

NBER WORKING PAPER SERIES

THE ENDOWMENT MODEL AND MODERN PORTFOLIO THEORY

Stephen G. Dimmock
Neng Wang
Jinqiang Yang

Working Paper 25559
<http://www.nber.org/papers/w25559>

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
February 2019

For helpful comments, we thank Andrew Ang, Winston Dou, Thomas Gilbert, Steve Grenadier, Lars Peter Hansen, Bob Hodrick, Steve Kaplan, Monika Piazzesi, Jim Poterba, Mark Schroder, Tom Sargent, Stijn Van Nieuwerburgh, Luis Viceira, and Steve Zeldes; seminar participants at Columbia University; and participants at the European Finance Association and NBER New Developments in Long-Term Asset Management conferences. We thank Matt Hamill and Ken Redd of NACUBO and John Griswold and Bill Jarvis of Commonfund for assistance with data. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

NBER working papers are circulated for discussion and comment purposes. They have not been peer-reviewed or been subject to the review by the NBER Board of Directors that accompanies official NBER publications.

© 2019 by Stephen G. Dimmock, Neng Wang, and Jinqiang Yang. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

The Endowment Model and Modern Portfolio Theory
Stephen G. Dimmock, Neng Wang, and Jinqiang Yang
NBER Working Paper No. 25559
February 2019
JEL No. G11,G23

ABSTRACT

We develop a dynamic portfolio-choice model with illiquid alternative assets to analyze conditions under which the “Endowment Model,” used by some large institutional investors such as university endowments, does or does not work. The alternative asset has a lock-up, but can be voluntarily liquidated at any time at a cost. Quantitatively, our model's results match the average level and cross-sectional variation of university endowment funds' spending and asset allocation decisions. We show that asset allocations and spending crucially depend on the alternative asset's expected excess return, risk unspanned by public equity, and investors' preferences for inter-temporal spending smoothing.

Stephen G. Dimmock
Division of Finance and Banking
Nanyang Technological University
Singapore
dimmock@ntu.edu.sg

Neng Wang
Columbia Business School
3022 Broadway, Uris Hall 812
New York, NY 10027
and NBER
nw2128@columbia.edu

Jinqiang Yang
Shanghai University of Finance
and Economics
Guoding Rd. 777
Shanghai, 200433
China
yang.jinqiang@mail.sufe.edu.cn

1. Introduction

University endowment funds allocate large fractions of their portfolios to illiquid alternative assets. Indeed, more than half of aggregate university endowment fund assets are currently allocated to illiquid assets such as hedge funds, private equity, venture capital, and natural resources. An investment strategy of high allocations to illiquid assets is often referred to as the Endowment Model, as it was popularized by the success of David Swensen of the Yale University endowment fund and other universities (Lerner, Schoar, and Wang (2008)).¹ The Endowment Model (e.g., Swensen, 2000) argues that long-term investors with access to valuable illiquid investment opportunities should have relatively high allocations to alternative assets, so as to earn illiquidity premiums and exploit the inefficiencies found in illiquid markets.

The Endowment Model, however, has its limitations. As Swensen (2005) himself notes, only sophisticated investors who have access and can identify superior active managers should invest in alternative assets. Further, illiquidity can be costly. For example, during the financial crisis the endowments of Harvard and Stanford Universities incurred significant losses due to their large allocations to illiquid assets.² That is, illiquidity can be particularly costly in bad times when financial reserves are most needed. Therefore, investors need to properly balance the costs and benefits of investing in illiquid assets. Although the Endowment Model provides general guidelines for investing in illiquid assets, these guidelines are based on static mean-variance modeling (Takahashi and Alexander (2002)) and rules of thumb. This is not surprising, as the Endowment Model is developed from practitioners' lore and not explicitly derived from a dynamic optimizing framework.

In this paper, we bridge the gap between the practitioners' Endowment Model and academic portfolio theory by developing a parsimonious yet sufficiently rich model to analyze a long-term investor's dynamic consumption and asset allocation decisions. The

¹See Swensen (2000), Takahashi and Alexander (2002), and Leibowitz, Bova, and Hammond (2010). Lerner, Schoar, and Wang (2008, figure 3) show that allocations to alternative assets increased sharply for selective institutions.

²For further discussion of the effects of endowment illiquidity on Harvard University see Ang (2012) and for Stanford see <http://www.nytimes.com/2009/10/06/business/06sorkin.html>.

key is to incorporate illiquid (alternative) investment opportunities into an otherwise standard Modern portfolio theory (MPT) framework developed by Markowitz (1952) and Merton (1969, 1971) where all assets are liquid.³ We use this dynamic portfolio choice model to illustrate the heterogeneity of investment strategies followed by investors with different preferences, investment horizons, and investment opportunities. We show the conditions under which portfolio allocations similar to those recommended by the Endowment Model can be optimal for investors, as well as the conditions under which the Endowment Model does not serve investors well.

Our model incorporates a key feature of alternative assets, illiquidity, in a manner that is sufficiently realistic and also analytically tractable. First, the alternative asset has automatic liquidity events on dates when the alternative asset (or a fraction of it) becomes fully liquid. For example, when a hedge fund lock-up expires or a private equity fund liquidates. This is an important feature of our model, as advocates of the Endowment Model argue that natural liquidity, such as private equity cash distributions, offsets much of the apparent illiquidity of alternative assets (see Swensen (2000) and Takahashi and Alexander (2002)). Our model show how an investor can engage in *liquidity diversification* by investing in multiple alternative asset funds with automatic liquidity events staggered over time.⁴ Second, the investor can voluntarily transact in the illiquid alternative asset at any time by paying a proportional transaction cost. For example, this cost would include the empirically observed discounts for alternative assets sold in secondary markets due to search and adverse selection costs. Third, the alternative asset's risk is not fully spanned by public equity, and this unspanned risk together with the illiquidity has a first-order effect on asset allocation.

Our qualitative and quantitative results significantly differ from the standard predictions implied by MPT. We show that $w_t = W_t/K_t$, the ratio between the value of liquid wealth W_t (e.g., stocks and bonds) and the value of the illiquid alternative asset holding

³Many research papers in the portfolio-choice literature analyze the impact of illiquidity on asset allocation, which we fully acknowledge. The key contribution of our model is to link to the widely-used Endowment Model and take a balanced perspective on when it is sensible to follow the model's recommendation, and to examine how to improve upon the Endowment Model.

⁴For an empirical examination of liquidity diversification, see Robinson and Sensoy (2016).

K_t , follows a double-barrier policy: the investor rebalances only when the value of the alternative asset rises or falls to the endogenous rebalancing boundaries.⁵ This result is in sharp contrast to the classic MPT prediction that the allocation ratio between any two assets is constant over time.

We provide an analytical characterization for the investor’s certainty equivalent wealth under optimality, $P(W_t, K_t, t)$, which is the time- t total wealth that makes the investor indifferent between: permanently forgoing the opportunity to invest in the illiquid asset and keeping the status quo with liquid wealth W_t and illiquid wealth K_t with the opportunity to invest in the illiquid asset. We exploit the model’s tractability to provide a quantitative yet intuitive analysis of a long-term investor’s optimal portfolio choice, spending rule, and welfare measured by $P(W_t, K_t, t)$.

We show that the two types of illiquidity – arising from lock-ups and from transaction costs – interact over time. As an automatic liquidity event approaches, the investor becomes less willing to liquidate alternative assets. The rebalancing policies are also strongly affected by liquidity diversification; investors who invest in more alternative asset funds and stagger the maturities of their investments can maintain much more stable portfolio allocations.

We calibrate our model using parameters drawn from the literature and compare the results with the portfolio allocations of actual university endowment funds. We find three main results.

First, our model matches the empirically observed variation in endowment funds’ allocations to alternative assets given reasonable variation in beliefs about its alpha. The model defines alpha relative to the benchmark of public equity, and thus “alpha” may include compensation for illiquidity, managerial skill, and the value created through improved corporate governance and incentive structures.⁶ With an alpha of 3% per year and

⁵The double-barrier policy is a standard feature in models with transaction costs. See Davis and Norman (1990) as an early example in the portfolio-choice literature.

⁶Franzoni, Nowak, and Phalippou (2012) show that private equity funds earn liquidity premia; Aragon (2007) and Sadka (2010) show similar findings for hedge funds. Cornelli and Karakas (2008) argue that private equity funds create value-add (i.e., alpha) by improving corporate governance and operating efficiency. See Kaplan and Strömberg (2009), Metrick and Yasuda (2011) and Agarwal, Mullally, and Naik (2015) for review studies of alternative asset performance.

unspanned volatility of 15% our model implies an alternative asset allocation of 60.2%, which is consistent with the actual allocations of the largest decile of endowment funds. With expected alphas of 1% and 2%, our model implies alternative asset allocations of 12.7% and 34.5%, respectively, which approximately match the interquartile range of endowment fund allocations.

Second, our calibration shows that asset allocations are very sensitive to the unspanned volatility of the alternative asset. For example, as the unspanned volatility changes from 10% to 15%, the implied allocation to alternative assets falls by more than half from 76.3% to 34.5%. We further show that, controlling for the level of the alternative asset's total risk, the spanned and unspanned risks have quantitatively very different effects on asset allocation. While the investor can offset the alternative asset's spanned risk of the by adjusting allocations to public equity, the alternative asset's unspanned volatility is by definition specific to the alternatives and cannot be hedged. Alternative asset performance metrics such as internal rates of returns (IRRs) and public market equivalent (PMEs), while useful, do not directly guide investors' asset allocation as these metrics ignore the distinction between spanned and unspanned volatilities.

Third, investors' preferences for inter-temporal spending smoothing have a first-order effect on their investment strategies due to the presence of illiquid alternative assets. We use non-expected utility preferences developed by Epstein and Zin (1989), which separates risk aversion from the elasticity of intertemporal substitution (EIS). This separation is economically and quantitatively important for our analysis as by varying the EIS we conveniently capture the heterogeneity in spending flexibility across institutional investors.⁷ For example, defined benefit pension plans have little spending flexibility and so have a low EIS. In contrast, family offices have high spending flexibility and university endowments have intermediate flexibility.

The EIS has large effects on the spending rate: changing the EIS from 0.5 to 1 decreases the spending rate from 5.3% to 4.0%. This closely matches the actual variation in university endowment fund spending rates (e.g., Sedlacek and Jarvis (2010) report that

⁷See Hayes, Primbs, and Chiquoine (2015) for a discussion of spending flexibility across investors.

policy payout rates typically range from 4.0% to 5.5%). The EIS also affects portfolio allocations when the alternative asset is illiquid. An investor with a high EIS is more willing to substitute consumption across periods, and so is more willing to accept portfolio illiquidity. For example, changing the EIS from 0.5 to 1 implies an increase in alternative asset holdings of 5.7 percentage points. This contrasts with the case of full spanning, in which the EIS does not affect asset allocation. Thus, a higher EIS results in lower spending and higher portfolio risk. Over time, this combination results in substantially greater wealth accumulation for high EIS investors.

We focus on the dynamic asset allocation problem faced by an institutional investor's Chief Investment Officer (CIO) who is assumed to behave in the interest of the investor, and leave the delegated management problem (between the CIO and asset managers who implement security selection) outside of our model for simplicity. However, Sharpe (1981), Elton and Gruber (2004), and Binsbergen, Brandt, and Koijen (2008) emphasize that this two-step delegated investment process can result in misaligned incentives. Binsbergen, Brandt, and Koijen (2008) show that these incentive issues between the CIO and external asset managers can result in large utility costs for institutional investors due to loss of incongruence differences in the preferences of the institution and the asset manager.

Related Literature. Ang, Papanikolaou, and Westerfield (2016) develop a parsimonious and tractable quantitative model of portfolio choice with an illiquid asset. They assume that the alternative asset cannot be traded for intervals of uncertain duration (modeled via a stochastic Poisson arrival process). In their model the asset can be fully illiquid for extended periods of time due to the long tails associated with exponential distributions. In contrast, our model has a secondary market that allows liquidation at all times by paying a proportional transaction cost.

Sorensen, Wang, and Yang (2014) use a dynamic portfolio choice model to value the cost of illiquidity and management compensation in private equity. For technical tractability they use constant absolute risk aversion (CARA) utility and hence their model generates a wealth-independent dollar allocation to private equity and hence is

not suited for analyzing asset allocation (which in practice is almost always framed in terms of percentage allocations). Also, the two papers model illiquidity differently. They assume that the investor must hold the alternative asset for a fixed T years without any option to exit and does not include a secondary market for the alternative asset.

Bollen and Sensoy (2016) develop a model for valuing illiquid private equity when secondary markets exist. They assume that investors must place their entire private equity allocation in a single fund, and they exogenously fix the spending rate and the allocation to the risk-free asset rather than allowing investors to make optimal portfolio rebalancing decisions. In contrast, our model allows for liquidity diversification as investors can stagger illiquid investments over time, and solves for optimal allocations to the risk free asset, the spending rate, and rebalancing decisions at all times.

In the portfolio-choice literature, illiquidity is often modeled via transaction costs⁸ or via trading restrictions in which the asset is freely tradable at certain points in time but cannot be traded at other times.⁹ Motivated by the structures of private equity and hedge funds, as well as the secondary markets for these illiquid alternatives, we combine the features of both types of models; the alternative asset becomes fully liquid at maturity (e.g., when a private equity fund is dissolved) but can be liquidated prior to maturity by paying a proportional cost (e.g., selling a private equity fund at a discount on the secondary market).

Merton (1992), Gilbert and Hrdlicka (2015), and Cejnek, Franz, and Stoughton (2017) model the spending and asset allocation decisions of university endowment funds. In these papers, endowments invest only in fully liquid assets.

