Optimal Mortgage Design*

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Abstract

This paper studies the optimal mortgage design in a continuous time setting with volatile and privately observable income, costly foreclosure, and a stochastic market interest rate. We show that the optimal mortgage takes the form of an option adjustable rate mortgage (option ARM). The default rates and interest rates on the optimal mortgage correlate positively with the market interest rate. Gains from using the optimal contract relative to simpler mortgages are substantial and are the biggest for those who face substantial income variability, buy pricey houses given their income level or make little or no downpayment. Our model thus may help to explain a high concentration of option ARMs among riskier borrowers.

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1 Introduction

Recent years have seen a rapid growth in originations of more sophisticated alternative mortgage products (AMPs), such as option adjustable rate mortgages (option ARMs) and interest only mortgages. In the United States, from 2003 through 2005, the originations of AMPs grew from less than 10% of residential mortgage originations to about 30%.$^1$ As of the first half of 2006, 37%$^2$ of mortgage originations were AMPs. Option adjustable rate mortgages experienced particularly fast growth. They accounted for as little as 0.5% of all mortgages written in 2003, but their share soared to at least 12.3% through the first five months of 2006.$^3$ Option ARMs have been frequently marketed to borrowers with substantial income variability and to less wealth borrowers to allow them to purchase homes they otherwise might not be able to afford with a conventional mortgage.$^4$ Consequently, this form of lending has been concentrated among those who buy pricey houses given their income level, make little or no downpayment, and face substantial income variability.

Unlike traditional fixed rate mortgages (FRMs) and adjustable rate mortgages (ARMs), option ARMs let borrowers pay only the interest portion of the debt or even less than that, while the loan balance can grow above the amount initially borrowed, resulting in negative amortization. Once the borrower’s debt balance reaches a maximum allowed level, a default occurs. Interest rates on such loans can increase as interest rates in the economy move higher, resulting in increased risk of delinquencies and defaults among borrowers.

Because of concern about increased delinquencies and defaults among borrowers, these forms of borrowing have generated great controversy and criticism. Critics contend that option ARMs can hurt borrowers by increasing their indebtedness over time and exposing them to high interest payments in the future.$^5$ On the other hand, proponents claim that option ARMs are more efficient than traditional mortgages because they allow both lenders and borrowers to manage their cash flows intelligently.$^6$

Surprisingly, despite the economic significance of AMPs and the extent of the surrounding controversy, there has been no attempt so far to formally address whether these new mortgages are welfare improving for borrowers and lenders relative to traditional mortgages.

In this paper, we formally approach this issue by addressing a more general normative question. Assuming rational behavior of borrowers and lenders, what is the best possible mortgage contract between a home buyer and a financial institution? Instead of considering a particular class of mortgages, we derive an optimal mortgage contract as a solution to a general dynamic contracting problem without imposing restrictive assumptions on the payments between the borrower and the lender or the circumstances under which the home is repossessed. We then compare features of existing mortgage contracts with the derived best possible contract.

We focus our attention on a simple setting that nevertheless allows us to capture three aspects that we

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$^3$ Data from LoanPerformance, an industry tracker unit of First American Real Estate Solutions (FARES).
$^6$ See for example Gerardi, Rosen and Willen (2007).
believe are central to address the optimality of AMPs. These are: (i) a risky borrower who needs to be given incentives to repay his debt, (ii) costly liquidation and (iii) stochastic interest rate. This setting allows us to focus on the fundamental feature of the borrowing-lending relationship with collateral in a stochastic interest rate environment, which is how to efficiently provide a borrower with incentives to repay his debt using a threat of a costly liquidation taking into account a time-varying cost of capital.

Specifically, we consider a continuous-time setting in which a borrower with limited liability needs outside financial support from a risk-neutral lender in order to purchase a house. Home ownership generates for the borrower a public and deterministic utility stream. We assume that the borrower must use his available funds to first cover the necessary expenses, which are given by an exogenous stochastic process, before spending on "discretionary" consumption or potential debt repayment. The distribution of the "excess" income, which the borrower can use to pay back his debt, is publicly known, but its realizations are privately observable by the borrower.

We assume the borrower and the lender have different discount rates. The borrower’s discount rate $\gamma$ represents his intertemporal consumption preferences and is constant over time. On the other hand, the lender, a big financial institution, discounts future cash flows using a stochastic market interest rate $r_t$. We assume that the borrower is more impatient than the lender, i.e., $\gamma > r_t$, reflecting that a borrowing-constrained household has a higher marginal rate of substitution than a financial institution. To the best of our knowledge, this is the first paper that allows for a stochastic interest rate in an optimal dynamic security design setting.

There is a liquidation technology that allows termination of the relationship and transfer of the house to the lender. This transfer of ownership leads to inefficiencies due to associated deadweight costs. For simplicity, we assume that the value of the home remains constant over time.

Before purchase of the house, the borrower and the lender sign a contract that will govern their relationship after the sale. The contract obligates the borrower to report his income to the lender and specifies transfers between the borrower and the lender and the circumstances under which the lender would foreclose the loan and seize the home, conditional on the history of the borrower’s reports. While the borrower’s reports cannot be verified, the threat of losing the home can be used to induce the borrower to pay his debt.

We solve for the optimal contract between the borrower and the lender, i.e., the contract that maximizes the borrower’s utility subject to a certain payoff to the lender. The optimal contract is characterized using two state variables: the market interest rate $r_t$, and the borrower’s continuation utility $a_t$, i.e., the expected payoff to the borrower at time $t$ provided he acts optimally given the terms of his contract with the lender. Under the optimal contact, the borrower finds it optimal to truthfully report his income. The borrower defaults on the loan and the home is repossessed when the borrower’s continuation utility $a_t$ hits the borrower’s reservation utility $A$ for the first time. The borrower consumes part of his excess income whenever $a_t$ reaches the upper boundary $a^1 (r_t)$. When $a_t \in [A, a^1 (r_t)]$, all the excess income of the borrower is transferred to the lender and so he enjoys no discretionary consumption in this region. The borrower’s continuation utility increases (decreases) when his excess income realization is high (low).

Interestingly, when the interest rate $r_t$ switches from high to low, the borrower’s continuation utility
jumps up under the optimal contract. On the other hand, when the interest rate $r_t$ switches from low to high, the borrower's continuation utility jumps down, which, in certain cases, can trigger immediate liquidation. This is optimal because the stream of borrower's payments is more valuable to the lender when the market interest rate is low, because they are discounted at the low interest rate. As a result, the chances of home repossession are reduced by moving the borrower's continuation utility further away from the default boundary $A$ when the interest rate switches to low. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, the optimal contract increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low state. This is done by moving the borrower's continuation utility closer to the default boundary $A$ when the interest rate switches to high.

The optimal option ARM mortgage takes the form of a credit line with a portion of the balance subject to a low preferential interest rate. On the remaining part of the balance, a variable rate is charged which positively correlates with the market interest rate. Reflecting the changes in the borrower's continuation utility, the balance subject to the preferential rate increases when the interest rate switches from high to low and decreases when the interest rate switches from low to high. The borrower can further indebt himself to finance the interest rate payments or his consumption as long as his debt balance is below the credit limit. In general, the borrower enjoys a higher credit limit when the market rate is low and vice versa. The borrower is in default if he is unable to make mortgage payments without exceeding the credit limit. In this case, the lender forecloses the loan and seizes ownership of the home.

Under the optimal option ARM, it is optimal for the borrower to use his excess income to make the current interest rate payments and to repay his debt balance. When the borrower realization of the excess income is low, the borrower increases his debt balance to finance interest payments. If the borrower repays a sufficient amount of debt, so that all his remaining balance is subject to the low preferential interest rate, he spends part of his excess income on discretionary consumption.

The features of the optimal mortgage contract can be explained by the incentive-compatibility constraints and the dual optimization objective of the contracting problem: minimization of liquidation inefficiencies and maximization of the gains of trade due to the differences in discount factors between the borrower and the lender.

The credit line feature of the optimal mortgage provides flexibility for the borrower to cover possible low income realizations, which in turn lowers chances of default inefficiencies. The credit limit is determined by the incentive compatibility constraints, so that the borrower is always indifferent between paying down the mortgage and drawing the credit line to its limit and defaulting immediately. The borrower does not need to maintain precautionary savings, because the credit commitments by the lender provide a safety net. The borrower defaults only after receiving a sufficient amount of negative shocks to his necessary spending or his total income.8

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7 If $a_t$ is substantially high, the borrower's continuation utility can jump in the opposite direction. However, efficiency gains from such jumps are small. As a result, our approximate implementation does not incorporate jumps in the opposite direction. 
8 This result is consistent with empirical evidence pointing that consumer delinquency problems are mainly the result of unexpected negative events, that neither the lender nor the borrower could have anticipated at the time the credit request was
The adjustable features of the mortgage - interest rates, the preferential balance and the credit limit - implement the adjustments of the borrower's continuation utility when the market interest rate changes. It is optimal to reduce the chances of default when the market interest rate is low because the stream of borrower's payments is more valuable to the lender. As a result, when the interest rate is low the borrower takes advantage of the low interest payments and the higher credit limit. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. Consequently, the borrower faces higher interest rate, lower credit limit, and as a result a higher risk of default when the market interest rate is high.

The parameters of the optimal contracts, such as interest rates and credit limits, do not depend on the competitive structure of the lending industry. However, a lender with more bargaining power extracts a bigger surplus by charging a higher underwriting fee. For demonstration purposes, we consider numerical examples in which the lender breaks even.

The optimal contract does not allow the borrower to refinance his mortgage with another lender. Offering this option would limit the ability to provide incentives to the borrower to repay his debt, resulting in a less efficient contract. Therefore, our results lend support to prepayment penalties on refinancing.\footnote{According to the Wall Street Journal (2005) between 40\% and 70\% of option ARMs now carry prepayment penalties.}

This paper demonstrates that the key properties of option ARMs are consistent with the properties of the optimal mortgage contract, which represents a Pareto improvement over traditional mortgages. Thus, our analysis provides a theoretical evidence that AMPs can benefit both lenders and borrowers by managing default timing intelligently and exploiting the gains of trade due to the differences in discount factors between the borrower and the lender in an environment with a stochastic interest rate.

