density to ensure continuity of the price coefficients in $\lambda$. A discrete distribution is simpler to work with numerically, and we achieve continuity by interpolating when we take expectations over state transitions of $f$, as discussed in Appendix C. The interpolation has a similar effect as smoothing the shock distribution; moreover, a discrete distribution can be approximated arbitrarily closely by a density.
the market due to a higher $f$, $a$ and thus the average price drop sharply (see Section 4.5).

Figure 6 shows the dependence on $f$ of prices, excess returns, return volatility, and expected excess return variance. The top-left panel shows the average price at each value of $f$, i.e., $P_0 \equiv a(f) + d\bar{D}$. The top-right panel shows the annualized dividend yield in excess of the risk-free rate $D/P_0 - (R_{12} - 1)$, as well as the expected excess return $12 \times \mu_\pi/P_0$, where $\mu_\pi$ (see equation 20) is the conditional mean of the net profit $\pi_{t+1}$. The bottom-left panel shows return volatility and the bottom-right panel shows the expected return variance for informed and uninformed investors (both are discussed further below). The effects are dramatic: a small increase in $f$ leads to a price drop of 10%, accompanied by a corresponding increase in the expected return and a spike in return volatility. All of this results from an increase in information precision and an endogenous response of the number of informed investors. In stress times, more information revelation (and, as we will see in Section 4.5 greater information asymmetry) can destabilize the market.

It is customary to associate large declines in market values with the arrival of bad news.
Following a 10% decline in an individual stock price or the overall market, one would expect media and expert accounts of what bit of bad news — a product failure, a CEO scandal, a change in government policy — triggered the fall. But in our setting it is simply more news — in the form of increased precision $f_t$ — that drives investors, not necessarily good or bad news.

In the model of Gorton and Ordoñez (2014), the onset of a crisis is defined by the release of information about collateral quality. But the additional information in Gorton and Ordoñez (2014) is negative information: collateral quality is inferior to what was previously believed. This revelation leads to either a collapse in lending or a diversion of productive resources to information acquisition, thus reducing growth in both cases. The mechanism in our model is entirely different and does not rely on adverse information about fundamentals.

In practice, an increase in the quantity of information without an accompanying positive or negative implication for fundamentals is rare, and this makes it difficult to disentangle a change in precision from a directional effect of news.\footnote{We study such an exogenous information shock in Section 5.} However, our model serves to isolate
the information component; our calibration indicates that this component alone can have a material effect. When the additional information is bad news as well as more precise news, we can expect the effects to be even greater.\textsuperscript{14}

The potential for increased volatility from increased information has policy implications. A regulatory change that leads to persistently higher information precision for informed investors is potentially destabilizing in times of market stress. Interestingly, in their analysis of disclosure of the results of regulatory stress tests for banks, Goldstein and Leitner (2018) conclude that disclosure is valuable only under adverse conditions. Our results are not in conflict but rather reflect different considerations, as the objective in Goldstein and Leitner (2018) is optimal risk sharing among banks, and the information disclosed separates banks with weak and strong fundamentals.\textsuperscript{15}

\section*{4.4 The Role of Volatility}

In this section, we investigate the role of volatility in greater detail. The bottom-left panel of Figure 6 shows the approximate volatility of excess returns as a function of $f$.\textsuperscript{16} The low information regime (in the vicinity of $f = 0.175$) is characterized by low return volatility, and the high information regime (in the vicinity of $f = 0.88$) is characterized by relatively high volatility. The area in the middle, however, is characterized by extremely high return volatility. This volatility spike is driven by the large variation in the price due to the drop in $a()$ curve, which can be be seen in Figure 5.

The bottom-right panel of Figure 6 shows the expected conditional variance of the net profit $\sigma$ from a share of the stock, where the expectation is taken over the information sets of the informed or the uninformed investors. This expected variance is the denominator in agents’ demand curves (the $q_D$ and $q_U$ in equation 25 of the appendix). Here we clearly see the tradeoff engendered by increased information precision. When $f$ is low, an increase in $f$ decreases the expected variance of net profits for informed and uninformed investors because more is known about next period’s dividend, the $D_{t+1}$ term in (7). As long as $f$ is low enough so that $\lambda(f) = 0$ (to the left of the vertical, dashed line in the graphs) this is the only effect, and higher information precision lowers uncertainty. However, past the no-informed point, i.e., for $f$ large enough so that $\lambda(f) > 0$, the uncertainty of next period’s net profit starts to increase, due to

\textsuperscript{14}In a macro context, Cesa-Bianchi and Fernandez-Corugedo (2018) find that an increase in economic uncertainty results in a decrease in the risk premium, which is consistent with our results.

\textsuperscript{15}The information disclosed about regulatory stress tests is disclosed publicly, but the design of scenarios and the interpretation of the results are technical matters that are arguably accessible only to informed investors who have acquired the necessary expertise.

\textsuperscript{16}The derivation is discussed in Appendix C. The return volatility measure is in equation (69).
the increasing variance of $P_{t+1}$ in (7). For high enough $f$ the increased information about next period’s dividend begins to dominate, and the expected conditional variance begins to fall.

Thus, more informed investors lead to more variable prices today, and because of the persistence of information regimes, i.e., the high $\kappa_f$ in (4), more informed today also increase tomorrow’s signal precision, thereby attracting more informed tomorrow and increasing tomorrow’s price volatility. Holding the stock may be riskier in the high-information regime, which in turn requires the stock to have a larger risk-premium, and thus a lower price. Indeed, the stock’s expected net profit is given by

$$E[\pi] \equiv \mu_\pi = \gamma \times \bar{X} \times \left( \frac{1}{q_D} \lambda + (1 - \lambda) \frac{1}{q_D} \right)^{-1},$$

(20)

which reflects the average return uncertainty faced by investors, weighted by the number of informed and uninformed in the economy (and scaled by $\gamma \bar{X}$). As a consequence, $\mu_\pi$ (shown in the bottom-right panel of Figure 5) inherits the shape of the $q_D$ (shown in the bottom-right panel of Figure 6). We turn next to investigating in greater detail how these quantities determine the price discount.

4.5 Decomposing the $a(\cdot)$ Curve

Changes in the price level and in expected excess returns, as seen in Figure 6, are driven by the drop in the $a(\cdot)$ curve in Figure 5. In Appendix A.2 we show that $a_t$ can be decomposed into two components,

$$a_t = \frac{(1 + d)\mu_D}{R - 1} - \frac{1}{R} \sum_{i=0}^{\infty} \frac{1}{R^i} E_t \mu_{\pi,t+i},$$

(21)

where, as in (20), $\mu_{\pi,t} = E_t[\pi_{t+1}]$ is the expectation of the stock net profit $\pi_{t+1}$ in (7), conditional on a given level of $f$ and $\phi$. This expression shows that $a_t$ is the present value of all future expected dividend payments minus a discount reflecting the expected present value of all future net profits. In the context of the Campbell (1991) and Vuolteenaho (2002) return variance decomposition, the second term in $a_t$ represents the effect of a time-varying discount rate on the stock price.\(^{18}\)

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\(^{17}\)This is shown in equation (41) of the appendix. This expression generalizes the corresponding quantity derived from equation (A10) in Grossman and Stiglitz (1980). In unreported results, we show that our model produces a nonlinear relationship between volatility and expected stock market returns, with some features of the nonlinear relationship documented empirically in Adrian, Crump, and Vogt (2017).

\(^{18}\)In our model all sources of discount rate variation, with the exception of the information channel effect, are absent. The second term in (21) induces low-frequency but very large changes in risk-premia, as can be seen in
To understand why this term experiences significant time-series variability as $f_t$ varies, consider the graph of $\mu_\pi$ in the lower-right panel of Figure 5. In the region to the left of the vertical dashed line, the number of informed is zero, and the $\mu_\pi$ curve reflects the falling uncertainty of the uninformed due to the public component of the revealed information, measured by $(1 - \phi) \times f_t$. When $f_t$ is just above 0.4, informed investors enter the market and the price begins to incorporate the private part $\phi \times f_t$ of the knowable part of the dividend. This increases the volatility of the next-period price, thereby increasing the aggregate net profit uncertainty. Only for very high levels of $f_t$ does the amount of information revealed by the price about the next period’s dividend begin to outweigh the increased volatility of the next-period price, thus acting to decrease $\mu_\pi$.

Changes in $\mu_\pi$ across different values of $f$ get amplified through (21) to produce the large drop in the $a()$ curve in Figure 5. To explain why, we first note that $\mu_\pi$ is larger near $f_t = 0.88$ (the upper fixed point in Figure 3) than it is near $f_t = 0.175$ (the lower fixed point). The spread between these values is amplified in (21) if $E_t[\mu_{\pi,t+i}|f_t = 0.88] > E_t[\mu_{\pi,t+i}|f_t = 0.175]$ for large $i$, which holds if transitions between regimes (meaning transitions between $f_t \approx 0.88$ and $f_t \approx 0.175$) are rare. If transitions were very frequent, then even if the expectations differed for $i = 1$, they would quickly converge as $i$ increased. This in turn would dampen the change in $a()$ across the two regimes.

The difference between $\mu_\pi$ at $f = 0.88$ and $f = 0.175$ is thus amplified into a large drop in $a()$ if the $f_t$ process spends long periods near each of these fixed points before moving near the other fixed point. The right panel of Figure 4 shows the low-to-high transition probability $P[f_{t+i} > 0.5|f_t = 0.175]$ (solid line) and the high-to-low transition probability $P[f_{t+i} < 0.5|f_t = 0.88]$ (dashed line) as functions of the number of months $i$. Both probabilities grow very slowly, reaching only 6-8% even after 240 months, confirming that transitions between regimes are infrequent. As a consequence, in the left panel of Figure 4, we get a bimodal steady-state distribution for $f_t$, with probability peaks at the low- and high-information regimes.

Are such infrequent regime transitions plausible? Barro (2009) estimates country level crises occur with a 1.7% per year probability. Assuming independence across time, a country has a 29% (i.e., $1 - (1 - 0.017)^{20}$) probability of experiencing at least one crisis over any 20-year period. From Figure 4, the probability of a low to high state transition in our model is approximately 7% over a 20-year period. Our calibration therefore suggests that one out of every four country-

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19 The same argument predicts values of $\mu_\pi$ just to the right of the unstable fixed point near $f = 0.48$ in Figure which will be much larger than values just to the left, placing the drop in the $a()$ curve near the unstable fixed point.
level crises may be accompanied by the information-driven price drop of our model. If crises are typically associated with positive shocks to \( f_t \), the actual ratio may be higher.

### 4.6 The Role of Time-Varying Information Asymmetry

The amount of knowable information that is private plays a key role in generating the effect we’ve identified. The top panel of Figure \[ \] compares the equilibrium \( a() \) curves for different values of \( \phi \); the case \( \phi = 0.35 \) is the one we have analyzed thus far.

When \( \phi = 0 \) and all knowable information is public, the economy is characterized by no information asymmetry — the knowable information is equally known to all agents. The \( a() \) curve corresponding to the no-asymmetry case is the highest one (represented as a solid line), indicating the smallest price discount relative to the present value of future dividends. The \( a() \) curve in this case is quite insensitive to \( f_t \). The \( \phi = 1 \) case represents the highest informational asymmetry possible in the model, and corresponds to the lowest \( a() \) curve, representing a large price discount needed to induce the informationally disadvantaged uninformed to participate in risk sharing, regardless of the information state \( f_t \). Indeed, when \( \phi \) is high, even a low \( f_t \) induces a large information asymmetry, and therefore prices in the high-\( \phi \) equilibria are low for all values of \( f_t \). Only for intermediate values of \( \phi \) does the economy transition over time from low-informational-asymmetry regimes (\( f_t \) low) to high-informational-asymmetry regimes (\( f_t \) high). Such regime shifts are accompanied by large changes in the prices.