2. Model

We analyze a long-term investor's dynamic spending (or equivalently consumption) and asset allocation decisions by incorporating an illiquid investment opportunity into the

⁸For example, see Constantinides (1986), Davis and Norman (1990), Grossman and Laroque (1990), Vayanos (1998), and Lo, Mamaysky, and Wang (2004).

⁹For example, see Kahl, Liu, and Longstaff (2003), Longstaff (2009), and Ang, Papanikolaou, and Westerfield (2016).

classic modern portfolio theory developed by Merton (1969, 1971) and Samuelson (1969). We interpret the illiquid investment opportunity in our model as the representative portfolio of *alternative assets* including private equity, hedge funds, private real estate, etc. For technical convenience, we develop our model in continuous time. Next, we summarize the standard liquid investment opportunities and then introduce the alternative asset.

Liquid Investment Opportunities: Bonds and Public Equity. The risk-free bond pays interest at a constant (annualized) risk-free rate r . Public equity can be interpreted as the market portfolio of publicly traded securities, and its cum-dividend market value, S_t , follows a geometric Brownian motion (GBM):

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S d\mathbb{B}_t^S, \quad (1)$$

where \mathbb{B}_t^S is a standard Brownian motion, and μ_S and σ_S are the constant drift and volatility parameters. The liquid investment opportunity in our model is the same as in Merton (1971). Next, we introduce the alternative asset, which is the investor's third investment opportunity and the key building block in our model.

2.1. The Alternative Asset

Adding the alternative asset expands the investment opportunity set and thus makes the investor better off. Additionally, provided the alternative asset is not perfectly correlated with public equity, it provides diversification benefits. Unlike public equity, however, alternative assets are generally illiquid and involve some form of lock-up. For example, investments in private equity typically have a life span of 10 years with extension options and hedge funds often have lock-up periods and gate provisions.

A key feature of alternative assets is that their illiquidity is not constant over time. For example, private equity funds are highly illiquid for much of their lives but eventually mature and return liquid capital to their investors. We model these liquidity events as follows. Let $\{A_t; t \geq 0\}$ denote the alternative asset's fundamental value process with a given initial stock A_0 . The fundamental value refers to the fully realizable value of the

asset if it is held to maturity. However, with illiquidity, at any time t prior to maturity the asset's fundamental value differs from its market value. Let $\{K_t; t \geq 0\}$ denote the accounting value of the alternative asset holding process with a given initial stock K_0 . To capture the target finite duration of the lock-up and holding period, we assume every mT years, where m is a positive integer, a δ_T fraction of the stock of illiquid alternative asset K_{mT} automatically becomes liquid at no cost. Naturally, the investor's liquid asset value at time mT increases by $\delta_T K_{mT-}$. Therefore, in the absence of any active acquisition or divestment of the illiquid asset at mT , we have $K_{mT} = (1 - \delta_T)K_{mT-}$.

The Fundamental Value Process A for the Alternative Asset. We assume that the fundamental value A , in the absence of a scheduled automatic liquidity event (at time mT) or any interim acquisition or divestment, evolves via the following GBM:

$$\frac{dA_t}{A_{t-}} = \mu_A dt + \sigma_A d\mathbb{B}_t^A - \delta_A dt, \quad (2)$$

where \mathbb{B}_t^A is a standard Brownian motion, μ_A is the cum-payout expected return (net of fees), σ_A is the constant volatility of returns, and δ_A is the alternative asset's payout rate. That is, the alternative asset pays dividends at the rate of $\delta_A A_t$ with an implied payout yield of δ_A . Intuitively, δ_A is one way for illiquid alternative assets to provide liquidity to investors. We use ρ to denote the correlation coefficient between the shocks to alternative assets, \mathbb{B}_t^A , and the shocks to public equity, \mathbb{B}_t^S .

Note that in complete markets, the investor can frictionlessly and dynamically trade the alternative asset without restrictions or costs. Therefore, the alternative asset's market value equals its fundamental value and the Modigliani-Miller theorem holds, meaning that whether we explicitly model the alternative asset's payout yield δ_A is irrelevant. In this ideal case, the alternative asset is conceptually no different than liquid public equity. In contrast, when the alternative asset is illiquid and not fully spanned by public equity, we must separately keep track of the payout yield δ_A and expected capital gains $\mu_A - \delta_A$. That is, the cum-dividend payout rate μ_A is no longer a sufficient measure of the total expected returns for the alternative asset as its (current) payout yield and expected

capital gains influence the investor's portfolio optimization problem differently.

Interim Acquisition and Liquidation of the Alternative Asset Holding. At any time, the investor can choose to change her alternative asset holdings through acquisitions or liquidations. Let dL_t denote the amount of the alternative asset that the investor liquidates at any time $t > 0$, and let dX_t denote the amount of the alternative asset that the investor purchases at time t . Then, we can incorporate the investor's acquisition and liquidation options into the alternative asset's fundamental value process as follows:

$$dK_t = (\mu_A - \delta_A)K_t dt + \sigma_A K_t d\mathbb{B}_t^A - dL_t + dX_t - \delta_T K_t \mathbb{I}_{\{t=mT\}}. \quad (3)$$

Here, $\mathbb{I}_{\{t=mT\}}$ is the indicator function, which is equal to one if and only if t is an integer multiple, m , of T . The first two terms correspond to the standard drift and volatility terms, the third and fourth terms give the liquidation and acquisition amounts, and the last term captures the lumpy payout to the investor at the scheduled liquidity event dates $t = mT$ where $m = 0, 1, \dots$

Although the acquisition and liquidation costs for the alternative asset do not appear in (3), they will appear in the liquid wealth accumulation process. We assume that the cost of voluntary liquidation is proportional. That is, by liquidating an amount $dL_t > 0$, the investor realizes only $(1 - \theta_L)dL_t$ in net, where the remaining amount $\theta_L dL_t$ is the liquidation cost. Similarly, if the investor acquires an amount $dX_t > 0$, the transaction cost $\theta_X dX_t$ is paid out of the liquid asset holding to the third party. Naturally, $0 \leq \theta_L \leq 1$ and $\theta_X \geq 0$. Higher values of θ_L or θ_X indicate that the alternative asset is less liquid.

Intuitively, θ_L can be interpreted as the illiquidity discount on secondary market sales of alternative assets (e.g., see Kleymenova, Tamor, and Vasvari (2012), Bollen and Sensoy (2016), and Nadauld, Sensoy, Vorkink, and Weisbach (2019)). Such discounts can arise to compensate buyers for search costs, asymmetric information risks, or due to market power when there are few buyers. The parameter θ_X can be interpreted as the transaction costs of purchasing alternative assets, such as search costs, legal fees, placement agent fees, consultant fees, and etc. The costs of interim liquidation (θ_L) and of purchases (θ_X)

can be asymmetric as voluntary liquidation is generally more costly, particularly when there are few buyers and many sellers such as during the recent financial crisis.

Alpha, Beta, and Epsilon (Unspanned Volatility). Suppose that the instantaneous return for the alternative asset, dA_t/A_{t-} , is perfectly measurable. We can then regress dA_t/A_{t-} on dS_t/S_t , and obtain the alternative asset's beta with respect to public equity, following the standard capital asset pricing model (CAPM) formula:

$$\beta_A = \frac{\rho\sigma_A}{\sigma_S}. \quad (4)$$

However, in reality, because investors cannot dynamically rebalance their holdings in the illiquid asset without incurring transaction costs, investors will demand compensation in addition to the standard risk premium implied by the CAPM.

We decompose the total volatility of the alternative asset, σ_A , into two orthogonal components: the part spanned by the public equity, $\rho\sigma_A$, and the remaining unspanned volatility, ϵ , given by:

$$\epsilon = \sqrt{\sigma_A^2 - \rho^2\sigma_A^2} = \sqrt{\sigma_A^2 - \beta_A^2\sigma_S^2}. \quad (5)$$

This volatility, ϵ , introduces an additional risk into the portfolio, as markets are incomplete and adjusting the alternative asset holding is costly. We will show that the spanned and unspanned volatilities play distinct roles in the dynamic asset allocation.¹⁰

Anticipating our subsequent risk-return tradeoff analysis in the context of dynamic portfolio construction, we next introduce α implied by the CAPM, where public equity is used as the aggregate market portfolio. That is, we define α as follows:

$$\alpha = \mu_A - (r + \beta_A(\mu_S - r)), \quad (6)$$

where β_A is the alternative asset's beta given by (4).

In frictionless capital markets where investors can continuously rebalance their portfolio without incurring any transaction costs, α measures the risk-adjusted excess return

¹⁰Although ϵ is unspanned by public equity, this does not necessarily imply it is purely idiosyncratic risk. We take no stand on whether public equity is the market portfolio in the sense of Roll (1977). Thus, ϵ may include systematic risks that are potentially compensated, but which are not correlated with the market portfolio (e.g., liquidity risk).

after benchmarking against the other risky assets, which in our model corresponds to public equity. However, in our framework with illiquid assets, α also includes the part of the excess return that compensates investors for bearing an illiquidity premium (or for bearing any other systematic risk that is unspanned by public equity).

2.2. The Optimization Problem

Liquid Wealth and Net Worth. We use W to denote the investor's liquid wealth and Π to denote the amount allocated to public equity. The remaining liquid wealth, $W - \Pi$, is allocated to the risk-free bond. Thus, liquid wealth evolves according to:

$$\begin{aligned} dW_t = & (rW_{t-} + \delta_A K_{t-} - C_{t-}) dt + \Pi_{t-} ((\mu_S - r)dt + \sigma_S d\mathbb{B}_t^S) \\ & + (1 - \theta_L)dL_t - (1 + \theta_X)dX_t + \delta_T K_{t-} \mathbb{I}_{\{t=mT\}}, \end{aligned} \quad (7)$$

where the first two terms in (7) are the standard ones in Merton's consumption/portfolio-choice problem. The third and fourth terms describe the effect on liquid wealth W due to the investor's interim liquidation and purchase of the alternative asset. Recall that θ_L and θ_X capture the proportional cost of interim liquidations and purchases of the alternative asset, respectively. Finally, the last term captures the lumpy payout to the investor at the automatic liquidity event dates $t = mT$.

Recursive Preferences and Value Functions. The investor's preferences allow for separation of risk aversion and the elasticity of intertemporal substitution (EIS). Epstein and Zin (1989) and Weil (1990) develop this utility in discrete time by building on early important work by Kreps and Porteus (1978). We use the continuous-time formulation of this non-expected utility, introduced by Duffie and Epstein (1992). That is, the investor has a recursive preference defined as follows:

$$V_t = \mathbb{E}_t \left[\int_t^\infty f(C_s, V_s) ds \right], \quad (8)$$

where $f(C, V)$ is known as the normalized aggregator for consumption C and the investor's utility V . Duffie and Epstein (1992) show that $f(C, V)$ for Epstein-Zin non-

expected homothetic recursive utility is given by:

$$f(C, V) = \frac{\zeta}{1 - \psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1-\gamma)V)^\chi}{((1-\gamma)V)^{\chi-1}}, \quad (9)$$

where

$$\chi = \frac{1 - \psi^{-1}}{1 - \gamma}. \quad (10)$$

The parameter $\psi > 0$ measures the EIS, and the parameter $\gamma > 0$ is the coefficient of relative risk aversion. The parameter $\zeta > 0$ is the investor's subjective discount rate.

This recursive, non-expected utility formulation allows us to separate the coefficient of relative risk aversion (γ) from the EIS (ψ), which is important for our quantitative analysis. For example, a key source of preferences heterogeneity among investors is the elasticity and flexibility of their spending. The expected CRRA utility is a special case of recursive utility where the coefficient of relative risk aversion, γ , equals the inverse of the EIS, $\gamma = \psi^{-1}$, implying $\chi = 1$.¹¹

There are three state variables for the optimization problem: liquid wealth W_t , the alternative asset's value K_t , and calendar time t . Let $V(W_t, K_t, t)$ denote the corresponding value function. The investor chooses consumption C , public equity investment Π , and the alternative asset's cumulative (undiscounted) liquidation L and cumulative (undiscounted) acquisition X to maximize (8).

Naturally, at each automatic liquidity event date iT , if $W_{iT} = W_{(i-1)T} = W$, and $K_{mT} = K_{(m-1)T} = K$, we must have:

$$V(W, K, mT) = V(W, K, (m-1)T). \quad (11)$$

Hence, it is sufficient for us to characterize our model over $(0, T]$, as the solution is stationary every T years.

3. Model Solution

We solve the model as follows. First, we analyze the investor's problem in the region where there is no voluntary adjustment of the alternative asset in the absence of automatic

¹¹For the special case of CRRA, $f(C, V) = U(C) - \zeta V$, where $U(C) = \zeta C^{1-\gamma}/(1-\gamma)$. By integrating Eq. (8), we obtain $V_t = \mathbb{E}_t \left[\int_t^\infty e^{-\zeta(s-t)} U(C_s) ds \right]$.

liquidity event (i.e., when $t \neq mT$.) Second, we characterize the investor's voluntary liquidation and acquisition decisions for the alternative asset when $t \neq mT$. Finally, we integrate the periodic liquidity event that occurs at $t = mT$ to complete our analysis.