The parametrized examples we consider indicate substantial utility gains from using the optimal mortgage contract compared with more traditional mortgages. Gains from using the optimal contract relative to simpler mortgages are substantial and are the biggest for those who face substantial income variability, buy pricey houses given their income level or make little or no downpayment. Our model thus may help to explain a high concentration of the option ARMs among riskier borrowers.

Related Literature

This paper belongs to the growing literature on dynamic optimal security design, which is a part of the literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990) and Phelan and Townsend (1991), among many others.\footnote{See Sannikov (2006a) for a recent treatment of continuous-time techniques for a principal-agent problem.} The two studies most closely related to ours are DeMarzo and Fishman (2007a) and its continuous-time formulation by DeMarzo and Sannikov (2006). These papers study long-term financial contracting in a setting with privately observed cash flows, and show that the implementation of the optimal contract involves a credit line with a constant interest rate and credit limit, long-term debt, and equity. Biais et al. (2006) study the optimal contract in a stationary version of DeMarzo and Fishman's (2007a) model...
and show that its continuous time limit exactly matches DeMarzo and Sannikov’s (2006) continuous-time characterization of the optimal contract. Tchistyi (2006) considers a setting with correlated cash flows and shows that the optimal contract can be implemented using a credit line with performance pricing. Sannikov (2006b) shows that an adverse selection problem, due to the borrower’s private knowledge concerning the quality of a project to be financed, implies that, in the implementation of the optimal contract, a credit line has a growing credit limit. He (2007) studies the optimal executive compensation in the continuous-time agency model where the manager privately controls the drift of the geometric Brownian motion firm size. Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2007b) offer theoretical analyses of optimal investment and security design in moral hazard environments.

Unlike this paper, none of the above studies considers an environment with a stochastic discount rate. We solve for the optimal contract in the stochastic discount rate environment and find that its implementation involves a variable interest rate charged on the borrower’s debt, as well adjustable preferential debt treatment or balance adjustments, or a combination of both. On the technical side, building on the martingale techniques developed for Lévy processes, we extend DeMarzo and Sannikov (2006) characterization of the optimal contract in a continuous-time setting to a stochastic discount rate environment.

There is a sizeable real estate finance literature that addresses the design of mortgages.\textsuperscript{11} In terms of this literature, to our knowledge, our paper is the first study of optimal mortgage design in a dynamic moral hazard environment, and the first study that addresses the optimality of alternative mortgage products.

There is also a growing literature that focuses on the choice of mortgage contracts and the risk associated with them (see for example, Campbell and Cocco (2003)). This literature restricts its attention to specific class of contracts, however, and thus, unlike this paper, does not address the question of efficient lending. Lustig and Van Nieuwerburgh (2005) focus on limited commitment of the households and study consumption insurance and risk premia in a model with housing collateral. Unlike this paper they do not consider a moral hazard problem and their model does not feature default in equilibrium.

The paper is organized as follows. Section 2 presents the continuous-time setting of the model. Section 3 introduces the dynamic contracting model with a stochastic discount rate. Section 4 derives the optimal contract. Section 5 presents the implementation of the optimal contract. Section 6 discusses the approximate implementation of the optimal contract. Section 7 studies the efficiency gains due to the optimal contract. Section 8 concludes.

\section{Set-up}

Time is continuous and infinite. There is one borrower (a homebuyer) and one lender (a big financial institution).\textsuperscript{12} The lender is risk neutral, has unlimited capital, and values a stochastic cumulative cash flow


\textsuperscript{12}Without loss of generality, we can think about the lender as a group of investors who maximize their combined payoff from the relationship with the borrower. How the investors divide proceeds among themselves is not relevant for the purpose of designing an optimal contract between the borrower and the investors.
where $R_t$ is the market cumulative interest rate at which the lender discounts cash flows that arrive at time $t$. We assume that

$$R_t = \int_0^t r_s ds,$$

where $r$ is an instantaneous interest rate process, which takes values in the set $\{r_L, r_H\}$, where $0 < r_L < r_H$. We assume that $r$ follows a two-state continuous Markov process. Let $N = \{N_t, \mathcal{F}_{1,t}; 0 \leq t < \infty\}$ be a standard compound Poisson process with the intensity $\delta(N_t)$ on a probability space $(\Omega_1, \mathcal{F}_1, m_1)$. Then for $t \geq 0$:

$$r_t(N_t) = \begin{cases} r_0 & \text{if } N_t \text{ is even} \\
r_0^c & \text{if } N_t \text{ is odd} \end{cases},$$

$$\delta(N_t) = \begin{cases} \delta(r_0) & \text{if } N_t \text{ is even} \\
\delta(r_0^c) & \text{if } N_t \text{ is odd} \end{cases},$$

where $r_0 \in \{r_L, r_H\}$ is given, and $r_0^c = \{r_L, r_H\} \setminus \{r_0\}$. The above formulation implies that

$$P[r_{t+s} = r_L \text{ for all } s \in [t, t+\Delta)|r_t = r_L] = e^{-\delta(r_L)\Delta},$$

$$P[r_{t+s} = r_H \text{ for all } s \in [t, t+\Delta)|r_t = r_H] = e^{-\delta(r_H)\Delta}.$$

We assume that the borrower and the lender are sufficiently small so that their actions have no effect on macroeconomic variables such as the market interest rate.\textsuperscript{13}

The borrower’s consumption consists of two categories. The first is "necessary" consumption, which includes grocery food, medicine, transportation, and other goods and services essential for the survival. The cumulative minimum level of necessary consumption is given by an exogenous stochastic process $\{f_t\}$ that incorporates shocks such as medical bills, auto repair costs, fluctuations of food and gasoline prices, and so on. The second is discretionary consumption, which, among many other things, may include such items as restaurant dining, vacation trips, buying a new car, et cetera. We assume that the borrower must use his available funds to first cover the necessary expenses $\eta_t$ before spending on "discretionary" consumption or potential debt repayment.\textsuperscript{14}

\textsuperscript{13}In a general equilibrium framework, actions of mortgage lenders and homebuyers on the aggregate level can affect macroeconomic variables. However, as long as the economic agents on the individual level have no market power, they should regard macroeconomic variables as exogenous in an equilibrium.

\textsuperscript{14}This specification is similar in flavor to the one used by Ait-Sahalia, Parker and Yogo (2004), who propose a partial resolution of the equity premium puzzle by distinguishing between the consumption of basic goods and that of luxury goods. In their model, households are much more risk averse with respect to the consumption of basic goods, of which a certain amount is required in every period, which is consistent with the subsistence aspect of basic goods and the discretionary aspect of luxuries.

7
The borrower values cumulative discretionary consumption $C_t$ as

$$E \left[ \int_0^\infty e^{-\gamma t} dC_t \right],$$

where $dC_t \geq 0$. The zero consumption ($dC_t = 0$) means that the borrower consumes only necessities. We assume that the borrower is more impatient than the lender, i.e., $\gamma > r_t$ for all $t$, reflecting in our setting that the intertemporal marginal rate of substitution for a borrowing-constrained household is greater than those of a financial institution.

Let $\tilde{Y}_t \geq 0$ denote the borrower’s total income up to time $t$. We will focus on the borrower’s "excess" income $Y_t = \tilde{Y}_t - \eta_t$, which represents a better measure of the borrower’s ability to pay for a house than the total income. A standard Brownian motion $Z = \{Z_t, \mathcal{F}_{2,t}; 0 \leq t < \infty \}$ on $(\Omega_2, \mathcal{F}_2, m_2)$ drives the borrower’s income process, where $\{\mathcal{F}_{2,t}; 0 \leq t < \infty \}$ is an augmented filtration generated by the Brownian motion. The borrower’s income up to time $t$, denoted by $Y_t$, evolves according to

$$dY_t = \mu dt + \sigma dZ_t,$$

where $\mu$ is the drift of the borrower’s disposable income and $\sigma$ is the sensitivity of the borrower’s income to its Brownian motion component. The excess income is negative when the necessary expense shock $\eta_t$ is greater than the total income $\tilde{Y}_t$. In this case, the borrower has to cover the deficit by drawing on his saving account or by borrowing more, and if he is no able to do so he will declare bankruptcy. From now on, we will refer to $Y_t$ and $C_t$ simply as the borrower’s income and borrower’s consumption.

We assume that the lender knows $\mu$ and $\sigma$, but does not know realizations the borrower’s excess income shocks $Z_t$, so the borrower has the ability to misrepresent his income. Thus, realizations of the borrower’s income are not contractible. These assumptions are motivated by the observation that lenders use a variety of methods\textsuperscript{15} to determine a type of the borrower (represented here by $(\mu, \sigma)$ pair) before the loan is approved, but henceforth do not condition the terms of the contract on the realizations of the borrower’s income, likely because the borrower’s necessary spending shocks and possibly his total income as well are too costly or impossible to monitor.

The borrower is allowed to maintain a private savings account. The private savings account balance $S$ grows at the interest rate $\rho$, which is adapted to the process $r$, and is such that for all $t$, $\rho_t \leq r_t$. The borrower must maintain a non-negative balance in his account.

The borrower wants to buy a home at date $t = 0$.\textsuperscript{16} We assume that the borrower intends to live in this home forever. The ownership of the home would generate him public and deterministic utility stream $\theta$.\textsuperscript{17} The price $P$ of the home is greater than the borrower’s initial wealth $Y_0$, i.e., $0 < Y_0 < P$.\textsuperscript{18} Thus, the

\textsuperscript{15}Like credit score, current job status and so on.
\textsuperscript{16}To justify the initial purchase of the home, we assume that the borrower extracts more utility from the house when he owns it than when he rents it.
\textsuperscript{17}For simplicity, we do not consider a possibility that the borrower can make modifications that can either increase or decrease the quality of the house.
\textsuperscript{18}The price $P$ is considered as a macroeconomic variable, which is not affected by actions of the borrower and the lender. It
borrower must obtain funds from the lender to finance the purchase of the home.