The reason that prices in the case of \( \phi \in \{0,1\} \) don’t change much across different values of \( f \) can be seen from the bottom panel of Figure \[ \] which shows the steady-state distribution of \( f \) in the different \( \phi \) models. When \( \phi = 0 \), there are no informed investors since all knowable information is public. With \( \lambda = 0 \) in (4), any positive \( \epsilon_{f,t+1} \) shock quickly decays, pulling \( f_t \) back to its low-information fixed point. This dynamic is seen in the unimodal distribution, with the peak centered at \( a_f \), when \( \phi = 0 \). In this case, \( E_t[\mu_{\pi,t+i}|f_t = f] \) will be close to \( E_t[\mu_{\pi,t+i}|f_t = 0.175] \) for large \( i \) for any \( f \), and \( a_t \) in (21) is consequently insensitive to \( f \).

Similarly, when \( \phi = 1 \), all knowable information is private, and \( \lambda \) is relatively large. Via the \( b_f \) term in the dynamics of \( f_{t+1} \) in (4), a relatively high \( \lambda \) produces a steady state distribution that is unimodal at \( f = 1 \). Any negative \( \epsilon_{f,t+1} \) quickly dissipates as \( f \) is pulled back to one. Regardless of starting \( f \), dynamics in the high \( \phi \) case tend to pull the market towards the high-information fixed point. Therefore, \( E_t[\mu_{\pi,t+i}|f_t = f] \) will be close to \( E_t[\mu_{\pi,t+i}|f_t = 1] \) for large \( i \) for any \( f \), and \( a_t \) in (21) is again insensitive to \( f \).

For intermediate values of \( \phi \) (0.35 in our calibration), the tendencies of \( f_t \) towards \( a_f \) and towards the high-information fixed-point are balanced. Therefore, a sequence of shocks can
Figure 7: Top panel shows $a(\cdot)$ as a function of $f$ for different values of $\phi$ in the $\phi$-fixed model. Bottom panel shows the steady-state model distribution for $f$ across different levels of $\phi$. 
occasionally push the economy from one information regime to the other. Both regimes turn out to be very persistent. This persistence ensures that $E_t[\mu_{\pi,t+i}|f_t = 0.88] > E_t[\mu_{\pi,t+i}|f_t = 0.175]$ even as $i$ grows large, thus leading to a large price sensitivity to $f$, as can be seen from the $a()$ representation in (21). The middle picture in the bottom panel of Figure 7 shows this bimodal distribution. In both the $\phi = 0$ and $\phi = 1$ cases, the transition probability from the non-peak-regime to the peak-regime is very high, while transition probabilities in the other direction are close to zero. As can be seen from Figure 4 when $\phi = 0.35$ the transition probabilities from the high-to-low information regimes, and vice versa, are very similar in magnitude — suggesting regime transitions in both directions in the steady-state.

Figure 8: The top left chart shows price $P_t$ and the bottom left chart shows price variance $V_B$ in the same simulated path of the model. The top right charts shows the evolution of $a(f_t)$ along this path. The bottom right chart shows the price $P_t$ vs price variance $V_B$ in this path.

Therefore large price level changes occur only in markets characterized by time-varying information asymmetry. Figure 8 shows an example of a simulated path of the model for 1,000 periods, or 83 years if each period is one month. In this example, the market starts in the low volatility and high price regime, transitions to the high volatility and low price regime,
and then transitions back.\footnote{The model of Gorton and Ordoñez (2018), based on a very different framework, also exhibits an equilibrium with information cycles. In their setting, low information booms alternate with high information crashes.} The figure shows a large drop in price driven by a fall in $a(f_t)$ and associated with a spike in variance; this price drop is much larger than the within-regime short-term price volatility, which is driven primarily by dividend fluctuations. The relationship between the levels of prices and variance across two regimes is further illustrated in the scatter plot at the bottom of Figure 8, which uses the values from the simulated path.

5 Exogenous Changes in Information

As discussed in the introduction, the loss of sell-side analyst coverage resulting from the closures or mergers of brokerage research departments provides a setting with a plausibly exogenous loss of public information. In an empirical setting, the effects of such closures have been studied in Hong and Kacperczyk (2010), Kelly and Ljungvist (2012), Balakrishnan et al. (2014), Johnson and So (2017), and Chen, Kelly and Wu (2018, CKW), among others. In particular, CKW study the effects of closures of brokerage research departments on the trading environment of affected stocks (i.e., those formerly covered by the closed research department). Importantly, the brokerage closures are driven by business conditions at the brokerages themselves and are unrelated to the prospects of the covered stocks. By focusing only on stocks that had five or fewer covering analysts prior to the brokerage closure, CKW identify securities whose information environment is particularly affected by the loss of coverage. They document the following effects: price information efficiency falls; hedge funds trade more aggressively in the stock around earnings; hedge funds experience better investment performance in the stock; sophisticated investors increase their information acquisition; conditional on a large hedge fund presence in the stock, the loss of coverage has a smaller effect on price efficiency.

CKW interpret their findings as indicative of “a substitution effect between sophisticated investors and providers of public information in facilitating market efficiency.” As far as we know, ours is the first theoretical analysis of this substitution effect in a dynamic setting. We model exogenous research department closures as positive shocks to $\phi_{t+1}$ in (5). Recall that $\phi$ measures the fraction of knowable cash flow information $\tilde{m}_t$ that is known only to the informed. As we will see, the CKW substitution effect arises endogenously in our model because some traders choose to acquire costly information when the quality of public information deteriorates.

We initially set $b\phi = 0$ in (5); in this case, an increase in informed investors affects the information sets of the uninformed investors only through the stock price. However, we have an additional channel for the substitution effect. With $b\phi < 0$, the presence of informed traders...
directly improves the public signal available to all investors in the next period. This can happen, for example, if hedge fund investors publicly disseminate their research ideas about a stock and thereby provide de facto research reports to the public.\footnote{The Ira Sohn conferences, and the frequent appearances of hedge fund managers on CNBC, are examples of this effect. We could additionally model \( \phi_t \) as a mean-reverting process to proxy for the possibility that firms either modify their information disclosures or get covered by the news media in a different way following exogenous \( \phi_t \) shocks, though we do not pursue this idea further in this paper.} We show below that the equilibrium effect of this feedback is to stabilize prices following exogenous brokerage closures.

### 5.1 Model Calibration

In calibrating (5), we use several of the parameters in Table 1. For the dividend process, we use the same values of \( \rho \) and \( \sigma_M \) as before, and we continue to set \( \bar{D} = \bar{X} = 1 \). Because we are modeling trading in a stock during a non-stressed market environment, the average turnover is lower. In Panel A of Figure 13 we show that the mean monthly turnover of the S&P 600 Small Cap index is 0.085 per month. Using (19), we get \( \sigma_X = 0.0533 \). We use the Small Cap index because the loss of an analyst is less significant for large companies.

With \( R \) and \( \gamma \) as before, the annualized excess return, \( 12 \times \mu_{\pi}/P_0 \), is roughly 9.5%. This level is plausible for small stocks, which combine their market beta with a small-minus-big factor exposure. We set the per month cost of being informed at \( c_I = 0.25 \). This leads to an equilibrium number informed between 0 and 6%.

With \( \phi_t \) changing over time, we fix \( f_t \equiv f_0 \). We use \( f_0 = 0.5 \), which is higher than the annual predictability of firm-level earnings (of 0.2) and of overall market earnings (of 0.3), but is lower than the annual predictability of S&P 500 dividends (of 0.64) (see Glasserman and Mamaysky 2018).

Finally, in the \( \phi \) dynamics in (5), we use a three-point distribution taking values \( \{-0.25, 0, 0.25\} \) with probabilities \( \{0.003, 0.994, 0.003\} \). A shock of \( \epsilon_\phi = 0.25 \) is meant to represent the closure of a single brokerage house for firms that are covered by four brokers, a case consistent with CKW (who look at five brokers or less). We assume a change in coverage (either a closure or research coverage by a new brokerage) occurs with a 0.3% probability per month.\footnote{Say there is a 1% probability per year of a brokerage either closing or opening. Then per month the probability is \( 1 - 0.99^{1/12} \), assuming independence across months. If a typical firm has 5 covering brokers, and their closure outcomes are independent, then the probability that no firm closes in a month is \( (1 - (1 - 0.99^{1/12}))^5 \approx 1 - 0.003 \). Similarly if there are four currently non-covering brokers with a 1% per year chance of initiating coverage, we get a 0.003 per month chance of coverage.} We first set \( b_\phi = 0 \) to focus only on the substitution-through-price channel. We then set \( b_\phi = -0.04 \) to explore the potential direct information provision role of informed speculators. Table 2 summarizes our model parameters.
Figure 9 summarizes our model equilibrium quantities as functions of the information state \( \phi \). The fraction informed \( \lambda(\phi) \) ranges from zero to six percent. With \( \phi_t = 0 \), we have no informed investors (since all of \( \tilde{m}_t \) is public), and the stock would need to experience two positive shocks to \( \phi \) to enter the region where \( \lambda \) becomes positive. The \( b \) coefficient of the price (10) increases in \( \phi \) since informed trade more aggressively on their signal when public information is worse. The \( c \) coefficient increases in \( \phi \) because with higher information asymmetry between informed and uninformed supply shocks command a high risk premium. Finally, the unconditional risk premium is also increasing in \( \phi \) (i.e., \( a \) is decreasing) and for the same reason. The scalloping in the \( a \) curve is due to the fact that with \( b_\phi = 0 \), conditional on a given starting point \( \phi_0 \), all
Figure 10: Equilibrium responses to a shock $\phi_{t+1} = \phi_t + 0.25$. The x-axis in the figures shows that value of $\phi_t$ immediately prior to the shock.

Future $\phi$’s can only take on one of four values.

5.2 Response to Brokerage Closures

We model a brokerage closure as a positive shock $\epsilon = 0.25$ to $\phi_{t+1}$. For a given model variable $V$ we then look at the difference in that variable as we jump from $\phi_t$ to $\phi_{t+1}$, i.e.,

$$\Delta V(\phi) \equiv V(\phi_t + \epsilon) - V(\phi_t).$$

Figure 10 shows the $\Delta V$’s for the value function of the informed investors, for the model trading intensity (see below), for $q_u$ (the expectation of the conditional variance of net profits for the uninformed, given by equation 25), and the equilibrium fraction informed $\lambda$. We now turn to CKW’s five main results about stocks that experience a closure in a covering brokerage.

1) CKW document that the magnitude of the post earnings announcement drift (PEAD) increases for affected stocks. They interpret PEAD as a proxy for the informational efficiency of the underlying stock. We interpret an increased PEAD in one of two ways. First, it indicates

23 For $b_0 < 0$ no such effects are visible.
a higher price response from informed order flow, as measured by $c_t$ in the model. We see that $c_t$ increases with $\phi$ and therefore market liquidity $1/c_t$ (a measure suggested by Goldstein and Yang 2017, among others) falls. Second, the higher PEAD indicates a higher level of uncertainty faced by uninformed investors as to whether the current price is “fair.” This is measured in our setting by the expected conditional variance of net profit $\pi$ faced by the uninformed, $q_D^U$. From Figure 10 we see that $\Delta q_D^U$ is always positive.

