Quantifying U.S. Treasury Investor Optimism.*

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Abstract

When the government commits to a debt policy, the future value of government primary surpluses at all horizons is dictated by the debt dynamics under the risk-neutral measure. We compare the present discounted value of future surpluses implied by the U.S. federal government debt dynamics in a no-arbitrage bond pricing model to the PDV of actual government surpluses. Since the late 1990s, the debt-implied PDV of surpluses have consistently and persistently exceeded realized surpluses. They have also exceeded surplus forecasts resulting from tax and spending policy rules. U.S. Treasury investors appear to have been overly optimistic when assessing future surpluses.

Key words: fiscal policy, term structure, debt maturity, convenience yield
JEL codes: H6, G1, E6

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1 Introduction

When the government commits to a debt policy and in the absence of a bubble on government debt, the debt dynamics imply an expectation of future surpluses under the risk neutral measure. We propose a flexible model of both U.S. federal government debt dynamics and asset prices and back out the implied dynamics of government surpluses. Since the late 1990s, the debt-implied forecast of future surpluses over the next 5 to 10 years has consistently exceeded the forecast from using the surplus dynamics directly, the cash-flow measure. U.S. bond market investors have systematically overestimated the surpluses generated by the U.S. federal government. This lends credence to the view that bond market investors have been overly optimistic.

We assume that the Treasury commits to a debt policy; the debt/output ratio is an affine function of a rich state vector consisting of macro-economic and financial variables such as bond yields. If this commitment is credible, then the spending and tax policy will have to adjust at some future date. As a result, the Treasury cannot also commit to a spending-and-taxation rule. We show how to back out the implied risk-neutral surplus dynamics from the estimated debt dynamics, and how to test the transversality (TVC) or no-bubble condition for debt.

The estimated model produces the bond market's forecast of the PDV of surpluses. We compare these forecasts to the cash-flow-based forecasts obtained when the Treasury commits to a spending and tax policy. Since the late 90s, the bond market's forecast systematically exceeds surplus forecasts that are based directly on cash flows from Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).

There is a large literature in macro-economics that addresses the question of government debt sustainability, starting with the seminal work by Hansen and Sargent (1980); Hansen, Roberds, and Sargent (1991) (see Sargent, 2012, for a comprehensive review). This literature largely sidesteps the issue of priced long-run cash flow risk. Instead, most of the models assume constant discount factors. From the asset pricing literature, we know that long-run cash flow risk is priced and accounts for a large fraction of risk premia (Alvarez and Jermann, 2005; Hansen and Scheinkman, 2009; Borovička, Hansen, and Scheinkman, 2016; Backus, Boyarchenko, and Chernov, 2018). We bring Hansen and Jagannathan (1991)'s stochastic discount factor machinery, a critical part of modern asset pricing, to bear on the valuation of government debt.

In order to uncover the risk-neutral debt dynamics, we need to price claims to debt outstanding in the future. We refer to these as debt strips. Debt strips are priced like GDP strips when debt and GDP are co-integrated, even when the debt itself is risk-free.[1] If we abstract from priced cash-flow risk in assessing debt sustainability, we are likely to find that the no-bubble or TVC condition is violated in government debt markets when the average growth rate exceeds the risk-free rate

[1]As a result, discounting future debt at the risk-free rate is akin to discounting future output at the risk-free rate. The ratio of aggregate firm value divided by GDP would be infinite if the average growth rate exceeds the risk-free rate.

(see Blanchard, 2019; Furman and Summers, 2020, for prominent examples). These assessments abstract from long-run GDP growth risk that is priced in securities markets. We show how to test the TVC in a dynamic asset pricing model that takes growth risk into account.

Like Cochrane (2019, 2020a), we decompose variation in the valuation of government debt, but we enforce no-arbitrage restrictions across asset classes when determining the appropriate discount rate for valuing future surpluses. When we use the debt dynamics to back out future surpluses, we need a model that properly prices risky cash flows, because the government's borrowing capacity in the future depends on future output. We make sure our model matches moments of other asset returns, including equities.

Our dynamic asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1991, 1993, 1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003); Lustig, Van Nieuwerburgh, and Verdelhan (2013); Gupta and Van Nieuwerburgh (2018); Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).[2] Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019)'s work focuses on directly pricing a claim to government surpluses. Similar to Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019), this paper explores novel no-arbitrage restrictions on the aggregate Treasury portfolio. Different from that paper, this paper considers the implications of the government's commitment to a debt policy by adding the debt/output ratio to the state variables. We compare the surpluses implied by the debt policy to the direct surplus forecasts in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019). This comparison provides evidence consistent with bond market optimism.

There is a mature literature on rational bubbles in asset markets, starting with the seminal work by Samuelson (1958); Diamond (1965); Blanchard and Watson (1982). As pointed out by Giglio, Maggiori, and Stroebel (2016), there is an ongoing debate about whether bubbles can be sustained in the presence of long-lived investors.[3] In the context of government debt, most authors simply compare the growth rate of the economy to the risk-free interest rate to test whether the no-bubble condition holds. We devise a rigorous test of the no-bubble condition when the government commits to a debt policy. This test requires comparing the risk-adjusted growth rate of the economy to the risk-free rate.

In section 2, we show model-free evidence that the relation between the valuation of U.S. government debt and the fundamentals is weak. In section 3, we derive the general relation between the risk-neutral debt and risk-neutral surplus dynamics. Section 4 explores the implications of a debt policy in the context of an affine pricing model. Section 5 explores the quantitative implications. The U.S. bond investors have systematically overestimated the surpluses generated by the U.S. federal government since the late 90s.

[2] Lustig, Van Nieuwerburgh, and Verdelhan (2013) study the properties of the price-dividend ratio of a claim to aggregate consumption, the wealth-consumption ratio, and Gupta and Van Nieuwerburgh (2018) evaluate the performance of private equity funds in similar settings.

[3] Giglio, Maggiori, and Stroebel (2016) devised a model-free test for bubbles in housing markets.

2 What Drives Variation in the U.S. Debt/Output Ratio? Model-free Evidence

Cochrane (2019, 2020a) develops a finite horizon version of the log-linear Campbell-Shiller decomposition of the government's debt/output ratio. Using Cochrane (2019)'s approach, we find that most of the variation in the debt/output ratio cannot be attributed to subsequent variation in fundamentals at short to medium horizons.

2.1 Campbell-Shiller decomposition of the debt/output ratio

Let r^d_{t+j} denote the return on the government debt portfolio, x_{t+j} real GDP growth (in logs), and π_{t+j} log inflation. We implement Cochrane (2019)'s version of a Campbell-Shiller decomposition for the log of the debt/GDP ratio $v_t = \log d_t$:

$$v_t = \sum_{j=1}^{T} \left(s_{t+j} - \widetilde{r}^d_{t+j} \right) + v_{t+T},$$

where $\widetilde{r}^d_{t+j} = r^d_{t+j} - x_{t+j} - \pi_{t+j}$ and

$$s_{t+j} = \frac{sy_{t+j}}{e^v} = v_{t+j-1} + r^d_{t+j} - x_{t+j} - \pi_{t+j} - v_{t+j},$$

where sy_t denotes the surplus/output ratio.[4] This decomposition expands the debt/output ratio around the unconditional average $r = x + \pi$. Cochrane's decomposition uses an effective discount rate of zero for future surplus/GDP ratios. This leaves a future output-to-debt ratio term in the Campbell-Shiller decomposition. Cochrane (2019)'s decomposition is silent on whether the debt is valued correctly. That requires taking a stand on what the right discount rate is to eliminate the v_{t+T} term. Taking covariances with v_t on both sides of the previous equation, we obtain the following :

$$var(v_t) = cov\left(\sum_{j=1}^{T} s_{t+j}, v_t\right) - cov\left(\sum_{j=1}^{T} \widetilde{r}^d_{t+j}, v_t\right) + cov(v_t, v_{t+T}). \quad (1)$$

The log/debt output ratio varies because it predicts future surpluses, future returns, or future debt/output ratios.

To compute the variance decomposition, we estimate a system of forecasting regressions for $(s_{t+1}, \widetilde{r}^d_{t+1}, v_{t+1})$ using the lagged debt/output ratio as a predictor. Based on the U.S. evidence on the partial autocorrelation function for the debt/output ratio, we choose an AR(2) model for

[4] Note that this is not the actual surplus/output ratio but an approximation constructed to ensure that $s_{t+1} + v_{t+1} = v_t + r^d_{t+1} - x_{t+1} - \pi_{t+1}$

debt/output v_{t+1}. Therefore, the predictors are (v_t, v_{t-1}):

$$\begin{aligned} s_{t+1} &= a_s + b_s^1 v_t + b_s^2 v_{t-1} + \epsilon^s_{t+1}, \\ \widetilde{r}_{t+1} &= a_r + b_r^1 v_t + b_r^2 v_{t-1} + \epsilon^r_{t+1}, \\ v_{t+1} &= \phi_0 + \phi_1 v_t + \phi_2 v_{t-1} + \epsilon^r_{t+1}. \end{aligned} \tag{2}$$

We obtain the following covariance terms for each horizon j:

$$\begin{aligned} \frac{cov(s_{t+j}, v_t)}{var(v_t)} &= \rho_1^{j-1}(b_s^1 + b_s^2 \rho_1), \\ \frac{cov(\widetilde{r}_{t+j}, v_t)}{var(v_t)} &= \rho_1^{j-1}(b_r^1 + b_r^2 \rho_1), \\ \frac{cov(v_{t+j}, v_t)}{var(v_t)} &= \rho_1^{j-1}(\phi_1 + \phi_2 \rho_1), \end{aligned}$$

where the first-order autocorrelations are given by: $\rho_1 = \phi_1/(1-\phi_2)$. We obtain the following expression for the terms in the variance decomposition (1):

$$\begin{aligned} \frac{cov(\sum_{j=1}^T s_{t+j}, v_t)}{var(v_t)} &= \frac{1-\rho_1^T}{1-\rho_1}(b_s^1 + b_s^2 \rho_1), \\ \frac{cov(\sum_{j=1}^T \widetilde{r}_{t+j}, v_t)}{var(v_t)} &= \frac{1-\rho_1^T}{1-\rho_1}(b_r^1 + b_r^2 \rho_1), \\ \frac{cov(v_{t+T}, v_t)}{var(v_t)} &= \rho_1^T. \end{aligned} \tag{3}$$

The cross-equation restriction $(b_s^1 + b_s^2\rho_1) - (b_r^1 + b_r^2\rho_1) + (\phi_1 + \phi_2\rho_1) = 1$ is automatically satisfied. This follows from the definition of s_t.

The upper panel of Figure 1 reports the decomposition of the variance into the component due to fundamentals (surpluses and returns combined) and future debt/output ratios for each horizon T. The bottom panel reports the fraction of variance in v_t that can be attributed to surpluses, discount rates, and future debt/GDP ratios. We compute this decomposition by estimating the regressions in Equation 2. Then we use Equation 3 to compute the variance decomposition. The standard errors were bootstrapped by drawing 50,000 random samples with replacement from the estimated system of equations in Equation 2, and re-estimating the coefficients. The cross-equation restrictions hold for each draw. We plot two standard error bands around the point estimates.