Before purchase of the house, the borrower and the lender sign a contract that will govern their relationship after the purchase is made. The borrower has to report his income to the lender. However, the lender has no way to verify the borrower’s reports. The contract specifies transfers between the borrower and the lender, conditional on the history of the borrower’s reports and the circumstances under which the lender would foreclose the loan and seize the home.

The borrower has the option to default at any time. If the borrower defaults or violates the terms of the contracts, he loses the home and receives his reservation value equal to $A$. Reservation value $A$ represents the borrower’s continuation utility after the loss of the home, which incorporates such factors as the consumption value $\xi$ of his expected future income, financial and intangible moving costs, losses associated with the damaged credit history, possible government aid, and the option to buy or rent another home in the future. The lender sells the repossessed house at a foreclosure auction and receives payoff $L$. We assume that the liquidation value $L$ of the house and the borrower’s reservation utility $A$ are low enough: $r_H L + \gamma A < \theta + \mu$, so that it is not efficient to repossess the house, and that $A \geq \frac{\theta}{\gamma}$.

Our objective is to derive an optimal contract between the borrower and the lender, and analyze how the fluctuations of the interest rate and personal income affect the properties of the optimal contract. For the sake of simplicity, our analysis is focused on a time homogeneous environment with $L$ and $A$ being constant over time. We also do not consider a possibility of refinancing or renegotiation of the contract, since this would negatively affect the efficiency of the contract.

3 Contracting Problem

At time 0, the funds needed to purchase the home in the amount of $P - Y_0$ are transferred from the lender to the borrower. A contract, $(\tau, I)$, specifies a termination time of the relationship, $\tau$, and the transfers $I$ between the lender and the borrower that are based on the borrower’s report of his income and the realized interest rate process. Without loss of generality, we assume that the borrower is required to pay the reported income to the lender.\(^\text{19}\)

Let $(\Omega, \mathcal{F}, m) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, m_1 \times m_2)$ be the product space of $(\Omega_1, \mathcal{F}_1, m_1)$ and $(\Omega_2, \mathcal{F}_2, m_2)$. Let $\hat{Y} = \{\hat{Y}_t : t \geq 0\}$ be the borrower’s report of his income, where $\hat{Y}$ is $(Y, r)$-measurable ($\mathcal{F}_t$-measurable).

At any time $0 \leq t \leq \tau$, the contract transfers the reported amount, $\hat{Y}_t$, from the borrower to the lender, and $I_t(\hat{Y}, r)$ from the lender to the borrower. Below we formally define a contract.

**Definition 1** A contract, $\xi = (\tau, I)$, specifies a termination time, $\tau$, and transfers from the lender to the borrower, $I = \{I_t : 0 \leq t \leq \tau\}$, that are based on $\hat{Y}$ and $r$. Formally, $\tau$ is a $(\hat{Y}, r)$-measurable stopping time, is reasonable to expect that the home price $P$ is increasing in its utility $\theta$, and the borrower optimizes over the set of available $(\theta, P)$ pairs. This optimization is not considered in the paper. This clearly does not lead to a loss of generality, since our analysis applies to any $(\theta, P)$ pair.

\(^{19}\)Since we do not impose restrictions on $I$, this assumption does not restrict the space of contracts.
and \( I \) is a \((\hat{Y}, r)\)-measurable continuous-time process, which is such that the process

\[
E \left[ \int_0^\tau e^{-\gamma s} dI_s | \mathcal{F}_t \right]
\]

is square-integrable for \( 0 \leq t \leq \tau \) and \( \hat{Y} = Y \).

The borrower can misreport his income. Consequently, under the contract \( \xi = (\tau, I) \), up to time \( t \leq \tau \), the borrower receives a total flow of income equal to

\[
(dY_t - d\hat{Y}_t) + dI_t,
\]

and his private savings account balance, \( S \), grows according to

\[
dS_t = \rho_t S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t,
\]

where \( dC_t \) is the borrower’s consumption at time \( t \), which must be non-negative. We remember that, for all \( t \geq 0 \), \( S_t \geq 0 \) and \( \rho_t \leq r \). The borrower has the right to terminate the relationship with the lender (default) at any time, in which case he loses the home and gets his outside option \( \hat{Y} \).

In response to a contract \((\tau, I)\), the borrower chooses his income reports \( \hat{Y} \), consumption \( C \), saving \( S \), and quitting time \( \tau^Q \). We remember that when the borrower quits the relationship with the lender he receives the outside value of \( A \).

**Definition 2** Given a contract \( \zeta = (\tau, I) \), a feasible strategy for the borrower is a pair \((\hat{Y}, C, S, \tau^Q)\) such that

(i) \( \hat{Y} \) is a continuous-time process adapted to \((Y, r)\),

(ii) \( C \) is a nondecreasing continuous-time process adapted to \((Y, r)\),

(iii) the savings process defined by (2) stays non-negative: \( S_t \geq 0 \),

(iv) quitting time \( \tau^Q \) is adapted to \((Y, r)\).

**Definition 3** Given a contract \( \zeta = (\tau, I) \), the borrower’s strategy \((\hat{Y}, C, S, \tau^Q)\) is incentive compatible if

(i) it is feasible,

(ii) it provides him with the highest expected utility among all feasible strategies, that is

\[
E \left[ \int_0^{\tau \wedge \tau^Q} e^{-\gamma t}(dC_t + \theta dt) + e^{-\gamma (\tau \wedge \tau^Q)} A | \mathcal{F}_0 \right] \geq E_0 \left[ \int_0^{\tau \wedge \tau^Q} e^{-\gamma t}(dC_t + \theta dt) + e^{-\gamma (\tau \wedge \tau^Q)} A | \mathcal{F}_0 \right]
\]

for all the borrower’s feasible strategies \((\hat{Y}', C', S', \tau'^Q)\), given a contract \( \zeta = (\tau, I) \).
The contract is optimal if for a given time-zero expected utility \( a_0 \) for the borrower no other contract can increase the expected payoff for the lender. Below we provide a formal definition of the optimal contract.

**Definition 4** Given the continuation utility to the borrower, \( a_0 \), a contract \( \zeta = (\tau^*, I^*) \) together with an incentive compatible strategy \((C, \hat{Y}, S, \tau^Q)\) is optimal, if it maximizes the lender’s expected payoff:

\[
b_0 = E \left[ \int_0^{\tau \wedge \tau^Q} e^{-R_t (d\hat{Y}_t - dI_t)} + e^{-R_{\tau \wedge \tau^Q} D_t} L | \mathcal{F}_0 \right],
\]

while the expected utility for the borrower is equal to:

\[
a_0 = E \left[ \int_0^{\tau \wedge \tau^Q} e^{-\gamma t (dC_t + \theta dt)} + e^{-\gamma \tau \wedge \tau^Q} A | \mathcal{F}_0 \right].
\]

The definition of the optimal contract implies maximization of the total surplus, and can be also reformulated in terms of maximizing the borrower’s expected utility for a given payoff of the lender. The initial payoffs \( a_0 \) and \( b_0 \) are determined by the bargaining powers of the borrower and the lender. While our subsequent analysis is valid for any \( a_0 > A \), for our numerical examples (in Section 7) we assume the competitive lending industry, i.e., the lender breaks even: \( b_0 = P - Y_0 \), and the optimal contract maximizes the borrower’s expected utility.

The payoffs for the investor and the borrower depend on minimum of the two stopping times: \( \tau \) and \( \tau^Q \). Without loss of generality, we can write a contract such that \( \tau \leq \tau^Q \). This is because the borrower can always underreport and steal at rate \( \gamma A \) until termination, and any such a strategy yields the borrower at least \( A \), the value he would get by quitting. Therefore, from now on, we will drop the time \( \tau^Q \).

In the following lemma, we show that searching for optimal contracts, we can restrict our attention to contracts in which truth telling and zero savings are incentive compatible.

**Lemma 1** There exists an optimal contract in which the borrower chooses to tell the truth and maintains zero savings.

**Proof** In the Appendix.

The intuition for this result is straightforward. The first part of the result is due to the direct-revelation principle. The second part follows from the fact that it is weakly inefficient for the borrower to save on his private account \((r_t \leq r)\) as any such contract can be improved by having the lender save and make direct transfers to the borrower. Therefore, we can look for an optimal contract in which truth telling and zero savings are incentive compatible.
4 Derivation of the Optimal Contract

In this subsection, we formulate recursively the dynamic moral hazard problem and determine the optimal contract. First, we consider a problem in which the borrower is not allowed to save and we determine the optimal contract\textsuperscript{20} in this environment. We know from Lemma 1 that it is sufficient to look for optimal contracts in which the borrower reports truthfully and maintains zero savings, and so the optimal contract of the problem with no private savings, for a given continuation utility to the borrower, yields to the lender at least as much utility as the optimal contract of the problem when the borrower is allowed to privately save. Finally, we show that the optimal contract of the problem with no private savings is fully incentive compatible, even when the borrower can maintain undisclosed savings.

Methodologically, our approach is based on continuous-time techniques used by DeMarzo and Sannikov (2006). We extend their techniques to a setting with Lévy processes.

4.1 The Optimal Contract without Hidden Savings

Consider for a moment the dynamic moral hazard problem in which the borrower is not allowed to save. First, we will find a convenient state space for the recursive representation of this problem. For this purpose, we define the borrower’s total expected utility received under the contract $V_t$ conditional on his information at time $t$, from transfers and termination utility, if he tells the truth:

$$V_t = E \left[ \int_0^\tau e^{-\gamma s} [dI_s + \theta ds] + e^{-\gamma \tau} A | {\mathcal{F}_t} \right].$$

Lemma 2 The process $V = \{V_t, {\mathcal{F}_t}; 0 \leq t < \tau\}$ is a square-integrable ${{\mathcal{F}_t}}$-martingale.