(2) CKW then show the hedge funds as a group trade more aggressively on stocks affected by brokerage closures around those stocks’ earnings releases. In our case, we would like to measure how aggressively informed investors trade on their signal $m_t$. Here we appeal to the Goldstein and Yang (2015) trading intensity measure. They define trading intensity as

$$I_t = \lambda \partial q_I^I(m_t, P_t) / \partial m_t,$$

which measures the sensitivity of all informed traders’ demand (from equation 27) to changes in their signal $m_t$. In our model this trading intensity is given by

$$I_t = \frac{b_t}{c_t}.$$

We see from Figure 10 that the trading intensity increases with reduced public information.

(3) Related to their increased trading intensity, CKW find that hedge funds experience better investment performance post closure. We have a direct measure of this in our model. From (59) and (60) we see that an investor’s value function, $J^I_t$, is equal to (scaled) expected wealth. Imagine the economy is in the no-informed region, but there is a small hedge fund whose cost of becoming informed is very low relative to other informed investors, and which therefore optimally participates in trading the stock. We assume further that the hedge fund trades with no price impact (to be consistent with being in the no-informed equilibrium). This small hedge fund’s expected profits meaningfully increase with positive $\phi$ shocks, as can be seen from the top-left panel of Figure 10, as long as $\phi_t$ is in the no-informed region. Once the economy is in an interior equilibrium with $\lambda > 0$ the effect of a positive $\phi$ shock is to slightly lower the expected profits to being informed: $\Delta J^I$ becomes negative, though it is indistinguishable from zero at the resolution of the figure. The decrease in $J^I$ is due to a loss of risk sharing with the loss of public information.

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$^{24}$Market clearing requires that $\lambda q_I^I(m, P) + (1 - \lambda)q_D^U(P) = X + \bar{X}$. Differentiating with respect to $X$ and using subscripts to indicate partial derivatives, we get $\lambda q_I^I_P X + (1 - \lambda)q_D^U_P X = 1$ from which we conclude $P_X = 1/\lambda q_I^I + (1 - \lambda)q_D^U$. Differentiating the market clearing condition with respect to $m$ we get $\lambda q_I^I_m + q_D^U_m + (1 - \lambda)q_D^U_m = 0$. This implies $\lambda q_I^I_m = -(\lambda q_I^I + (1 - \lambda)q_D^U)P_m = -P_m/P_X = b/c$. 

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(4) Furthermore, CKW find that sophisticated investors scale up their information acquisition following a brokerage research department closure. Following a positive $\phi$ shock we see that $\lambda$ increases, as informed investors enter the market given the higher fraction of private information.

(5) Finally, CKW find that conditional on a large hedge fund presence in the stock, brokerage closure has a smaller effect on PEAD. As we see from Figure 9, there is a monotonic relationship between $\phi_t$ and $\lambda_t$. Therefore we can interpret the x-axis in Figure 10 as a proxy for the number of informed market participants. Conditional on having more informed participants, the impact of a $\phi$ shock on all equilibrium quantities we examine is decreasing.

Our model is therefore able to reproduce all five of the major findings in CKW. While our model is much more general than this particular example, its ability to capture a variety of empirical findings associated with an exogenous information shock is significant. We now examine more closely the implications of the dynamics of $\phi_t$.

5.3 The Effect on Prices: The Stabilizing Role of Hedge Funds

Kelly and Ljungqvist (2012) study the short-term effects of a brokerage closure on the affected firms’ stock prices. In an event study they find that affected firms experience cumulative abnormal returns (CAR) of $-112$ basis points on the day of an exogenous termination and of $-261$ basis points over the ensuing 5 days. We study the price response to a shock of $\epsilon_{\phi} = 0.25$ shock in our model under alternative $\phi_t$ dynamics.

To set a benchmark, we examine first a fully static model with $b_{\phi} = \epsilon_{\phi,t+1} = 0$. A $\phi$ shock is a one time, permanent and completely unanticipated change in the information environment. Figure 11 shows the percent drop in price at $\phi + \epsilon_{\phi}$ when the economy is at the steady-state level of all state variables. From the price function in (10), we see this is given by

$$100 \times \frac{a(\phi + \epsilon_{\phi}) - a(\phi)}{a(\phi) + dD}.$$  

Here the maximal next period (one-month in the calibration) price response is a price drop of 400 basis points (Panel A in Figure 11) at $\phi$ somewhere near 0.3.

We next examine the case from the prior section with $P[\epsilon_{\phi,t+1} \neq 0] = 2 \times 0.003$ and $b_{\phi} = 0$. The maximal price response in this case is a drop of 140 basis points, as shown in Panel B. Therefore a static analysis meaningfully overstates the expected price drop due to exogenous information shocks, as accounting for the potential of offsetting future shocks meaningfully decreases the anticipated information asymmetry. Interestingly, the static ($b_{\phi} = \epsilon_{\phi,t+1} = 0$)
Figure 11: Equilibrium price responses to exogenous information shocks to $\phi$. The x-axis in the charts shows the value of $\phi_t$ immediately prior to the shock.

and persistent ($b_\phi = 0$) cases, with maximal impacts of 400 and 140 basis points respectively, straddle the empirical finding in Kelly and Ljungqvist (2012) of a maximal impact of 261 basis points.

Finally, we allow for the feedback version of the model by setting $b_\phi = -0.04$. We interpret this as hedge funds sharing their research ideas with the general public either through investor presentations, appearance on news outlets, or unintended leakage of information. With $b_\phi = -0.04$, a 0.25 increase in $\phi$ from a starting point of $\phi = 0.25$ has a half-life along the equilibrium path of over six years (we iterate on equation 5 setting $\epsilon_{\phi,t+1} = 0$). So the information transmission from hedge funds to the public that we anticipate happens over very long periods of time. Yet even this very slow degree of reversion of $\phi$ back to its initial state meaningfully decreases the price impact of a $\phi$ shock, as can be seen in Panel C of Figure 11. The maximal percentage price drop is now only 50-60 basis points.

The intuition for this result is similar to our discussion about the price drop in Section 4.3. $\phi_t$ can be interpreted as a measure of information asymmetry in the model conditional on the fixed level $f$ of disclosure about the underlying dividend process. As before, agents are very averse to high information asymmetry states — though information asymmetry in Section 4.3
was induced by an increase in \( f_t \) while here \( \phi_t \) directly determines the asymmetry. In the static case of \( b_\phi = \epsilon_{\phi,t+1} = 0 \) agents believe the new higher level of asymmetry will last forever, and therefore the price concession demanded by the uninformed is very large. In the persistent case of \( b_\phi = 0 \) shocks to \( \phi_t \) are persistent but can be reversed by future shocks in the opposite direction, and therefore are not permanent. In the feedback case of \( b_\phi < 0 \), shocks structurally diminish over time in response to an endogenous increase in informed. This meaningfully diminishes the anticipated information asymmetry, and therefore also the commensurate price impact.

Perhaps the large \(-261\) basis point weekly CARs observed by Kelly and Ljungqvist (2012) reflects a lack of appreciation by the investment community of the stabilizing feedback effects from hedge fund participation that we posit here. Indeed our analysis suggests that hedge funds can play a stabilizing role in the case of exogenous shocks that increase information asymmetry by slowly disseminating their own research to the public.

6 Conclusions

We have developed a model of a financial market in a dynamic information environment. The model combines exogenous shocks to the level of potentially available information, an endogenous response by investors, and feedback from investor information choices to the information environment.

We study two types of feedback affecting two dimensions of the information environment: a positive feedback mechanism in which more information becomes available as more investors choose to become informed; and a negative feedback mechanism in which an increase in privately acquired information produces leakage and greater public information. In both cases, changes in the level of information are accompanied by changes in the degree of information asymmetry between informed and uninformed investors.

We show that the equilibrium dynamics under the positive feedback mechanism is characterized by two regimes, one with high prices and low volatility, and one with low prices and high volatility. A transition from the first regime to the second is reminiscent of a financial crisis but with no change in fundamentals — the price drop is driven by the dynamics of information and an increase in information asymmetry.

We use the negative feedback mechanism to explain empirical results on the effect on stock prices of the loss of analyst coverage. In particular, we examine the substitution effect documented by Chen et al. (2018), showing that hedge funds respond to the loss of public information
with greater private information acquisition. In this setting, negative feedback describes leakage from private to public information. We show that accounting for information dynamics and incorporating feedback effects meaningfully change the predicted response to a loss of coverage when compared to a static model.

A Market Equilibrium

A.1 Proof of Proposition 3.1 (Existence of a Market Equilibrium)

Investor Demands for the Risky Asset

We prove Proposition 3.1 by solving explicitly for the coefficients of the price in (10). To allow for arbitrary variance beliefs, we write the investor optimization problem (8) as

\[ \hat{J}_i^t \equiv \max_q \mathbb{E} \left[ \mathbb{E} \left[ W_{t+1} | I_i^t, f_{t+1}, \phi_{t+1} \right] - \frac{\gamma}{2} \hat{\text{var}}(W_{t+1} | I_i^t, f_{t+1}, \phi_{t+1}) \mid I_i^t \right], \quad i \in \{I, U\}, \tag{22} \]

where, using (15),

\[ \hat{\text{var}}(W_{t+1} | I_i^t, f_{t+1}, \phi_{t+1}) = q^2 (1 + d)^2 \left[ \text{var}(m_t | I_i^t) + (1 - f_t) \sigma_M^2 \right] + q^2 V_B(f_{t+1}, \phi_{t+1}). \tag{23} \]

If the variance belief \( V_B \) is self-consistent, then (23) yields the conditional variance, but (22) makes explicit investors’ objectives with arbitrary variance beliefs.