At the one-year horizon, 99% of the variance is attributed to the future debt/output ratio. The log debt/output ratio is highly persistent. The first-order autocorrelation ρ_1 is 0.98.[5] To develop intuition for how this persistence impacts the variance decomposition, consider the simpler case

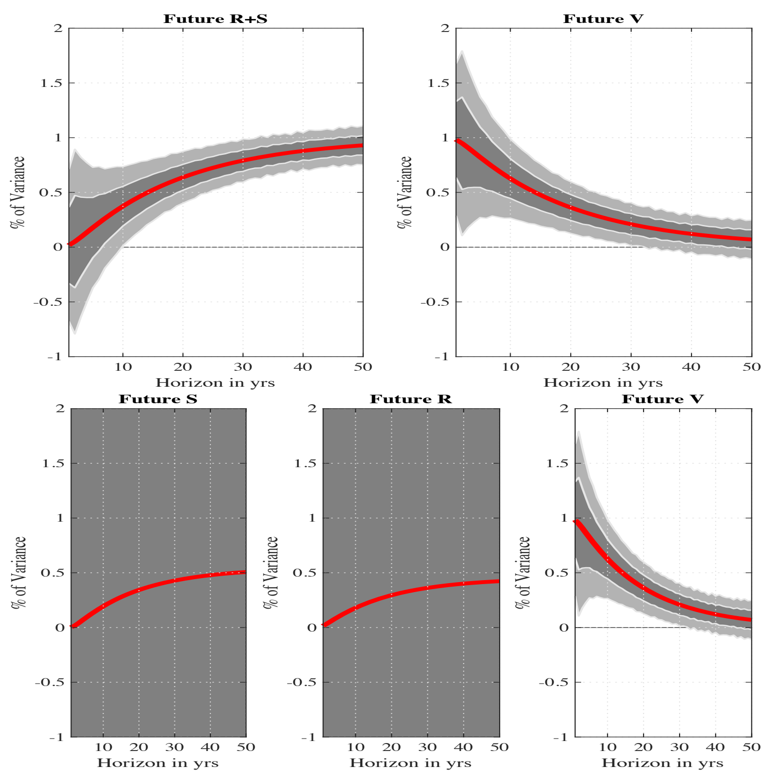
[5] Using the Augmented Dickey-Fuller test, we cannot reject the null hypothesis of the presence of the unit root in the log debt/output ratio in our sample period.

in which the debt/ouput ratio follows an AR(1): $\phi_2 = b_r^2 = b_s^2 = 0$. The variance decomposition is given by $b_s(1-\phi)^T/(1-\phi_1)$, $-b_r(1-\phi_1)^T/(1-\phi_1)$ and ϕ_1^T. In this case, as $\phi_1 \to 1$, v_{t+T} accounts for all of the variation in v_t at horizon T.

Even at the 5-year (10-year) horizon, more than 82% (62%) is attributed to the future debt/output ratio. These numbers reflect the slow mean-reversion in the debt/output ratio.

The two most left panels in the bottom row of Figure 1 decompose the contribution to the variance of fundamentals into its constituent components. The data are silent about whether the adjustment happen through adjustments in future returns or surpluses. There is no statistical evidence that the debt/output ratio predicts either the surplus or returns. The larger contribution of fundamentals at longer horizons is completely driven by the mean-reversion in the debt/output ratio.

Figure 1: Variance Decomposition of log debt/output ratio



Variance decomposition of the log debt/output ratio v_t into components due to $(\sum_{j=1}^T s_{t+j}, \sum_{j=1}^T \widetilde{r}_{t+k}, v_{t+T})$ using Equation 3 where $\widetilde{r}_{t+1}^d = r_{t+1}^d - x_{t+1} - \pi_{t+1}$ and $s_{t+1} = \frac{sy_{t+1}}{e^v}$ on v_t in Equation 2. Annual data. Sample: 1947-2019. Standard errors by bootstrapping 50,000 samples of the same size by drawing with replacement from the innovations in Equation 2 .

There is a large literature in asset pricing which tests the present value equation for long-lived assets, including stocks and bonds, starting with the seminal work by Shiller (1981); LeRoy and Porter (1981); Campbell and Shiller (1988); Giglio and Kelly (2018). The prices of these long-lived assets seem excessively volatile relative to their fundamentals. In contrast, the valuation of the entire U.S. government debt portfolio does not respond enough to the fundamentals.

Mechanically, the small role of fundamentals is the combined result of the persistence of the debt/output ratio and the lack of return and surplus predictability. The log of the U.S. debt/output ratio does not forecast returns or surpluses. Table 1 reports predictability results for horizons from 1 to 5 years by regressing cumulative returns or surpluses directly on the log of the debt/output ratio, following Jordà (2005). There is no statistical evidence that the log debt/output ratio predicts either returns or surpluses. The null hypothesis that there is no predictability in returns cannot be rejected at any horizon. Similarly, the null that there is no predictability in future surpluses cannot be rejected. At the 5-year horizon, we cannot even reject the joint null that the debt/output ratio does not predict the sum of surpluses and returns. At the 5-year horizon, 82% of the debt/output ratio fluctuations can be attributed to the future debt/output ratio.

Table 1: Forecasting Returns and Surpluses with log debt/output ratio

Regression of $\sum_{j=1}^{T} s_{t+j}, \sum_{j=1}^{T} \tilde{r}_{t+j}, v_{t+T}$ on v_t: Annual data. Sample: 1947-2019. HAC standard errors.

Horizon	1	2	3	4	5
			Forecasting $\sum_{j=1}^{T} \tilde{r}_{t+j}$		
$-b_r^T$	0.012	0.034	0.048	0.058	0.070
s.e.	[0.015]	[0.025]	[0.032]	[0.040]	[0.047]
R^2	0.010	0.027	0.032	0.032	0.034
			Forecasting $\sum_{j=1}^{T} s_{t+j}$		
b_s^T	-0.007	0.012	0.043	0.073	0.110
s.e.	[0.019]	[0.035]	[0.050]	[0.063]	[0.076]
R^2	0.0017	0.0018	0.0111	0.02	0.0308
			Forecasting $\sum_{j=1}^{T} (s_{t+j} - \tilde{r}_{t+j})$		
$-b_r^T + b_s^T$	0.0059	0.047	0.091	0.131	0.181
s.e.	[0.027]	[0.049]	[0.067]	[0.081]	[0.097]
R^2	0.00065	0.0131	0.027	0.037	0.0509
			Forecasting v_{t+T}		
ϕ^T	0.994	0.953	0.908	0.869	0.819
s.e.	[0.027]	[0.049]	[0.066]	[0.081]	[0.096]
R^2	0.949	0.847	0.734	0.632	0.525

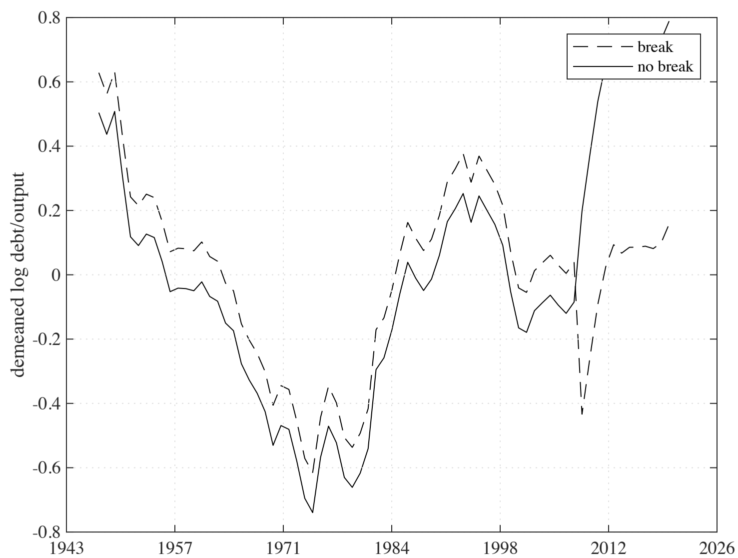
2.2 Structural Breaks

There may have been structural shifts in the relation between the valuation of debt and the fundamentals. A major contributor to the small role of fundamentals is the large run-up in government debt during the GFC which was not followed by commensurate increases in surpluses or decreases in returns. There is statistical evidence of a structural break in the log debt/output ratio in 2007. The Chow test for structural breakpoints rejects the null hypothesis of no structural break at the 1% level in 2007 (and at no other date).

We allow for a structural break in the log debt/output ratio by demeaning the log debt/output ratio $\widetilde{v}_t$ with a lower pre-2007 sample mean (0.30) and a higher post-2007 sample mean (0.66). The structural break introduces a 36 percentage point permanent increase in the debt/output ratio. Figure 2 plots the resulting series. It is clearly less persistent than the original series.

Following Lettau and Van Nieuwerburgh (2008), we demean the log debt-to-output ratio with two different means (before and after 2007), and then re-estimate the forecasting regressions. As reported in Table 2, the fundamentals now explain 45% of the variation in the log debt/output ratio with a structural break in 2007 at the 5-year horizon. This approach restores a role for fundamentals, but obviously does not explain the permanent increase in the debt/output ratio in the post-2007 sample.

Figure 2: Debt/Output Ratio with Break



The full line is the demeaned log debt/output ratio. The dashed line is the demeaned log debt/output ratio, demeaned by two different sub-sample means before and after 2007. The null hypothesis of no structural break is rejected in 2007 at the 1% level with F-statistic of 8.81.

In summary, this model-free exercise suggests that the bond market's assessment of future surpluses may diverge from the econometrician's. The latter does not anticipate larger surpluses when the debt/output ratio rises. In section 4, we use a no-arbitrage model to tackle this issue more carefully.

Table 2: Forecasting Returns and Surpluses with break in log debt/output ratio

Regression of $\sum_{j=1}^{T} s_{t+j}, \sum_{j=1}^{T} \widetilde{r}_{t+j}, v_{t+T}$ on $\widetilde{v}_t$ with structural break in 2007. Annual data. Sample: 1947-2019. HAC standard errors .

Horizon	1	2	3	4	5
			Forecasting $\sum_{j=1}^{T} \widetilde{r}_{t+j}$		
$-b_r^T$	0.015	0.04	0.0483	0.0399	0.0396
s.e.	[0.0309]	[0.0618]	[0.0876]	[0.116]	[0.142]
R^2	0.00712	0.0201	0.0183	0.0091	0.00681
			Forecasting $\sum_{j=1}^{T} s_{t+j}$		
b_s^T	0.0784	0.178	0.269	0.346	0.41
s.e.	[0.0361]	[0.0592]	[0.0778]	[0.0947]	[0.106]
R^2	0.0636	0.155	0.24	0.309	0.375
			Forecasting $\sum_{j=1}^{T} (s - \widetilde{r}_{t+j})$		
$-b_r^T + b_s^T$	0.0934	0.218	0.317	0.385	0.45
s.e.	0.0421	0.0613	0.0737	0.0817	0.0881
R^2	0.0665	0.156	0.216	0.252	0.286
			Forecasting v_{t+T}		
ϕ^T	0.907	0.782	0.683	0.615	0.55
s.e.	[0.0546]	[0.094]	[0.125]	[0.158]	[0.181]
R^2	0.87	0.706	0.562	0.462	0.375

3 Risk-Neutral Debt Dynamics and Risk-Neutral Surplus Dynamics

We follow Cochrane (2020b)'s suggestion to include the market value of debt in the econometrician's information set when assessing government debt sustainability. However, we show that the government commits to a debt policy if we include the debt in the state vector and impose no-arbitrage conditions. This debt policy fully determines the risk-neutral dynamics of the surpluses.

We use T_t to denote government revenue, and G_t to denote government spending. $M_{t,t+j}$ denotes the cumulative stochastic discount factor (SDF) used to discount risky cash flows j periods in the future. D_t denotes the market value of debt at the end of the period. We consider an investor who buys the entire government debt portfolio at the end of period t. Her total return is given by the value of the debt at the end of the period $t+1$ plus the cash flow, which consists of the primary surplus:

$$R_{t+1}^D = \frac{D_{t+1} + T_{t+1} - G_{t+1}}{D_t}.$$

The standard no-arbitrage condition for the entire debt portfolio is given by:

$$\mathbb{E}_t[M_{t,t+1} R_{t+1}^D] = 1.$$

Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) do not impose this no-arbitrage condition.

Instead, they allow for the possibility that debt is mis-priced:

$$D_t \neq \mathbb{E}_t[M_{t+1}(T_{t+1} - G_{t+1})] + \mathbb{E}_t[M_{t+1}D_{t+1}],$$

We use η_t to denote the wedge in the Euler equation for the debt portfolio such that:

$$\mathbb{E}_t[M_{t,t+1}R^D_{t+1}] = 1 + \eta_t.$$

Proposition 3.1. The PDV of k future surpluses is implied by current debt, the PDV of future debt, and the PDV of k future wedges:

$$\mathbb{E}_t \sum_{j=1}^{k} M_{t,t+j}S_{t+j} = D_t + \mathbb{E}_t \sum_{j=0}^{k} M_{t,t+j}D_{t+j}\eta_{t+j} - \mathbb{E}_t[M_{t,t+k}D_{t+k}],$$

where the primary surplus $S_t = T_t - G_t$.

The present discounted value of future surpluses cannot be simply inferred from the dynamics of the debt value. However, if we impose no-arbitrage conditions on the entire debt portfolio, and debt is in the state vector, then the PDV of future surpluses is dictated by the dynamics of debt under the risk-neutral measure.