Dynamic Programming and First-Order Conditions (FOCs). Fix time t within the time interval $((m-1)T, mT)$, where m is a positive integer. Using the standard dynamic programming approach, we have the following standard Hamilton-Jacobi-Bellman (HJB) equation for the investor's value function $V(W_t, K_t, t)$ in the interior region:

$$0 = \max_{C, \Pi} f(C, V) + (rW + \delta_A K + (\mu_S - r)\Pi - C)V_W + \frac{(\Pi\sigma_S)^2}{2}V_{WW} + V_t + (\mu_A - \delta_A)KV_K + \frac{\sigma_A^2 K^2}{2}V_{KK} + \rho\Pi K\sigma_S\sigma_A V_{WK}. \quad (12)$$

The first three terms on the right side of (12) capture the standard effects of consumption and asset allocation (both drift and volatility effects) on the investor's value function, $V(W_t, K_t, t)$ as in Merton (1971). The investor's opportunity to invest in the illiquid alternative asset generates three additional effects on asset allocation: 1) the effect of target holding horizon T captured by V_t ; 2) the risk-return and volatility effects of changes in the value of the alternative asset K ; and 3) the additional diversification/hedging benefits due to the correlation between public equity and the alternative asset. By optimally choosing C and Π , the investor equates the right side of (12) to zero in the interior region where there is no interim liquidation nor acquisition.

The optimal consumption C is characterized by the following standard FOC:

$$f_C(C, V) = V_W(W, K, t), \quad (13)$$

which equates the marginal benefit of consumption with the marginal value of savings V_W . The optimal investment in public equity is given by:

$$\Pi = -\frac{\mu_S - r}{\sigma_S^2} \frac{V_W}{V_{WW}} - \frac{\rho\sigma_A}{\sigma_S} \frac{KV_{WK}}{V_{WW}}. \quad (14)$$

The first term gives the classical Merton's mean-variance demand and the second term captures the investor's hedging demand with respect to the illiquid alternative asset.

Note that the hedging demand depends on the cross partial V_{WK} , and is proportional to $\rho\sigma_A/\sigma_S$ (which is equal to β_A as shown in (4)). Both results are intuitive and follow from the standard hedging arguments in Merton (1971); the investor chooses her public equity allocation to fulfill two objectives: to obtain the desired mean-variance exposure and to hedge the fraction of the alternative asset's risk spanned by public equity.

Certainty Equivalent Wealth $P(W, K, t)$. We express the investor's value function $V(W, K, t)$ during the time period $t \in ((m-1)T, mT)$ as:

$$V(W, K, t) = \frac{(b_1 P(W, K, t))^{1-\gamma}}{1-\gamma}, \quad (15)$$

where b_1 is a constant given by:

$$b_1 = \zeta^{\frac{\psi}{\psi-1}} \phi_1^{\frac{1}{1-\psi}}, \quad (16)$$

and ϕ_1 is the constant given by:

$$\phi_1 = \zeta + (1-\psi) \left(r - \zeta + \frac{(\mu_S - r)^2}{2\gamma\sigma_S^2} \right). \quad (17)$$

Guided by MPT, we can interpret $P(W, K, t)$ as the investor's certainty equivalent wealth, which is the minimal amount of total wealth required for the investor to permanently give up the opportunity to invest in the alternative asset. That is, in the interim period where $(m-1)T < t < mT$,

$$V(W, K, t) = J(P(W, K, t)). \quad (18)$$

Here, $J(\cdot)$ is the value function for an investor who can invest only in liquid public equity and risk-free bonds. We show that $J(\cdot)$ is given by

$$J(W) = \frac{(b_1 W)^{1-\gamma}}{1-\gamma}, \quad (19)$$

where b_1 is given in (16).

Homogeneity Property. In our model, the certainty equivalent wealth $P(W, K, t)$ has the homogeneity property in W and K , and hence it is convenient to work with the

liquidity ratio $w_t = W_t/K_t$ and the scaled certainty equivalent wealth function $p(w_t, t)$ defined as follows:

$$P(W_t, K_t, t) = p(w_t, t) \cdot K_t. \quad (20)$$

This homogeneity property is due to the Duffie-Epstein-Zin utility and the value processes for public equity and the alternative asset. Importantly, this homogeneity property allows us to conveniently interpret the optimal portfolio rule and target asset allocation.

Endogenous Effective Risk Aversion γ_i . To better interpret our solution it is helpful to introduce the following measure of endogenous relative risk aversion for the investor, denoted by $\gamma_i(w, t)$ and defined as follows:

$$\gamma_i(w, t) \equiv -\frac{V_{WW}}{V_W} \times P(W, K, t) = \gamma p_w(w, t) - \frac{p(w, t)p_{ww}(w, t)}{p_w(w, t)}. \quad (21)$$

In (21) the first identity sign gives the definition of γ_i and the second equality follows from the homogeneity property.

What economic insights does $\gamma_i(w, t)$ capture and what is the motivation for introducing it? First, recall the standard definition of the investor's coefficient of absolute risk aversion is $-V_{WW}/V_W$. To convert this to a measure of relative risk aversion, we need to multiply absolute risk aversion $-V_{WW}/V_W$ with an appropriate economic measure for the investor's total wealth. Under incomplete markets, although there is no market-based measure of the investor's economic well being, the investor's certainty equivalent wealth $P(W, K, t)$ is a natural measure of the investor's welfare. This motivates our definition of γ_i in (21).¹² We will show that the illiquidity of alternative assets causes the investor to be effectively more risk averse, meaning $p_w(w, t) > 1$ and $p_{ww}(w, t) < 0$, so that $\gamma_i(w, t) > \gamma$. In contrast, if the alternative asset is publicly traded (and markets are complete), $\gamma_i(w, t) = \gamma$ as $p_w(w, t) = 1$ and $p_{ww}(w, t) = 0$.

¹² See Wang, Wang, and Yang (2012) and Bolton, Wang and Yang (2018) for similar definitions involving endogenous risk aversion but for very different economic applications.

Optimal Policy Rules. Again, by using the homogeneity property, we may express the scaled consumption rule $c(w_t, t) = C(W_t, K_t, t)/K_t$ as follows:

$$c(w, t) = \phi_1 p(w, t) p_w(w, t)^{-\psi}. \quad (22)$$

Because illiquidity makes markets incomplete, the investor's optimal consumption policy is no longer linear and depends on both the certainty equivalent wealth $p(w, t)$ and also the marginal certainty equivalent value of liquid wealth $p_w(w, t)$.

The allocation to public equity is $\Pi_t = \pi(w_t, t)K_t$ where $\pi(w, t)$ is given by:

$$\pi(w, t) = \frac{\mu_S - r}{\sigma_S^2} \frac{p(w, t)}{\gamma_i(w, t)} - \frac{\rho\sigma_A}{\sigma_S} \left(\frac{\gamma p(w, t)}{\gamma_i(w, t)} - w \right), \quad (23)$$

where $\gamma_i(\cdot)$ is the investor's effective risk aversion given by (21). Intuitively, the first term in (23) reflects the mean-variance demand for the market portfolio, which differs from the standard Merton model in two ways: 1) risk aversion γ is replaced by the effective risk aversion $\gamma_i(w, t)$ and 2) net worth is replaced by certainty equivalent wealth $p(w, t)$. The second term in (23) captures the dynamic hedging demand, which also depends on $\gamma_i(w, t)$ and $p(w, t)$.

PDE for $p(w, t)$. Substituting the value function (15) and the policy rules for c and π into the HJB equation (12) and using the homogeneity property and the definition of the investor's effective risk aversion, γ_i , given by (21), we obtain the following PDE for $p(w, t)$ at time t , for the liquidity ratio w_t in the interior region, and when $(m-1)T < t < mT$:

$$0 = \left(\frac{\phi_1 (p_w(w, t))^{1-\psi} - \psi\zeta}{\psi - 1} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2} \right) p(w, t) + p_t(w, t) + \frac{\epsilon^2 w^2}{2} p_{ww}(w, t) \quad (24)$$

$$+ (\delta_A - \alpha + \gamma\epsilon^2) w p_w(w, t) - \frac{\gamma\epsilon^2 w^2 (p_w(w, t))^2}{2 p(w, t)} + \frac{\left(\frac{\mu_S - r}{\sigma_S} - \gamma\rho\sigma_A \right)^2 p_w(w, t) p(w, t)}{2\gamma_i}.$$

Because of incomplete spanning (e.g., $\epsilon \neq 0$), unlike Black-Scholes, (24) is a nonlinear PDE, and moreover, $p_w(w, t) > 1$, as we will show. The numerical solution for $p(w, t)$ involves the standard procedure.

Rebalancing the Illiquid Alternative Asset during the Interim Period. Although under normal circumstances the investor plans to hold the alternative asset until an automatic liquidity event occurs at date mT , under certain circumstances the investor may find it optimal to actively rebalance even at time $t \neq mT$.

As acquisition and voluntary liquidation are costly, we have an inaction region at all time including $t = mT$. Let \underline{w}_t and \overline{w}_t denote the lower liquidation boundary and the upper acquisition boundary for the liquidity ratio w_t at time t , respectively. We show that it is optimal for the investor to keep the liquidity ratio w_t within the boundary $(\underline{w}_t, \overline{w}_t)$ by voluntarily liquidating a portion of the illiquid alternative asset if w_t is too high and acquiring the illiquid alternative asset if w_t is too low.

We show in Appendix B that the following conditions hold at \underline{w}_t and \overline{w}_t , respectively,

$$p(\underline{w}_t, t) = (1 - \theta_L + \underline{w}_t) p_w(\underline{w}_t, t), \quad (25)$$

$$p(\overline{w}_t, t) = (1 + \theta_X + \overline{w}_t) p_w(\overline{w}_t, t). \quad (26)$$

These two equations at the boundaries are implied by the continuity of the value function, the linear transaction acquisition and liquidation cost functions (i.e., constant θ_L and θ_X), and the homogeneity property of our model.¹³

Next, we provide the conditions that describe the investor's *optimal* liquidation and acquisition decisions. By differentiating (25) with respect to \underline{w}_t and (26) with respect to \overline{w}_t , we obtain the following boundary conditions:

$$p_{ww}(\underline{w}_t, t) = 0, \quad (27)$$

$$p_{ww}(\overline{w}_t, t) = 0, \quad (28)$$

which are often referred to as the “super contact” conditions as in Dumas (1991).

Until now, we have characterized the investor's optimal decision ignoring the potential automatic liquidity event. Next, we analyze the investor's decision at $t = mT$ when the

¹³As $p \geq 0$ and $p_w \geq 0$, equation (25) implies $\underline{w}_t \geq -(1 - \theta_L)$, meaning that the investor can borrow only a fraction of the alternative asset's fundamental value. As a result, the investor can repay the liability with probability one by liquidating the alternative asset. Thus, the investor's debt capacity is endogenously determined by the liquidation value of the alternative asset. Although the investor can borrow, in our numerical exercise, as in reality, the investor rarely does borrow.

portfolio's liquidity changes discretely due to the automatic liquidity event, i.e., when an alternative investment in the investor's portfolio pays a lumpy liquidating dividend.

Value and Decisions when there is an Automatic Liquidity Event at $t = mT$.

At time $t = mT$, a fraction δ_T of the alternative asset automatically becomes fully liquid without any voluntary liquidation. We use \widehat{W}_{mT} and \widehat{K}_{mT} to denote the corresponding levels of liquid wealth and the alternative asset at $t = mT$ if the investor chooses not to do any voluntary rebalancing. It is immediate to see $\widehat{W}_{mT} = \lim_{t \rightarrow mT} (W_t + \delta_T K_t)$ and $\widehat{K}_{mT} = \lim_{t \rightarrow mT} (K_t - \delta_T K_t)$. Let \widehat{w}_{mT} denote the corresponding liquidity ratio:

$$\widehat{w}_{mT} \equiv \frac{\widehat{W}_{mT}}{\widehat{K}_{mT}} = \lim_{t \rightarrow mT} \frac{w_t + \delta_T}{1 - \delta_T}. \quad (29)$$

By now, we have outlined the procedures for calculating both \overline{w}_{mT} (ignoring the automatic liquidity event) and \widehat{w}_{mT} (focusing only the automatic liquidity event.) Of course, the investor optimizes her decision by considering both the ‘‘marginal analysis’’ for the liquidity ratio and the automatic liquidity event at $t = mT$. As a result, we have two cases to consider at $t = mT$: Case (i) where $\widehat{w}_{mT} \leq \overline{w}_{mT}$ and Case (ii) where $\widehat{w}_{mT} > \overline{w}_{mT}$. As the automatic liquidity event always increases liquid asset holdings, \widehat{w}_{mT} is always larger than \underline{w}_{mT} . Hence, we need only consider these two cases.

In Case (i) when $\widehat{w}_{mT} \leq \overline{w}_{mT}$, the optimal liquidity ratio at mT is \widehat{w}_{mT} as it is optimal for the investor not to voluntarily rebalance the illiquid alternative asset holding. The intuition is that, even with the automatic increase in liquidity at mT , the liquidity ratio will still lie within the inaction range $\lim_{t \rightarrow mT} (\underline{w}_t, \overline{w}_t)$. Therefore, the continuity of the value function implies $P(W_{mT}, K_{mT}) = P(\widehat{W}_{mT}, \widehat{K}_{mT})$, which can be simplified as:

$$\lim_{t \rightarrow mT} p(w, t) = p(\widehat{w}_{mT}, t)(1 - \delta_T), \quad (30)$$

where \widehat{w}_{mT} is given in (29).

In Case (ii) when $\widehat{w}_{mT} > \overline{w}_{mT}$, the optimal liquidity ratio at mT is \overline{w}_{mT} as it is optimal for the investor to voluntarily acquire the illiquid alternative asset. In this case, the automatic liquidity events results in $w_{mT} = \widehat{w}_{mT} > \overline{w}_{mT}$, which means the

investor holds too much of the liquid asset. To bring the portfolio liquidity ratio back into the inaction region, the investor must acquire more of the alternative asset, so that $w_{mT} = \bar{w}_{mT}$. In Appendix B, we show

$$\lim_{t \rightarrow mT} p(w, t) = p(\bar{w}_{mT}, mT) (1 - \delta_T + \lambda), \quad (31)$$

where λ reflects the effect of rebalancing and is given by

$$\lambda = \lim_{t \rightarrow mT} \frac{w_t + \delta_T - \bar{w}_{mT}(1 - \delta_T)}{1 + \theta_X + \bar{w}_{mT}}. \quad (32)$$

Finally, the homogeneity property allows us to express the value-matching condition (11) in terms of $p(w, t)$ at $t = mT$:

$$p(w, mT) = p(w, (m - 1)T). \quad (33)$$

Next, we summarize the main results of our model.