We can write the terminal wealth in (6) as \( W_{t+1} = RW_t + q \pi_{t+1} \). Recalling that \( W_t \) is known to time-\( t \) investors, we set \( \hat{\text{var}}(\pi_{t+1} | I_i^t, f_{t+1}, \phi_{t+1}) = \hat{\text{var}}(W_{t+1} | I_i^t, f_{t+1}, \phi_{t+1})/q^2 \) to get

\[ \hat{\text{var}}(\pi_{t+1} | I_i^t, f_{t+1}, \phi_{t+1}) = (1 + d)^2 \left[ \text{var}(m_t | I_i^t) + (1 - f_t) \sigma_M^2 \right] + V_B(f_{t+1}, \phi_{t+1}). \tag{24} \]

The first-order condition for optimality in (22) becomes

\[ q_i^t = \frac{1}{\gamma} \mathbb{E} \left\{ \mathbb{E} [\pi_{t+1} | I_i^t, f_{t+1}, \phi_{t+1}] | I_i^t \} \right\} \equiv \frac{1}{\gamma} q_i^N + q_i^D, \tag{25} \]

where \( q_i^N \) is the conditional expectation of the net profit, and \( q_i^D \) is the expectation of its conditional variance, given a price variance of \( V_B \). Through (24), the conditional variances, reflecting the cash-flow component and the time-\( t \) price variance, take the form

\[ q_i^D = (1 + d)^2 (1 - f_t) \sigma_M^2 + \mathbb{E}_t V_B(f_{t+1}, \phi_{t+1}), \]

\[ q_i^U = q_i^D + (1 + d)^2 \text{var}(m_t | P_t, \theta_t). \tag{26} \]
Evaluating the conditional mean in the numerator of (25) as in (13), the demands for time-t informed and uninformed agents become

\[ q^I = \frac{q^I_N}{\gamma q^I_D} = \frac{1}{\gamma q^I_D} \left[ (1 + d)(\mu_D + \rho D_t + \theta_t + m_t) + E_t a(f_{t+1}, \phi_{t+1}) - RP_t \right], \]

\[ q^U = \frac{q^U_N}{\gamma q^U_D} = \frac{1}{\gamma q^U_D} \left[ (1 + d)(\mu_D + \rho D_t + \theta_t + E[m_t|P_t, D_t, \theta_t]) + E_t a(f_{t+1}, \phi_{t+1}) - RP_t \right]. \]

For the informed, we have used the fact that \( E[m_t|\mathcal{I}_t^I] = m_t \) and \( \text{var}(m_t|\mathcal{I}_t^I) = 0 \). For the uninformed, we evaluate (27) using \( ^\gammaD \)

\[ E[m_t|P_t, \theta_t] = K_t(b_t m_t - c_t X_t), \]

\[ \text{var}(m_t|P_t, \theta_t) = \phi_t f_t \sigma_M^2 (1 - K_t b_t) = \phi_t f_t \sigma_M^2 (1 - \mathcal{R}_t^2), \]

with

\[ K_t = \frac{\text{cov}(m_t, P_t|\theta_t, D_t)}{\text{var}(P_t|\theta_t, D_t)} = \frac{b_t \phi_t f_t \sigma_M^2}{b_t^2 \phi_t f_t \sigma_M^2 + c_t^2 \sigma_X^2} \quad \text{and} \quad \mathcal{R}_t^2 \equiv K_t b_t. \]

**Market Clearing and Price Coefficients**

We now impose market clearing (9), taking \( \lambda = \) as given. We substitute investor demands \( q^I \) in (5), use the price function from (10), and collect terms. We do not have to expand \( q^I_D \) or \( q^U_D \) in the following because these depend on \( (f_t, \phi_t) \) but not on \( D_t, m_t, \theta_t, \) or \( X_t \). Equation (9) becomes

\[ \lambda q^U_D \left[ (1 + d)(\mu_D + \rho D_t + \theta_t + m_t) + E_t a(f_{t+1}, \phi_{t+1}) - RP_t \right] \\
+(1 - \lambda) q^I_D \left[ (1 + d)(\mu_D + \rho D_t + \theta_t + E[m_t|P_t, \theta_t]) + E_t a(f_{t+1}, \phi_{t+1}) - RP_t \right] \\
= \gamma q^I_D q^U_D X_t + \gamma q^I_D q^U_D \mathcal{X}. \]

Collecting the \( D_t \) terms and then the \( \theta_t \) terms yields the constants

\[ d = \frac{\rho}{R - \rho} \quad \text{and} \quad g = \frac{1}{R - \rho}. \]

\[ ^{25} \text{Say} \ E[m|P] = K(b_m + cX) \text{for } K = \text{cov}(m, P)/\text{var}(P) \text{ and } \text{var}(m|P) \equiv \text{var}(m - E[m|P]). \text{Since } m - E[m|P] = (1 - K b_m - K c X) \text{ then } \text{var}(m - E[m|P]) = (1 - K b)^2 \text{var}(m) + K^2 c^2 \text{var}(X). \text{This equals } (1 - 2 K b + K^2 b^2) \text{var}(m) + K^2 c^2 \text{var}(X) = (1 - 2 K b) \text{var}(m) + K^2 (b^2 \text{var}(m) + c^2 \text{var}(X)). \text{Note that } K = b \text{var}(m)/(b^2 \text{var}(m) + c^2 \text{var}(X)) \text{ and therefore } K^2 (b^2 \text{var}(m) + c^2 \text{var}(X)) = b^2 \text{var}(m)/(b^2 \text{var}(m) + c^2 \text{var}(X)) = K b \text{var}(m). \text{And therefore } \text{var}(m|P) = (1 - K b) \text{var}(m). \]
Collecting the constant terms in (30) yields

$$a_t = \frac{1}{R} \left( (1 + d)\mu - \frac{\gamma q_t \bar{q}^U}{\lambda q + (1 - \lambda)q_f} X + E_t a(f_{t+1}, \phi_{t+1}) \right). \quad (32)$$

The function $a$ appears on both sides. Assuming for a moment that a solution $a_t = a(f_t, \phi_t)$ exists, we can proceed to solve for $b$ and $c$ because $a$ plays no role in the inference the uninformed make from the price in (29). We return to solve (32) after solving for $b$ and $c$.

Collecting the $m_t$ terms in (30) we get

$$b_t = 1 + d \times \frac{\lambda q + (1 - \lambda)q_f K_t b_t}{\lambda q + (1 - \lambda)q_f}. \quad (33)$$

Collecting the $X_t$ terms and simplifying — mainly dividing the resulting equation by (33) — we find

$$b_t/c_t = \frac{\lambda(1 + d)}{\gamma q_I D}, \quad \text{with} \quad c_t = \frac{\gamma q_I D^U}{R} \text{ if } \lambda = 0. \quad (34)$$

We can now combine these equations to solve for $b$ and $c$ through the following steps, each of which involves only known quantities on the right side:

$$q_I^L(f, \phi) = (1 + d)^2(1 - f)\sigma^2_M + E[V_B(f_{t+1}, \phi_{t+1})|f_t = f, \phi_t = \phi], \quad (35)$$

$$r(f, \phi) = \lambda(f, \phi)(1 + d)/(\gamma q_I D(f, \phi)), \quad (36)$$

$$R^2(f, \phi) = \frac{r^2(f, \phi)f \sigma^2_M}{r^2(f, \phi)} + \sigma^2_X, \quad (37)$$

$$q_D^U(f, \phi) = q_I^U(f, \phi) + (1 + d)^2 f \phi \sigma^2_M (1 - R^2(f, \phi)), \quad (38)$$

$$b(f, \phi) = \frac{1 + d \lambda(f, \phi) q_I^U(f, \phi) + (1 - \lambda(f, \phi))q_D^U(f, \phi) R^2(f, \phi)}{R \lambda(f, \phi) q_I^U(f, \phi) + (1 - \lambda(f, \phi))q_D^U(f, \phi)}, \quad (39)$$

$$c(f, \phi) = \begin{cases} b(f, \phi)/r(f, \phi), & \lambda(f, \phi) > 0; \\ \gamma q_D^U(f, \phi)/R, & \lambda(f, \phi) = 0. \end{cases} \quad (40)$$

Equation (35) restates the first line of (26); (36) is the ratio in (34); (37) rewrites the expression for $R^2$ in (29); (38) follows from the second line of (26); (39) and (40) come from (33) and (34).
A.2 Solving for the \( a() \) curve

We now return to (32). Using the price function from (10) and the net profit from (7), we see that

\[
\mu_{\pi,t} \equiv \mathbb{E}\left[ \pi_{t+1} | f_t, \phi_t \right] = \mathbb{E}_t[D_{t+1} + P_{t+1} - RP_t]
\]

\[
= \mu_D + \rho D_t + \mathbb{E}_t[a_{t+1}] + d(\mu_D + \rho D_t) - R\pi_t - RdD_t
\]

\[
= (1 + d)\mu_D + \mathbb{E}_t[a_{t+1}] - R\pi_t
\]

\[
= \frac{\gamma \mathcal{X}}{\lambda q_D^U + (1 - \lambda)q_D^P}
\]

where the third step follows from the definition of \( d \) in (31) and the fourth step follows from \( a_t \) in (32). Note that \( \mu_{\pi,t} \) is a function of \( f_t, \phi_t \). Using (41) we can rewrite \( a_t \) in (32) as

\[
a_t = \frac{1}{R} \left[ (1 + d)\mu_D - \mu_{\pi,t} + \mathbb{E}_t[a(t + 1)] \right]
\]

\[
= \frac{1}{R} \left[ (1 + d)\mu_D - \mu_{\pi,t} + \mathbb{E}_t \left\{ \frac{1}{R} \left[ (1 + d)\mu_D - \mu_{\pi,t+1} + \mathbb{E}_{t+1}[a(f_{t+2}, \phi_{t+2})] \right] \right\} \right]
\]

\[
= \cdots = \frac{(1 + d)\mu_D}{R - 1} - \frac{1}{R} \sum_{i=0}^{\infty} \frac{1}{R^i} \mathbb{E}_t a_{\pi,t+i}.
\]

If the variance belief \( V_B \) is bounded above and bounded away from zero, then \( |\mu_{\pi,t}| \) is bounded and the expression in (42) is well-defined and finite. The quantities in (35)–(40) and (41) are all functions solely of the information state \((f, \phi)\), so the conditional expectations in (32) and (42) are taken purely with respect to the evolution of the information state in (4) or (5), for given \( \lambda \). Equation (42) shows \( a_t \) is equal to the present value of all future expected dividend payments minus a discount reflecting the expected present value of all future net profits.

A.3 Statement and Proof of Proposition 3.2 (Existence of a Rational Expectations Equilibrium)

In this section, we prove the existence of a rational expectations equilibrium by showing that the variance belief updating mapping has a fixed point. This demonstrates the existence of self-consistent variance beliefs given an exogenously specified \( \lambda() \) curve.\(^{26}\)

We now precisely state Proposition 3.2 for model (4) with fixed \( \phi \) and varying \( f \). With \( f_t \) restricted to a finite set \( D \), we represent any function of \( f_t \) as a vector of dimension \( n = |D| \). We

\(^{26}\)Note that this exogenous \( \lambda() \) curve need not result in equivalent utilities for informed and uninformed investors. We endogenize the \( \lambda() \) curve in Section B.
suppose $\lambda$ is fixed (not necessarily constant) with $0 \leq \lambda(f, \varphi) \leq 1$ for all $f$ and $\varphi$. Let $F(b, c)$ be the mapping that sends the initial coefficients $(b, c)$ to updated coefficients $(b', c')$ through (16) and (35)–(40). A fixed point refers to the coefficients $b$ and $c$ such that $(b, c) = F(b, c)$.

Assume there exists a scalar $\bar{c} > 0$ satisfying the following four polynomial conditions:

$$
\gamma \sigma_X (2\bar{q} + (1 + d)^2 \sigma_M^2 \phi) - \bar{c} R \sigma_X \left( 4 - (1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 \phi \right) \leq 0, 
$$

(43)

$$
4\gamma R \sigma_X^2 \bar{q} - 4R^2 \sigma_X^2 \bar{c}^2 + (1 + d)^2 \sigma_M^2 \phi \left( 1 + \gamma R \sigma_X^2 \bar{c} \right)^2 \leq 0, 
$$

(44)

$$
\gamma^2 \sigma_X^2 \left( \bar{q} + (1 + d)^2 \sigma_M^2 \phi \right) \leq 1, 
$$

(45)

$$
\gamma \left( (1 + d)^2 \sigma_M^2 \eta + \bar{c}^2 \sigma_X^2 \right) - R \bar{c} \leq 0, 
$$

(46)

where $\bar{q}$ in (43), (44), and (45) is given by

$$
\bar{q} = (1 + d)^2 \sigma_M^2 \delta + \bar{c}^2 \sigma_X^2, 
$$

(47)

and the quantities $\eta$ and $\delta$ in (46) and (47) are defined by

$$
\eta := \max_{f \in \mathcal{D}} \left\{ \frac{1}{R^2} \mathbb{E} \left[ f_{t+1} | f_t = f \right] + 1 - (1 - \phi) f \right\}, 
$$

$$
\delta := \max_{f \in \mathcal{D}} \left\{ \frac{1}{R^2} \mathbb{E} \left[ f_{t+1} | f_t = f \right] + 1 - f \right\}. 
$$

(48)

Our simplest conditions will require $\eta, \delta \leq 2$, which is natural for $f \in [0, 1]$ and $R > 1$. The precise statement of Proposition 3.2 is that the mapping $F$ has a fixed point in $[0, \bar{b}]^n \times [0, \bar{c}]^n$ with $n = |\mathcal{D}|$, $\bar{c}$ satisfying (43)–(46), and

$$
\bar{b} = \frac{1 + d}{R}. 
$$

The existence of a $\bar{c}$ satisfying (43)–(46), and thus the existence of a fixed point for mapping $F$, only depends on model parameters. Note that (31) implies $\bar{b}$ equals $g$, the loading of $P_t$ in (10) on the publicly known portion of the dividend $\theta_t$. So $b \in [0, \bar{b}]^n$ is equivalent to $0 \leq b_t \leq g$, which says that $P_t$ is more informative about $\theta_t$ than it is about $m_t$.