Corollary 3.2. If the debt is fairly priced and the no-arbitrage restrictions hold for the entire debt portfolio, for $j = 1, \ldots, k$

$$E_{t+j-1}[M_{t+j-1,t+j}R^D_{t+j}] = 1,$$

then, for any j, the PDV of k future surpluses is implied by current debt and the PDV of future debt:

$$\mathbb{E}_t[\sum_{j=1}^{k} M_{t,t+j}S_{t+j}] = D_t - \mathbb{E}_t[M_{t,t+k}D_{t+k}].$$

If the debt is currently above its risk-adjusted discounted mean, then the future surpluses under the risk-neutral measure can be inferred from the predicted mean-reversion under the risk-neutral measure. Put differently, the government can commit to running deficits over short horizons only if it convinces the debt markets that future debt will be higher than current debt: $\mathbb{E}_t[M_{t,t+k}D_{t+k}] > D_t$. This is true even if the debt is not risk-free.

This corollary has important implications. Under the risk-neutral measure, the future surpluses are implied by expected value of future outstanding debt. When the government commits to a process for debt dynamics, the process for future surpluses is implied. Hence, there is no need to forecast future surpluses. They are already implied by the dynamics of the debt variable in the state vector. So, if the null hypothesis is that debt is fairly priced, then there is no point in including surpluses and debt in the same VAR, because the debt dynamics imply the surplus dynamics.

If we allow for mispricing of government debt, then the gap between these 2 forecasts reveals the PDV of the wedges:

$$\mathbb{E}_t \sum_{j=1}^{k} M_{t,t+j} S_{t+j} - (D_t - \mathbb{E}_t[M_{t,t+k} D_{t+k}]) = \mathbb{E}_t \sum_{j=0}^{k} M_{t,t+j} D_{t+j} \eta_{t+j}. \quad (4)$$

Convenience yields would imply negative wedges ($\eta_t < 0$).[6] In what follows, we assume that the wedges are zero.

4 No-Arbitrage Implications of the Government Debt Policy

We explore the cross-equation restrictions in affine asset pricing models when debt is included in the state vector. When we include the debt or the debt/output ratio in the state vector, we effectively assume that the government commits to a debt policy that is an affine function of the state vector z_t. We devise a test of the transversality condition.

4.1 A Simple Risk-Neutral Example

We start with a simple model in which investors are risk-neutral. There is no growth in this economy. Suppose the SDF $M_{t,t+j} = \beta^j$ and the state vector $z_t = (D_t, S_t)$ follows VAR(1) dynamics:

$$z_t = \Psi z_{t-1} + \Gamma \varepsilon_t,$$

then Proposition 3.2 implies the following restrictions on the expected present-discounted value (PDV) or future primary surpluses:

$$e_s' \sum_{j=1}^{k} \beta^j \Psi^j z_t = \mathbb{E}_t[\sum_{j=1}^{k} \beta^j S_{t+j}] = D_t - \beta^k \mathbb{E}_t[D_{t+k}] = e_d'(I - \beta^k \Psi^k) z_t,$$

[6]There is a large literature on the specialness of U.S. government bonds, which finds that U.S. government bonds trade at a premium relative to other risk-free bonds (Longstaff, 2004; Krishnamurthy and Vissing-Jorgensen, 2012; Fleckenstein, Longstaff, and Lustig, 2014; Krishnamurthy and Vissing-Jorgensen, 2015; Nagel, 2016; Bai and Collin-Dufresne, 2019). Greenwood, Hanson, and Stein (2015) study the government debt's optimal maturity in the presence of such premium, and Jiang, Krishnamurthy, and Lustig (2018) study this premium in international finance. We tackle the question of how expensive a portfolio of all Treasuries is relative to the underlying collateral, a claim to surpluses. Using the standard convenience yield estimates of Krishnamurthy and Vissing-Jorgensen (2012), Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) finds that convenience yields help to partly explain why government debt appears to be valued more than the PDV of future surpluses. This leaves open the possibility that convenience yields are much larger, as suggested by Jiang, Krishnamurthy, and Lustig (2018).

at each horizon $k = 1, \ldots, \infty$, where e_s select the first row of z_t, and e_d select the second row of z_t. These restrictions can be restated without the state vector:

$$\boldsymbol{e}_s' \sum_{j=1}^{k} \beta^j \Psi^j = \boldsymbol{e}_d'(I - \beta^k \Psi^k), \text{ for } k = 1, \ldots, \infty.$$

We can restate this expression as follows:

$$\boldsymbol{e}_s'(I - \beta\Psi)^{-1}\beta\Psi(I - \beta^k\Psi^k) = \boldsymbol{e}_d'(I - \beta^k\Psi^k), \text{ for } k = 1, \ldots, \infty. \tag{5}$$

which collapses to a single restriction:

$$\boldsymbol{e}_s'(I - \beta\Psi)^{-1}\beta\Psi = \boldsymbol{e}_d'. \tag{6}$$

The long-run discounted forecast of the future surpluses, a linear combination of the state variables, has to equal the debt at all times. As a result, this imposes a cross-equation restriction on the dynamics of the state variables. If this cross-equation restriction is not imposed, then the dynamics of the surplus will violate this restriction.

We can back out the forecast of future surpluses implied by the debt dynamics, $\boldsymbol{e}_d'(I - \beta^k\Psi^k)\boldsymbol{z}_t$, and compare it to actual realized surpluses, even in a model without surpluses in the state vector.

In their seminal paper on testing the government budget constraint, Hansen, Roberds, and Sargent (1991) derive a different condition for the case of risk-free debt when devising an econometric approach to testing the budget constraint:

$$(\mathbb{E}_t - \mathbb{E}_{t-1})[\sum_{j=1}^{\infty} \beta^j S_{t+j}] = \boldsymbol{e}_s' \sum_{j=1}^{\infty} \beta^j \Psi^j \boldsymbol{\Gamma} \varepsilon_t = 0,$$

for all innovations. If the debt is truly risk-free, its value cannot respond to the news about the surplus at time t. This implies that:

$$\boldsymbol{e}_s'(I - \beta\Psi)^{-1}\boldsymbol{\Gamma} = 0$$

Our restriction in Equation 6 applies regardless of whether the debt is risk-free.

4.2 Dynamic Asset Pricing Model

Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) use a conditionally log-normal Gaussian model to price a claim to future surpluses. This model is equiped to price a menu of assets, including bond and stocks. We assume that the $N \times 1$ vector of state variables $\boldsymbol{z}$ follows a Gaussian first-order VAR:

$$\boldsymbol{z}_t = \boldsymbol{\Psi} \boldsymbol{z}_{t-1} + \boldsymbol{u}_t = \Psi \boldsymbol{z}_{t-1} + \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\varepsilon}_t, \tag{7}$$

with $N \times N$ companion matrix Ψ and homoskedastic innovations $\boldsymbol{u}_t \sim i.i.d. \mathcal{N}(0, \Sigma)$. The Cholesky decomposition of the covariance matrix, $\boldsymbol{\Sigma} = \Sigma^{\frac{1}{2}} \left(\Sigma^{\frac{1}{2}} \right)'$, has non-zero elements on and below the diagonal. In this way, shocks to each state variable u_t are linear combinations of its own structural shock $\boldsymbol{\varepsilon}_t$, and the structural shocks to the state variables that precede it in the VAR, with $\boldsymbol{\varepsilon}_t \sim i.i.d. \mathcal{N}(0, I)$.

By including the log of debt/output ratio in the state vector $\boldsymbol{z}_t$, we assume that the government commits to a debt policy, and that this commitment is credible. The log of the debt/output ratio is affine in the state variables. As we will show, this rules out affine spending and tax policies when we impose the government's intertemporal budget constraint.

Motivated by affine term structure model developed by Ang and Piazzesi (2003), we specify an exponentially affine stochastic discount factor (SDF). The nominal SDF $M^{\$}_{t+1} = \exp(m^{\$}_{t+1})$ is conditionally log-normal:

$$m^{\$}_{t+1} \quad = \quad -y^{\$}_t(1) - \frac{1}{2} \boldsymbol{\Lambda_t}' \boldsymbol{\Lambda_t} - \boldsymbol{\Lambda_t}' \boldsymbol{\varepsilon_{t+1}}, \tag{8}$$

The real SDF is $M_{t+1} = \exp(m_{t+1}) = \exp(m^{\$}_{t+1} + \pi_{t+1})$; it is also conditionally Gaussian. The priced sources of risk are the structural innovations in the state vector ε_{t+1} from equation (7). These aggregate shocks are associated with a $N \times 1$ market price of risk vector $\boldsymbol{\Lambda_t}$ of the affine form:

$$\boldsymbol{\Lambda_t} = \boldsymbol{\Lambda_0} + \boldsymbol{\Lambda_1} z_t,$$

The $N \times 1$ vector $\boldsymbol{\Lambda_0}$ collects the average prices of risk while the $N \times N$ matrix $\boldsymbol{\Lambda_1}$ governs the time variation in risk premia. Asset pricing in this model amounts to estimating the market prices of risk in $\boldsymbol{\Lambda_0}$ and $\boldsymbol{\Lambda_1}$.

4.3 Testing the No-Bubble Condition

We need to solve for the debt dynamics under the risk-neutral measure. Given the government's commitment to a debt policy, we can price debt strips, which are claims to debt at a future date. The implied present discounted value of the future debt divided by current output is affine in the state vector $\boldsymbol{z}_t$ in this model:

$$\log P^D_t(k) = \frac{\mathbb{E}_t[M_{t,t+k} D_{t+k}]}{Y_t} = A^d_0(k) + A^d_1(k) \boldsymbol{z}_t.$$

This result is derived by pricing debt strips. For the current debt strip (at maturity zero), we have that:

$$\log P_t^D(0) = \log \frac{D_t}{Y_t} = A_0^d(0) + A_1^d(0) z_t,$$

where $A_0^d(0) = \mu_d$ is the unconditional mean log debt/output ratio and $A_1^d(0) = \boldsymbol{e}_d$, a vector of zeros except for a one in the location of log debt/output in the VAR. The coefficients $A_0^d(k+1)$ and $A_1^d(k+1)$ follow a system of ODEs obtained by verifying the Euler equation:

$$\begin{aligned} A_0^d(k+1) &= -y_0^{\$}(1) + x_0 + \pi_0 + A_0^d(k) + \frac{1}{2}(e_x + e_\pi + A_1^d(k))'\Sigma(e_x + e_\pi + A_1^d(k)) \\ &\quad -(e_x + e_\pi + A_1^d(k))'\Sigma^{\frac{1}{2}}\Lambda_0 \\ A_1^d(k+1) &= (e_x + e_\pi + A_1^d(k))'\Psi - e_{yn}' - (e_x + e_\pi + A_1^d(k))'\Sigma^{\frac{1}{2}}\Lambda_1 \end{aligned}$$

The derivation is in subsection A.3 of the Appendix.

In the model version in which we allow for structural break in the mean debt/output ratio μ_d before and after 2007, we have two sets of coefficients $(A_0^d(k), A_1^d(k))$, one before and one after the break.

To fix ideas, consider the case in which the government commits to a constant debt/output ratio. The loadings solve the following recursions:

$$\begin{aligned} A_0^d(1) &= -y_0^{\$}(1) + x_0 + \pi_0 + \mu_d + \frac{1}{2}(e_x + e_\pi + \mathbf{0})'\Sigma(e_x + e_\pi + \mathbf{0}) \\ &\quad -(e_x + e_\pi + \mathbf{0})'\Sigma^{\frac{1}{2}}\Lambda_0, \\ A_1^d(1) &= (e_x + e_\pi + \mathbf{0})'\Psi - e_{yn}' - (e_x + e_\pi + \mathbf{0})'\Sigma^{\frac{1}{2}}\Lambda_1. \end{aligned}$$

The one-period debt strip has an exposure of 1 to real GDP growth and to inflation; see the expression for $A_1^d(1)$. Hence, nominal output risk is priced in the debt strip, even though the debt itself is risk-free. The risk premium of the k-horizon debt strip is given by:

$$\begin{aligned} \mathbb{E}_t\left[r_{t+1}^d(k)\right] - y_{t,1}^{\$} + \frac{1}{2}V_t\left[r_{t+1}^d(k)\right] &= -Cov_t\left[m_{t+1}^{\$}, r_{t+1}^d(k)\right] \\ &= (A_1^d(k) + e_x + e_\pi)\Sigma^{\frac{1}{2}}\Lambda_t. \end{aligned}$$

This is the same risk premium as the one that obtains for a nominal output strip k periods hence.