Proof follows directly from the definition of process $V$ and the fact that this process is square-integrable, which is implied by Definition 1. \hfill $\blacksquare$

Below is a convenient representation of the borrower’s total expected utility received under the contract $\xi = (\tau, I)$ conditional on his information at time $t$, from transfers and termination utility, if he tells the truth. Let $M = \{M_t = N_t - t\delta(N_t), {\mathcal{F}_{1,t}}; 0 \leq t < \infty\}$ be a compensated compound Poisson process.

**Proposition 1** There exists ${{\mathcal{F}_t}}$-predictable processes $(\beta, \psi) = \{(\beta_t, \psi_t); 0 \leq t \leq \tau\}$ such that

$$V_t = V_0 + \int_0^t e^{-\gamma s} \beta_s dZ_s + \int_0^t e^{-\gamma s} \psi_s dM_s =$$

$$= V_0 + \int_0^t e^{-\gamma s} \beta_s \left( \frac{dY_s - \mu ds}{\sigma} \right) + \int_0^t e^{-\gamma s} \psi_s (dN_s - \delta(N_s) ds). \quad (3)$$

\textsuperscript{20}That is the allocation satisfying the properties of Definition 4 and the additional constraint that $S = 0$. 

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Proof We note that the couple \((Z,N)\) is a Brownian-Poisson process, and it is an independent increment process, which is a \(\text{Lévy processes, on the space } (\Omega, \mathcal{F}, m)\). Then, Theorem III.4.34 in Jacod and Shiryaev (2003) gives us the above martingale representation for a square-integrable martingale adapted to \(\mathcal{F}_t\) taking values in a finite dimensional space (the process \(V\)).

According to the martingale representation (3), the total expected utility of the borrower under the contract \(\xi = (\tau, I)\) and truth telling conditional on his information at time \(t\) equals its unconditional expectation plus two terms that represent the accumulated effect on the total utility of, respectively, the income uncertainty revealed up to time \(t\) (Brownian motion part), and the interest rate uncertainty that has been revealed up to time \(t\) (compensated compound Poisson part).

According to Proposition 1, when the borrower reports truthfully, his total expected utility under the contract \(\xi = (\tau, I)\) conditional on the termination time \(\tau\) equals

\[
V_\tau = V_0 + \int_0^\tau e^{-\gamma s} \left( \frac{dY_s - \mu ds}{\sigma} \right) + \int_0^\tau e^{-\gamma s} \psi_s dM_s.
\]

If the borrower deviates and reports \(\hat{Y}\), the contract will result in

\[
I \text{ and } \tau \text{ depend exclusively on the borrower’s report } \hat{Y} \text{ and the public interest rate process } r. \text{ If the borrower misreports his income, he pockets the difference between } Y \text{ and } \hat{Y}, \text{ but the contract treats the borrower as if } \hat{Y} \text{ is his true income. Hence, his expected utility, } a_0(\hat{Y}), \text{ is given by}
\]

\[
a_0(\hat{Y}) = E \left[ V_0 + \int_0^\tau e^{-\gamma t} \hat{Y}_t \left( \frac{dY_t - \mu dt}{\sigma} \right) + \int_0^\tau e^{-\gamma t} \psi_t dM_t + \int_0^\tau e^{-\gamma t} (dY_t - d\hat{Y}_t) | \mathcal{F}_0 \right] - \int_0^\tau e^{-\gamma t} \psi_t dM_t | \mathcal{F}_0.
\]

Note that because the process \((\beta, \psi) = \{(\beta_t, \psi_t); 0 \leq t \leq \tau\}\) is \(\mathcal{F}_t\)-predictable, as for any \(t \geq 0, s \geq 0, E_0[Z_{t+s} - Z_t | \mathcal{F}_0] = E_0[M_{t+s} - M_t | \mathcal{F}_0] = 0\), and given that \(E[V_0 | \mathcal{F}_0] = V_0\), we have that

\[
a_0(\hat{Y}) = V_0 + E \left[ \int_0^\tau e^{-\gamma t} \left( 1 - \frac{\beta_t}{\sigma} \right) (dY_t - d\hat{Y}_t) | \mathcal{F}_0 \right].
\]

(4)

Representation (5) leads us to the following formulation of incentive compatibility.

**Proposition 2** If the borrower cannot save, truth telling is incentive compatible if and only if \(\beta_t \geq \sigma (m - a.s.) \) for all \(t \leq \tau\).

**Proof** Immediately follows from (5).
strategy of the borrower does affect the magnitude of these adjustments, from the perspective of the borrower such adjustments have zero effect on the borrower’s expected utility, whatever his reporting strategy. This property considerably simplifies the formulation of incentive compatibility.

To characterize the optimal contract recursively, we define the borrower’s continuation utility at time \( t \) if he tells the truth as

\[
a_t = E \left[ \int_t^\tau e^{-\gamma(s-t)} \left( dI_s + \theta ds \right) + e^{-\gamma(\tau-t)} A \mathcal{F}_t \right].
\]

Since the borrower can quit at any time, the stopping time \( \tau \) must be chosen so that \( a_t \geq A \) for any \( t \leq \tau \).

Note that for \( t \leq \tau \) we have that

\[
V_t = \int_0^t e^{-\gamma s} (dI_s + \theta dt) + e^{-\gamma t} a_t.
\] (6)

Differentiating (6) and taking into account (3) gives the following law of motion of the borrower’s continuation utility:

\[
da_t = \gamma a_t dt - \theta dt - dI_t + \beta_t dZ_t + \psi_t dM_t = (\gamma a_t - \theta - \psi_t \delta(r_t)) dt - dI_t + \beta_t dZ_t + \psi_t dN_t.
\] (7)

Here we discuss informally, using the dynamic programming approach, how to find out the most efficient way to deliver to a borrower any continuation utility \( a \geq A \). The proof of Proposition 3 formalizes our discussion below. Let \( b(a,r) \) be the highest expected utility of the lender that can be obtained from an incentive compatible contract that provides the borrower with utility equal to \( a \) given that the current interest rate is equal to \( r \). To simplify our discussion we assume that the function \( b \) is concave and \( C^2 \) in its first argument. Let \( b' \) and \( b'' \) denote, respectively, the first and the second derivative of \( b \) with respect to the borrower’s continuation utility \( a \).

The lender can always make a payment \( dI \) to the borrower. The efficiency implies that

\[
b(a,r) \geq b(a - dI,r) - dI,
\] (8)

which shows that for all \( (a,r) \in [A,\infty) \times \{ r_L, r_H \} \) the marginal cost of delivering the borrower his continuation utility can never exceed the cost of an immediate transfer in terms of the lender’s utility, that is

\[
b'(a,r) \geq -1.
\]

Define \( a^1(r), r \in \{ r_L, r_H \} \), as the lowest value of \( a \) such that \( b'(a,r) = -1 \). Then, it is optimal to pay the borrower as follows

\[
dI(a,r) = \max(a - a^1(r),0).
\]

These transfers, and the option to terminate, keep the borrower’s continuation utility between \( A \) and \( a^1(r) \). This implies that when \( a \in [A,a^1(r)] \), and when the borrower is telling the truth, his continuation utility
We need to characterize the optimal choice of process \((\beta_t, \psi_t)\), where \(\frac{\beta}{\sigma}\) determines the sensitivity of the borrower’s continuation utility with respect to his report, and \(\psi_t\) determines the adjustment of the borrower’s continuation utility due to a change in the interest rate. Using Ito’s lemma, we find that

\[
\frac{db(a_t, r_t)}{dt} = (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) dt + \frac{1}{2} \beta_t^2 b''(a_t, r_t) dt + \beta_t b'(a_t, r_t) dZ_t + [b(a_t + \psi_t, r_t^c) - b(a_t, r_t)] dN_t,
\]

where \(r_t^c = \{r_L, r_H\} \setminus \{r_t\}\). Using the above equation, we find that the lender’s expected cash flows and the change in the value he assigns to the contract are given as follows:

\[
E[dY_t + db(a_t, r_t) | F_t] = \left[ \mu + (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \frac{1}{2} \beta_t^2 b''(a_t, r_t) + \delta(r_t) (b(a_t + \psi_t, r_t^c) - b(a_t, r_t)) \right] dt.
\]

From Proposition 2, we know that if \(\beta_t \geq \sigma\) for all \(t \leq \tau\) then the borrower’s best response strategy is to report the truth, that is, \(\hat{Y} = Y\). Because at the optimum, at any time \(t\), the lender should earn an instantaneous total return equal to the interest rate, \(r_t\), we have the following Bellman equation for the value function of the lender

\[
\begin{aligned}
\max_{\beta_t \geq \sigma, \psi_t \geq A - a_t} & \left[ \mu + (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \frac{1}{2} \beta_t^2 b''(a_t, r_t) + \delta(r_t) (b(a_t + \psi_t, r_t^c) - b(a_t, r_t)) \right] \\
\end{aligned}
\]

Given the concavity of the function \(b(\cdot, r_t)\), \(b''(a_t, r_t) = \frac{d^2 b(a_t, r_t)}{da_t^2} \leq 0\), setting

\(\beta_t = \sigma\)

for all \(t \leq \tau\) is optimal. The concavity of the objective function in \(\psi_t\) in the RHS of the Bellman equation (10) also implies that the optimal choice of \(\psi_t\) is given as a solution to

\[
b'(a_t, r_t) = b'(a_t + \psi_t, r_t^c),
\]

provided that \(\psi_t > A - a_t\), and otherwise \(\psi_t = A - a_t\). This implies that \(\psi_t\) is a function of \(a_t\) and \(r_t\). We can write that \(\psi_t = \psi(a_t, r_t)\).

The lender’s value function therefore satisfies the following differential equation

\[
r_t b(a_t, r_t) = \mu + (\gamma a_t - \theta - \psi(a_t, r_t) \delta(r_t)) b'(a_t, r_t) + \frac{1}{2} \sigma^2 b''(a_t, r_t) + \delta(r_t) (b(a_t + \psi(a_t, r_t), r_t^c) - b(a_t, r_t))
\]
with \( b(a_t, r_t) = b(a_t^1(r_t), r_t) - (a - a_t^1(r_t)) \) for \( a_t > a_t^1(r_t) \) and the function \( \psi \) specified above.