Proposition 1 *The scaled certainty equivalent wealth $p(w, t)$ in the interim period when $(m - 1)T < t \leq mT$ solves the PDE (24) subject to the boundary conditions (25), (26), (27), (28), and (33). Additionally, $p(w_{mT-}, mT-)$ satisfies (30), if $\hat{w}_{mT} \leq \bar{w}_{mT}$ where \hat{w}_{mT} is given by (29), and satisfies (31) if $\hat{w}_{mT} > \bar{w}_{mT}$.*

4. Data and Calibration

4.1. Data and Summary Statistics

The university endowment fund data come from the annual NACUBO-Commonfund Endowment Survey (NCES) conducted by the National Association of College and University Business Officers (NACUBO) and Commonfund. See Brown, Garlappi, and Tiu (2010), Dimmock (2012), and Brown, Dimmock, Kang, and Weisbenner (2014) for detailed discussions of these data. We focus on the cross-section of 774 university endowment funds¹⁴ as of the end of the 2014-2015 academic year.

¹⁴If a university reports multiple endowment funds we aggregate to form a single observation.

Asset Allocation. The NCES provides annual snapshots of endowment funds’ portfolio allocations. To link the NCES data to the model, we aggregate endowment allocations into the three asset classes featured in our model: (1) *the risk-free asset*, which aggregates cash and fixed income, (2) *public equity*, which aggregates public equity and REITs, and (3) *the alternative asset*, which aggregates hedge funds, private equity, venture capital, private real estate, and illiquid natural resources. In the summary statistics, and for generating some of the calibration parameters, we use the disaggregated sub-asset classes (e.g., hedge funds, etc.).

Table 1 shows the summary statistics as of the end of the 2014-2015 academic year. The first and second columns show the equal and value weighted averages, respectively. The remaining columns show averages within various size categories of endowment funds (e.g., “0-10%” summarizes the variables for the smallest decile of funds). Panel A shows that the average *Endowment Size* is \$677 million, but the distribution is highly positively skewed and the average size for the median decile is \$116 million. On an equal weighted basis, *Public Equity* has the largest average allocation at 50.7%. On a value weighted basis, *Alternative Allocations* has the largest average allocation at 51.7%, compared with 35.6% for public equity and 12.7% for cash and fixed income. The average equally weighted spending rate is 4.2%.

Panel B of Table 1 shows the summary statistics for the more detailed sub-asset categories. Within *Alternative Allocations*, hedge funds has the largest allocation with an equal weighted average allocation of 16.7%. For all of the sub-asset classes the allocations increase with endowment size, particularly for the least liquid categories: private equity, venture capital, private real estate, and illiquid natural resources.

Prior studies show a positive relation between endowment size and alpha, which they attribute to access to superior alternative asset investments (Lerner, Schoar, and Wang (2008), Brown, Garlappi, and Tiu (2010), Barber and Wang (2013), and Ang, Ayala, and Goetzmann (2018)). We follow these papers and also interpret size as a proxy for the endowment’s ability to access alternative assets with alphas.

Table 1: SUMMARY OF ENDOWMENT FUND ASSET ALLOCATION

This table summarizes endowment fund portfolios as of the end of the 2014-2015 academic year for 774 endowments. The first three columns show the mean, value weighted mean, and standard deviation of the variables, respectively. The columns 0-10% to 90-100% show averages within size-segmented groups of endowment funds. For example, the column “0-10%” shows the value weighted average portfolio allocation for the smallest decile of endowment funds. The final column shows results for the 20 largest endowment funds. Panel A shows summary statistics for endowment fund size (reported in millions of dollars), asset class allocations and spending rates (reported in percentages), the number of alternative asset funds that the endowment holds, and the average target horizon for the alternative assets. *Cash & Fixed Income* includes cash, cash equivalents, and fixed income securities (except for distressed securities). *Public Equity* includes domestic and foreign equity as well as REITs. *Alternative Allocations* includes hedge funds, private equity, venture capital, private real estate, and illiquid natural resources. Panel B reports summary statistics for the more detailed sub-asset classes. *Hedge Funds* includes managed futures. *Natural Resources* includes illiquid natural resources, such as timberland and oil & gas partnerships.

	Mean	VW Mean	Std Dev	0-10%	20-30%	45-55%	70-80%	90-100%	Top 20
Panel A									
<i>Endowment Size (\$M)</i>	677		2,483	17	50.1	116	408	13,409	18,585
<i>Cash & Fixed Income</i>	21.0%	12.7	11.3	33.1	21.7	22.4	15.2	10.8	9.7
<i>Public Equity</i>	50.7%	35.6	14.6	60.1	57.9	54.7	45.9	32.0	29.3
<i>Alternative Allocations</i>	28.3%	51.7	19.3	6.3	20.4	22.9	38.9	57.1	61.0
<i>Spending Rate</i>	4.2%	4.4	2.2	4.5%	3.7	3.9	3.9	4.5	4.5
<i>No. Alternative Funds</i>	16.9	56.0	28.6	1.1	5.6	7.2	22.5	86.5	152.2
<i>Alt. Target Horizon</i>	4.2	5.5	2.5	3.6	4.1	4.0	4.6	5.9	6.3
Panel B									
<i>Cash & Equivalents</i>	5.1%	4.0	8.3	7.2	3.5	5.6	3.8	3.5	2.6
<i>Fixed Income</i>	15.9%	8.7	9.3	25.8	18.2	16.8	11.4	7.4	7.1
<i>Public Equity</i>	50.7%	35.6	14.6	60.1	57.9	54.7	45.9	32.0	29.3
<i>Hedge Funds</i>	16.7%	23.4	12.2	4.6	13.0	14.3	22.3	23.8	23.7
<i>Private Equity</i>	4.6%	10.9	6.6	0.2	3.1	3.1	7.1	12.3	14.0
<i>Venture Capital</i>	1.7%	5.7	3.3	0.3	0.5	0.4	2.2	6.9	7.3
<i>Private Real Estate</i>	2.7%	6.1	3.8	0.4	1.8	2.7	3.1	7.0	8.4
<i>Natural Resources</i>	2.7%	5.9	3.5	0.8	1.9	2.0	4.2	6.7	7.5

Portfolio Illiquidity and Target Horizons. We estimate target holding periods for alternative assets based on investors' portfolio allocations and the horizons of each sub-asset class within alternatives. For hedge funds, we assume a horizon of six months, which approximately equals the sum of the average redemption, advance notice, and lockup periods reported in Getmansky, Liang, Schwarz, and Wermers (2015). For private equity and venture capital, we assume a horizon of 10 years, based on the average commitment period reported in Metrick and Yasuda (2010). For private real estate and illiquid natural resources, we also assume horizons of 10 years, based on the holding periods reported in Collet, Lizieri, and Ward (2003) and Newell and Eves (2009). Panel A of Table 1 reports *Alt. Target Horizon*, which is the value-weighted average target horizon.

Table 1 also reports the average number of alternative asset funds held by the endowments, which is an important component of liquidity management. Suppose Endowment A invests its entire alternative asset allocation into a single private equity fund with a 10 year lock-up. Suppose Endowment B spreads its alternative asset allocation across 120 different private equity funds, each of which has a 10 year lock-up. Further suppose that Endowment B staggers its investments across time, so that one lock-up expires every month. Although both endowments have the same allocation to private equity, their liquidity exposures are very different. Endowment A can only reduce its exposure to private equity through the secondary market. In contrast, Endowment B can costlessly reduce its exposure to private equity as lock-ups expire each month. Thus, by holding multiple funds with staggered maturities, the endowment can enhance the liquidity of its portfolio, which we refer to as *liquidity diversification*. In our quantitative analysis, we explore the relation between liquidity diversification and investor welfare.

As Panel A of Table 1 shows, there is a strong positive relation between endowment size and the number of alternative asset funds. On average, endowments in the largest decile hold 86.5 alternative asset funds; endowments in the smallest decile own only a single fund. Thus, illiquidity diversification is more effective for larger endowments, lowering the unspanned risk.

4.2. Parameter Choices and Calibration

Calculating Unspanned Volatility ϵ . Calibrating the model requires the standard deviation, beta, and unspanned volatility of the representative alternative asset. To obtain these parameters, we build up from the standard deviations and correlations of the sub-asset classes comprising the representative alternative asset. For each sub-asset class a , we combine its β_a and R_a^2 with the standard deviation of the market $\sigma_S = 20\%$ to obtain the implied standard deviation for the asset class: $\sigma_a = \sqrt{\beta_a^2 \sigma_S^2 / R_a^2}$.

Panel A of Table D1 in Appendix D shows the β_a , R_a^2 , and σ_a for each of the alternative sub-asset classes. For hedge funds, the β and R^2 are taken from Getmansky, Lo, and Makarov (2004) and account for return smoothing. For private equity and venture capital, the β and R^2 are taken from Ewens, Jones, and Rhodes-Kropf (2013). For private real estate and illiquid natural resources, the variables are based on Pedersen, Page, and He (2014) and account for return smoothing. Panel B of Appendix Table D1 shows the pairwise correlations between the asset classes, which are calculated using index returns over the period 1994-2015.¹⁵ We combine the asset allocations from Table 1 with the data from Appendix Table D1 to impute portfolio β , σ , and unspanned volatility (ϵ). Table 2 shows the imputed variables for the cross-section of endowment funds.

Parameter Choices. We set the alternative asset $\beta_A = 0.6$ and the unspanned volatility of the alternative asset to $\epsilon = 15\%$. We set the horizon of the representative alternative asset $H = 6$ years. These values closely match the value weighted endowment fund portfolio values reported earlier.

Following the literature, we choose the following standard parameter values: the annual risk-free rate $r = 4\%$ and the investor's coefficient of relative risk aversion $\gamma = 2$. We set the EIS to be $\psi = 0.5$, so that it corresponds to expected utility with $\gamma = 1/\psi = 2$. We also set the investor's discount rate equal to the risk-free rate, $\zeta = r$. For public equity,

¹⁵The indexes are: Bloomberg/Barclays US Aggregate Bond Index, CRSP value weighted index, Credit Suisse/Tremont Aggregate Hedge Fund Index, Cambridge Associates U.S. Private Equity Index, Cambridge Associates U.S. Venture Capital Index, NCREIF Property Index (unsmoothed), and the S&P Global Timber and Forestry Index. For private equity, venture capital, private real estate, and illiquid natural resources the returns are quarterly; the other returns are monthly.

Table 2: IMPLIED ALTERNATIVE-ASSET RISK AND UNSPANNED VOLATILITY

This table summarizes the implied parameters of the alternative asset: β_A is the beta, σ_A is the standard deviation, and ϵ is the unspanned volatility. The first two columns show results for the equal-weighted and value-weighted average portfolios. The remaining columns show allocations for size-segmented groups of endowments. e.g., the column “0-10%” shows the value-weighted statistics for the smallest decile of endowment funds.

	Avg.	VW	0-10%	20-30%	45-55%	70-80%	90-100%	Top 20
β_A	0.58	0.61	0.53	0.55	0.54	0.57	0.62	0.62
σ_A	18.1%	18.7	17.7%	17.7	17.3	18.2	18.9	18.8
ϵ	13.9%	14.2	14.1	13.9	13.5	14.2	14.3	14.2

we use an annual volatility of $\sigma_S = 20\%$ and a public equity risk premium of $\mu_S - r = 6\%$.

In our model, the alpha of the illiquid alternative investment includes compensation for skill, liquidity risk, and other risks unspanned by public equities. We set $\alpha = 2\%$, which we view as reasonable given the empirical findings in the literature. For example, Franzoni, Nowak, and Phalippou (2012) find that private equity earns a net-of-fees liquidity risk premium of 3% annually. Aragon (2007) and Sadka (2010) find similar net-of-fees liquidity risk premia for hedge funds. Empirical studies of endowment funds also find a positive relation between performance and allocations to illiquid assets (e.g., Lerner, Schoar, and Wang (2008), Brown, Garlappi, and Tiu (2010), Barber and Wang (2013), and Ang, Ayala, and Goetzmann (2018)). Given this assumed alpha, the expected overall return on the alternative asset is $\mu_A = 0.02 + 0.04 + 0.6 \times (0.10 - 0.04) = 0.096 = 9.6\%$.

For voluntary liquidations, we assume that the proportional transaction cost is $\theta_L = 10\%$, based on empirical findings and the following back-of-the-envelope calculation: For secondary market liquidations of private equity, Kleymenova, Talmor, and Vasvari (2012) and Nadauld, Sensoy, Vorkink, and Weisbach (2019) find average discounts of 25.2% and 13.8%, respectively. For secondary market liquidations of hedge funds, Ramadorai (2012) finds an average discount of 0.9%, which rises to 7.8% during the financial crisis. Therefore, we combine the aggregate endowment fund portfolio weights with liquidation costs of 20% for PE and VC, 1% for hedge funds, and 10% for private real estate and timber-

land, to obtain a proportional liquidation cost of 9.3% for the representative alternative asset. For acquisitions, we assume that the proportional acquisition cost is $\theta_X = 2\%$, which is equal to the average placement agent fee reported by Rikato and Berk (2015) and Cain, McKeon, and Solomon (2016).