With the debt strip prices in hand, we can check the TVC. The TVC requires that:

$$\lim_{k\to\infty} P_t^D(k) = \lim_{k\to\infty} A_0^d(k) + \lim_{k\to\infty} A_1^d(k) z_t = 0.$$

Blanchard (2019) has argued that the U.S. has ample debt capacity to fund additional spending by rolling over its debt because interest rates are below GDP growth rates. Because of the risk premia, the TVC can be satisfied even when the average nominal GDP growth rate exceeds the average short rate: $x_0 + \pi_0 > y_0^{\$}(1)$.

Proposition 4.1. When the risk prices of all innovations are zero, $\Lambda_0 = \mathbf{0}$, then the TVC is violated $\lim_{k\to\infty} P_t^D(k) \to \infty$ when $x_0 + \pi_0 > y_0^{\$}(1)$.

To see why, note that:

$$A_0^d(k+1) \quad \geq \quad (k+1)(x_0 + \pi_0 - y_0^{\$}(1)) + \mu_d.$$

The right hand side explodes as $k \to \infty$ if $x_0 + \pi_0 > y_0^{\$}(1)$. When none of the risk is priced, Blanchard's condition a sufficient condition for violations of the TVC. However, if output risk is not priced, then the value of a claim to GDP divided by current GDP would explode as well.

4.4 Debt-Implied PDV of Surpluses

In the first model, which we label DAPM-D, the government commits to a particular debt policy, and the spending/taxes will adjust.

Proposition 4.2. When the government commits to an affine debt policy and the debt is priced fairly (DAPM-D), the present value of future surpluses at any horizon k implied by the debt dynamics can be stated as a function of the state vector $\boldsymbol{z}_t$:

$$\frac{\mathbb{E}_t \sum_{j=1}^{k} M_{t,t+j} S_{t+j}}{Y_t} = D_t/Y_t - P_t^D(k) = \exp(\boldsymbol{e}_d' \boldsymbol{z}_t) - \exp(A_0^d(k) + A_1^d(k)\boldsymbol{z}_t). \tag{9}$$

This is the bond market's view of future surpluses when the government commits to an affine policy for the debt/output ratio. We can back out the PDV of future surpluses at all horizons. We refer to this as the debt-implied PDV of surpluses.

At the infinite horizon, we obtain the following result.

Corollary 4.3. When the TVC holds and the government commits to a debt policy (DAPM-D), the value of debt reveals PDV of surpluses.

$$\begin{aligned}
\frac{\mathbb{E}_t \sum_{j=1}^{\infty} M_{t,t+j} S_{t+j}}{Y_t} &= D_t/Y_t - \lim_{k\to\infty} \mathbb{E}_t[M_{t,t+k} D_{t+k}]/Y_t \\
&= D_t/Y_t - \lim_{k\to\infty} P_t^D(k) \\
&= D_t/Y_t = \exp(\boldsymbol{e}_d' \boldsymbol{z}_t)
\end{aligned}$$

Today's value of debt (relative to current output) reveals the PDV of all future surpluses (relative to current output).

4.5 Cash Flow-Implied PDV of Surpluses

In the second model, which we label DAPM-TG, the government commits to a spending and a tax policy. We stipulate government tax and spending rules as functions of the state, and directly measure the PDV of future surpluses. This is the approach followed by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019). In this approach, the debt/output ratio is merely included in the state vector as a predictor variable.

Let $PD_t^j(h)$ denote the price-dividend ratio of the tax revenue strip ($j = T$) or spending strip ($j = G$) with maturity h (Wachter, 2005; van Binsbergen, Brandt, and Koijen, 2012). Then, the price-dividend ratios on the tax claim and the spending claim are the sum of the price-dividend ratios of their strips, whose logs are affine in the state vector z_t:

$$PD_t^T = \frac{P_t^T}{T_t} = \sum_{h=0}^{\infty} \exp(A_\tau(h) + B_\tau'(h) z_t), \tag{10}$$

$$PD_t^G = \frac{P_t^G}{G_t} = \sum_{h=0}^{\infty} \exp(A_g(h) + B_g'(h) z_t). \tag{11}$$

The proof is in Appendix A.5 and A.6.

Proposition 4.4. When the government commits to spending and tax policies that are affine in the state vector and the debt is priced fairly (DAPM-TG), the PDV of surpluses can be stated for each $k = 1, \ldots, \infty$ as:

$$\frac{\mathbb{E}_t \sum_{j=1}^{k} M_{t,t+j} S_{t+j}}{Y_t} = = \frac{T_t}{Y_t} \sum_{h=1}^{k} (\exp(A_\tau(h) + B_\tau'(h) \mathbf{z}_t) - \frac{G_t}{Y_t} \sum_{h=1}^{k} (\exp(A_g(h) + B_g'(h) \mathbf{z}_t). \tag{12}$$

We refer to this as the cash flow-implied PDV of surpluses.

4.6 Contrasting the Two Approaches

Corollary 4.5. Under the null that the debt is priced fairly, when the government commits to affine debt, spending and tax policies, the following cross-equation restrictions need to hold, for all $\mathbf{z}_t$:

$$\begin{aligned} &\exp(\mathbf{e}_d' \mathbf{z}_t) - \exp(A_0^d(k) + A_1^d(k) \mathbf{z}_t) \\ &= \exp(\log \tau_t) \sum_{h=1}^{k} (\exp(A^\tau(h) + B^{\tau\prime}(h) \mathbf{z}_t) - \exp(\log g_t) \sum_{h=1}^{k} (\exp(A^g(h) + B^{g\prime}(h) \mathbf{z}_t). \end{aligned} \tag{13}$$

In our log-linear model, each of these cross-equation restrictions depend on $\mathbf{z}_t$. When the government commits to a debt policy in the DAPM-D, then the spending and tax policies cannot both be log-linear in the state. Conversely, when the government commits to these spending and tax policies in the DAPM-TG, the process for the debt/output ratio cannot be affine. This means that we cannot then also assume that the government commits to an independent debt policy.

As we take the limit as $k \to \infty$ and impose the TVC, we end up with the following condition:

$$\exp(e_d' z_t) = \exp(\log \tau_t) \sum_{h=1}^{k} (\exp(A^{\tau}(h) + B^{\tau\prime}(h) z_t) - \exp(\log g_t) \sum_{h=1}^{k} (\exp(A^{g}(h) + B^{g\prime}(h) z_t). \quad (14)$$

In this log-normal model, the horizon-dependent restrictions in Equation 13 do not collapse the single restriction in Equation 14 because the model is log-linear rather than linear. Equation 14 is the implication tested by Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).

5 Quantitative Implications

To explore the model's quantitative implications, we need to (i) take a stance on the time-series properties of revenue and spending, and (ii) a stochastic discount factor $M_{t,t+j}$ to discount these cash flows. We specify the quantitative model as in Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019).

5.1 Cash Flow Dynamics

Table 3 summarizes the variables we include in the state vector, in order of appearance of the VAR.

Table 3: State Variables

Position	Variable	Mean	Description
1	π_t	π_0	Log Inflation
2	x_t	x_0	Log Real GDP Growth
3	$y_t^{\$}(1)$	$y_0^{\$}(1)$	Log 1-Year Nominal Yield
4	$yspr_t^{\$}$	$yspr_0^{\$}$	Log 5-Year Minus 1-Year Nominal Yield Spread
5	pd_t	$\overline{pd}$	Log Stock Price-to-Dividend Ratio
6	Δd_t	μ_d	Log Stock Dividend Growth
7	$\Delta \log \tau_t$	μ_τ	Log Tax Revenue-to-GDP Growth
8	$\log \tau_t$	$\log \tau_0$	Log Tax Revenue-to-GDP Level
9	$\Delta \log g_t$	μ_g	Log Spending-to-GDP Growth
10	$\log g_t$	$\log g_0$	Log Spending-to-GDP Level
11	$\Delta \log d_t$	μ_d	Log Debt-to-GDP Growth
12	$\log d_t$	$\log d_0$	Log Debt-to-GDP Level

The vector z contains the state variables demeaned by their respective sample averages. The VAR estimates for the companion matrix Ψ and $\gamma^{\frac{1}{2}}$ are reported in Appendix .

This VAR imposes co-integration between debt and output, spending and output, and taxes and output. We allow for a structural break in 2007 in the the debt/output ratio by including the demeaned log debt/output ratio, where the demeaning takes out a different mean pre-2007 and post-2007. Recall Figure 2. In section 2, we found that the introducing a break restores some of the link between fundamentals and the debt/output ratio.

Figure 3 and Figure 4 show the VAR-implied forecasts of the log tax revenue/output ratio and the log government spending/output ratio. The first columns is the VAR model without debt, the second column is for the VAR model that adds the debt/output equations to the VAR, and the third column is for the model with debt/output and the structural break in the mean debt/output ratio. The legend in each panel reports the root mean-squared forecast error at horizons of 5-years (top row), 10-years (middle row), and 20-years (bottom row). Adding debt to the VAR does not meaningfully improve the forecast errors of tax revenues in the long-run, while it improves the forecasts of spending somewhat. The forecasting performance for $\Delta \log \tau$ and $\Delta \log g$ improves slightly when using the VAR system with debt and structural break compared to the VAR system with debt and no structural break.

Figure 5 shows the predictions for future debt/output ratios. The VAR with a break in the debt/output ratio in 2007 substantially outperforms the VAR with debt and the VAR without debt when it comes to forecasting the long-run evolution of debt. The long-run forecast errors for $\Delta \log(debt/GDP)$ in the VAR with debt and structural break are much lower than those in the VAR with debt.

5.2 SDF

We estimate the market prices of risk parameters to best fit a set of bond and stock moments, following Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019). The resulting point estimates for the market prices of risk are reported in Appendix B.2.

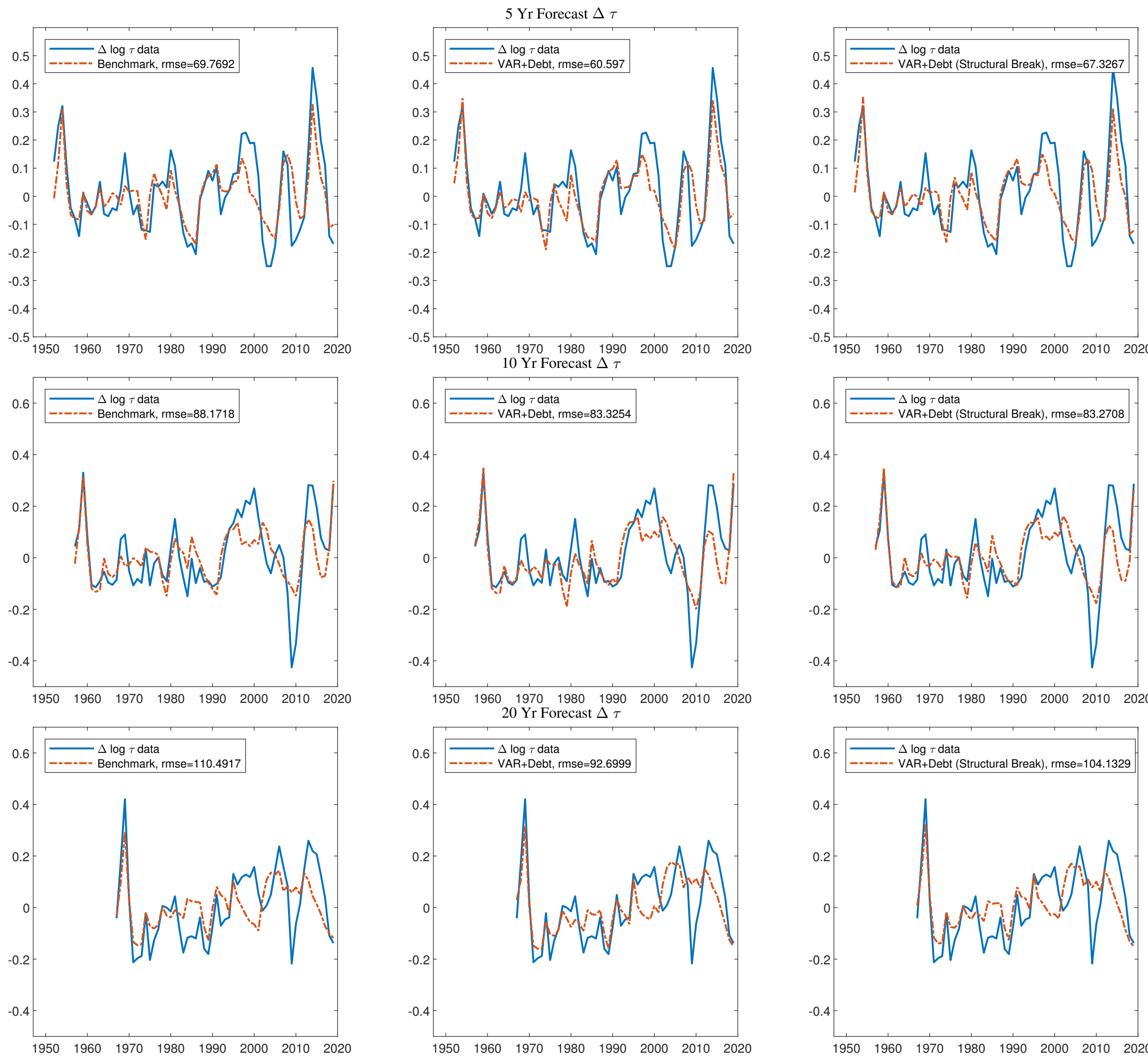
5.3 Comparing PDV of Surpluses

Based on the estimation of the VAR system with debt (with structural break in the mean debt/output ratio) and the estimated market prices of risks, we can now compare the debt-implied PDV of surpluses in Equation 9 in the DAPM-D model to the cash flow-implied PDV of surpluses in Equation 12 in the DAPM-TG model.