The home is repossessed when the expected utility for the borrower becomes equal to his reservation value \( A \). This determines the first boundary condition: \( b(A, r_t) = L \). The second boundary condition comes from the fact that the first derivatives must agree at the boundary \( a_t^1(r_t) \): \( b'(a_t^1(r_t), r_t) = -1 \). The final boundary condition is the condition for the optimality of \( a_t^1(r_t) \), which requires that the second derivatives match at the boundary. This condition implies that \( b''(a_t^1(r_t), r_t) = 0 \), or equivalently, using equation (12), that

\[
 r_t b(a_t^1(r_t), r_t) = \mu + \theta - \gamma a_t^1(r_t) + \delta(r_t) \left[ \psi(a_t^1(r_t), r_t) + b(a_t^1(r_t), r_t, r_t^c) - b(a_t^1(r_t), r_t) \right]. \tag{13}
\]

By definition, \( a_t^1(r) \) is the lowest value of \( a \) such that \( b'(a, r) = -1 \), thus

\[
 \psi(a_t^1(r_L), r_L) = -\psi(a_t^1(r_H), r_H) = a_t^1(r_H) - a_t^1(r_L). \]

This, combined with (13) implies that

\[
 \begin{align*}
 \mu + \theta &= r_L b(a_t^1(r_L), r_L) + 2 \gamma a_t^1(r_L) - \delta(r_L) \left[ b(a_t^1(r_H), r_H) - b(a_t^1(r_L), r_L) + a_t^1(r_H) - a_t^1(r_L) \right], \\
 \mu + \theta &= r_H b(a_t^1(r_H), r_H) + 2 \gamma a_t^1(r_H) - \delta(r_H) \left[ b(a_t^1(r_L), r_L) - b(a_t^1(r_H), r_H) + a_t^1(r_L) - a_t^1(r_H) \right].
\end{align*}
\]

The proposition below formalizes our findings.

**Proposition 3** Let \( b \) be a \( C^2 \) function (in \( a \)) that solves:

\[
 rb(a, r) = \mu + (\gamma a - \theta - \psi(a, r)\delta(r))b'(a_t, r_t) + \frac{1}{2} \sigma^2 b''(a, r) + \delta(r_t) \left( b(a_t + \psi(a, r), r^c) - b(a, r) \right), \tag{14}
\]

for \( a \in [A, a_t^1(r)] \), and \( b'(a, r) = -1 \) for \( a > a_t^1(r) \), with boundary conditions \( b(A, r) = L \) and

\[
 \begin{align*}
 \mu + \theta &= r_L b(a_t^1(r_L), r_L) + 2 \gamma a_t^1(r_L) - \delta(r_L) \left[ b(a_t^1(r_H), r_H) - b(a_t^1(r_L), r_L) + a_t^1(r_H) - a_t^1(r_L) \right], \\
 \mu + \theta &= r_H b(a_t^1(r_H), r_H) + 2 \gamma a_t^1(r_H) - \delta(r_H) \left[ b(a_t^1(r_L), r_L) - b(a_t^1(r_H), r_H) + a_t^1(r_L) - a_t^1(r_H) \right],
\end{align*}
\]

where

\[
 \psi(a, r) = \begin{cases} 
 \text{is a } C^1 \text{ (in } a \text{) solution to } b'(a, r) = b'(a + \psi, r^c) \text{ for all } (a, r) \\
 \text{for which the solution is such that } \psi(a, r) > A - a \\
 \text{otherwise it is equal to } A - a
\end{cases}, \tag{15}
\]

where \( r \in \{r_L, r_H\} \) and \( r^c = \{r_L, r_H\} \setminus \{r\} \).

Then the optimal contract that delivers to the borrower the value \( a_0 \) takes the following form:

\[ (i) \text{ If } a_0 \in [A, a_t^1(r_0)], \text{ then } a_t \text{ evolves as } \\
 da_t(r_t) = (\gamma a_t dt - \theta dt - dI_t) + (dY_t - \mu dt) + \psi(a_t, r_t)(dN_t - \delta(r_t)dt), \tag{16} \]
and

- when \( a_t \in [A, a^1(r_t)) \), \( dI_t = 0 \),
- when \( a_t = a^1(r_t) \) the transfers \( dI_t \) cause \( a_t \) to reflect at \( a^1(r_t) \).

(ii) If \( a_0 > a^1(r_0) \) an immediate transfer \( a_0 - a^1(r_0) \) is made.

The relationship is terminated at time \( \tau \) when \( a_t \) hits \( A \). The lender’s expected payoff at any time \( t \) is given by the function \( b(a_t, r_t) \) defined above, which is strictly concave in \( a_t \) over \([A, a^1(r_1)]\).

Proof In the Appendix. ■

The evolution of the continuation utility (16) implied by the optimal contract serves three objectives: promise keeping, incentives, and efficiency. The first component of (16) accounts for promise keeping. In order for \( a_t \) to correctly describe the lender’s promise to the borrower, it should grow at the borrower’s discount rate, \( \gamma \), less the payment, \( \theta dt \), he receives from owning the home, and less the flow of payments, \( dI_t \), from the lender.

The second term of (16) provides the borrower with incentives to report his income truthfully to the lender. Because of inefficiencies resulting from liquidation, reducing the risk in the borrower’s continuation utility lowers the probability that the borrower’s expected utility reaches \( A \), and thus lowers the probability of costly liquidation. Therefore, it is optimal to make the sensitivity of the borrower’s continuation utility with respect to its report as small as possible provided that it does not erode his incentives to tell the truth. The minimum volatility of the borrower’s continuation utility with respect to his report of income required for truth-telling equals 1. To understand this, note that, under this choice of volatility, underreporting income by one unit would provide the borrower with one additional unit of current utility through increased consumption, but would also reduce the borrower’s continuation utility by one unit, so that this volatility provides the borrower with just enough incentives to report a true realization of income. Note that when the borrower reports truthfully, the term \( dY_t - \mu dt \) is driftless and equals to \( \sigma dZ_t \).

The last term of (16) captures the effects of changes in the lender’s interest rate process on the borrower’s continuation utility. The optimal adjustments, \( \psi \), in the borrower’s continuation utility, which are applicable when there is a change in the lender’s interest rate, are such that the sensitivity of the lender’s expected utility, \( b \), with respect to the borrower’s continuation utility, \( a \), is equalized just before and after an adjustment is made.\(^{21}\) This sensitivity represents an instantaneous marginal cost of delivering the borrower his continuation utility in terms of the lender’s utility, and so the efficiency calls for equalizing this cost across the states. We note that these adjustments imply the compensating trend in the borrower’s continuation utility, \( -\delta(r_t)\psi(a_t, r_t)dt \), which exactly offsets the expected effect these adjustments have on the borrower’s expected utility.

Below we state a useful lemma that characterizes the behavior of the optimal contract when the borrower’s continuation utility is close to liquidation and there is an interest rate change.

\(^{21}\)Provided that the solution to (11) is interior.
Lemma 3 At the optimal contract, there exists $\tilde{a} \in (A, a^1(r_L)]$ such that
- $\psi(A, r_H) = \tilde{a} - A$,
- $\psi(a, r_L) = A - a$ for $a \in [A, \tilde{a}]$.

Proof From the definition of function $b$ and the fact that $r_L < r_H$ it follows that, for any $a > A$, $b(a, r_L) > b(a, r_H)$. This, together with $b(A, r_L) = b(A, r_H) = L$, implies that $b'(A, r_L) > b'(A, r_H)$. Let $\tilde{a}$ be the smallest $a > A$ such that $b'(a, r_L) = b'(A, r_H)$. The existence of such $\tilde{a}$ follows from the fact that, for any $a \in [A, a^1(r_t)]$, $b'(a, r_t) \geq -1$ and $b'(a^1(r_t), r_t) = -1$. This combined with (15) yields us the alleged properties of function $\psi$. ■

Lemma 3 implies that under the optimal contract, an increase in the interest rate, $r_t$, triggers instantaneous liquidation whenever $a_t \in (A, \tilde{a}]$.

4.2 The Optimal Contract with Hidden Savings

So far we have characterized the optimal contract under the assumption that the borrower cannot save. Now we show that, given the optimal contract, the borrower has no incentive to save, and thus the contract of Proposition 3 is also optimal in the environment where the borrower can privately save.

Proposition 4 Suppose that the process $a_t$ is bounded above and solves

$$da_t = \gamma a_t dt - \theta dt - dI_t + (d\tilde{Y}_t - \mu dt) + \psi_t dM_t$$

until stopping time $\tau = \min \{t | a_t = A\}$, where $\psi_t$ is an $\mathcal{F}_t$-predictable process. Then the borrower’s expected utility from any feasible strategy in response to a contract $(\tau, I)$ is at most $a_0$. Moreover, the borrower attains the expected utility $a_0$ if the borrower reports truthfully and maintains zero savings.

Proof In the Appendix. ■

Intuitively, the optimal contract provides enough insurance against low income realizations, so that the borrower does not need to save on his own.

4.3 An Example

In this section we illustrate the features of the optimal contract in a parametrized example. Table 1 shows the parameters of the model.
Table 1. Parameters of the model

<table>
<thead>
<tr>
<th>Interest rate process</th>
<th>Borrower’s discount rate</th>
<th>Income process</th>
<th>Utility flow from home</th>
<th>Liquidation values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_L$</td>
<td>$r_H$</td>
<td>$\delta (r_L)$</td>
<td>$\delta (r_H)$</td>
</tr>
<tr>
<td></td>
<td>1.5%</td>
<td>6.5%</td>
<td>0.12</td>
<td>0.12</td>
</tr>
</tbody>
</table>

The left-hand side of Figure 1 shows the lender’s value function at both interest rates as a function of the borrower’s continuation utility. For a given continuation utility to the borrower, the value function of the lender at the low interest rate is always above the one at the high interest rate, except at termination when they are equal, as the lender attaches more value to the proceeds from the continuation of the relationship when his discount rate is lower. As we observe, it is optimal to allow the borrower to consume his disposable income earlier when the interest rate is low, that is $a^1(r_L) < a^1(r_H)$. Intuitively, when the lender’s interest rate is low, it is more costly to postpone borrower’s consumption, as tension between the borrower’s valuation of future payoffs and that of the lender is larger. To reduce this cost, it is optimal to allow the borrower to consume his excess disposable income earlier.