Calibrating the model also requires a payout parameter, which determines the liquidity generated by automatic liquidity events (e.g., liquidity from funds maturing and paying out capital). The payout rate depends on the number of alternative asset funds held by the investor. For example, given the target horizon of $H = 6$ years, an investor with a single alternative asset fund would receive a large payout once every six years. In contrast, an investor with a large number of funds would receive smaller but more frequent payouts. For any given number of funds, denoted by i , Appendix E shows how it is possible to impute the payout rate using the previously described parameter values. For our baseline calibration we use $i \rightarrow \infty$, which implies a continuous payout rate of $\delta_A = 4.0\%$. For comparison, we also consider the cases with $i = 1$ and $i = 6$.

Table 3 summarizes the baseline parameter values.

5. Quantitative Results

In this section, we analyze the model using the parameter values from Table 3. As a benchmark, we also analyze the case when the alternative asset is fully liquid.¹⁶

5.1. Certainty Equivalent Wealth and Net Worth

We introduce the widely-used net worth as the accounting value of the investor's portfolio:

$$N_t \equiv W_t + K_t. \quad (34)$$

In general, due to illiquidity N_t is not an economic measure of the investor's true welfare.

Figure 1 plots $P(W, K, t)/N_t$, the ratio of the certainty equivalent wealth to the portfolio's book value (net worth) N_t , as a function of

$$z_t = \frac{K_t}{N_t} = \frac{K_t}{W_t + K_t} = \frac{1}{w_t + 1}, \quad (35)$$

¹⁶In this case, the alternative asset simply expands the investment opportunity set. Thus, as Appendix B.1 shows, the value function is clearly higher than when the alternative asset is illiquid.

Table 3: SUMMARY OF KEY PARAMETERS

This table summarizes the baseline parameter values. For completeness, the table also reports values of implied parameter values, i.e., those parameters whose values are determined by other parameters.

Parameter	Symbol	Value
Coefficient of relative risk aversion	γ	2
Elasticity of intertemporal substitution	ψ	0.5
Subjective discount rate	ζ	4%
Risk-free rate	r	4%
Public equity expected return	μ_S	10%
Volatility of market portfolio	σ_S	20%
Beta of the alternative asset	β_A	0.6
Alternative asset alpha	α	2%
Alternative asset expected return	μ_A	9.6%
Volatility of alternative asset	σ_A	19.2%
Alternative asset target horizon	H	6
Proportional cost of liquidation	θ_L	0.1
Proportional cost of acquisition	θ_X	0.02
<i>Implied Parameter Values</i>		
Correlation between risky assets	ρ	0.625
Unspanned volatility	ϵ	15%
Payout rate	δ_A	4.00%

the proportion of the portfolio allocated to alternative assets. Recall that we use the liquidity ratio, w_t , as the effective state variable when analyzing the model and its solution in Sections 2 and 3. Here, we use z_t to exposit our quantitative results, as practitioners typically work with portfolio allocations. Also, note that z_t is typically between zero and one making the results easier to interpret.

As the optimal w is a range (\underline{w}, \bar{w}) and z decreases with w , the corresponding range for the optimal z is (\underline{z}, \bar{z}) , where

$$\underline{z} = \frac{1}{\bar{w} + 1} \quad \text{and} \quad \bar{z} = \frac{1}{\underline{w} + 1}. \quad (36)$$

Therefore, the lower liquidation boundary \underline{w} maps to the upper liquidation boundary \bar{z} and the upper acquisition boundary \bar{w} maps to the lower acquisition boundary \underline{z} .

Figure 1 includes the case of $i \rightarrow \infty$, and for comparison it also includes the cases of $i = 1$ and $i = 6$. For the case where $i \rightarrow \infty$, we see that z lies between

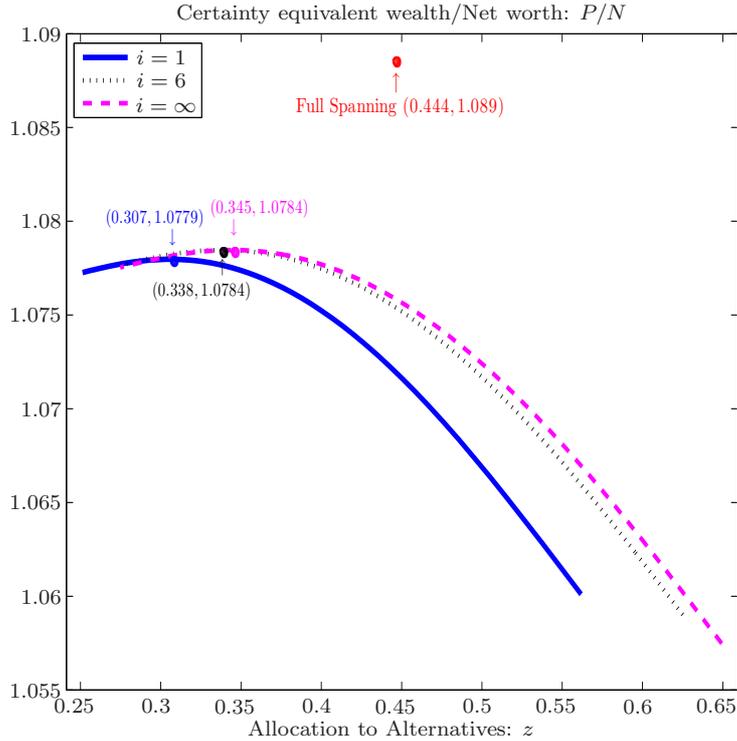


Figure 1: This figure plots $P/N = p(w)/(w + 1)$, the ratio of the certainty equivalent wealth $P(W, K) = p(w)K$ and net worth $N = W + K$, on the y -axis and the portfolio's percentage allocation to alternative assets $z = K/N$ on the x -axis. For the case with $i = \infty$ and $\delta_A = 4\%$, the optimal range for the allocation is $(\underline{z}, \bar{z}) = (0.275, 0.649)$. Other parameter values are given in Table 3. Note that $\hat{z} = 0.345$ and the maximand is 1.078, highlighted in the figure. For $i = 1$ and $i = 6$ the figure shows results at $t = mT$.

$(\underline{z}, \bar{z}) = (27.5\%, 64.9\%)$. That is, if the allocation to alternatives z falls to the endogenous acquisition boundary, $\underline{z} = 27.5\%$, the investor immediately sells just enough units of the liquid assets and invests the proceeds in the illiquid alternative asset to keep $z \geq 27.5\%$. If the allocation to alternatives rises to the endogenous liquidation boundary, $\bar{z} = 64.9\%$, the investor sells just enough units of the illiquid asset so that z falls back to 64.9%

Hypothetically, if the investor could costlessly choose z , she would choose the “desired” target $\hat{z} = 34.5\%$. At this point, her certainty equivalent wealth is 7.8% higher than her net worth. The curve is noticeably asymmetric and declines more rapidly to the right of the maximum, as the investor approaches the voluntary liquidation boundary, because liquidating alternative assets is more costly than acquiring them (i.e., $\theta_L = 10\% > \theta_X = 2\%$).

In sharp contrast, when the alternative asset is perfectly liquid, as in the case of full

spanning, the admissible illiquid alternative asset holding is not a range, but instead is a singleton with the value of $z^* = 44.4\%$.

For the case of $i = 1$, the rebalancing boundaries are further to the left, indicating that the investor holds less of the alternative asset when there is less liquidity diversification from staggering maturities across time. The curve for $i = 6$ is similar to that for $i \rightarrow \infty$ indicating that even a moderate number of alternative asset funds provide benefits from liquidity diversification.

5.2. Rebalancing Boundaries

Figure 2 shows the rebalancing boundaries over the period $0 < t < T$. First, recall that for the case with $i = \infty$, the rebalancing boundaries are constant over time which correspond to two horizontal lines at $\underline{z} = 27.5\%$ and $\bar{z} = 64.9\%$. Next, we turn to the case with time-varying rebalancing boundaries. We use the case with $i = 1$ (the investor owns only a single alternative asset fund) to highlight the main results.

In this case, there is an automatic liquidity event every six years at which time the alternative asset becomes fully liquid (see Appendix E for more details). The differences between the cases of $i \rightarrow \infty$ and $i = 1$ highlights one of the unique features of our model – that it can accommodate the *liquidity diversification* from investing in illiquid assets with staggered lock-up expirations.

The initial rebalancing boundaries, at time $t = 0$, are lower for the case of $i = 1$ than that of $i \rightarrow \infty$ because the effective cost of illiquidity due to trading restrictions is greater, resulting in lower demand for the illiquid asset. The comparison between the cases of $i = 1$ and $i \rightarrow \infty$ illustrates the interrelation of illiquidity from transactions costs and illiquidity from trading restrictions (i.e., lockups). For the case of $i = 1$, both boundaries increase as $t \rightarrow mT$. This means that the investor becomes less willing to liquidate alternative assets and more willing to voluntarily acquire alternative assets as the automatic liquidity event at $t = mT$ approaches. This is intuitive, as the investor can simply wait until the automatic liquidity event rather than incurring the liquidation cost. Indeed, the liquidation boundary \bar{z}_t becomes very large, indicating that as $t \rightarrow mT$ the

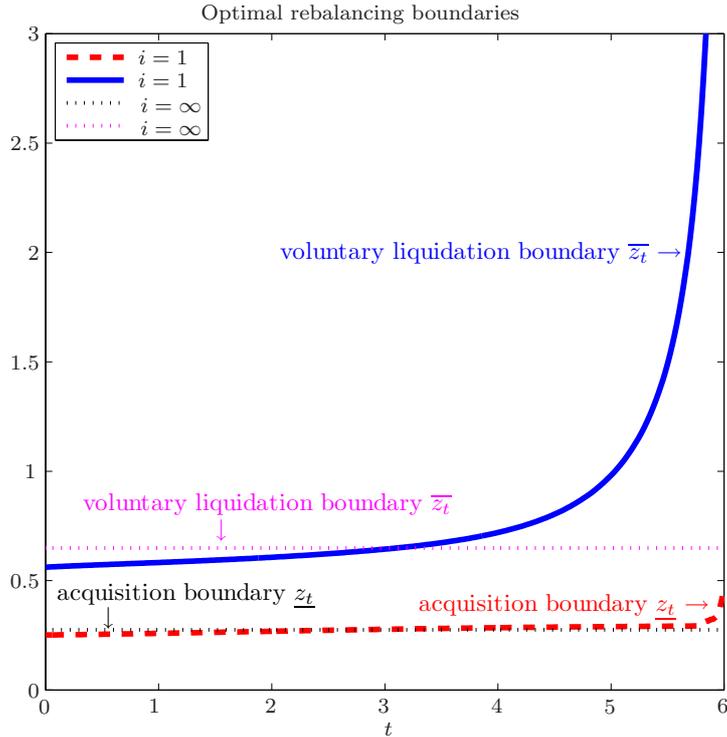


Figure 2: This figure plots the rebalancing boundaries for the portfolio's percentage allocation to alternative assets over time: $z = K/N$, where $N = W + K$. The optimal (lower) acquisition boundary is termed \underline{z}_t and the (upper) liquidation boundary is termed \bar{z}_t . The input parameter values are given in Table 3. For the case with $i \rightarrow \infty$ and $\delta_A = 4\%$, the optimal z lies between $\underline{z}_t = 0.275$ and $\bar{z}_t = 0.649$ for all t (see the two horizontal lines). The blue solid and red dashed lines correspond to the case with $i = 1$ and the implied payout $\delta_T = 21.34\%$ every six years, i.e., $H = 6$.

investor prefers to wait for the automatic liquidity event to adjust her portfolio rather than incur transaction costs.

Similarly, the acquisition boundary, \underline{z}_t , rises as $t \rightarrow mT$ for the case of $i = 1$. However, the quantitative effects for the acquisition boundary are smaller than for the liquidation boundary, because the acquisition cost is smaller than the liquidation cost ($\theta_L > \theta_X$.)

5.3. Comparative Statics

This section reports calibrated results for $i \rightarrow \infty$. As shown earlier, because of the transaction costs the model generates a range of history-dependent optimal allocation, (\underline{z}, \bar{z}) . For ease of interpretation, we do not report comparative static results for the admissible range (\underline{z}, \bar{z}) , but instead only for the desired target, \hat{z} , shown in Figure 1. This single number, \bar{z} , gives the highest possible utility for the investor. For each table,

the row in bold font shows results using the baseline parameter values from Table 3.

5.3.1. The Effect of α

Table 4 reports comparative static results for α . Panel A shows results for the general case where the alternative asset is illiquid. For comparison, Panel B shows results for the case of full spanning. For the general case with the baseline parameters, the investor allocates 53.93% of the portfolio to public equity, 34.48% to alternative assets, and the remaining 11.59% to bonds. These values are similar to the equal weighted average endowment fund portfolio allocations found in the data (see Table 1), which has allocations of 50.7% to public equity, 28.3% to alternative assets, and 21.0% to bonds. In the case of full spanning, the model implies approximately a 10 percentage point higher allocation to alternative assets and the spending rate is slightly higher.

Table 4 shows that asset allocations are quite sensitive to changes in α . For example, increasing α from 2% to 3% increases the alternative asset allocation from 34.48% to 60.24%. As the allocation to the alternative asset increases, the allocation to public equity falls from 53.93% to 37.91% to manage the overall portfolio β and because of the additional liquidity risk. If α rises to 4% the investor will optimally borrow 8.12% of net worth to invest in the alternative asset. It is worth noting that some endowment funds, such as Harvard, have occasionally taken on debt to invest in public and private equities. For example, during the financial crisis period, Harvard chose not to liquidate its endowment but rather to issue bonds (see Ang (2012)).