As a shortcut for checking that these conditions hold, we show in Appendix A.3.1 that if $R \in [1, 1.2]$ and condition (17) holds, then $F$ has a fixed point in $[0, \bar{b}]^n \times [0, \bar{c}]^n$ with $\bar{b} = (1 + d)/R$ and $\bar{c} = R/(2\gamma \sigma_X^2)$.

For model (5) with fixed $f$ and varying $\varphi$, a similar argument ensures the existence of a
fixed point. Redefine the quantities \( \delta \) and \( \eta \) as

\[
\eta := \frac{1}{R^2} f + 1 \quad \text{and} \quad \delta := \frac{1}{R^2} f + 1 - f.
\]

Then we claim a fixed point exists in \([0, \bar{b}]^n \times [0, \bar{c}]^n\) as long as there exists \( \bar{c} > 0 \) satisfying (43)–(46) with \( \phi \) replaced by \( f \) in these equations. Furthermore, assuming \( R \in [1, 1.2] \), we can show using arguments similar to those in Section A.3.1 that (17) is a sufficient condition for the existence of a fixed point.

**Proof of Proposition 3.2**

We prove the proposition for model (4) with fixed \( \phi \) and varying \( f \), then the result for model (5) with fixed \( f \) and varying \( \phi \) follows in similar way. To prove the result, we use Brouwer’s fixed point theorem, which states that if \( F \) is a continuous function mapping a compact convex set \( S \) to itself, then \( F \) has a fixed point in this set, meaning there exists \( x \in S \) for which \( x = F(x) \); see, for example, p.29 of Border (1989).

We show conditions for the Brouwer’s fixed point theorem are satisfied with the belief updating mapping \( F \) and the compact convex set \([0, \bar{b}]^n \times [0, \bar{c}]^n, n = |D|\). First, it is evident that \( F \) is continuous as each mapping in (35)–(40) is continuous in its input. It is also evident that \( F(b, c) \geq 0 \) since each step in (35)–(40) returns a nonnegative value. Next, for any input \((b, c)\), we have \( b'(f, \phi) \leq (1 + d)/R = \bar{b} \) for all \( f \) and \( \phi \): it is easy to see \( q_D(f, \phi) \geq 0 \), \( R^2(f, \phi) \in [0, 1] \), and \( q_{U_D}(f, \phi) \geq 0 \), so the second factor in (39) is in \([0, 1]\) and the bound on \( b'(f, \phi) \) follows.

It only remains to show that if \((b, c) \in [0, \bar{b}]^n \times [0, \bar{c}]^n\) and \((b', c') = F(b, c)\), then \( c'(f, \phi) \leq \bar{c} \) for all \( f \) and \( \phi \). For this we first establish two useful bounds. To lighten notation, in the following we abbreviate conditional expectations of the form \( \mathbb{E}[V_B(f_{t+1}, \phi) | f_t = f] \) as \( \mathbb{E}_t[V_B(f, \phi)] \).

Recalling (16), we get the bound

\[
\mathbb{E}_t[V_B(f, \phi)] = \mathbb{E}_t \left[ b(f, \phi)^2 f \right] \phi \sigma_M^2 + g^2 (1 - \phi) \mathbb{E}_t[f] \sigma_M^2 + \mathbb{E}_t \left[ c(f, \phi)^2 \right] \sigma_X^2 \\
\leq \bar{b}^2 \mathbb{E}_t[f] \phi \sigma_M^2 + g^2 (1 - \phi) \mathbb{E}_t[f] \sigma_M^2 + \bar{c}^2 \sigma_X^2 \\
= (1 + d)^2 \frac{1}{R^2} \mathbb{E}_t[f] \sigma_M^2 + c^2 \sigma_X^2. \tag{49}
\]

The inequality uses the assumption \((b, c) \in [0, \bar{b}]^n \times [0, \bar{c}]^n\). The last equality uses the fact that
\( \tilde{b} = g. \) Combining this with (35), we can bound \( q_D(f, \phi) \) by

\[
q_D(f, \phi) = (1 + d)^2(1 - f)\sigma_M^2 + \mathbb{E}_t[V_B(f, \phi)] \\
\leq (1 + d)^2\sigma_M^2 \delta + \bar{c}^2\sigma_X^2 = \bar{q},
\]

(50)

where \( \bar{q} \) and \( \delta \) are given in (47) and (48). With these preparations, we proceed to prove the desired bound \( c'(f, \phi) \leq \bar{c} \) (for all \( f \) and \( \phi \)) by the following two cases.

I. The Case Without Informed Investors: \( \lambda(f, \phi) = 0 \)

We first prove the bound for \( c'(f, \phi) \) for the case \( \lambda(f, \phi) = 0. \) By the second case in (40), \( c'(f, \phi) \) is now given by

\[
c'(f, \phi) = \gamma q_U(f, \phi)/R.
\]

With \( \lambda(f, \phi) = 0, \) (36), (37), and (39) lead to

\[
r(f, \phi) = R^2(f, \phi) = \tilde{u}(f, \phi) = 0.
\]

Thus by (35) and (38) we have

\[
q_U(f, \phi) = q_{ID}(f, \phi) \leq (1 + d)^2\sigma_M^2 \delta + \bar{c}^2\sigma_X^2,\]

where the last inequality follows (48). Then the updated coefficient \( c'(f, \phi) \) satisfies

\[
c'(f, \phi) = \frac{\gamma q_U(f, \phi)}{R} \leq \frac{\gamma}{R} ((1 + d)^2\sigma_M^2 \delta + \bar{c}^2\sigma_X^2) \leq \bar{c},
\]

which follows from condition (46).

II. The Case With Informed Investors: \( \lambda(f, \phi) > 0 \)

Next, we consider the case \( \lambda(f, \phi) > 0. \) Applying the first case in (40), we get

\[
c'(f, \phi) = \frac{b(f, \phi)}{r(f, \phi)} = \frac{\gamma}{R} \left( q_D(f, \phi) \cdot \frac{\lambda(f, \phi)q_U(f, \phi) + (1 - \lambda(f, \phi))q_D(f, \phi)R^2(f, \phi)}{\lambda(f, \phi)q_U(f, \phi) + (1 - \lambda(f, \phi))q_D(f, \phi)} \right).
\]

(51)

We now outline the steps to establish the desired bound \( c'(f, \phi) \leq \bar{c} \) from (51):

1. Express \( q_U(f, \phi) \) and \( R^2(f, \phi) \) in terms of \( q_D(f, \phi). \)

2. Show that condition (46) is sufficient for the case \( f = 0 \) (or \( \phi = 0 \)).
3. Derive an upper bound for (51) by maximizing it over $\lambda(f, \phi) \in (0, 1]$ and using the bounds $q_D^I(f, \phi) \leq \bar{q}$ and $f \leq 1$.

4. Show that conditions (43)–(45) suffice to establish $c'(f, \phi) \leq \bar{c}$ for all $f$ and $\phi$ through the bound derived in step 3.

As the first step, we make substitutions in (51) by plugging in (38) for $q^I_U(f, \phi)$ and using (36) and (37) for $R^2(f, \phi)$. With these substitutions, the quantity in parentheses in (51) can be expressed as

$$
\frac{\lambda q_D^I(1 + d)^2 \sigma^2_M f \phi + (q_D^I)^2 \gamma \sigma^2_X (1 + d)^2 \sigma^2_M f \phi + (q_D^I)^3 \gamma \sigma^2_X}{\lambda^2 (1 + d)^2 \sigma^2_M f \phi + \lambda q_D^I \gamma \sigma^2_X (1 + d)^2 \sigma^2_M f \phi + (q_D^I)^2 \gamma \sigma^2_X},
$$

where we have dropped the dependence on $f$ and $\phi$ to simplify notation. In the following, we will establish the needed bound for (51) by combining expression (52), the bound for $q^I_D(f, \phi)$ in (50), and conditions (43)–(45). We need a bound that is valid for all $\lambda \in (0, 1]$.

We first consider the case $f = 0$ (or $\phi = 0$). If this holds, (52) directly reduces to $q^I_D(f, \phi)$ itself. By (51) and (50), $c'(f, \phi)$ satisfies

$$
c'(f, \phi) = \frac{\gamma}{R} q^I_D(f, \phi) \leq \frac{\gamma}{R} \bar{q} = \frac{\gamma}{R} ((1 + d)^2 \sigma^2_M \delta + \bar{c} \sigma^2_X).
$$

By (48), it is easy to verify $\delta \leq \eta$ always holds given $\phi \geq 0$. Thus the desired bound for $c'(f, \phi)$ can be established as

$$
c'(f, \phi) \leq \frac{\gamma}{R} ((1 + d)^2 \sigma^2_M \delta + \bar{c} \sigma^2_X) \leq \frac{\gamma}{R} ((1 + d)^2 \sigma^2_M \eta + \bar{c} \sigma^2_X) \leq \bar{c},
$$

where the last inequality directly follows from condition (46).

Next, we bound (52) for $f > 0$ and $\phi > 0$. Here we need to consider the maximum of (52) over all $\lambda \in (0, 1]$. To achieve this, we consider the derivative of (52) with respect to $\lambda$. As the numerator and denominator of (52) are linear and quadratic functions in $\lambda$ respectively, the numerator of its derivative is a quadratic function in $\lambda$. Furthermore, it is easy to check this quadratic function is concave and the denominator of the derivative is always positive. Thus the maximum of (52) is attained at the larger root of its derivative, as long as that root falls in $(0, 1]$. Through algebraic manipulations, the larger root of the derivative is given by

$$
\hat{\lambda} = \frac{-\tau + q^I_D \gamma \sigma_X \sqrt{\tau + (1 + d)^2 \sigma^2_M f \phi}}{(1 + d)^2 \sigma^2_M f \phi},
$$

where the last inequality directly follows from condition (46).
where \( \tau = q_D^2 \gamma^2 \sigma_X^2 \left( q_D^2 + (1 + d)^2 \sigma_M^2 f \phi \right) \). Also, \( \tilde{\lambda} \in (0, 1] \) is guaranteed by condition (45) and the bounds \( q_D^I \leq \bar{q} \) and \( f \leq 1 \). Thus the maximum of (52) is indeed attained at \( \lambda = \tilde{\lambda} \).