We start with the DAPM-D model. The risk-neutral debt dynamics pin down the surplus dynamics under the risk-neutral surpluses. Figure 6 plots the present value of future surpluses implied by the debt dynamics over different horizons k ranging from 1 to 500 years. All variables plotted are divided by current GDP. At short horizons of 5 years, the model predicts surpluses until 1960, after which the model predicts deficits between 1970 and the early 1980s, followed by predicted surpluses until the late 90s. At the start of the financial crisis in 2008, the debt dynamics imply large deficits; the PDV of future debt increases by more than the current debt.

As we examine longer horizons by increasing k, $P_t^D(k)$ converges to zero, because of the TVC, and this risk-neutral forecast of future surpluses converges to the actual market value of debt. What this means is that bond market investors expect large future surpluses.

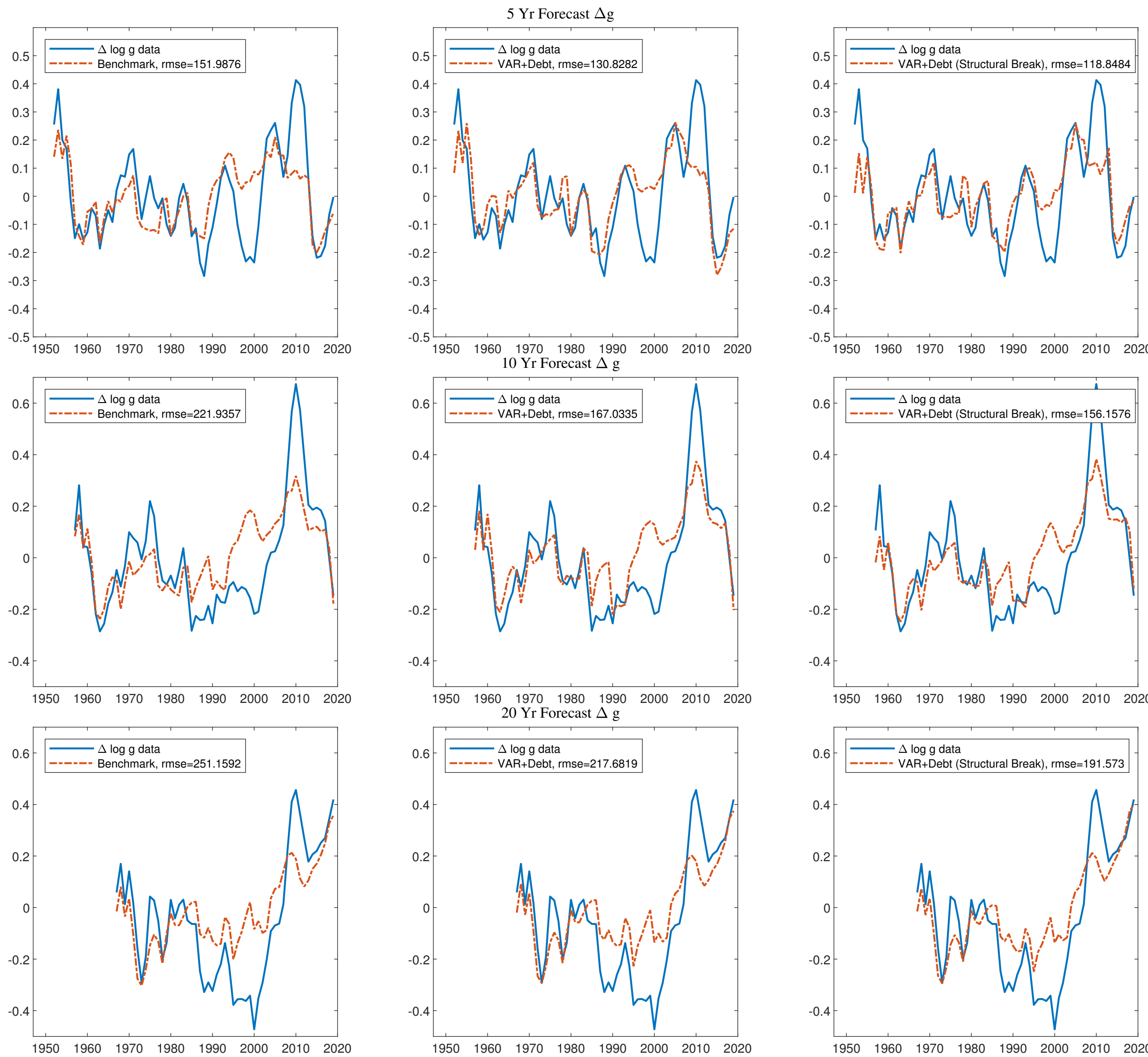
Figure 3: Long Term Forecast for $\Delta\tau$



We plot the actual log tax growth rates over 5-year, 10-year and 20-year rolling windows in solid blue lines. The value at each year represents the k-year growth rates that end at that year. We also plot these rates as forecasted by the benchmark model (the first column), the rates forecasted by the VAR with debt/gdp ratio (the middle column), and these rates as forecasted by the VAR with debt/gdp ratio and structural change (the last column). The value at each year represents the k-year growth rates condition on the information k years ago.

Figure 7 plots the cash flow-implied PDV of surpluses from the DAPM-TG model (solid line), and compares those against the debt-implied PDV of surpluses from the DAPM-D model (dashed line). At the 5-year horizon, the debt dynamics seems to track the direct valuation of the surpluses rather well. However, after the 2008 crisis, the debt-implied PDV of surpluses is 20% of GDP larger than the cash flow-implied PDV of surpluses. For the past decade, debt markets have been

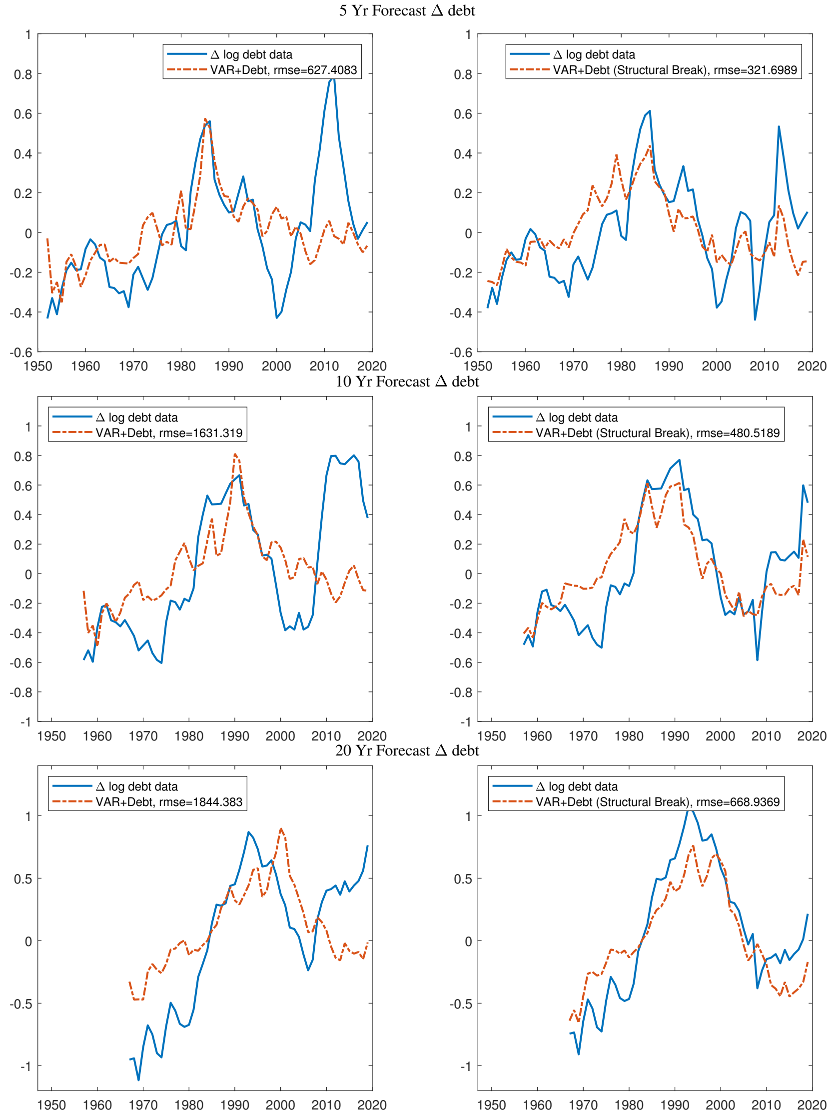
Figure 4: Long Term Forecast for Δg



We plot the actual log spending growth rates over 5-year, 10-year and 20-year rolling windows in solid blue lines. The value at each year represents the k-year growth rates that end at that year. We also plot these rates as forecasted by the benchmark model (the first column), the rates forecasted by the VAR with debt/gdp ratio (the middle column), and these rates as forecasted by the VAR with debt/gdp ratio and structural change (the last column). The value at each year represents the k-year growth rates condition on the information k years ago.

forecasting higher surpluses over the next 5 years than those implied by the cash flow dynamics. Because the debt/output ratio is above its risk-adjusted mean, risk-neutral mean reversion in the debt/output ratio generates a fast return to positive surpluses coming out of the 2008 crisis. But the cash flow dynamics tell a different story; they suggest surpluses will remain depressed in negative territory for years to come. If we had not allowed for a structural break in the mean

Figure 5: Long Term Forecast for $\Delta \log(debt/GDP)$

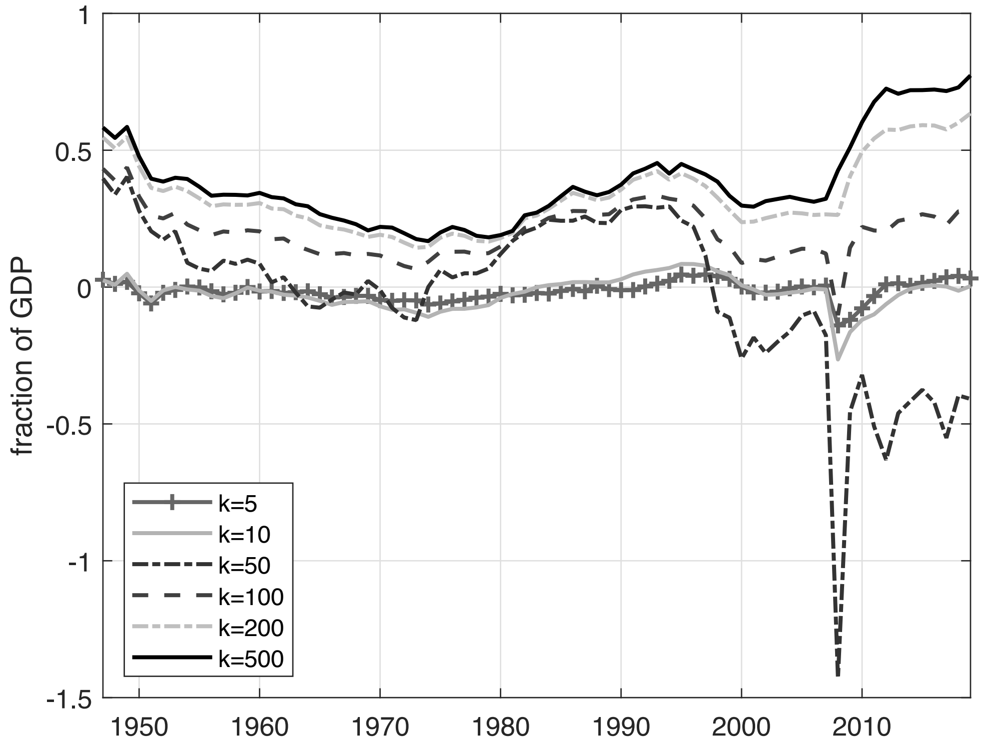


We plot the actual log debt/gdp growth rates over 5-year, 10-year and 20-year rolling windows in solid blue lines. The value at each year represents the k-year growth rates that end at that year. We also plot these rates as forecasted by the benchmark model (the first column), the rates forecasted by the VAR with debt/gdp ratio (the middle column), and these rates as forecasted by the VAR with debt/gdp ratio and structural change (the last column). The value at each year represents the k-year growth rates condition on the information k years ago.

debt/output ratio in 2007, the DAPM-D model would have implied mean-reversion to a lower mean and even larger implied surpluses.