The right-hand side of Figure 1 shows the optimal adjustments in the borrower’s continuation utility, $\psi$, applicable when there is a change in the market interest rate. The borrower’s continuation utility increases with a decrease in the interest rate and decreases with an interest rate increase, except in the area close to the reflection barriers when this relationship is reversed. The size of these adjustments is proportional to the distance of the borrower’s continuation utility from the termination cutoff of $A$.

The optimal adjustment of the borrower’s continuation utility, $\psi$, is shaped by two competing forces stemming from, respectively, the costly termination of the relationship and the difference in the discount rates. The closer the borrower’s continuation utility is to the termination boundary $A$, the bigger is the role played by the costly termination in shaping the optimal adjustment function. It is efficient to reduce the chances of costly termination when the interest rate falls, as the stream of transfers from the borrower is more valuable for the lender when the interest rate is low. A reduction in the likelihood of termination is engineered by influencing the borrower’s continuation utility in two ways. First, it is optimal to instantaneously increase the borrower’s continuation utility if the market interest rate falls, and this is even more so the more likely the relationship is to be terminated. Second, it is optimal to introduce a positive trend in the law of motion of the borrower’s continuation utility, which reinforces the first adjustment over time to the extent that the interest rate stays low. As a result of these adjustments, the chances of costly home repossession are reduced by moving the borrower’s continuation utility further away from the termination boundary $A$. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, the optimal contract increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low-interest state, both by instantaneously decreasing the borrower’s continuation utility and by introducing a negative trend in its law of motion.

If the borrower’s continuation utility is distant from the termination boundary $A$, then, intuitively, the discrepancy in the discount rates begins to play the dominant role in shaping the optimal adjustment.
function, as the likelihood of termination is small. When the lender’s interest rate switches to low, there is more tension between the borrower’s valuation of future payoffs and that of the lender, and thus it is more costly to postpone the borrower’s consumption, the more so the bigger is his continuation utility. To reduce this cost, it is optimal to decrease the borrower’s continuation utility when the interest rate falls, by both an instantaneous adjustment and a negative time trend, provided that his prior continuation utility was sufficiently large. In order to compensate for this reduction in the borrower’s continuation utility when the interest rate switches to low, his continuation utility is increased to a range of high values of the borrower’s continuation utility when the interest rate increases. It is important to observe that the adjustment of the borrower’s continuation utility in this region has second order welfare effects. This is because there is less difference between the slopes of the lender’s value function at the low and at the high interest rate state, the further away the borrower’s continuation utility is from the termination boundary $A$. We will use this fact in Section 6, where we simply ignore the adjustments of the borrower’s continuation utility in a region close to the reflection barriers.

5 Implementation of the Optimal Contract: Option ARM

So far, we have characterized the optimal contract in terms of the transfers between the borrower and the lender and liquidation timing. In this section, we show that the optimal contract can be implemented using financial arrangements that resemble the ones used in the residential mortgage market. In particular, we show that the optimal contract can be implemented using an option adjustable rate mortgage (option ARM).
We start with the following definition.

**Definition 5** The mortgage contract is optimal if it implements the optimal contract of Proposition 3.

In order to implement an optimal contract we consider an option ARM. This is an adjustable rate mortgage with no minimum payment, where the borrower has discretion of how much to pay until his debt reaches a certain limit. An important feature of this contract is that it allows for negative amortization; that is, a loan payment for any given period can be less than the interest charged over that period. Once the borrower’s debt balance reaches a maximum allowed level a default happens. The option ARM is essentially a revolving line of credit collateralized with the value of home. It can also be interpreted as home equity line of credit (HELOC) with variable interest rate. The definition below provides a formal description of this class of mortgage contracts.

**Definition 6** An option adjustable rate mortgage with a preferential interest rate consists of:

- Mortgage loan with a time-t credit limit equal to $C^t_L$. If the balance of the loan exceeds the credit limit, default occurs, in which case the lender repossesses the home.

- At any time $t$, an instantaneous interest rate on the balance, $B_t$, is equal to a preferential rate $\tilde{r}_t^P$ on a part of the balance below $p_t$, and $\tilde{r}_t$ on the part of the balance above $p_t$.

Figure 2 graphically demonstrates features of option ARM.
The proposition below shows that the optimal contract can be implemented with an option ARM with a preferential interest rate.

**Proposition 5** There exists an optimal option adjustable rate mortgage with a preferential interest rate that has the following features

\[
\hat{r}_t(B_t - p_t, r_t) = \gamma + \delta(r_t) \left[ \psi(a^1(r_t) - (B_t - p_t), r_t) - \psi(a^1(r_t), r_t) \right], \quad \text{if } B_t \geq p_t
\]  

\[
\tilde{r}_t^p(p_t, r_t) = \frac{\theta + \mu - \gamma a^1(r_t) + \delta(r_t) \psi(a^1(r_t), r_t)}{p_t}
\]  

\[
C_t^L(p_t, r_t) = p_t + a^1(r_t) - A
\]  

\[
dp_t = \begin{cases} 
[\psi(a^1(r_t) - (B_t - p_t), r_t) - (a^1(r_t^*) - a^1(r_t))] dN_t, & \text{if } B_t \geq p_t \\
0, & \text{if } B_t < p_t
\end{cases}
\]  

Under the terms of this mortgage, it is optimal for the borrower to use all available cash flows to pay down balance \(B_t\) when \(B_t > p_t\), and consumes all excess cash flows once the balance drops to \(p_t\). For balance \(B_t \geq p_t\), the borrower’s continuation utility \(a_t\) is equal to

\[
a_t = A + \left[ C_t^L(p_t, r_t) - B_t \right] = a^1(r_t) - (B_t - p_t)
\]  

If the preferential rate reaches its upper boundary \(\gamma\), the mortgage is reset to a contract that implements the continuation of the optimal contract.

**Proof** In the Appendix. ■

**Remark 1** The initial balance \(B_0\) should be greater than or equal to \(p_0\). If \(B_0 < p_0\), it is optimal for the borrower to consume \(p_0 - B_0\) immediately by increasing the balance to \(p_0\).

**Remark 2** The credit limit \(C_t^L\), preferential balance \(p_t\), and the preferential interest rate \(\tilde{r}_t^p\) are reset only when the interest rate in the economy changes.

**Remark 3** When default happens, the lender receives the liquidation value \(L\) of the home, and the borrower obtains the value \(A\) of his outside option.

**Remark 4** In order for the mortgage to remain incentive compatible, the preferential rate \(\tilde{r}_t^p\) must stay below \(\gamma\). Although it is theoretically possible, our simulations show that for most parameters chances of the preferential rate reaching its upper boundary \(\gamma\) over a period of 30 years are extremely small.

**Remark 5** In the proposed implementation, parameter \(p_0\) at time zero can be chosen arbitrarily, provided interest rate \(\tilde{r}_0^p\) given by (19) is no greater than \(\gamma\). One way to initiate the mortgage is to have the market value of the mortgage equal to the book value:

\[
B_0 = b \left( r_0, p_0 + a^1(r_0) - B_0 \right).
\]
How does the optimal option adjustable rate mortgage implement the optimal contract? The debt balance above the debt limit subject to the preferential interest rate can be considered as a memory device that summarizes all the relevant information regarding the past cash flow realizations revealed by the borrower through repayments. The interest rates charged on the balance, along with the preferential debt limit, and the credit limit are chosen so that equation (22) always holds. This ensures incentive compatibility of the mortgage. Indeed, at any time $t$ the borrower can consume all his available credit $C_L(p_t, r_t) - B_t$ and default immediately. However, (22) implies that the payoff from this strategy is equal to the expected utility $a_t$ the borrower would obtain by postponing consumption until his debt balance is reduced to the preferential debt limit.

The adjustable features of the above mortgage contract are needed to implement the effects of the changes in the interest rate on the borrower’s continuation utility. In the optimal option ARM, the adjustments of the debt subject to the preferential rate (21) and the adjustments to the credit limit implement all instantaneous adjustments in the borrower’s promised utility that are applicable when the lender’s interest rate changes. The variable component of the interest rate (18) guarantees that a change in the borrower’s promised utility implied by the mortgage contract includes the trend that compensates the borrower, in expectation, for the instantaneous adjustments in his promised utility that happen when the interest rate changes.

The fixed component of the interest rate (18) on the debt above the preferential debt limit insures that under the optimal strategy of the borrower, given the above mortgage contract, the borrower’s promised utility would be increased at the rate of $\gamma$ as in the optimal contract. The preferential interest rate insures that an above-average income realization and thus an above-average repayment increases the borrower’s promised utility, which corresponds here to a decrease in his debt balance, and vice versa.

To further characterize the above mortgage contract, we will restrict our attention to the environment in which the optimal contract satisfies the following condition.\textsuperscript{22}

**Condition 1** Function $\psi$ implied by the optimal contract is such that $\psi(a, r_L)$ is strictly increasing in $a$ for $a \in [\bar{a}, a^1(a_L)]$, and so $\psi(a, r_H)$ is strictly decreasing in $a$ for $a \in [A, a^1(a_H)]$, where $\bar{a}$ is defined as in Lemma 3.

Parameters $C^L_t, p_t, \bar{r}_t^p$ and $\bar{r}_t$ are reset each time the interest rate in the economy changes. This is needed to take into account the effects that the interest rate in the economy has on the borrower’s continuation utility. As the following corollary shows, under the optimal option ARM, whenever the debt balance is close to the credit limit, an increase in the interest rate would cause default on the mortgage. If the optimal adjustment function, $\psi$, satisfies the properties of Condition 1, a decrease in the lender’s interest rate results in an increase in the amount of debt subject to the preferential rate and vice versa. In addition, interest rate $\bar{r}_t$ positively correlates with the lender’s interest rate.