The sensitivity of the implied portfolio allocations to changes in α is consistent with the large cross-sectional dispersion in endowment funds' allocations to alternative assets. An α of 0% can explain non-participation, while an α of 3% implies allocations that are broadly consistent with those of large endowments such as Yale and Stanford. Thus, with reasonable parameter values, our model is consistent with both the average allocation and also the cross-sectional dispersion of allocations to alternative assets.

The sensitivity of allocations to α is also consistent with the empirically observed strong relation between endowment fund size and allocations to alternative assets. Lerner,

Table 4: THE EFFECT OF α ON ASSET ALLOCATION AND SPENDING RATES

This table reports the comparative static effect of α on asset allocation and spending. The three columns, Public Equity, Alternatives (Alternative Assets), and Bonds, report Π/N , K/N , and $(W - \Pi)/N$, respectively, evaluated at the desired target highlighted in Figure 1. The Spending column reports the corresponding desired target spending rate, C/N . These four columns are presented in percent (%), which are omitted for simplicity. Panel A reports results for the case with illiquidity. Panel B reports results for the case of full spanning. The baseline parameter values are given in Table 3.

A. General case				
	Public Equity	Alternatives	Bonds	Spending
$\alpha = 0\%$	75.00	0.00	25.00	5.13
$\alpha = 1\%$	67.29	12.69	20.02	5.16
$\alpha = 2\%$	53.93	34.48	11.59	5.32
$\alpha = 3\%$	37.91	60.24	1.87	5.60
$\alpha = 4\%$	20.41	87.72	-8.12	6.00
B. Full-spanning case				
$\alpha = 2\%$	48.33	44.44	7.22	5.35

Schoar, and Wang (2008), Brown, Garlappi, and Tiu (2010), Barber and Wang (2013), and Ang, Ayala, and Goetzmann (2018) find that large endowment funds persistently earn significant alphas, which they attribute to superior alternative asset investments, while small endowments do not earn significant alphas. Lerner, Schoar, and Wang (2008) discuss how large endowments typically have better investment committees, better access to elite managers, and can exploit economies of scale in selecting alternative assets.

5.3.2. Unspanned Volatility ϵ

Table 5 shows that the unspanned volatility of the alternative asset, ϵ , has a quantitatively large effect on asset allocation. This table also reports the implied value of ϵ^2/σ_A^2 , the fraction of the alternative asset's total variance from unspanned variance (which changes in response to the change of ϵ). We use two panels to demonstrate how the unspanned volatility affects asset allocation.

In Panel A, we fix $\beta_A = 0.6$, which implies that the systematic volatility of the alternative asset, $\rho\sigma_A$, is also fixed. In our calibration, the spanned variance is $(\rho\sigma_A)^2 =$

Table 5: THE EFFECT OF ϵ ON ASSET ALLOCATION AND SPENDING

This table reports the comparative static effect of ϵ on asset allocation and spending. The three columns, Public Equity, Alternatives (Alternative Assets), and Bonds, report Π/N , K/N , and $(W - \Pi)/N$, respectively, evaluated at the desired target highlighted in Figure 1. The Spending column reports the corresponding desired target spending rate, C/N . These four columns are presented in percent (%), which are omitted for simplicity. Panel A show the effect of changing ϵ while β_A is fixed at $\beta_A = 0.6$ and the column “Implied σ_A ” shows the total volatility of the alternative asset. Panel B shows the effect of changing ϵ while the total volatility of the alternative asset is fixed at $\sigma_A = 19.2\%$ and the column “Implied β ” shows the implied beta of the alternative asset. In both panels, the column “ ϵ^2/σ_A^2 ” shows the alternative asset’s unspanned variance as a percentage of its total variance. The baseline parameter values are given in Table 3.

Panel A. Fixing $\beta_A = 0.6$						
	Implied σ_A	ϵ^2/σ_A^2	Public Equity	Alternatives	Bonds	Spending
$\epsilon = 10\%$	15.6%	41.0%	27.79	76.34	-4.13	5.59
$\epsilon = 15\%$	19.2%	61.0%	53.93	34.48	11.59	5.32
$\epsilon = 17.5\%$	21.2%	68.0%	60.17	24.33	15.50	5.26
$\epsilon = 19.2\%$	22.6%	71.9%	63.07	19.61	17.33	5.24
Panel B. Fixing $\sigma_A = 19.2\%$						
	Implied β_A	ϵ^2/σ_A^2	Public Equity	Alternatives	Bonds	Spending
$\epsilon = 10\%$	0.82	27.1%	-2.12	95.24	6.88	5.62
$\epsilon = 15\%$	0.60	61.0%	53.93	34.48	11.59	5.32
$\epsilon = 17.5\%$	0.40	83.0%	66.41	20.58	13.01	5.25
$\epsilon = 19.2\%$	0	100.0%	74.57	11.82	13.61	5.20

$(\beta_A \sigma_s)^2 = (0.6 \times 0.2)^2 = 0.0144$. (This calculation follows from the CAPM equation given in (4).) Thus, the total variance of the alternative asset, $\sigma_A^2 = (\rho \sigma_A)^2 + \epsilon^2 = 0.12^2 + \epsilon^2$, changes one-to-one with ϵ^2 , which implies that ϵ^2/σ_A^2 , also changes with ϵ^2 . For example, as we decrease ϵ from 15% (the baseline case) to 10%, the ratio of unspanned to total variance decreases from 61.0% to 41.0%. The investor reacts by substantially increasing the allocation to alternative assets from 34.48% to 76.34%, decreasing public equity from 53.93% to 27.79%, and changing a long position of 11.59% in the bond to a short position (-4.13%). These changes are quantitatively very large. Moreover, the difference between 10% and 15% for unspanned volatility is not that substantial, particularly given the noisiness of empirical estimates of the unspanned volatility of

alternative assets. Our quantitative results suggest it is worth devoting much more work to improve the empirical estimates of unspanned volatility.

In Panel B of Table 5, we fix the total volatility of the alternative asset σ_A at 19.2%. As we increase the unspanned variance, ϵ^2 , the alternative's β_A must decrease to adjust the spanned variance, $(\rho\sigma_A)^2 = (\beta_A\sigma_S)^2$, so that $\epsilon^2 + (\rho\sigma_A)^2 = \sigma_A^2 = 19.2\%^2$. Now consider again the exercise of decreasing ϵ from 15% (the baseline case) to 10%. Unlike in Panel A, β_A must increase from 0.60 to 0.82 to offset the reduction in unspanned volatility and keep $\sigma_A^2 = 19.2\%^2$. The investor reacts to this change in the composition of total variance by increasing the allocation to alternative assets from 34.48% to 95.24% and decreasing public equity from 53.93% to a short position of -2.12%. As in the earlier panel, changes in the unspanned volatility have quantitatively large effects on asset allocation.

Finally, we study the effect of increasing the spanned variance, $\sigma_A^2 - \epsilon^2 \equiv (\rho\sigma_A)^2$, while holding the unspanned variance, ϵ^2 , fixed. We do this by comparing the same numbered rows in Panels A and B. For example, ϵ is 10% in the first row of both panel, but $\beta_A = 0.6$ and $\sigma_A = 15.6\%$ in Panel A, which are lower than $\beta_A = 0.82$ and $\sigma_A = 19.2\%$ in Panel B. As a result, by decreasing ϵ^2/σ_A^2 from 41% in Panel A to 27.1% in Panel B, the investor significantly increases the allocation to the alternative asset from 76.34% to 95.24%, decreases public equity from 27.79% to a short position (-2.12%), and increases bonds from a short position (-4.13%) to a long position of 6.88%. The intuition for this dramatic shift in asset allocation is as follows. As ϵ^2/σ_A^2 decreases, a higher allocation to the alternative asset is less costly in terms of the exposure to the alternative asset's unspanned risk. The investor therefore increases the allocation to alternatives and reduces public equity and increases bonds in order to keep the portfolio's β in a target range.

In sum, the alternative asset's unspanned and spanned risk have conceptually and quantitatively very different effects on asset allocation. The investor can offset the public-market-spanned risk of the alternative asset by adjusting allocations to public equity, while the alternative asset's unspanned volatility is by definition specific to the alternative asset and cannot be hedged.

Table 6: THE EFFECT OF THE EIS ψ on Asset Allocation and Spending

This table reports the comparative static effect of ψ on asset allocation and spending. The three columns, Public Equity, Alternatives (Alternative Assets), and Bonds, report Π/N , K/N , and $(W - \Pi)/N$, respectively, evaluated at the desired target highlighted in Figure 1. The Spending column reports the corresponding desired target spending rate, C/N . These four columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 3. For the results in this table, we fix risk aversion at $\gamma = 2$.

	Public Equity	Alternatives	Bonds	Spending
$\psi = 0.1$	56.20	30.77	13.03	6.36
$\psi = 0.5$	53.93	34.48	11.59	5.32
$\psi = 1$	50.47	40.16	9.37	3.95
$\psi = 2$	44.39	50.25	5.36	1.33

5.3.3. The EIS ψ

Table 6 shows that varying the EIS has large quantitative effects on the spending rate. In this panel, we fix risk aversion at $\gamma = 2$, a widely used value. An investor who is unwilling to substitute spending over time (e.g., $\psi = 0.1$) has a spending rate of 6.36%, which is relatively high (in light of the permanent-income argument). In contrast, an investor who is willing to substitute spending over time, (e.g., $\psi = 2$ as in the long-run risk literature following Bansal and Yaron (2004)), has a spending rate of only 1.33%. The intuition is that an investor with a high EIS defers spending to exploit the investment opportunity.

As the EIS increases, the investor increases allocations to the illiquid alternative asset and reduces allocations to the liquid asset classes (public equity and bonds). For example, an investor with $\psi = 0.1$ allocates 56.20% of net worth to public equity and 30.77% to alternatives, compared to an investor with $\psi = 2$ who allocates 44.39% to public equity and 50.25% to illiquid alternatives. A high EIS increases the investor's willingness to shift consumption across periods, which allows a high EIS investor to respond to return shocks by deferring consumption rather than engaging in costly liquidation of the alternative asset. The strong dependence of asset allocation on the EIS in our model is due to the interactive effect between asset allocation and optimal spending policies in our

Table 7: THE EFFECT OF γ ON ASSET ALLOCATION AND SPENDING RATES

This table reports the comparative static effect of γ on asset allocation and spending. The three columns, Public Equity, Alternatives (Alternative Assets), and Bonds, report Π/N , K/N , and $(W - \Pi)/N$, respectively, evaluated at the desired target highlighted in Figure 1. The Spending column reports the corresponding desired target spending rate, C/N . These four columns are presented in percent (%), which are omitted for simplicity. The baseline parameter values are given in Table 3. For the results in this table, we fix the EIS at $\psi = 0.5$.

	Public Equity	Alternatives	Bonds	Spending
$\gamma = 1$	112.75	53.76	-66.51	6.57
$\gamma = 2$	53.93	34.48	11.59	5.32
$\gamma = 4$	27.21	17.36	55.43	4.66

incomplete-markets model. In contrast, for the case of full spanning, the asset allocation rule is the same as that in Merton (1971) and it is risk aversion, not the the EIS, that influences asset allocation (see equations (C.1) and (C.2)).

Our model-implied results for the relation between spending flexibility and portfolio liquidity are consistent with empirical facts. Hayes, Primbs, and Chiquoine (2015) argue that pension funds have little spending flexibility and family offices have a great deal of flexibility. Rose and Seligman (2016) find that the average allocation to alternative assets for public pension plans is only 3.3%. In contrast, a UBS/Campden survey found that family offices hold more than 50% of their wealth in illiquid asset classes.¹⁷ Over a medium or long horizon, the combined effect of a high EIS – reducing spending and tilting investments towards illiquid alternatives which deliver alpha – will have a significant impact on the accumulation of net worth.

5.3.4. Risk Aversion γ

Table 7 shows that the coefficient of relative risk aversion has a large effect on asset allocation. For a fixed EIS of $\psi = 0.5$, if risk aversion decreases from $\gamma = 2$ to $\gamma = 1$ the investor increases the portfolio allocation to alternative assets from 34.48% to 53.76%. Even more strikingly, the investor changes the portfolio allocation to the risk-free asset

¹⁷See <http://www.globalfamilyofficereport.com/investments/>.

from a long position of 11.59% to a short position (borrowing 66.51% of net worth). As a result, the investor increases the portfolio allocation to public equity from 53.93% to a levered position (112.75% of net worth). As risk aversion increases from $\gamma = 2$ to $\gamma = 4$, allocations to bonds significantly increase from 11.59% to 55.43%, allocations to alternative assets decrease by about half from 34.48% to 17.36%, and allocations to public equity decrease from 53.93% to 27.21%.

6. Conclusion

The “Endowment Model” is widely used by many university endowment funds and other institutional investors. We build on the framework of Modern Portfolio Theory to develop a dynamic portfolio-choice model with illiquid alternative assets to analyze conditions under which the Endowment Model does and does not work. We capture the illiquidity of the alternative asset as follows. First, a fraction of the alternative asset periodically matures and becomes fully liquid, and the investor can benefit from liquidity diversification by holding alternative assets maturing at different dates. Second, the investor can voluntarily trade in the illiquid asset at any time by paying a proportional transaction cost. Third, the alternative asset’s risk is not fully spanned by public equity.

Quantitatively, our model’s results are broadly consistent with the average level and the cross-sectional variation of actual university endowment funds’ asset allocation and spending decisions. We show that asset allocations and spending decisions crucially depend on the alternative asset’s expected excess return, its risk unspanned by public equity, and investors’ preferences for inter-temporal spending smoothing.