Letting \( \lambda = \tilde{\lambda} \) in (52) and plugging the resulting maximum into (51), we can derive

\[
c'(f, \phi) \leq \frac{\gamma \sigma_X (2q_D^I + (1 + d)^2 \sigma_M^2 f \phi) + 2\sqrt{(1 + d)^2 \sigma_M^2 f \phi + q_D^I \gamma^2 \sigma_X^2 \left( q_D^2 + (1 + d)^2 \sigma_M^2 f \phi \right)}}{R\sigma_X (4 - (1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 f \phi)}. \tag{54}
\]

Denote the right side by \( \kappa(f, q_D^I) \). It is positive as condition (43) directly implies the denominator is greater than zero. As \( \kappa(f, q_D^I) \) clearly increases in both \( q_D^I \) and \( f \), it can be further bounded by setting \( q_D^I \) and \( f \) at their upper bounds \( \bar{q} \) and 1, respectively. Thus to establish the needed bound \( c'(f, \phi) \leq \bar{c} \), it suffices to show

\[
c'(f, \phi) \leq \kappa(f, q_D^I) \leq \kappa(1, \bar{q}) \leq \bar{c}. \tag{55}
\]

Setting \( q_D^I = \bar{q} \) and \( f = 1 \) on the right side of (54), the needed inequality \( \kappa(1, \bar{q}) \leq \bar{c} \) becomes

\[
2\sqrt{(1 + d)^2 \sigma_M^2 \phi + \bar{q} \gamma^2 \sigma_X^2 \left( \bar{q} + (1 + d)^2 \sigma_M^2 \phi \right)} \leq \bar{c} R\sigma_X (4 - (1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 \phi) - \gamma \sigma_X (2\bar{q} + (1 + d)^2 \sigma_M^2 \phi). \tag{56}
\]

By condition (43), the right side of (56) is positive, so this inequality is equivalent to the one obtained by squaring both sides. Taking squares and simplifying, we can show all the higher order terms of \( \bar{q} \) cancel out, and inequality (56) is equivalent to

\[
\bar{q} \leq \frac{4R^2 \sigma_X^2 \bar{c}^2 - (1 + d)^2 \sigma_M^2 \phi (1 + \gamma R \bar{c}^2)^2}{4\gamma R \bar{c}^2},
\]

which holds as long as \( \bar{c} \) and \( \bar{q} \) satisfy condition (44). By (55), this proves the needed bound \( c'(f, \phi) \leq \bar{c} \) for \( \lambda(f, \phi) > 0 \). Combining the two cases with \( \lambda(f, \phi) = 0 \) and \( \lambda(f, \phi) > 0 \), we have proved \( c' \leq \bar{c} \) holds when conditions (43)–(46) are satisfied. The existence of a fixed point for the variance belief updating mapping now follows by Brouwer's theorem.

### A.3.1 The Simplified Condition in (17)

Finally, we prove that (17) is a simple sufficient condition for the existence of fixed point. We follow the general conclusions established above and show that as long as \( R \in [1, 1.2] \) and condition (17) hold, then the value \( \bar{c} = R/(2\gamma \sigma_X^2) \) satisfies conditions (43)–(46). Thus by the proposition, a fixed point of belief updating exists in \([0, (1 + d)/R]^n \times [0, R/(2\gamma \sigma_X^2)]^n\).
Plugging in $\bar{c} = R/(2\gamma \sigma_X^2)$ into conditions (43)-(46), we find they simplify to following equivalent conditions:

$$(1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 \leq \frac{3R^2}{3R^2 + 4\delta + 6}, \quad (1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 \leq \frac{2R^4}{(4 + 4R^2 + R^4)\phi + 8R^2\delta},$$

as well as

$$(1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 \leq \frac{4 - R^2}{4(\delta + \phi)}, \quad \text{and} \quad (1 + d)^2 \gamma^2 \sigma_M^2 \sigma_X^2 \leq \frac{R^2}{4\eta},$$

respectively. A nice property of these new conditions is that they are all upper bounds for the product $(1 + d) \gamma \sigma_M \sigma_X$. Thus it suffices to find a bound for $(1 + d) \gamma \sigma_M \sigma_X$ that is small enough such that all the four conditions are satisfied. Since all the bounds in the right-hand sides of (57) and (58) are clearly decreasing in $\phi$, $\delta$, and $\eta$, it suffices to consider their values at $\phi = 1$ and $\delta = \eta = 2$, as by (48) we clearly have $\delta \leq \eta \leq 2$. On the other hand, as the bounds do not monotonically depend in the risk-free return $R$, we impose a relatively mild condition $R \in [1, 1.2]$. Plugging these values into the right-hand sides of (57) and (58), the minimum of the four upper bounds approximately equals to 0.283, thus (17) is sufficient for the existence of fixed point.

**B Information Equilibrium**

**B.1 Equating Expected Utilities of Informed and Uninformed given $V_B$**

In this section we discuss the procedure for solving for an endogenous $\lambda$ given variance beliefs. Given the demands in (26) and variance beliefs, we have, for $\iota \in \{I, U\}$,

$$E[E[W_{t+1}|I_t, f_{t+1}, \phi_{t+1}]|I_t] = q^I \times q_N + RW_t = \frac{(q^I_N)^2}{\gamma q_D^I} + RW_t,$$

$$E[\text{var}(W_{t+1}|I_t, f_{t+1}, \phi_{t+1})|I_t] = (q^I)^2 \times q_D^I = \frac{(q^I_N)^2}{\gamma^2 q_D^I}.$$  

(59)

We can therefore write the agent’s value function in (8), conditional on $I_t$ as

$$J^\iota_t = RW_t + \frac{1}{2\gamma} \frac{(q^I_N)^2}{q_D^I}, \quad \iota \in \{I, U\}.$$  

(60)

To find an endogenous $\lambda$ in the sense of Definition 2.1, we need to evaluate the conditional expectation of $J^I_t - J^U_t$ given the information state $(f_t, \phi_t)$. We can pull the denominators $q_D^I$ out of the conditional expectation because we know from (35) and (38) that they are purely
functions of the information state. For the numerator terms, using the demands from (27), the price process from (10) and the condition on \( a_t \) in (32), it is straightforward to show that

\[
q_N = \mu + (1 + d)E[m_t|I^t] - Rb_t m_t + Rc_t X_t
\]

where \( \mu \) — which is a function of \( \lambda \) — is defined in (41). The \( \theta_t \) and \( D_t \) terms drop out, as do the terms involving \( a_t \).

Note that \( q_N \) equals the expected net profit \( \mu \), which only conditions on \( f_t \) and \( \phi_t \), adjusted for the information set of agent \( \iota \).

Since \( E[m_t|I^t] = m_t \) we have

\[
E[(q_N^I)^2|f_t, \phi_t] = \mu^2 + (1 + d - Rb_t)^2 \phi_t f_t \sigma_M^2 + R^2 \phi^2 \sigma_X^2.
\]

(61)

And from (28) we have \( E[m_t|I^U_t] = K_t b_t m_t - K_t c_t X_t \). From this we have that

\[
E[(q_N^U)^2|f_t, \phi_t] = \mu^2 + (1 + d)K_t b_t - Rb_t)^2 \phi_t f_t \sigma_M^2 + [Rc_t - (1 + d)K_t c_t]^2 \sigma_X^2
\]

\[
= \mu^2 + [(1 + d)K_t - R]^2 (b_t^2 \phi_t f_t \sigma_M^2 + c_t^2 \sigma_X^2).
\]

(62)

Combining these expressions with \( J_I^I \) in (60), we get an expression for the difference in conditional expectations

\[
\Delta_{(f,\phi)} = E[J_I^I - Rc_t|f_t = f, \phi_t = \phi] - E[J_I^U|f_t = f, \phi_t = \phi].
\]

(63)

When \( \Delta_{(f,\phi)} \) is positive, the marginal investor has an incentive to become informed. For a given \( \{f, \phi\} \) we then numerically solve for the \( \lambda \in [0,1] \) which sets \( \Delta_{f,\phi} = 0 \). If this difference is always strictly positive we set \( \lambda = 1 \), and if it is always strictly negative we set \( \lambda = 0 \).

### B.2 Proof of Proposition 3.3 (Existence of \( \lambda() \) given Variance Beliefs)

To simplify notation, we focus on the case of dynamic \( f_t \), as in (4) with \( \phi_t \equiv \phi_0 \) fixed. The argument for varying \( \phi_t \) with fixed \( f_t \) works the same way. We therefore write \( \Delta_{f} \) in (63) and omit \( \phi_t \) from the conditioning information on the right.

Recall that we have restricted \( f_t \) to a finite set \( D \). We need to be more explicit about the mapping to \( D \) in (4). Suppose \( D = \{s_1, \ldots, s_n\} \subset [0,1] \). Partition the extended real line using

\[ -\infty = c_0 < c_1 < \cdots < c_n < c_{n+1} = \infty, \]

27In particular, we do not need to evaluate \( a(\cdot) \) to find the endogenous \( \lambda(\cdot) \), which is useful in solving the model numerically.
and let \( \Pi_D : (c_j, c_{j+1}] \mapsto s_{j+1}, j = 0, 1, \ldots, n \). We prove the proposition for this choice of \( \Pi_D \).

We can write the difference in expected utilities (63) as

\[
\Delta_f = \frac{1}{2\gamma} E\left[ \frac{q_N^{L2}}{q_D^{L2}} - \frac{q_N^{U2}}{q_D^{U2}} \right] f_t = f - Rc_I = \frac{1}{2\gamma} \left( \frac{E[q_N^{L2}f_t = f]}{q_D^{L}(f)} - \frac{E[q_N^{U2}f_t = f]}{q_D^{U}(f)} \right) - Rc_I. \tag{64}
\]

The terms on the right depend on the mapping \( \lambda : D \mapsto [0, 1] \). However, for each \( f \), \( \Delta_f \) depends on \( \lambda \) only through \( \lambda(f) \). This follows from the expressions in (35)–(40). We may therefore write \( \Delta_f \) as \( \Delta_f(\ell) \), with the interpretation that \( \ell \) is the value of \( \lambda(f) \). The proposition will follow once we show that \( \Delta_f(\cdot) \) is continuous: given continuity, either \( \Delta_f(\ell^*) = 0 \) at some \( \ell^* \in [0, 1] \) (in which case we set \( \lambda(f) = \ell^* \)), or \( \Delta_f(\ell) < 0 \) for all \( \ell \in [0, 1] \) (in which case we set \( \lambda(f) = 0 \)), or \( \Delta_f(\ell) > 0 \) for all \( \ell \in [0, 1] \) (in which case we set \( \lambda(f) = 1 \)). This specification satisfies the conditions in Definition 2.3.

To establish continuity of \( \Delta \), we use the representation in (64). It is evident that, holding \( f \) fixed, each of the operations in (36)–(39) is continuous in \( \ell = \lambda(f) \). But \( \lambda \) is also implicit in (35) through the conditional expectation of the variance belief, which takes the form

\[
E[V_B(f_{t+1}|f_t = s_i)] = \sum_{s_j \in D} P(f_{t+1} = s_j|f_t = s_i)V_B(s_j).
\]

With \( \lambda(s_i) = \ell \), the transition probabilities take the form

\[
P(f_{t+1} = s_j|f_t = s_i) = P(a_f + b_f \ell + \kappa_f(f_t - a_f) + \epsilon_{f,t+1} \in (c_j, c_{j+1}]|f_t = s_i)
\]

\[
= P(\epsilon_{f,t+1} \in (c_j - [a_f + b_f \ell + \kappa_f(s_i - a_f)], c_{j+1} - [a_f + b_f \ell + \kappa_f(s_i - a_f)]),
\]

which is the integral of the density of \( \epsilon_{f,t+1} \) over the indicated interval and is therefore continuous in the endpoints and in \( \ell \). It follows that \( E[V_B(f_{t+1}|f_t = f)] \) is continuous in \( \lambda(f) \), and therefore that the mapping (35)–(39) is continuous in \( \lambda(f) \), including, in particular, \( q_D^U(f) \) and \( q_D^L(f) \).