At longer horizons ($k \geq 25$), the two measures diverge even earlier, in the late 1990s. The debt dynamics in DAPM-D imply small deficits over the next 50 years, but the cash flow dynamics in DAPM-TG imply deficits that grow to five times GDPs at the end of the sample. At the 200-year

Figure 6: PDV of Surpluses Implied by Debt Dynamics



Plot of the PDV of future surpluses divided by GDP, measured by the debt-to-GDP ratio minus the present value of the debt in k years $D_t/Y_t - P_t^D(k)$ divided by GDP in Equation 9. (DAPM-D)

horizon, the gap widens to ten GDPs.

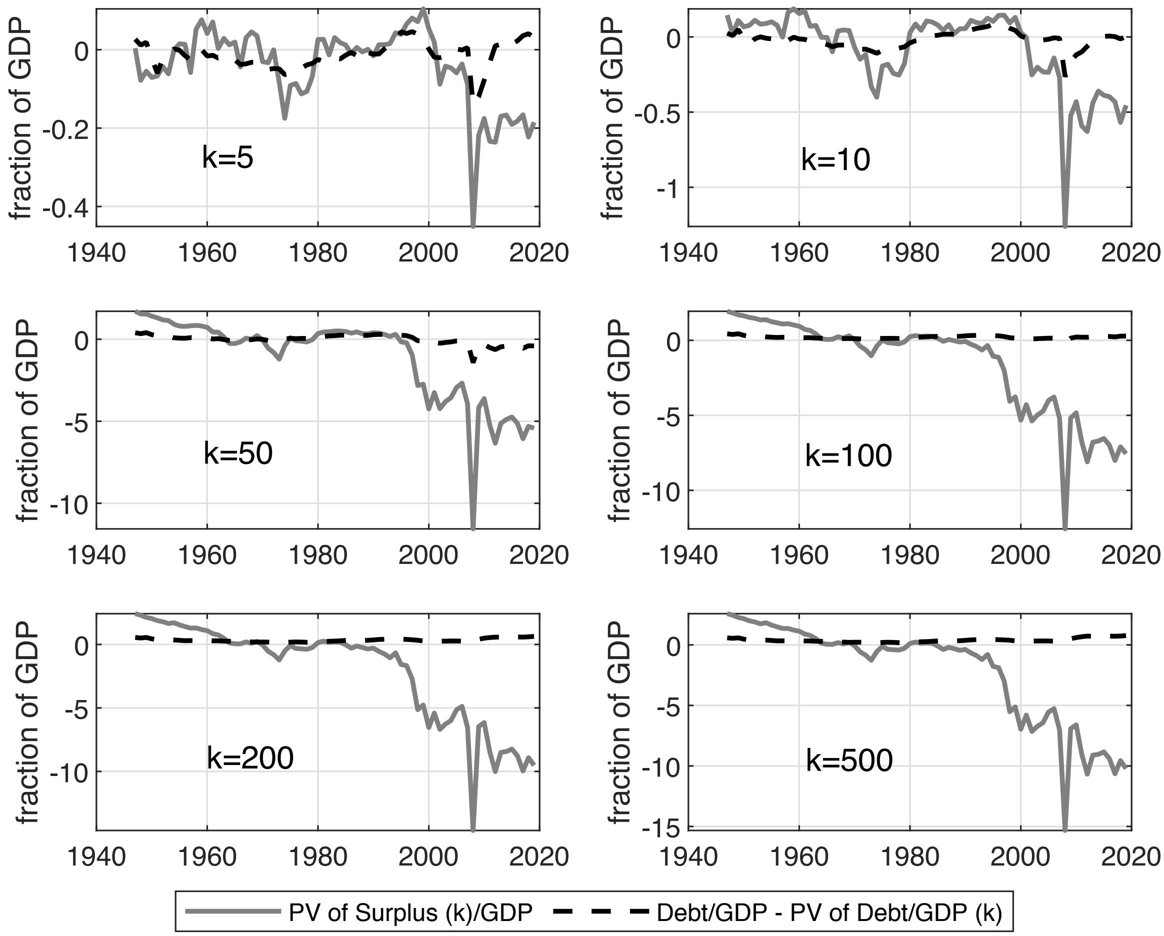
Figure 8 plots the gap between the PDV of surpluses from the DAPM-D and DAPM-TG models over horizons of 1, 5, 2, 50, 100, and 500 years. The gap is increasing in absolute value in horizon k. As the horizon k increases, the size of the violation converges to the size of what Jiang, Lustig, Van Nieuwerburgh, and Xiaolan (2019) call the government debt valuation puzzle.

5.4 Comparing to Realized Surpluses

How do these surplus forecasts in the DAPM-D and DAPM-TG models stack up against the *realized* surpluses? To construct the PDV of realized surpluses, we discount the realized surpluses off the Treasury yield curve, and we fill in the data between 2020 and 2030 using the CBO budget forecasts.

Figure 9 plots the PDV of realized surpluses (fill line) against the PDV of forecasted surpluses (dashed and dashed-dotted lines) over the next five years. The red line is constructed using the CBO forecasts to fill in missing data. All variables are scaled by current GDP. The PDV of forecasts implied by cash flow dynamics in the DAPM-TG tracks the PDV of realized surpluses fairly well, and more closely than the debt-implied PDV in DAPM-D. The latter is too low in the first half of

Figure 7: PDV of Surpluses Implied by Debt and Cash Flow Dynamics

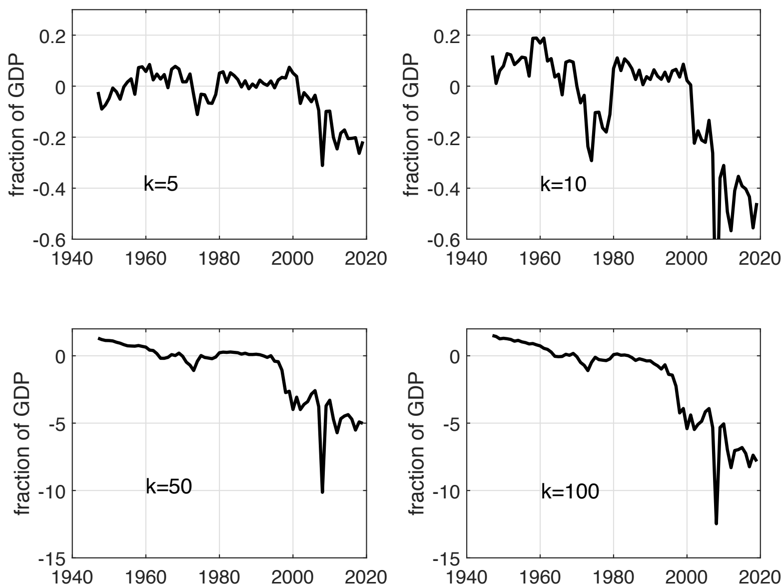


The dashed line is the present value of future surpluses over a k-year horizon implied by the debt dynamics: the debt-to-GDP ratio minus the present value of the debt in k years $D_t/Y_t - P_t^D(k)$ in Equation 9 (DAPM-D). The solid line is the present value of future surpluses over a k-year horizon implied by the cash flow dynamics: $\tau_t \sum_{h=1}^{k} P_t^{\tau}(h) - g_t \sum_{h=1}^{k} P_t^{G}(h)$ in Equation 12 (DAPM-TG). Sample: 1947 to 2019.

the sample, and arguably too high in the second half of the sample. In the late 1990s and the 2000s, even the cash flow-implied forecasts systematically over-predict realized surpluses in the run-up to the financial crisis. Following the 2008 crisis, there is a marked gap between the debt-implied and the cash-flow-implied forecasts. The debt measure predicts a speedy return to surpluses. The cash flow-implied measure does not, but rather stays at a 5-year cumulative deficit of 20% of GDP. Like the cash flow-implied surpluses, the realized surpluses (full line) do not feature the sharp reversal implied by the debt dynamics.

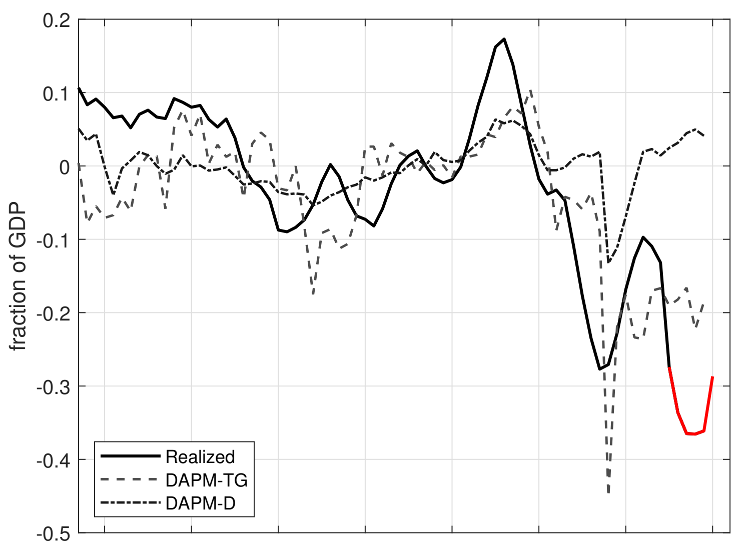
Next, we plot the 10-year PDV of realized surpluses (full line) against the PDV of forecasted surpluses (dashed and dashed-dotted lines) in Figure 10. We observe the same pattern. The cash flow-implied forecasts do rather well when plotted against the PDV of realized 10-year cumulative (discounted) surpluses. Again, the debt-based measure implies a reversal after the 2008 crisis to

Figure 8: PDV of Wedges over Horizon k



The figure shows the PDV of wedges over horizon of k years: $\mathbb{E}_t \sum_{j=0}^{k} M_{t,t+j} D_{t+j} \eta_{t+j}$ in **??**.

Figure 9: PDV of 5-Year Discounted Realized and Model-Implied Surpluses

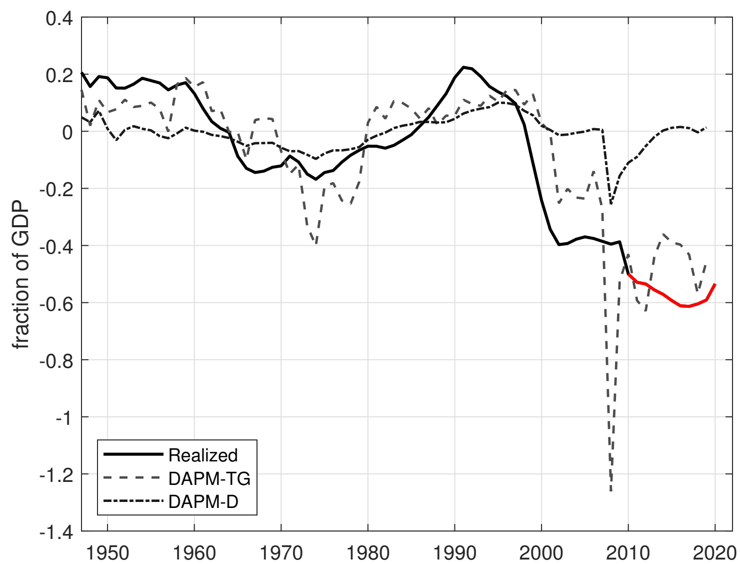


The full line plots the realized discounted surpluses over the next 5 years. The red line is constructed using the Sept 2020 CBO forecasts. The dashed-dotted line is the present value of future surpluses over a k-year horizon implied by the debt dynamics: the debt-to-GDP ratio minus the present value of the debt in 5 years $D_t/Y_t - P_t^D(k)$ in Equation 9 (DAPM-D). The dashed line is the present value of future surpluses over a 5-year horizon implied by the cash flow dynamics: $\tau_t \sum_{h=1}^{k} P_t^{\tau}(h) - g_t \sum_{h=1}^{k} P_t^{G}(h)$ in Equation 12 (DAPM-TG). Sample: 1947 to 2019.