Due to space considerations we have not included several conceptually interesting and generalizations. For example, stochastic calls and distributions are potentially important as they create uncertainty about the precise timing of cash flows making illiquid alternative assets less attractive, *ceteris paribus*. Extending the model to allow for time-variation in the liquidation cost is also an interesting direction for future research, particularly given the rise in illiquidity observed during the financial crisis.

Appendices

A Public Equity and Bonds with No Alternatives

First, we summarize the solution for the complete-markets special case of our model where an investor with Duffie-Epstein-Zin recursive preferences has the standard investment opportunities defined by the public equity's risky return process given by (1) and a risk-free bond that pays a constant rate of interest r . The investor dynamically adjusts her consumption/spending and frictionlessly rebalances her portfolio to maximize her recursive preferences given in (8)-(9). Note that the investor only has liquid wealth W . The following proposition summarizes the solution for this frictionless benchmark.

Proposition 2 *The investor allocates a constant fraction, denoted by π , of her wealth W_t to public equity, i.e., the total investment amount in public equity is $\Pi = \pi W$ where*

$$\pi = \frac{\mu_S - r}{\gamma \sigma_S^2}. \quad (\text{A.1})$$

Note that the optimal asset allocation rule is the same as that in Merton (1969, 1971). Specifically, the EIS has no effect on π in this frictionless benchmark. The optimal spending C_t is proportional to wealth W_t : $C_t = \phi_1 W_t$ where

$$\phi_1 = \zeta + (1 - \psi) \left(r - \zeta + \frac{(\mu_S - r)^2}{2\sigma_S^2 \gamma} \right). \quad (\text{A.2})$$

Note that the optimal spending rule depends on both risk aversion γ and the EIS ψ , which is different from Merton (1969, 1971). The investor's value function $J(W)$ is given by:

$$J(W) = \frac{(b_1 W)^{1-\gamma}}{1-\gamma}, \quad (\text{A.3})$$

where b_1 is a constant given by:

$$b_1 = \zeta^{\frac{\psi}{\psi-1}} \phi_1^{\frac{1}{1-\psi}}. \quad (\text{A.4})$$

Next, we analyze the general case where the investor can also invest in illiquid alternative assets in addition to public equity and bonds.

B Proof for Proposition 1

Optimal Policy Functions and PDE for $p(w, t)$. We conjecture that the value function $V(W, K, t)$ takes the following form:

$$V(W, K, t) = \frac{(b_1 P(W, K, t))^{1-\gamma}}{1-\gamma} = \frac{(b_1 p(w, t) K)^{1-\gamma}}{1-\gamma}, \quad (\text{B.1})$$

where b_1 is given in (A.4). Substituting (B.1) into the consumption FOC given in (13) and the asset-allocation FOC given in (14), we obtain (22) for the scaled consumption rule $c(w, t)$ and (23) for the scaled asset allocation in public equity $\pi(w, t)$, respectively. Finally, substituting the conjectured value function given in (B.1) and the consumption and asset-allocation policy rules, given in (22) and (23), into the HJB equation (12), we obtain the PDE (24) for the certainty equivalent wealth $p(w, t)$.

Lower Liquidation Boundary \underline{W}_t and Upper Acquisition Boundary \overline{W}_t . Let (W_t, K_t) denote the investor's time- t holdings in public equity and the alternative asset, respectively. We use Δ to denote the amount of the illiquid alternative asset that the investor is considering to liquidate. The investor's post-liquidation holdings in public equity and the alternative asset, are equal to $K_t - \Delta$ and $W_t + (1 - \theta_L)\Delta$, respectively. Because the investor's value function is continuous before and after liquidation, we have

$$V(W_t + (1 - \theta_L)\Delta, K_t - \Delta, t) - V(W_t, K_t, t) = 0. \quad (\text{B.2})$$

Dividing (B.2) by Δ and letting $\Delta \rightarrow 0$, we obtain under differentiability:

$$\begin{aligned} 0 &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} [V(W_t + (1 - \theta_L)\Delta, K_t - \Delta, t) - V(W_t + (1 - \theta_L)\Delta, K_t, t)] \\ &\quad + \lim_{\Delta \rightarrow 0} \frac{1 - \theta_L}{\Delta(1 - \theta_L)} [V(W_t + (1 - \theta_L)\Delta, K_t, t) - V(W_t, K_t, t)] \\ &= -V_K(W_t, K_t, t) + (1 - \theta_L)V_W(W_t, K_t, t). \end{aligned} \quad (\text{B.3})$$

The preceding equation implicitly defines the boundary \underline{W}_t , in that

$$V_K(\underline{W}_t, K_t, t) = (1 - \theta_L)V_W(\underline{W}_t, K_t, t). \quad (\text{B.4})$$

The optimality of \underline{W}_t implies that the derivatives on both sides of (B.6) are equal. Therefore,

$$V_{KW}(\underline{W}_t, K_t, t) = (1 - \theta_L)V_{WW}(\underline{W}_t, K_t, t). \quad (\text{B.5})$$

Substituting the value function given by (B.1) into (B.4), we obtain:

$$P_K(\underline{W}_t, K_t, t) = (1 - \theta_L)P_W(\underline{W}_t, K_t, t). \quad (\text{B.6})$$

Similarly, substituting the value function given by (B.1) into (B.5), we obtain:

$$P_{KW}(\underline{W}_t, K_t, t) = (1 - \theta_L)P_{WW}(\underline{W}_t, K_t, t). \quad (\text{B.7})$$

By using the homogeneity property, we obtain the following: $P_W(W_t, K_t, t) = p_w(w_t, t)$, $P_{WW}(W_t, K_t, t) = p_{ww}(w_t, t)/K_t$, $P_K(W_t, K_t, t) = p(w_t, t) - p_w(w_t, t)w_t$, and $P_{WK}(W_t, K_t, t) = -W_t p_{ww}(w_t, t)/K_t^2$. Substituting these expressions into (B.6) and (B.7), we obtain

$$p(\underline{w}_t, t) - p_w(\underline{w}_t, t)\underline{w}_t = (1 - \theta_L)p_w(\underline{w}_t, t) \quad (\text{B.8})$$

$$-p_{ww}(\underline{w}_t, t)\underline{w}_t/K_t = (1 - \theta_L)p_{ww}(\underline{w}_t, t)/K_t. \quad (\text{B.9})$$

Simplifying these two equations, we obtain (25) and (27).

We can derive the boundary conditions for \overline{W}_t and \overline{w}_t by using essentially the same procedure as the above.

The preceding proof is applicable to the upper and lower barriers for all t such that $t \neq mT$. To complete our analysis for $t = mT$, we need to incorporate the automatic liquidity event that takes place $t = mT$.

Value and Decisions at $t = mT$. When there is an automatic liquidity event at $t = mT$, it is possible that without active rebalancing, the automatic liquidity can cause the portfolio to be overly exposed to liquid assets. In this case, i.e., when $\widehat{w}_{mT} > \overline{w}_{mT}$, the investor may choose to reduce her liquid asset holding.

Suppose that the investor optimally purchases Λ units of the alternative asset such that $\widehat{W}_{mT} - (1 + \theta_X)\Lambda = \overline{W}_{mT}$ and the liquidity ratio is then equal to

$$\overline{w}_{mT} = \frac{\widehat{W}_{mT} - (1 + \theta_X)\Lambda}{\widehat{K}_{mT} + \Lambda} = \lim_{t \rightarrow mT} \frac{W_t + \delta_T K_t - (1 + \theta_X)\Lambda}{K_t - \delta_T K_t + \Lambda}, \quad (\text{B.10})$$

By solving the above equation, we obtain the investor's demand for the alternative asset at $t = mT$, $\Lambda = \lambda K_{mT-}$, where λ is the scaled demand and is given by

$$\lambda = \lim_{t \rightarrow mT} \frac{w_t + \delta_T - \bar{w}_{mT}(1 - \delta_T)}{1 + \theta_X + \bar{w}_{mT}}. \quad (\text{B.11})$$

C Full Spanning with Liquid Alternative Asset

In this appendix, we summarize the full-spanning case where the alternative asset is fully liquid. An investor with Duffie-Epstein-Zin recursive preferences has three investment opportunities: (a.) the public equity whose return process is given by (1), (b.) a risk-free bond that pays a constant rate of interest r , and (c.) the risky liquid alternative asset. The investor dynamically adjusts her consumption/spending and frictionlessly rebalances her portfolio to maximize her recursive preferences given in (8)-(9). Note that the investor's wealth is fully liquid. The following proposition summarizes the solution for this frictionless benchmark.

Proposition 3 *The investor continuously rebalances the portfolio so the investment in public equity, Π , and in the alternative asset, K , are proportional to net worth N , i.e.*

$$\Pi = \frac{1}{\sigma_S \gamma (1 - \rho^2)} \left(\frac{\mu_S - r}{\sigma_S} - \rho \frac{\mu_A - r}{\sigma_A} \right) N, \quad (\text{C.1})$$

$$K = \frac{\alpha}{\gamma \epsilon^2} N. \quad (\text{C.2})$$

The remaining wealth, $N - (\Pi + K)$, is allocated to the risk-free bond. The optimal consumption C is proportional to the net worth, N : $C = \phi_2 N$ where

$$\phi_2 = \zeta + (1 - \psi) \left[r - \zeta + \frac{1}{2\gamma(1 - \rho^2)} \left(\frac{(\mu_S - r)^2}{\sigma_S^2} - \frac{2\rho(\mu_S - r)(\mu_A - r)}{\sigma_S \sigma_A} + \frac{(\mu_A - r)^2}{\sigma_A^2} \right) \right]. \quad (\text{C.3})$$

The investor's value function $\bar{V}(N)$ is given by:

$$\bar{V}(N) = \frac{(b_2 N)^{1-\gamma}}{1-\gamma} = J((b_2/b_1)N), \quad (\text{C.4})$$

where b_2 is a constant given by

$$b_2 = \zeta^{\frac{\psi}{\psi-1}} \phi_2^{\frac{1}{1-\psi}}, \quad (\text{C.5})$$

and $J(\cdot)$ is the value function given in (A.3) for an investor who only has access to public equity and bonds.

By comparing ϕ_2 given in (C.3) and ϕ_1 given in (A.2), we see that diversification ($|\rho| < 1$) and an additional risk premium $\frac{\mu_A - r}{\sigma_A} > 0$ both make the investor better off. By introducing a new risky (alternative) asset into the investment opportunity set, the investor is better off because $b_2 > b_1$. The second equality in (C.4) implies that $b_2/b_1 - 1$ is the fraction of wealth that the investor would need as compensation to permanently give up the opportunity to invest in the liquid alternative asset and instead live in the environment where she can only invest in public equity and the risk-free asset.

Proof for the Case of Full Spanning with the Liquid Alternative Asset. Using the standard dynamic programming method, we have:

$$0 = \max_{C, \Pi, K} f(C, \bar{V}) + [rN + (\mu_S - r)\Pi + (\mu_A - r)K - C]\bar{V}_N + \frac{(\Pi\sigma_S)^2 + 2\rho\Pi\sigma_S K\sigma_A + (K\sigma_A)^2}{2}\bar{V}_{NN}, \quad (\text{C.6})$$

and using the FOCs for Π , K and C , we have:

$$f_C(C, \bar{V}) = V_N, \quad (\text{C.7})$$

$$\Pi = -\frac{\mu_S - r}{\sigma_S^2} \frac{V_N}{V_{NN}} - \frac{\rho\sigma_A}{\sigma_S} K, \quad (\text{C.8})$$

$$K = -\frac{\mu_A - r}{\sigma_A^2} \frac{V_N}{V_{NN}} - \frac{\rho\sigma_S}{\sigma_A} \Pi. \quad (\text{C.9})$$

We conjecture and verify that the value function takes the following form:

$$\bar{V}(N) = \frac{(b_2 N)^{1-\gamma}}{1-\gamma}. \quad (\text{C.10})$$

Substituting (C.10) into the FOCs, we obtain $C = \zeta^\psi b_2^{1-\psi} N$, (C.1), and (C.2). Finally, substituting them into the HJB equation (C.6) and simplifying the expression, we obtain (C.3).

D Inputs for Calculating Unspanned Volatility

Table D1 shows summary statistics for the sub-asset classes within the illiquid asset class, which are used to calculate the unspanned volatility of the endowment fund portfolios.

Table D1: SUMMARY OF ASSET CLASS RISK AND CORRELATIONS

Panel A shows β_a , R_a^2 , and σ_a for each alternative asset class a . Panel B shows the pairwise correlations between these sub-asset classes.

Panel A							
	β_a	R_a^2	σ_a				
<i>Hedge Funds (HF)</i>	0.54	0.32	19.1%				
<i>Private Equity (PrivEqu)</i>	0.72	0.32	25.4%				
<i>Venture Capital (VC)</i>	1.23	0.30	45.1%				
<i>Private Real Estate (PrivRE)</i>	0.50	0.49	16.0%				
<i>Natural Resources (NatRes)</i>	0.20	0.07	17.0%				

Panel B							
	<i>FixedInc</i>	<i>PubEqu</i>	<i>HF</i>	<i>PrivEqu</i>	<i>VC</i>	<i>PrivRE</i>	<i>NatRes</i>
<i>FixedInc</i>	1						
<i>PubEqu</i>	0.02	1					
<i>HF</i>	0.16	0.64	1				
<i>PrivEqu</i>	-0.23	0.78	0.73	1			
<i>VC</i>	-0.18	0.46	0.52	0.66	1		
<i>PrivRE</i>	-0.13	0.35	0.31	0.51	0.17	1	
<i>NatRes</i>	0.04	0.87	0.67	0.70	0.46	0.44	1

E Calibrating the Payout Rates: δ_A and δ_T

We focus on the steady state in which the investor always has i funds at any time t . This is feasible provided the investor immediately replaces each fund that exits.