Next we turn to (40) and verify that \( c(f) \) is continuous at \( \lambda(f) = 0 \). Using (51), we can write, for \( \lambda(f) > 0 \),

\[
c(f) = \frac{\gamma}{R} q_D^L(f) \left( \frac{\lambda(1 + d)^2 \sigma_M^2 f \phi + q_D^L(f) \gamma^2 \sigma_X^2 (1 + d)^2 \sigma_M^2 f \phi + (q_D^L(f))^2 \gamma^2 \sigma_X^2}{\lambda^2 (1 + d)^2 \sigma_M^2 f \phi + \lambda q_D^U(f) \gamma^2 \sigma_X^2 (1 + d)^2 \sigma_M^2 f \phi + (q_D^U(f))^2 \gamma^2 \sigma_X^2} \right).
\]

As \( \lambda(f) \to 0 \), we have \( R^2(f) \to 0 \) and

\[
c(f) \to \frac{\gamma}{R} ((1 + d)^2 \sigma_M^2 f \phi + q_D^L(f)) = \frac{\gamma}{R} q_D^U(f),
\]

48
which coincides with the value specified for \( c(f) \) in (40) at \( \lambda(f) = 0 \).

B.3 Proof of Proposition 3.4 (Existence of Information Equilibrium)

To prove the result, we apply Kakutani’s fixed point theorem, which states the following (see, for example, p.72 of Border 1989). Let the domain \( S \) be a non-empty, compact and convex set, and let \( F : S \mapsto 2^S \) be a set-valued function on \( S \). Suppose \( F(x) \) is non-empty and convex for all \( x \in S \), and suppose that \( F \) has a closed graph (as defined shortly). Then \( F \) has a fixed point, meaning a point \( x \in S \) for which \( x \in F(x) \).

Let \( G(b, c, \lambda) \) be the mapping that sends initial coefficients \((b, c)\) to updated coefficients \((b', c')\) using \( \lambda \) through (35)–(40). Let \( F(b, c, \lambda) = (b', c', \Lambda(b', c', \lambda)) = (G(b, c, \lambda), \Lambda(G(b, c, \lambda))) \) be the mapping that returns the updated \((b', c')\) and the set of \( \lambda \)s in \( \Lambda(b', c') \). This representation is consistent with our algorithm: we first update \((b, c)\) given \( \lambda \); we then solve for the endogenous \( \lambda \) given the new \((b', c')\). The mapping \( F \) returns all of \( \Lambda(b', c') \), rather than a single element.

Based on the proof of Proposition 3.2, we can restrict \((b, c)\) to a domain \([0, \bar{b}] \times [0, \bar{c}]\), in the sense that \( G(\cdot, \cdot, \lambda) \) maps this set into itself for any \( \lambda \). We may therefore take the domain of \( F \) to be \( S = [0, \bar{b}]^n \times [0, \bar{c}]^n \times [0, 1]^n \), which is compact and convex. Moreover, for any \((b, c, \lambda) \in S\), \( F(b, c, \lambda) \) is non-empty (by Propositions 3.1 and 3.3), and it is convex by the definition of \( \Lambda(b, c) \).

It only remains to show that \( F \) has a closed graph. The closed graph property states that for any sequences \( x_n \to x \), \( y_n \to y \) with \( y_n \in F(x_n) \) we have \( y \in F(x) \). Because \( G \) is single-valued and continuous (as in the proof of Proposition 3.2), it suffices to show the following:

if \( \lambda' \notin \Lambda(b', c') \), then \( \lambda \notin \Lambda(b, c) \) for all \((b, c, \lambda)\) in a neighborhood of \((b', c', \lambda')\). \hspace{1cm} (65)

We detail the case of model (4). Recall from the discussion surrounding (63) that when we solve for a \( \lambda \in \Lambda_o(b, c) \), we may solve for each \( \lambda(f), f \in \mathcal{D} \), separately; the conditions on \( \lambda(f) \) for different values of \( f \) do not interact. For each \( f \), we look for a point at which

\[
\Delta_f(\ell) \equiv \Delta_{b,c,f}(\ell) = E[J^f_t - R_c | f_t = f] - E[J^U_t | f_t = f]
\]

crosses zero and set \( \lambda(f) = \ell \); if zero is never crossed, we get a boundary case of \( \lambda(f) = 0 \) or 1.

We have written \( \Delta_{b,c,f} \) to emphasize that the utilities on the right are evaluated using \((b, c)\).
We know from Section 3.3 that $\Delta_{b,c,f}(\ell)$ is continuous in \( \ell \) for each \( f \), and continuity in \((b,c)\) follows similarly from (63). If $\Delta_{b,c,f}(\cdot)$ crosses zero, then we may define the first and last zero crossings by

$$
\ell_{\text{min}}(f) = \min\{\ell \in [0,1] : \Delta_{b,c,f}(\ell) = 0\}, \quad \ell_{\text{max}}(f) = \max\{\ell \in [0,1] : \Delta_{b,c,f}(\ell) = 0\};
$$

otherwise, set $\ell_{\text{min}}(f) = \ell_{\text{max}}(f) = 0$ if $\Delta_{b,c,f}(\ell) < 0$ for all $\ell$, and $\ell_{\text{min}}(f) = \ell_{\text{max}}(f) = 1$ if $\Delta_{b,c,f}(\ell) > 0$ for all $\ell$.

Returning to (65), it now follows that if $\lambda' \notin \Lambda(b',c')$ then it must be that $\lambda'(f) \notin [\ell_{\text{min}}(f), \ell_{\text{max}}(f)]$ for some $f$, again because the constraints on each $\lambda'(f)$ depend only on that $f$. In particular, then, it must be that $\Delta_{b',c',f}(\ell) \neq 0$ for all $\ell \in [0,\lambda'(f)]$ or for all $\ell \in [\lambda'(f),1]$; it suffices to consider the first case because a symmetric argument works for the second case. Suppose $\Delta_{b',c',f}(\lambda'(f)) < 0$; a symmetric argument applies if $\Delta_{b',c',f}(\lambda'(f)) > 0$. Then

$$
\Delta_{b',c',f}(\ell) < 0 \text{ for all } \ell \in [0,\lambda'(f)]. \tag{66}
$$

We claim that this holds in a neighborhood of \((b',c',\lambda')\). To argue by contradiction, suppose not. In other words, suppose that in any $\epsilon$ neighborhood of \((b',c',\lambda')\) we can find a point \((b_{\epsilon},c_{\epsilon},\lambda_{\epsilon})\) and an $\ell_{\epsilon} \in [0,\lambda_{\epsilon}(f)]$ with $\Delta_{b_{\epsilon},c_{\epsilon},f}(\ell_{\epsilon}) = 0$. Taking a sequence of $\epsilon$ decreasing to zero, gives us a sequence of such \((b_{\epsilon},c_{\epsilon},\lambda_{\epsilon})\) and $\ell_{\epsilon}$, with \((b_{\epsilon},c_{\epsilon},\lambda_{\epsilon}) \to (b',c',\lambda')\). As the $\ell_{\epsilon}$ take values in the compact set $[0,1]$, they have a convergent subsequence. So, by taking a subsequence $\epsilon'$ of the original $\epsilon$ values, we get, for some $\ell_0$, \((b_{\epsilon'},c_{\epsilon'},\lambda_{\epsilon'},\ell_{\epsilon'}) \to (b',c',\lambda',\ell_0)\). And since $\ell_{\epsilon'} \leq \lambda_{\epsilon'}(f)$ for all $\epsilon'$, we have $\ell_0 \leq \lambda'(f)$. By the continuity of $\Delta$,

$$
\Delta_{b',c',f}(\ell_0) = \lim_{\epsilon' \to 0} \Delta_{b_{\epsilon'},c_{\epsilon'},f}(\ell_{\epsilon'}) = \lim_{\epsilon' \to 0} 0 = 0,
$$

which contradicts (66). We have thus shown that (66) holds in a neighborhood of \((b',c',\lambda')\). But then $\lambda \notin \Lambda(b,c)$, for all \((b,c,\lambda)\) in a neighborhood of \((b',c',\lambda')\), which is what we needed to show to prove the closed graph property.

## C Numerical Implementation

To solve the model numerically, we take the state space $\mathcal{D}$ for $f_t$ (or $\phi_t$) to be the grid \(\{0,1/(n-1),2/(n-1),\ldots,1\}\), with $n = 101$. Functions of $f$ or $\phi$ are then $n$-dimensional vectors.
Market equilibrium. For this step, we need to find a self-consistent variance belief, meaning the fixed point in Proposition 3.2. We start from a flat variance belief $V_B$ with a small and constant value on $D$ — smaller than the smaller of the two roots in footnote 5. We then iteratively apply equations (35)–(40) until the difference between consecutive variance beliefs becomes small throughout the state space. We have not proved convergence of this iterative procedure, but in all of our experiments we have found numerically that the method converges very quickly, in roughly seven iterations, provided the initial variance belief is small.

In each iteration, we need to evaluate the conditional expectation of $V_B$ over $f_{t+1}$ given $f_t$ (or $\phi_{t+1}$ given $\phi_t$) in (35). To reduce the impact of our discretization of the state space and approximate the results we would obtain with larger $n$, we use linear interpolation. In more detail, with $\lambda(\cdot)$ given, we approximate the evolution of $f_t$ (or $\phi_t$) through an $n$-state Markov chain with a transition matrix $P$, where $P_{ij}$ is the probability of transitioning from state $f_i \in D$ to state $f_j \in D$. With $f_t = f_i$ on the grid $D$, a shock $\epsilon_{f,t+1}$ may map $f_{t+1}$ off the grid without the projection $\Pi_D$. Rather than round to the nearest grid point, we interpolate to reduce discretization error. If a shock $\epsilon_{f,t+1}$ puts probability $p$ on a point $f_{t+1} = \alpha f_i + (1-\alpha)f_{j+1}$, $0 \leq \alpha \leq 1$, between two grid points, we assign probability $\alpha p$ to $P_{ij}$ and $(1-\alpha)p$ to $P_{i,j+1}$. We use these interpolated probabilities in calculating the conditional expectation $E[V_B(f_{t+1},\phi)|f_t = f_i]$ and similar expressions. The steady-state distribution of $f_t$ in Figure 4 is calculated from the transition matrix $P$. We also use linear interpolation in plotting functions of $f$ or $\phi$ in Figures 3–7 and Figures 9–11.