0% of GDP. However, the realized surpluses and the cash-flow based forecasts remain at around -50% of GDP.

Since the mid-1990's, the debt-implied surpluses have been too optimistic. In other words, the realized surpluses have been consistently lower than what bond market investors would have expected. This points to irrationality, beliefs that deviate from rational expectations, on the part of bond investors.

Figure 10: PDV of 10-Year Discounted Realized and Model-Implied Surpluses



The full line plots the realized discounted surpluses over the next 10 years. The red line is constructed using the Sept 2020 CBO forecasts. The dashed-dotted line is the present value of future surpluses over a k-year horizon implied by the debt dynamics: the debt-to-GDP ratio minus the present value of the debt in 10 years $D_t/Y_t - P_t^D(k)$ in Equation 9 (DAPM-D). The dashed line is the present value of future surpluses over a 10-year horizon implied by the cash flow dynamics: $\tau_t \sum_{h=1}^{k} P_t^{\tau}(h) - g_t \sum_{h=1}^{k} P_t^{G}(h)$ in Equation 12 (DAPM-TG). Sample: 1947 to 2019.

5.5 Reverse Engineering PDV of Taxes

We can impose the cross-equation restrictions in Equation 13 and use them to back out the $PDV(taxes)$:

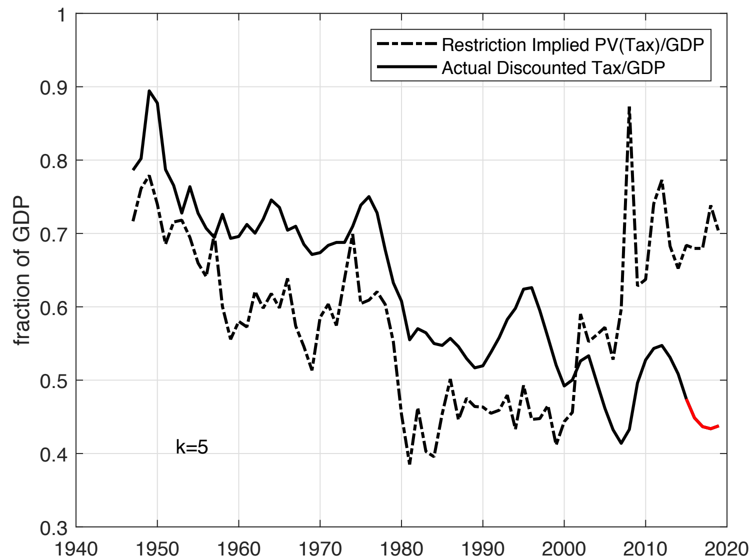
Corollary 5.1. If the government commits to a policy for spending and debt (DAPM-DG) and the debt is fairly priced, then the implied PDV of taxes is given by:

$$\begin{aligned} PDV_t[T]/Y_t &= \exp(e_d' z_t) - \exp(A_0^d(k) + A_1^d(k) z_t) \\ &+ \exp(\log g_t) \sum_{h=1}^{k} (\exp(A^g(h) + B^{g\prime}(h) z_t). \end{aligned} \tag{15}$$

This means we back out the right tax process to enforce the government's budget constraint. We call this the DAPM-DG model.

Figure 11 plots the implied PDV of tax revenues over the next five years divided by current output against the discounted value of realized tax revenues to current output. Figure 12 does the same for ten-year cumulative tax revenues.

Figure 11: PDV of 5-Year Discounted Realized and Model-Implied Taxes



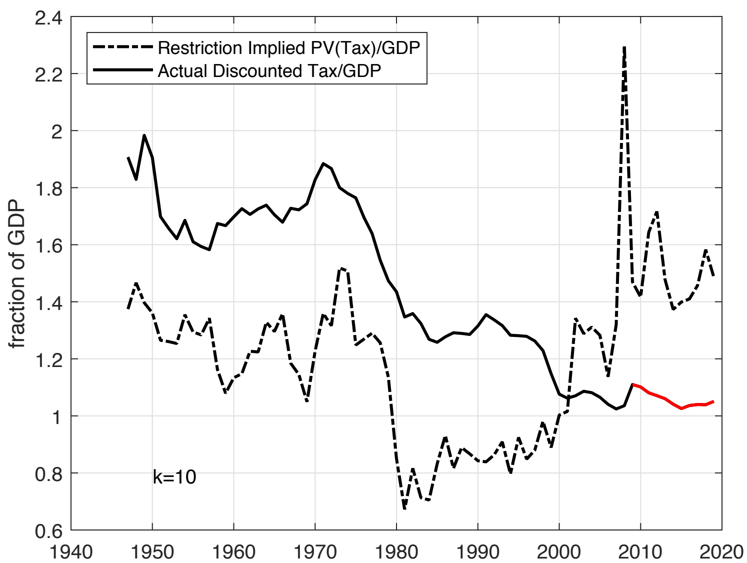
The dashed line plots the realized discounted surpluses over the next 5 years, discounted off the Treasury yield curve. The red line is constructed using the Sept 2020 CBO forecasts. The full line is the present value of future taxes over a 5-year horizon reverse engineered from Equation 15 to enforce the cross-equation no arbitrage restriction.

Over the first six decades of our sample, the PDV of realized taxes exceeds the valuation to the tax stream imputed by our DAPM-DG model. The tax claim is quite risky, due to its pro-cyclicality and its cointegration with GDP in the long run. As a result, the model tends to overstate the PDV of taxes when discounting it off the risk-free yield curve. However, starting in the late 90s, the gap between these 2 measures closes. The implied PDV of taxes over the next 10 years surpasses the realized tax measure by almost 40% of GDP at the end of 2019. This is an understatement of the true gap since realized future tax revenues should not be discounted at the risk-free rate but at a higher rate that reflects their risk.

6 Conclusion

Bond market investors pricing the debt of the federal government of the United States appear to systematically overstate future primary surpluses, even over long horizons. We find little evidence in the post-WW II sample that higher debt/output ratios are followed by subsequent surpluses.

Figure 12: PDV of 10-Year Discounted Realized and Model-Implied Taxes



The dashed line plots the realized discounted surpluses over the next 10 years, discounted off the Treasury yield curve. The red line is constructed using the Sept 2020 CBO forecasts. The full line is the present value of future taxes over a 10-year horizon reverse engineered from Equation 15 to enforce the cross-equation no arbitrage restriction.

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A Proofs

A.1 Proof of Proposition 3.1

Proof. We consider an investor who buys the entire government debt portfolio at the end of period t. Her total return is given by the value of the debt at the end of the period $t+1$ plus the cash flow that accrues to the investor which consists of the primary surplus:

$$R^D_{t+1} = \frac{D_{t+1} + T_{t+1} - G_{t+1}}{D_t}.$$

As a result, we can state the primary surplus at $t+1$ as follows: $S_{t+1} = D_t R^D_{t+1} - D_{t+1}$, and the primary surplus at $t+2$ is given by: $S_{t+2} = D_{t+1} R_{t+2} - D_{t+2}$. Next, we can compute the present discounted value of surpluses in the current and next period:

$$S_{t+1} + \mathbb{E}_{t+1}[M_{t+2} S_{t+2}] = D_t R^D_{t+1} - D_{t+1} + D_{t+1}\mathbb{E}_{t+1}[M_{t+2} R_{t+2}] - \mathbb{E}_t[M_{t+2} D_{t+2}].$$

If there is an Euler equation wedge for the entire debt portfolio $\mathbb{E}_{t+1}[M_{t+1,t+2} R_{t+2}] = 1 + \eta_{t+1}$, we end up with the following expression:

$$S_{t+1} + \mathbb{E}_{t+1}[M_{t+1,t+2} S_{t+2}] = D_t R^D_{t+1} + \eta_{t+1} D_{t+1} - \mathbb{E}_{t+1}[M_{t+1,t+2} D_{t+2}].$$

By the same logic, the PDV of the first 3 surpluses is given by:

$$\begin{aligned} & S_{t+1} + \mathbb{E}_{t+1}[M_{t+1,t+2} S_{t+2}] + \mathbb{E}_{t+1}[M_{t+1,t+3} S_{t+3}] \\ = \; & D_t R^D_{t+1} + \eta_{t+1} D_{t+1} + \mathbb{E}_{t+1}[M_{t+1,t+2} \eta_{t+2} D_{t+2}] - \mathbb{E}_{t+1}[M_{t+1,t+3} D_{t+3}], \end{aligned}$$

where we have used $\mathbb{E}_{t+2}[M_{t+2,t+3} R_{t+3}] = 1 + \eta_{t+2}$.

Multiplying by $M_{t,t+1}$ and taking expectations at time t, we obtain the following general result:

$$\mathbb{E}_t \sum_{j=1}^{k} M_{t,t+j} S_{t+j} = D_t - \mathbb{E}_t[M_{t+1,t+k} D_{t+k}] + \mathbb{E}_t \sum_{j=0}^{k-1} M_{t,t+j} D_{t+j} \eta_{t+j}.$$

□

A.2 Proof of Corollary 3.2:

Proof. Follows immediately from Proposition 3.1. □

A.3 Present Value of Future Debt

We conjecture that the log discounted value of the future debt to output ratio is affine in the state vector $\boldsymbol{z}_t$:

$$\log P^D_t(k) = \frac{\mathbb{E}_t[M_{t,t+k} D_{t+k}]}{Y_t} = A^d_0(k) + A^d_1(k)\boldsymbol{z}_t.$$

For the strip at zero, we have that:

$$\log P^D_t(0) = \log \frac{D_t}{Y_t} = A^d_0(0) + A^d_1(0)\boldsymbol{z}_t.$$

Hence, note that $A^d_0(0) = \mu(d)$ and $A^d_1(0) = e_d$, where we assume that d is demeaned in the state vector, and $\mu(d)$ denotes the mean.

We solve for the coefficient $A_0^d(k+1)$ and $A_1^d(k+1)$ by verifying the Euler equation:

$$\begin{aligned} P_t^D(k+1) &= \mathbb{E}_t\left[M_{t,t+1}P_{t+1}^D(k)\frac{Y_{t+1}}{Y_t}\right] = \mathbb{E}_t\left[\exp\{m_{t+1}^{\$}+x_{t+1}+\pi_{t+1}+\log(P_{t+1}^D(k))\}\right] \\ &= \exp\{-y_0^{\$}(1)-e_{yn}'z_t-\frac{1}{2}\Lambda_t'\Lambda_t+x_0+\pi_0+(e_x+e_\pi+A_1^d(k))'\Psi z_t+A_0^d(k)\} \\ &\quad\times\mathbb{E}_t\left[\exp\{-\Lambda_t'\epsilon_{t+1}+(e_x+e_\pi+A_1^d(k))'\Sigma^{\frac{1}{2}}\epsilon_{t+1}\}\right] \end{aligned}$$

We substitute for the affine expression for Λ_t and use the log-normality of ϵ_{t+1}, and obtain:

$$\begin{aligned} P_t^D(k+1) &= \exp\{-y_0^{\$}(1)-e_{yn}'z_t+x_0+\pi_0+(e_x+e_\pi+A_1^d(k))'\Psi z_t+A_0^d(k) \\ &\quad+\frac{1}{2}(e_x+e_\pi+A_1^d(k))'\Sigma(e_x+e_\pi+A_1^d(k))-(e_x+e_\pi+A_1^d(k))'\Sigma^{\frac{1}{2}}(\Lambda_0+\Lambda_1 z_t)\} \end{aligned}$$

Taking logs and collecting terms, we get

$$\begin{aligned} A_0^d(k+1) &= -y_0^{\$}(1)+x_0+\pi_0+A_0^d(k)+\frac{1}{2}(e_x+e_\pi+A_1^d(k))'\Sigma(e_x+e_\pi+A_1^d(k)) \\ &\quad-(e_x+e_\pi+A_1^d(k))'\Sigma^{\frac{1}{2}}\Lambda_0 \qquad (16) \\ A_1^d(k+1) &= (e_x+e_\pi+A_1^d(k))'\Psi-e_{yn}'-(e_x+e_\pi+A_1^d(k))'\Sigma^{\frac{1}{2}}\Lambda_1 \qquad (17) \end{aligned}$$

Given that we have a structural break with 2 different values for $\mu(d)$, we will have two solutions for (A_0^d, A_1^d), one before and one after the break. In the case of constant debt/output ratio, we need to change the initial conditions to $A_0^d(0)=\mu(d)$ and $A_1^d(0)=\mathbf{0}$. Without re-estimating the model parameters, we can also solve for (A_0^d, A_1^d) using the unconditional mean for the entire sample $\mu(d)$.