To simplify the exposition, assume that each fund's payoff structure involves only one contribution at its inception and one distribution upon its exit, and the horizon (or equivalently the lockup period) of each fund is H . At the steady state, i/H funds mature each year, which means that there is one liquidity event every $T = H/i$ years. For example, if the lockup period for each fund is $H = 6$ and there are three funds at the steady state ($i = 3$), then every two years ($T = 6/3 = 2$) an automatic liquidity event occurs. To ensure that the investor has three funds at the steady state, the investor immediately replaces the exited fund by investing in a new fund with a 6-year lock-up.

To ensure growth stationarity, we assume that both the growth rate of each fund, g_A , and the growth rate of the inception size for each fund (vintage), g_I , are constant. Consider a vintage- t fund, which refers to the fund that enters the portfolio at time t . Let IS_t denote the fund's initial size (IS) at inception. Its size at $(t + jT)$ is then $e^{g_A j T} IS_t$

where $j = 1, 2, \dots, i$ and hence the fund's size when exiting at time $t + H$ is $e^{g_A H} IS_t$.

At time $(t + H)$ the investor holds a total of i illiquid alternative funds ranging from vintage- t to vintage- $(t + (i - 1)T)$. Note that the value of the vintage- $(t + j)$ fund is $e^{g_A(i-j+1)T} \times (IS_t e^{g_I(j-1)T})$ as its inception size is $IS_t e^{g_I(j-1)T}$ and has grown at the rate of g_A per year for $(i - j + 1)T$ years. Summing across all vintages, we obtain:

$$\sum_{j=1}^i e^{g_A(i-j+1)T} \times (IS_t e^{g_I(j-1)T}) = e^{g_I H} IS_t \times \sum_{j=1}^i e^{(g_A - g_I)jT}.$$

The net payout at time $(t + H)$ is given by the difference between $e^{g_A H} IS_t$, the size of the exiting vintage- t fund, and $e^{g_I H} IS_t$, the size of the new vintage- $(t + H)$ fund. As the payout occurs once every T years, the annualized net payout rate is then:

$$\frac{1}{T} \frac{e^{g_A H} IS_t - e^{g_I H} IS_t}{e^{g_I H} IS_t \times \sum_{j=1}^i e^{(g_A - g_I)jT}} = \frac{1}{T} \frac{e^{(g_A - g_I)H} - 1}{\sum_{j=1}^i e^{(g_A - g_I)jT}} = \frac{1}{T} (1 - e^{-(g_A - g_I)T}). \quad (\text{E.1})$$

Next, we use this annualized net payout rate to calibrate δ_A and δ_T . Although, for the sake of generality, the model includes both δ_A and δ_T , in any single calibration we use only one of either δ_A or δ_T . Next, we provide three examples.

First, consider the case when $i \rightarrow \infty$, and with fixed finite holding period H for each fund, $T \equiv H/i \rightarrow 0$. Therefore, the investor continuously receives payout at a constant rate. This maps to the parameter δ_A in our model. By applying L'Hopital's rule to (E.1), we obtain, as one may expect, the following simple expression for the net payout rate,

$$\delta_A = g_A - g_I, \quad (\text{E.2})$$

which is simply the difference between the incumbent fund growth rate g_A and the growth of the new fund's initial size g_I . For our calibration, we set $\mu_A = g_A = 9.6\%$ and $g_I = 5.6\%$ (approximately equal to the average endowment fund growth rate over the past 20 years) resulting in $\delta_A = 4\%$.

Second, consider the case when the investor has only one fund outstanding at each point in time. Then, $T = H$, the payout occurs once every H years, and we use δ_T to capture the payout rate for this case. That is, when T is relatively large, δ_T is given by:

$$\delta_T = 1 - e^{-(g_A - g_I)T}. \quad (\text{E.3})$$

Note that δ_T as defined in the model is not annualized. Thus, with $g_A = 9.6\%$ and $g_I = 5.6\%$, for a single fund ($i = 1$) in the portfolio and $H = 6$, $\delta_T = 21.34\%$, which is equivalent to an annualized payout of 3.28% .

Third, consider an intermediate case when the investor has six funds at each point in time. Then, we have $T = H/i = 6/6 = 1$, and we could use $\delta_T = \delta_1 = 1 - e^{-0.04} = 3.92\%$ to capture the payout. Alternatively, we could approximate with a continuous constant dividend yield by annualizing δ_T and using this annualized value as δ_A in the calibration. In this case, we would have $\delta_A \approx (1 + \delta_T)^{1/T} - 1 = \delta_T = \delta_1 = 3.92\%$ when $T = 1$. As one can see, the difference between the two approximations is not noticeable.

References

- Agarwal, V., Mullally, K.A., Naik, N.Y., 2015. The economics and finance of hedge funds: A review of the academic literature. *Foundations and Trends in Finance*, 10, 1-111.
- Ang, A., 2012. *Liquidating Harvard*. CBS #100312. New York, NY: Columbia CaseWorks Publishing.
- Ang, A., Ayala, A., Goetzmann, W.N., 2018. Investment beliefs of endowments. *European Financial Management*, 24, 3-33.
- Ang, A., Papanikolaou, D., Westerfield, M.M., 2016. Portfolio choice with illiquid assets. *Management Science*, 60, 2737-2761.
- Aragon, G.O., 2007. Share restrictions and asset pricing: Evidence from the hedge fund industry. *Journal of Financial Economics*, 83, 33-58.
- Bansal, R., Yaron, A., 2004. Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance*, 59, 1481-1509.
- Barber, B.M., Wang, G., 2013. Do (some) university endowments earn alpha? *Financial Analysts Journal*, 69, 26-44.
- van Binsbergen, J.H., Brandt, M.W., Koijen, R.S.J., 2008. Optimal decentralized investment management. *Journal of Finance*, 63, 1849-1895.
- Bollen, N.P.B., Sensoy, B.A., How much for a haircut? Illiquidity, secondary markets, and the value of private equity. Working paper, Vanderbilt University.
- Bolton, P., Wang, N., Yang, J., 2018, Liquidity and risk management: Coordinating investment and compensation policy. *Journal of Finance*, forthcoming.
- Brown, J.R., Dimmock, S.G., Kang, J.K., Weisbenner, S., 2014. How university endowments respond to financial market shocks: Evidence and implications. *American Economic Review*, 104, 931-962.
- Brown, K.C., Garlappi, L., Tiu, C., 2010. Asset allocation and portfolio performance: Evidence from university endowment funds. *Journal of Financial Markets*, 13, 268-294.
- Cain, M.D., McKeon, S.B., Solomon, S.D., 2016. Intermediation in private equity: The role of placement agents. Working paper, University of Oregon.
- Cejnek, G., Franz, R., Stoughton, N.M., 2017. An integrated model of university endowments. Working paper, WU Vienna University.
- Collett, D., Lizieri, C., Ward, C., 2003. Timing and holding periods in institutional real estate. *Real Estate Economics*, 31, 205-222.

- Constantinides, G.M., 1986. Capital market equilibrium with transaction costs. *Journal of Political Economy*, 94, 842-862.
- Cornelli, F., Karakas, O., 2008. Private equity and corporate governance: Do LBOs have more effective boards? Working paper, London Business School.
- Davis, M.H.A., Norman, A.R., 1990. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15, 676-713.
- Dimmock, S.G., 2012. Background risk and university endowment funds. *Review of Economics and Statistics*, 94, 789-799.
- Duffie, D., Epstein, L.G., 1992. Stochastic differential utility. *Econometrica*, 60, 353-394.
- Dumas, B., 1991. Super contact and related optimality conditions. *Journal of Economic Dynamics and Controls*, 15, 675-685.
- Elton, E.J., Gruber, M.J., 2004. Optimum centralized portfolio construction with decentralized portfolio management. *Journal of Financial and Quantitative Analysis*, 39, 481-494
- Epstein, L.G., Zin, S.E., 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57, 937-969.
- Ewens, M., Jones, C.M., Rhodes-Kropf, M., 2013. The price of diversifiable risk in venture capital and private equity. *Review of Financial Studies*, 26, 1853-1889.
- Franzoni, F., Nowak, E., Phalippou, L., 2012. Private equity performance and liquidity risk. *Journal of Finance*, 67, 2341-2373.
- Getmansky, M., Liang, B., Schwarz, C., Wermers, R., 2015. Share restrictions and investor flows in the hedge fund industry. Working paper, University of Massachusetts.
- Getmansky, M., Lo, A.W., Makarov, I., 2004. An econometric model of serial correlation and illiquidity in hedge fund returns. *Journal of Financial Economics*, 74, 529-609.
- Gilbert, T., Hrdlicka, C., 2015. Why are university endowments large and risky? *Review of Financial Studies*, 28, 2643-2686.
- Grossman, S.J., Laroque, G., 1990. Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods. *Econometrica*, 58, 25-51.
- Hayes, M., Primbs, J.A., Chiquoine, B., 2015. A penalty cost approach to strategic asset allocation with illiquid asset classes. *Journal of Portfolio Management*, 41, 33-41.
- Kahl, M., Liu, J., Longstaff, F.A., 2003. Paper millionaires: How valuable is stock to a stockholder who is restricted from selling it? *Journal of Financial Economics*, 67, 385-410.

- Kaplan, S.N., Strömberg, P., 2009. Leveraged buyouts and private equity. *Journal of Economic Perspectives*, 23, 121-146.
- Kleymenova, A., Talmor, E., Vasvari, F.P., 2012. Liquidity in the secondaries private equity market. Working paper, London Business School.
- Kreps, D. M., and E. L. Porteus., 1978. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, 46, 185-200.
- Leibowitz, M.L., Bova, A., Hammond, P.B., 2010. The endowment model of investing: Return, risk, and diversification. Hobokon, NJ, John Wiley & Sons.
- Lerner, J., Schoar, A., Wang, J., 2008. Secrets of the academy: The drivers of university endowment success. *Journal of Economic Perspectives*, 22, 207-222.
- Lo, A.W., Mamaysky, H., Wang, J., 2004. Asset prices and trading volume under fixed transaction costs. *Journal of Political Economy*, 112, 1054-1090.
- Longstaff, F.A., 2009. Portfolio claustrophobia: Asset pricing in markets with illiquid assets. *American Economic Review*, 99, 1119-1144.
- Markowitz, H.M., 1952. Portfolio selection. *Journal of Finance*, 7, 77-91.
- Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, 51, 247-257.
- Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3, 373-413.
- Merton, R.C., 1992. Optimal investment strategies of university endowment funds. In *Continuous Time Finance*. ed. R.C. Merton, 649-674. Malden, MA, Blackwell.
- Metrick, A., Yasuda, A., 2010. The economics of private equity funds. *Review of Financial Studies*, 23, 2303-2341.
- Metrick, A., Yasuda, A., 2011. Venture capital and other private equity: A survey. *European Financial Management*, 17, 619-654.
- Nadauld, T.D., Sensoy, B.A., Vorkink, K., Weisbach, M.S., 2019. The liquidity cost of private equity investments: Evidence from secondary market transactions. *Journal of Financial Economics*, forthcoming.
- Newell, G., Eves, C., 2009. The role of U.S. timberland in real estate portfolios. *Journal of Real Estate Portfolio Management*, 15, 95-106.
- Pedersen, N., Page, S., He, F., 2014. Asset allocation: Risk models for alternative investments. *Financial Analysts Journal*, 70, 34-45.

- Ramadorai, T., 2012. The secondary market for hedge funds and the closed hedge fund premium. *Journal of Finance*, 67, 479-512.
- Rikato, M., Berk, A.S., 2015. Costliness of placement agents. *Journal of Financial Services Research*, 48, 263-287.
- Robinson, D.T., Sensoy, B.A. 2016. Cyclical, performance measurement, and cash flow liquidity in private equity. *Journal of Financial Economics*, 122, 521-543.
- Roll, R., 1977. A critique of the asset pricing theory's tests Part I: On past and potential testability of the theory. *Journal of Financial Economics*, 4, 129-176.
- Rose, P., Seligman, J., 2016. Are alternative investments prudent? Public sector pension use and fiduciary duty. *Journal of Alternative Investments*, 18, 5-20.
- Sadka, R., 2010. Liquidity risk and the cross-section of hedge-fund returns. *Journal of Financial Economics*, 98, 54-71.
- Samuelson, P.A., 1969. Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics*, 51, 239-246.
- Sedlacek, V.O., Jarvis, W.F. 2010. Endowment spending: Building a stronger policy framework. Commonfund Institute White Paper
- Sharpe, W.F., 1981. Decentralized investment management. *Journal of Finance*, 36, 217-234.
- Sorensen, M., Wang, N., Yang, J., 2014. Valuing private equity. *Review of Financial Studies*, 27, 1977-2021.
- Swensen, D.F., 2000. *Pioneering portfolio management: An unconventional approach to institutional investing*. New York, NY, The Free Press.
- Swensen, D.F., 2005. *Unconventional success: A fundamental approach to personal investment*. New York, NY, The Free Press.
- Takahashi, D., Alexander, D., 2002. Illiquid alternative asset fund modeling. *Journal of Portfolio Management*, 28, 90-100.
- Vayanos, D., 1998. Transaction costs and asset prices: A dynamic equilibrium model. *Review of Financial Studies*, 11, 1-58.
- Wang, C., Wang, N., Yang, J., 2012. A unified model of entrepreneurship dynamics. *Journal of Financial Economics*, 106, 1-23.
- Weil, P., 1990. Nonexpected utility in macroeconomics. *Quarterly Journal of Economics*, 105, 29-42.