**Corollary 1** The optimal option ARM with preferential interest rate has the following properties:

\textsuperscript{22}This condition holds in all parametrized examples we considered.
i) Let $B_t = p_t + a^1(r_L) - \bar{a}$ where $\bar{a}$ is defined as in Lemma 3. Then, whenever $B_t \in [\bar{B}_t, C^L_t(p_t, r_L))$, an instantaneous increase in the interest rate in the economy triggers mortgage default;

ii) Suppose further that function $\psi$ corresponding to the optimal contract satisfies the properties of Condition 1, then,

- $dp_t < 0$, whenever interest rate $r_t$ increases,
- $dp_t > 0$, whenever interest rate $r_t$ decreases,
- for any $B' \in [p_t, C^L_t(r_L)]$ and $B'' \in [p_t, C^L_t(r_H)]$,

$$\bar{r}_t(B' - p_t, r_L) < \gamma < \bar{r}_t(B'' - p_t, r_H)$$

**Proof** Corollary 1 follows from Proposition 5 and Lemma 3.

Since it is optimal for the borrower to use all available cash flows to pay down balance $B_t$ as long as $B_t > p_t$, lower rates on the mortgage do not necessarily reduce the cash flow to the lender. It is optimal to reduce interest rates, and as a result default rates, when the interest rate in the economy is low because the stream of borrower’s payments is more valuable for the lender when the lender discounts them with lower interest rate. On the other hand, the threat of repossession must be real, otherwise the borrower would not share his income with the lender. As a result the optimal option ARM increases the chances of repossession by charging higher rates when the interest rate in the economy is high in order to compensate for the weakened threat of repossession when the interest rate in the economy is low. Also, unless the borrower’s
balance is sufficiently close to the preferential debt limit, the borrower enjoys an increase of the credit limit when the market rate switches to low and vice versa.

Figure 3 presents the variable interest rate charged on the balance of the optimal option ARM above the preferential debt limit in the parametrized environment of Section 4.3.

6 Approximate Implementation

In this section we consider a simpler mortgage contract that implements the optimal contract approximately. Although the approximation may lead to some efficiency losses, it considerably simplifies the terms of the contract.

Figure 4: The approximately optimal function $\hat{\psi}$ and $\hat{a}^1$.

The idea of approximate implementation is based on the observation that the results of Proposition 5 concerning the implementation of the optimal contract do not rely on any particular properties of functions $\psi$ and $a^1$. In other words, if we replace the optimal function $\psi$ from Proposition 3 by any function $\hat{\psi} : [A, \infty) \times \{r_L, r_H\} \to R$, such that $\hat{\psi}(a, r) + a \geq A$ for any $a \geq A$, and replace the reflection barriers $a^1(r)$ by any finite $\hat{a}^1(r) \geq A$, then the resulting suboptimal contract will remain incentive compatible.

In what follows, we will focus on the following approximation of the optimal functions $\psi$ and $a^1$.

**Definition 7** The approximately optimal function $\hat{\psi}$ and $\hat{a}^1$ satisfy

- $\hat{a}^1 \equiv \hat{a}^1(r_L) = \hat{a}^1(r_H) = \inf \{a \geq A : \psi(a, r_L) = \psi(a, r_H) = 0\}$,
- $\hat{\psi}(a, r_L) = \begin{cases} A - a & \text{for } a \in [A, \bar{a}] \\ \left(\frac{\bar{a} - A}{\alpha^2 - \bar{a}}\right)(\hat{a}^1 - a) & \text{for } a \in [\bar{a}, \hat{a}^1] \end{cases}$
- $\hat{\psi}(a, r_H) = \left(\frac{\bar{a} - A}{\alpha^2 - \bar{a}}\right)(\hat{a}^1 - a) $ for $a \in [A, \hat{a}^1]$, where $\psi$ and $\hat{a}^1$ are the functions from the optimal contract of Proposition 3.

Figure 4 presents the approximately optimal functions $\hat{\psi}$ and $\hat{a}^1$, together with their optimal counterparts in the parametrized environment of Section 4.3. We note that function $\hat{\psi}$ satisfies Condition 1. We also note that $\hat{\psi}$ approximates $\psi$ reasonably well except for the region with high continuation utility, which is much less important in terms of contract optimality than the region near the bankruptcy threshold $A$. As we will show next, this approximation leads to simpler mortgage contracts that do not sacrifice much in terms of their payoff efficiency.

In Section 4, we demonstrated that the optimal contract can be implemented using the optimal option ARM. Replacing optimal function $\psi$ with approximately optimal function $\hat{\psi}$ defined in Section 6 gives us an approximately optimal option ARM with the following parameters:

\[
C_i^L(p_t) = p_t + \hat{a}^1 - A \tag{23}
\]
\[
\hat{r}_i(p_t) = \frac{\theta + \mu - \gamma \hat{a}^1}{p_t} \tag{24}
\]
\[
r_i(B_t - p_t, r_t) = \begin{cases} 
\gamma - \delta(r_L) \left(\frac{\hat{a} - A}{\alpha^2 - \hat{a}}\right) & \text{for } B_t \in [p_t, \hat{B}_i] \text{ and } r_t = r_L \\
\gamma - \delta(r_L) \left(\frac{\hat{a}^1 - A - B_t + p_t}{\alpha^2 - \hat{a}}\right) & \text{for } B_t \in [\hat{B}_i, C_i^L] \text{ and } r_t = r_L \\
\gamma + \delta(r_H) \left(\frac{\bar{a} - A}{\alpha^2 - \bar{a}}\right) & \text{for } B_t \in [p_t, C_i^L] \text{ and } r_t = r_H \\
- \left(\frac{\bar{a} - A}{\alpha^2 - \bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, \hat{B}] \text{ and } r_t = r_L \\
\left(\frac{\bar{a} - A}{\alpha^2 - \bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, C_i^L] \text{ and } r_t = r_H \\
0 & \text{for } B_t < p_t
\end{cases} \tag{25}
\]
\[
dp_t = \begin{cases} 
-\left(\frac{\bar{a} - A}{\alpha^2 - \bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, \hat{B}] \text{ and } r_t = r_L \\
0 & \text{for } B_t < p_t
\end{cases} \tag{26}
\]

where $\hat{B}_t = p_t + \hat{a}^1 - \bar{a}$.

**Proposition 6** Under the terms of option ARM given by (23-26), it is optimal for the borrower to use all available cash flows to pay down balance $B_t$ when $B_t > p_t$, and to consume all excess cash flows once the balance drops to $p_t$. The borrower’s continuation utility $\hat{a}_t$ is determined by the balance above the preferential debt limit as follows:

\[
\hat{a}_t = A + C_i^L - B_t = \hat{a}^1 - (B_t - p_t) \tag{27}
\]

**Proof** follows from the proof of Proposition 5 by replacing function $\psi$ with $\hat{\psi}$ and the reflection barriers $a^1(r)$ with $\hat{a}^1$.

The intuition behind incentive compatibility of the postulated strategy of the borrower under the above mortgage contract is the same as in the case of the optimal mortgage contract of Proposition 5. The credit
limit (23), the interest rates (24) and (25), and the preferential debt limit (26) play the same role in the approximate implementation as their counterparts from Proposition 5 in the exact implementation of the optimal contract.

**Corollary 2** The approximately optimal option ARM has the following properties:

(i) \( \bar{r}_t(B_t - p_t, r_L) < \gamma < \bar{r}_t(B_t - p_t, r_H) \), for any \( B_t' \in [p_t, C^L_t(r_L)], B''_t \in [p_t, C^L_t(r_H)] \).

(ii) \( dp_t < 0, dC^L_t < 0, d\bar{r}^p_t > 0 \) whenever \( r_t \) increases and \( B_t > p_t \),

\( dp_t > 0, dC^L_t > 0, d\bar{r}^p_t < 0 \) whenever \( r_t \) decreases and \( B_t > p_t \).

As the above corollary shows, a decrease in the interest rate in the economy results in smaller mortgage interest rates, higher amount of debt subject to the preferential rate, and higher credit limit.

Figure 5 shows the interest rate charged on the balance of the approximate option ARM above the preferential debt limit.

### Figure 5: Variable interest rate charged on the approximately optimal option ARM's balance above the preferential debt limit.

**7 Efficiency Gains Due to Optimal Mortgage**

In this section, we will investigate the efficiency gains implied by the adjustable features of the optimal mortgage. For that purpose we will evaluate and compare the highest expected utility the borrower would get under the optimal and approximately optimal mortgage with the highest expected utility he would get under a simpler mortgage, provided that the lender breaks even.
This simpler mortgage is the same as the optimal mortgage, except no adjustments in the borrower’s continuation value due to changes in the lender’s interest rate are allowed: $\psi_t = 0$ for all $t$. It is easy to show that this is a fixed rate mortgage combined with a line of credit.

The borrower who can downpay $Y_0$ will decide to buy a house whenever the total utility he gets from homeownership is at least as big as the value $R(Y_0)$ he could get by not buying. In the example below we set $R(Y_0) = \frac{b}{r} + Y_0$, which implies that the outside value of the prospective borrower, if he decides not to purchase a house, is equal to the sum of his initial wealth and the expected value of his "excess" disposable income. Let $a_0$ be the expected utility for the borrower such that the lender breaks even. Then the borrower’s net utility gain from homeownership equals to $a_0 - R(Y_0)$.

Figure 6 presents the percentage increase in the utility gain from homeownership the borrower would obtain by switching from a FRM with a line of credit to, respectively, the approximately optimal and the optimal mortgage. The computations are performed across the volatility of the borrower’s income keeping all the other parameters as in the example of Section 4.3. The price of the home is set to be equal to 17 and the current interest rate is set to be low ($r_0 = 1.5\%$). The left-hand side show the gains for the borrower with no downpayment, while the right-hand side shows the corresponding gains for a borrower who downpays 5% of the home value.

As we observe from Figure 6 the gains in the borrower’s utility due to approximately optimal mortgage are very close to the optimal one. Therefore, our comparison suggests that the mortgage terms can be considerably simplified, with little loss of efficiency, by implementing the approximately optimal contract.
instead of the optimal one. Both of these contracts yield substantial utility gains to the borrowers with more volatile income who make no downpayment. The borrower’s utility gain from homeownership can increase by as much as 13 percent if he switches from FRM with a line of credit to the optimal mortgage.