A.4 Proof of Proposition 4.1

Proof. We consider the case of a constant debt/output ratio. We set $\Lambda_0=\mathbf{0}$ and $\Lambda_1=\mathbf{0}$. From the recursions for these coefficients, it follows that:

$$\begin{aligned} A_0^d(0) &= \mu_d, \\ A_1^d(0) &= \mathbf{0} \\ A_0^d(1) &= -y_0^{\$}(1)+x_0+\pi_0+\mu_d+\frac{1}{2}(e_x+e_\pi+\mathbf{0})'\Sigma(e_x+e_\pi+\mathbf{0}) \\ A_0^d(k+1) &= -y_0^{\$}(1)+x_0+\pi_0+A_0^d(k)+\frac{1}{2}(e_x+e_\pi+A_1^d(k))'\Sigma(e_x+e_\pi+A_1^d(k)) \end{aligned}$$

Next, note that the variance terms are positive. This implies that:

$$A_0^d(k+1) \geq (k+1)(x_0+\pi_0-y_0^{\$}(1))+\mu_d.$$

This term tends to $+\infty$ when $x_0+\pi_0>y_0^{\$}(1)$. Note that the proof generalizes to any debt policy.

It is easy to how that the second loading is given by:

$$A_1^d(k)=(e_x+e_\pi)'\sum_{j=1}^{k}\Psi^j-ke_{yn}'.$$

This keeps track of forecasted nominal GDP growth minus the nominal short rates over the horizon, conditional on the state variable. However, the state variables $\boldsymbol{z}_t$ have mean zero. Hence, on average, these terms will not matter for the TVC.

□

A.5 Spending Claim

Nominal government spending growth equals

$$\Delta \log G_{t+1} = \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^g + \left(e_{\Delta g} + e_x + e_\pi\right)' z_{t+1}. \tag{18}$$

We conjecture the log price-dividend ratios on spending strips are affine in the state vector:

$$P_t^G(h) = \log\left(P_t^G(h)\right) = A^g(h) + B^{g\prime}(h) z_t.$$

We solve for the coefficients $A^g(h+1)$ and $B^g(h+1)$ in the process of verifying this conjecture using the Euler equation:

$$\begin{aligned}
P_t^G(h+1) &= \mathbb{E}_t\left[M_{t+1} P_{t+1}^G(h) \frac{G_{t+1}}{G_t}\right] = \mathbb{E}_t\left[\exp\{m_{t+1}^{\$} + \Delta \log g_{t+1} + x_{t+1} + \pi_{t+1} + P_{t+1}^G(h)\}\right] \\
&= \exp\{-y_0^{\$}(1) - e_{yn}' z_t - \frac{1}{2}\Lambda_t' \Lambda_t + \mu^g + x_0 + \pi_0 + (e_{\Delta g} + e_x + e_\pi + B^g(h))' \Psi z_t + A^g(h)\} \\
&\quad \times \mathbb{E}_t\left[\exp\{-\Lambda_t' \varepsilon_{t+1} + \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)' \Sigma^{\frac{1}{2}} \varepsilon_{t+1}\right].
\end{aligned}$$

We use the log-normality of ε_{t+1} and substitute for the affine expression for Λ_t to get:

$$\begin{aligned}
P_t^G(h+1) &= \exp\{-y_0^{\$}(1) + \mu^g + x_0 + \pi_0 + ((e_{\Delta g} + e_x + e_\pi + B^g(h))' \Psi - e_{yn}') z_t + A^g(h) \\
&\quad + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)' \Sigma \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\
&\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)' \Sigma^{\frac{1}{2}} (\Lambda_0 + \Lambda_1 z_t)\}
\end{aligned}$$

Taking logs and collecting terms, we obtain

$$\begin{aligned}
A^g(h+1) &= -y_0^{\$}(1) + \mu^g + x_0 + \pi_0 + A^g(h) + \frac{1}{2}\left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)' \Sigma \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right) \\
&\quad - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \\
B^g(h+1)' &= (e_{\Delta g} + e_x + e_\pi + B^g(h))' \Psi - e_{yn}' - \left(e_{\Delta g} + e_x + e_\pi + B^g(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_1,
\end{aligned}$$

and the price-dividend ratio of the cum-dividend spending claim is

$$\sum_{h=0}^{\infty} \exp(A^g(h+1) + B^g(h+1)' z_t)$$

A.6 Revenue Claim

Nominal government revenue growth equals

$$\Delta \log T_{t+1} = \Delta \log \tau_{t+1} + x_{t+1} + \pi_{t+1} = x_0 + \pi_0 + \mu_0^\tau + \left(e_{\Delta\tau} + e_x + e_\pi\right)' z_{t+1}. \tag{19}$$

where $\tau_t = T_t / GDP_t$ is the ratio of government revenue to GDP. Note that this ratio is assumed to have a long-run growth rate of zero. This imposes cointegration between government revenue and GDP. The growth ratio in this ratio can only temporarily deviate from zero.

The remaining proof exactly mirrors the proof for government spending, with

$$p\tau_t \equiv \log\left(\frac{P_t^{\tau,ex}}{T_t}\right) = \log\left(\frac{P_t^T}{T_t} - 1\right) = \sum_{h=0}^{\infty} \exp(A^\tau(h+1) + B^\tau(h+1)' z_t), \tag{20}$$

where A^{τ} and B^{τ} are as follows:

$$\begin{aligned} A^{\tau}(h+1) &= -y_0^{\$}(1) + \mu^{\tau} + x_0 + \pi_0 + A^{\tau}(h) + \frac{1}{2}\left(e_{\Delta\tau} + e_x + e_{\pi} + B^{\tau}(h)\right)' \Sigma \left(e_{\Delta\tau} + e_x + e_{\pi} + B^{\tau}(h)\right) \\ &\quad - \left(e_{\Delta\tau} + e_x + e_{\pi} + B^{\tau}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_0, \\ B^{\tau}(h+1)' &= \left(e_{\Delta\tau} + e_x + e_{\pi} + B^{\tau}(h)\right)'\Psi - e'_{yn} - \left(e_{\Delta\tau} + e_x + e_{\pi} + B^{\tau}(h)\right)' \Sigma^{\frac{1}{2}} \Lambda_1. \end{aligned}$$

B Quantitative Model

B.1 VAR Estimates

The log(debt/GPD) series may not be stationary. Using the augmented Dickey-Fuller test cannot reject the null hypothesis of the presence of the unit root in the log debt-to-GDP ratio in our sample period. In the OLS estimates of the VAR, the coefficient on lagged debt in the debt equation exceeds 1 ($\Psi_{12,12} = 1.046 > 1$). To deal with the issue of non-stationarity, we allow for structural breaks. We detect a structural break in the debt/output ratio around 2007 (See Figure 2). Chow tests for several potential structural breakpoints for the debt-to-GDP ratio from 1947 to 2019 show that we can only reject the null hypothesis of no structural break in 2007 at the 1% level. We demean the debt-to-GDP ratio with two different sample (before and after 2007) average, and then re-estimate the VAR, following recent work by Lettau and Van Nieuwerburgh (2008). The Ψ and $\gamma^{\frac{1}{2}}$ matrices for the VAR with the structural break-adjusted debt series are reported in Table 4. Note that now $\Psi_{12,12} = 0.96 < 1$.

Table 4: VAR with debt-to-GDP Ratio

Ψ =

	$l.\pi$	$l.x$	$l.r$	$l.tp$	$l.dp$	$l.dd$	$l.\Delta\tau$	$l.\tau$	$l.\Delta g$	$l.g$	$l.\Delta debt$	$l.debt$
π	0.375	-0.085	0.028	-0.302	-0.008	0.034	0.066	-0.035	-0.015	-0.016	0.001	-0.036
x	-0.095	0.191	0.376	0.236	0.021	0.060	-0.028	-0.064	0.057	0.077	-0.037	0.042
r	0.042	0.031	0.888	0.027	0.004	0.033	0.002	-0.037	-0.011	0.021	-0.019	-0.004
tp	-0.072	-0.085	-0.044	0.501	-0.008	-0.022	0.013	0.018	0.013	-0.024	0.018	-0.007
dp	-2.588	-1.209	0.278	2.397	0.773	-0.212	-0.075	0.191	0.062	-0.255	-0.111	-0.028
dd	0.315	0.257	-0.425	-1.651	0.053	0.157	-0.085	-0.167	-0.125	0.103	0.254	0.060
$\Delta\tau$	-0.689	0.470	0.622	-3.898	0.106	0.106	0.267	-0.539	0.154	0.256	0.131	0.091
τ	-0.689	0.470	0.622	-3.898	0.106	0.106	0.267	0.461	0.154	0.256	0.131	0.091
Δg	-0.445	0.089	-2.176	-1.207	-0.170	-0.223	0.173	0.136	0.278	-0.535	0.123	-0.214
g	-0.445	0.089	-2.176	-1.207	-0.170	-0.223	0.173	0.136	0.278	0.465	0.123	-0.214
$\Delta debt$	0.191	-0.850	0.830	4.622	-0.022	-0.237	0.070	0.013	-0.006	0.026	-0.071	-0.040
$debt$	0.191	-0.850	0.830	4.622	-0.022	-0.237	0.070	0.013	-0.006	0.026	-0.071	0.960

$100 \times \gamma^{\frac{1}{2}}$

	$l.\pi$	$l.x$	$l.r$	$l.tp$	$l.dp$	$l.dd$	$l.\Delta\tau$	$l.\tau$	$l.\Delta g$	$l.g$	$l.\Delta debt$	$l.debt$
π	0.918	0	0	0	0	0	0	0	0	0	0	0
x	0.569	1.843	0	0	0	0	0	0	0	0	0	0
r	0.369	0.503	1.208	0	0	0	0	0	0	0	0	0
tp	-0.104	-0.194	-0.277	0.428	0	0	0	0	0	0	0	0
dp	-3.403	-2.162	1.242	1.142	14.914	0	0	0	0	0	0	0
dd	-0.090	1.245	0.865	-1.315	-0.961	4.649	0	0	0	0	0	0
$\Delta\tau$	2.340	1.764	0.104	-0.460	1.083	0.246	4.514	0	0	0	0	0
τ	2.340	1.764	0.104	-0.460	1.083	0.246	4.514	0	0	0	0	0
Δg	-0.895	-2.178	-1.313	-0.425	0.030	-1.170	-0.054	0	3.326	0	0	0
g	-0.895	-2.178	-1.313	-0.425	0.030	-1.170	-0.054	0	3.326	0	0	0
$\Delta debt$	-2.524	-3.298	-1.321	-0.284	2.656	2.105	-0.669	0	0.284	0	6.396	0
$debt$	-2.524	-3.298	-1.321	-0.284	2.656	2.105	-0.669	0	0.284	0	6.396	0

$\Sigma^{\frac{1}{2}}$ is the Cholesky decomposition of the residual variance-covariance matrix. It is multiplied by 100 for readability.

B.2 Market Prices of Risks

We estimate the market prices of risks and report the estimates in Table 5.

Table 5: Market Prices of Risks

Λ_0 =	0	0.173	-0.375	0.034	0	1.558	0	0	0	0	0	0

	0	0	0	0	0	0	0	0	0	0	0	0
	0	33.428	0	0	0	0	0	0	0	0	0	0
	0	0	-14.182	-95.476	0	0	0	0	0	0	0	0
	-4.181	-15.777	-6.165	-3.443	-0.672	1.859	-0.300	-4.767	-0.181	-2.262	-3.995	-3.132
	0	0	0	0	0	0	0	0	0	0	0	0
Λ_1=	-39.426	-16.107	-18.200	50.462	-4.426	-0.246	-1.965	-0.592	-1.701	-3.654	2.985	-0.197
	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0