Prices. In calculating self-consistent variance beliefs, we get the price coefficients $b(\cdot)$ and $c(\cdot)$, and the coefficients $d$ and $g$ are constants. It only remains to find $a(\cdot)$, for which we use (42) and the matrix $P$ of transition probabilities. The expectation of the $m$-step ahead vector $\mu_{\pi,t} \in \mathbb{R}^n$ satisfies $\mu_{\pi,t+m} = P^m \mu_{\pi,t}$. The series in (42) can therefore be written in vector form as

$$
\frac{1}{R} \sum_{i=0}^{\infty} \frac{1}{R^i} E_t \mu_{\pi,t+i} = \frac{1}{R} \left( I + \frac{1}{R} P + \frac{1}{R^2} P^2 + \cdots \right) \mu_{\pi,t} \\
= \frac{1}{R} \left( I - \frac{1}{R} P \right)^{-1} \mu_{\pi,t}.
$$

(67)

Endogenous $\lambda$. For each $f$ (or $\phi$) we search numerically for a point $\lambda(f)$ (or $\lambda(\phi)$) at which the difference of expected utilities in (63) is zero, using linear interpolation between grid points.

---

28This mechanism differs slightly from that used in Proposition 3.3. Assuming the shocks $\epsilon_{f,t}$ have a density simplifies the continuity argument needed there, but for numerical calculations it is simpler to assume they have finite support.
That difference does not depend on \(a(\cdot)\), so we can find \(\lambda(\cdot)\) without calculating \(a(\cdot)\). If the difference in (63) is always positive, we set \(\lambda(f, \phi) = 1\); if it is always negative, we set \(\lambda(f, \phi) = 0\).

**Information equilibrium.** For a complete model solution we proceed as follows. We start with a small flat variance belief \(V_B\) and an arbitrary \(\lambda\); e.g., \(\lambda \equiv 0\). We do one update of \(V_B\) using (35)–(40) and then calculate the endogenous \(\lambda\). We repeat these updates iteratively, each time updating \(V_B\) and then solving for the new \(\lambda\). Once \(V_B\) has converged, we evaluate \(a(\cdot)\).

### C.1 Approximating Return Moments

Returns are given by \(r_{t+1} = (P_{t+1} + x_D D_{t+1})/P_t - 1\), where \(x_D \in \{0, 1\}\) indicates whether we want total (with dividends) or net returns. Because prices can become negative, we cannot calculate return moments in the usual sense. We instead normalize by \(P_0 = a(f) + d\bar{D}\) and calculate returns moments conditional on \(I = \{D_t = \bar{D}, f_t = f, \phi_t = \phi, m_t = \theta_t = X_t = 0\}\); \(P_0\) is the mean price given \(I\). The corresponding expected excess return for a given level of \(f\) or \(\phi\) is given by \(12 \times \mu_\pi / P_0\).

Using (10), we make the approximation

\[
r_{t+1} \approx \frac{a_{t+1} + b_{t+1}m_{t+1} + g\theta_{t+1} - c_{t+1}X_{t+1} + (x_D + d)(\bar{D} + \epsilon_{t+1})}{a(f, \phi) + d\bar{D}} - 1,
\]

where all quantities are conditioned on \(I\). The variability in the numerator (we refer to it as \(\nu\)) above will be driven by shocks \(m_{t+1}, \theta_{t+1}, X_{t+1}\) and \(\epsilon_{t+1}\), but also by changes in the coefficients \(a_{t+1}, b_{t+1}, c_{t+1}\) which are functions of \(f_{t+1}\) and \(\phi_{t+1}\). We will use the relationship that

\[
\text{var}(\nu) = \mathbb{E}[\text{var}(\nu|f_{t+1}, \phi_{t+1})] + \text{var}(\mathbb{E}[\nu|f_{t+1}, \phi_{t+1}]). \tag{68}
\]

The moments in (68) are conditional on \(I\), but we omit \(I\) to avoid clutter. We see that

\[
\text{var}(\nu|f', \phi') = b(f', \phi')^2 \phi' f' \sigma_M^2 + c(f', \phi')^2 \sigma_X^2 + g^2(1 - \phi') f' \sigma_M^2 + (x_D + d)^2(1 - f') \sigma_M^2.
\]

The first expectation in (68) is

\[
\sum_{f', \phi'} P(f_{t+1} = f', \phi_{t+1} = \phi'|f_t = f, \phi_t = \phi) \times \text{var}(\nu|f', \phi'),
\]

using the transition probabilities of \(f_t\) or \(\phi_t\).
We note that \( E[\nu|f', \phi'] = a(f', \phi') + (1 + d)\bar{D} \) and therefore

\[
\text{var}(E[\nu|f', \phi']) = \left\{ \sum_{f', \phi'} P(f_{t+1} = f', \phi_{t+1} = \phi'|f_t = f, \phi_t = \phi) \times a(f', \phi')^2 \right\} - \bar{a}^2,
\]

where \( \bar{a} = \sum_{f', \phi'} P(f_{t+1} = f', \phi_{t+1} = \phi'|f_t = f, \phi_t = \phi) \times a(f', \phi') \).

The return volatility at every \((f, \phi)\), conditional on \(I\) is given by

\[
\text{vol}(r_{t+1}) \approx \sqrt{12} \times \frac{\text{sd}(\nu)}{a(f, \phi) + d\bar{D}}. \tag{69}
\]

The return moments derived in this section entail two approximations required by the possibility of negative prices: we evaluate moments conditional on \(I\) and we normalize by the conditional mean price in calculating returns. We have verified via simulation (where negative prices are very rare) that the conditional return volatility above is close to the realized return volatility from simulated data.

### D Calibration Details

#### D.1 Dynamics of S&P 500 Dividends

We interpret \( D_t \) as an unobserved dividend process that would obtain if all S&P 500 companies paid a monthly dividend in each month. This \( D_t \) does not correspond to the actual, observed monthly S&P 500 dividend which represents the quarterly dividend payments of only a subset of the S&P 500 companies. To be consistent with \([1]\), we model \( D_t \) as an AR(1) process where

\[
D_{t+1} = \mu_D + \rho D_t + M_{t+1},
\]

\[
D_{t+2} = \mu_D + \rho \mu_D + \rho^2 D_t + \rho M_{t+1} + M_{t+2},
\]

\[
D_{t+3} = \mu_D + \rho \mu_D + \rho^2 \mu_D + \rho^3 D_t + \rho^2 M_{t+1} + \rho M_{t+2} + M_{t+3}.
\]

If we interpret a time period as a single month then the quarterly dividend \( Q_i = D_{t+1} + D_{t+2} + D_{t+3} \) can be written (expressing each \( D \) as in the last equation above) as an ARMA(1,1) process:

\[
Q_{i+1} = \mu_Q + \rho_Q Q_i + u_{i+1} + \theta u_i. \tag{D.1}
\]
where $E[u_{i+1}u_i] = 0$ and

$$
\begin{align*}
\mu_Q &= 3\mu_D(1 + \rho + \rho^2) \\
\rho_Q &= \rho^3 \tag{D.2} \\
u_{i+1} + \theta u_i &= \eta_{t+3} + \eta_{t+2} + \eta_{t+1} \\
&= \rho^2 M_{t+1} + \rho M_{t+2} + M_{t+3} + \rho^2 M_t + \rho M_{t+1} + M_{t+2} + \rho^2 M_{t-1} + \rho M_t + M_{t+1}.
\end{align*}
$$

The parameters $\theta$ and $\sigma_u^2$ are chosen to match the variance and autocorrelation of the error term $\eta_{t+3} + \eta_{t+2} + \eta_{t+1}$. The variance of the error term yields the restriction that

$$
\begin{align*}
\text{var}(u_{i+1} + \theta u_i) &= \sigma_u^2 + \theta^2 \sigma_u^2 \\
&= \sigma_M^2 \left[ 1 + (1 + \rho)^2 + (1 + \rho + \rho^2)^2 + (\rho + \rho^2)^2 + \rho^4 \right] \\
&= \sigma_M^2 [3 + 4 \rho + 5 \rho^2 + 4 \rho^3 + 3 \rho^4]. \tag{D.3}
\end{align*}
$$

Note that the quarter $i$ dividend contains innovations from months $t-4, t-3, t-2, t-1, t$, and therefore the correlation of the error terms from quarter $i+1$ and $i$ yields the restriction that

$$
\begin{align*}
\text{cov}(u_{i+1} + \theta u_i, u_i + \theta u_{i-1}) &= \theta \sigma_u^2 \\
&= (\rho + \rho^2) \sigma_M^2 + \rho_2 (1 + \rho) \sigma_M^2 = \sigma_M^2 (3 + 2 \rho^2 + \rho). \tag{D.4}
\end{align*}
$$

We use (D.2) and (D.3) to pin down $\sigma_M$. Equation (D.4) is then an overidentifying restriction on the model parameters, which together with (D.3) implies

$$
\theta \left(1 + \theta^2\right) = \frac{\rho + 2 \rho_2 + \rho^3}{3 + 4 \rho + 5 \rho^2 + 4 \rho^3 + 3 \rho^4} \tag{D.5}
$$

### D.1.1 Estimating AR(1) Parameters

Estimating (D.1) yields four parameter estimates: $\hat{\mu}_Q, \hat{\rho}_Q, \hat{\theta}, \hat{\sigma}_u^2$. Our monthly dividend process has parameters $\mu_D, \rho, \sigma_M^2$. From (D.2) and (D.3) we see that

$$
\begin{align*}
\hat{\rho} &= \hat{\rho}_Q^{1/3}, \\
\hat{\sigma}_M^2 &= \frac{\hat{\sigma}_u^2 + \hat{\theta}^2 \hat{\sigma}_u^2}{3 + 4 \hat{\rho} + 5 \hat{\rho}^2 + 4 \hat{\rho}^3 + 3 \hat{\rho}^4}.
\end{align*}
$$

\footnote{We use the \texttt{arima} function in R.}
Also the long-run level of the quarterly dividend is related to the long-run level of the monthly dividend via
\[
\bar{Q} = \frac{\mu_Q}{1 - \rho_Q} = 3\frac{\mu_D(1 + \rho + \rho^2)}{1 - \rho^3} = 3\frac{\mu_D}{1 - \rho},
\]
where the last step follows from \((1 + \rho + \rho^2)(1 - \rho) = 1 - \rho^3\). Therefore the long-run level of the monthly dividend is equal to \(\bar{D} = \bar{Q}/3\). Since we are interested in a monthly dividend process with a mean of 1, we set \(\mu_D = 1 - \hat{\rho}\) and normalize the innovation volatility by \(\hat{\sigma}_M/\hat{\bar{D}}\).

Our system of equations (involving \(\mu_Q, \rho_Q, \sigma_u^2, \theta\)) for \(\mu_D, \rho, \sigma_D^2\) is overidentified. In fact, (D.5) places an additional restriction on \(\hat{\rho}\) coming from \(\hat{\theta}\). In our estimates (below) we find \(\hat{\theta} < 0\) suggesting the model is misspecified.

To estimate the model we use an exponentially detrended real (deflated using the ex-food and energy seasonally adjusted PCE) quarterly dividend from Jan 1998 (when our daily S&P 500 dividend series starts) until Mar 2018. We also use a seasonally adjusted series that involves regressing quarterly dummies out of the log dividend. Estimates from these two models are given in Table 3. Figure 12 shows the S&P 500 dividend series used in this estimation.
D.2 Dynamics of Turnover

Panel A

Index turnover for SML 6/8/2018

Firm level monthly turnover for SML 6/8/2018

Panel B

Index turnover for DOW 5/25/2018

Firm level monthly turnover for DOW 5/25/2018

Figure 13: Turnover, defined as share trading volume divided by the number of shares outstanding, for the Dow Industrial and the S&P 600 SmallCap (SML) indexes.
References