This result is intuitive as adjustable features allow for the more intelligent management of default timing, and so the efficiency gains they present are bigger for borrowers who are more likely to default. These are the borrowers who make little or no downpayment and/or face more risk in their future ability to service debt.23

So far we have assumed that there is no difference in the cost of issuing a given mortgage contract. If origination of more complex contracts like option ARMs would be more costly, due for example to higher costs of managing such a contract or obtaining funds to finance it (for example by making securitization of such contracts more difficult), then efficiency would call for issuing complex mortgages only to those borrowers for whom the benefits outweigh this additional cost. In this case, we would expect a high concentration of option ARMs among the borrowers with a higher risk of default. Safer borrowers, like those who make substantial downpayment or face little risk in their future ability to repay, would be given a simpler product (like a fixed rate mortgage). Thus, our model may help to explain a high concentration of option ARMs among riskier borrowers.

8 Concluding Remarks

Recent years have seen a rapid growth in originations of more sophisticated alternative mortgage products (AMPs), such as option adjustable rate mortgages (option ARMs). This paper shows that the key properties of option ARMs are consistent with the properties of the optimal contract governing the relationship between the borrower and the lender, which represents a Pareto improvement over traditional mortgages. As a consequence, it is possible that both lenders and borrowers can benefit from option ARMs. Critics of this contract have raised the concern that low minimum payments can result in substantially higher mortgage payments and, as a consequence, higher default rates when the market interest rate increases. Nevertheless, this paper demonstrates that this does not necessarily contradict the optimality of option ARMs. Under the optimal mortgage contract, mortgage payments and default rates are indeed higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the market interest rate is low.

Our analysis is based on the assumption of full rationality of borrowers. Thus, it cannot be applied to a situation with irrational borrowers. Borrowers lacking self control may abuse such features of option ARMs such as low minimum payments, which would lead to inefficiently high default rates.

The optimal contracts do not allow borrowers to refinance their mortgages with another lender. In our setting, allowing refinancing without penalties would break the exclusivity of the borrowing-lending relationship, which could only decrease the efficiency of the contract. Therefore, our model lends support to prepayment penalties on refinancing. Introduction of borrowers’ mobility would result in a "soft" prepayment

23These results hold across all parameterizations we considered.
penalty; borrowers could sell their home at anytime without penalty, but if they wished to refinance the mortgage, they would pay the prepayment penalty sufficiently large to discourage it.

In this paper, we ignored inflation, which is an important consideration for home buyers choosing between ARMs and FRMs. However, as long as inflation equally affects both the borrower’s income and the liquidation value of the home, it will not change the properties of the optimal mortgage in real terms. We also did not allow for contract renegotiations, because a possibility of renegotiation would lead to a suboptimal contract. In practice, lenders should be able to commit to the terms of a mortgage contract, as the competition among them would drive those who are unable to do so out of the market.

We assumed that the liquidation values do not depend on the interest rate. Even if house prices were sensitive to the interest rate, a documented significant delay from initiation of the foreclosure process till foreclosure sale would make the liquidation values much less sensitive to the interest rate compared with house prices. The independence of liquidation values from the interest rate was meant to capture this reality in our setting. We also assumed that the borrower’s income process does not depend on the interest rate. This assumption is motivated by our view of the borrower as a borrowing constrained household as well as the empirical evidence indicating that most household income shocks are correlated only weakly with asset returns.

For the sake of tractability of our dynamic contracting problem, we had to assume risk-neutrality of the borrower with respect to discretionary consumption. The properties of the optimal mortgage are determined by the conflict of interest between the borrower and the lender, and by the gains from trade. In particular, the adjustable features of the optimal mortgage are driven by the fact that the lender values his relationship with the borrower more when his discount rate is low. Therefore, we expect that a more general form of risk aversion on the borrower’s side would weaken but not completely eliminate the adjustable features of the optimal mortgage.

There are a number of research directions one might pursue from here. In this paper we have considered a time-homogeneous setting, in which agents are infinitely lived and the borrower’s average income and the liquidation values of the home do not change over time. Relaxing these assumptions would allow us to study the effects of home prices and households’ life-cycle income profiles on optimal mortgage design. While these extensions would allow us to capture some additional characteristics of the residential mortgage borrowing, they should preserve optimality of the optional payments and adjustable features and their properties we derived in our setting. This is because the forces and trade-offs that shape them would also be present in a richer environment. Another avenue of research would be to extend our analysis to a general equilibrium framework and to study what effects the presence of private information in the mortgage origination market has on equilibrium home prices, and how this varies over the business cycle.

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24 See, for example, Campbell and Cocco (2003).
25 Ambrose et al. (1997) report that borrowers whose delinquency continues to foreclosure spend an average of 13.8 total months in default before their property rights are terminated.
26 See for example Campbell et al. (2000).
Appendix

A.1 Proofs of Lemmas and Propositions

Proof of Lemma 1

Consider any incentive-compatible contract \((\tau, I, C, \check{Y})\). We prove the lemma by showing the existence of the new incentive-compatible contract that has the following properties:

(i) the borrower gets the same expected utility as under the old contract \((\tau, I)\),

(ii) the borrower chooses to reveal the cash flows truthfully,

(iii) the borrower maintains zero savings,

(iv) the lender gets the same or greater expected profit as under the old contract \((\tau, I)\).

Consider the candidate incentive-compatible contract \((\tau', I', C, Y)\) where

\[
\tau'(Y, r) = \tau(\check{Y}(Y, r), r), \quad I'(Y, r) = C(Y, r).
\]

We observe that the borrower’s consumption and the termination time under the new contract and the proposed borrower’s response strategy, \((C, Y)\), are the same as under the old contract, so he earns the same expected utility, which establishes property (i). Also, by construction, the proposed response of the borrower to the contract \((\tau', I')\) involves truth-telling and zero savings, which establishes properties (ii) and (iii).

Now we will show that \((C, Y)\) is the borrower’s incentive compatible strategy under the contract \((\tau', I')\). We note that the strategy \((C, Y)\) yields the same utility to the borrower under the contract \((\tau', I')\) as the incentive compatible strategy associated with the contract \((\tau, I)\). Therefore, to show that \((C, Y)\) is the borrower’s incentive compatible strategy under the contract \((\tau', I')\), it is enough to show that if any alternative strategy \((C', Y')\) is feasible under the contract \((\tau', I')\), then \(C'\) is also feasible under the old contract \((\tau, I)\).

It follows that if \(C'\) is feasible under the new contract, then the borrower has nonnegative savings if he reports \(\check{Y}(Y'(Y, r), r)\) and consumes \(C'\) under the old contract, and thus \(C'\) is also feasible under the old contract \((\tau, I)\). To see this, we note that the borrower’s savings at any time \(t \leq \tau(\check{Y}(Y'(Y, r), r) = \)
Finally, to complete the proof, we need to show that under the new contract \((\tau', I')\) the lender gets the same or greater expected profit as under the contract \((\tau, I)\). Note that under the new contract the lender does savings for the borrower. As by assumption the lender’s interest rate process is always greater than or equal to the savings interest rate available to the borrower (i.e., for all \(t, r_t \geq \rho_t\)), the lender’s expected profit improves by

\[
E_0 \left[ \int_0^r e^{-\rho_t (r_t - \rho_t)} S_t dt \right] \geq 0,
\]

which shows (iv). □

**Proof of Proposition 3**

Let \(b\) be a \(C^2\) function (in \(a\)) that solves:

\[
rb(a, r) = \mu + (\gamma a - \theta - \psi(a, r)\delta(r))b'(a_t, r_t) + \frac{1}{2} \sigma^2 b''(a, r) + \delta(r) (b(a_t + \psi(a, r), r^c) - b(a, r)) \tag{28}
\]

when \(a\) is in the interval \([A, a^1(r)]\), and \(b'(a, r) = -1\) when \(a > a^1(r)\), with boundary conditions \(b(A, r) = L\) and

\[
\begin{align*}
\mu + \theta &= r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_L) \left\{ b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L) \right\}, \\
\mu + \theta &= r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_H) \left\{ b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H) \right\},
\end{align*}
\]

where

\[
\psi(a, r) = \begin{cases} 
\text{is a } C^1 \text{ (in } a\text{) solution to } b'(a, r) = b'(a + \psi, r^c) \text{ for all } (a, r) \\
\text{for which the solution is such that } \psi(a, r) > A - a \\
\text{otherwise it is equal to } A - a
\end{cases}
\]
Lemma 4 The function $b$ is strictly concave over $[a,a^1(r)]$.

Proof We will proceed in the series of steps below.

Step 1. Strict concavity in the neighborhood of the reflection barriers: Define the social surplus function as:
\[ F(a,r) = a + b(a,r). \] (29)
Then the social surplus function satisfies the following differential equation:
\[
\begin{align*}
    r(F(a,r) - a) &= \mu + (\gamma a - \theta - \psi(a,r)\delta(r))(F'(a,r) - 1) + \frac{1}{2}\sigma^2 F''(a,r) \\
    &\quad + \delta(r) (F(a + \psi(a,r), r^c) - F(a,r) + \psi(a,r))
\end{align*}
\]
which equals
\[
\begin{align*}
    rF(a,r) &= -(\gamma - r)a + \mu + (\gamma a - \theta - \psi(a,r)\delta(r))F'(a,r) + \frac{1}{2}\sigma^2 F''(a,r) \\
    &\quad + \delta(r) (F(a + \psi(a,r), r^c) - F(a,r)) \tag{30}
\end{align*}
\]
with the boundary conditions:
\[
\begin{align*}
    F(A,r) &= A + L, \\
    F'(a^1(r),r) &= 0, \\
    F''(a^1(r),r) &= 0.
\end{align*}
\]
Let’s now focus on $a \in [\bar{a}(r),a^1(r)]$ when $\bar{a}(r) \geq A$ is the smallest $a$ such that $b'(a,r) = b'(a + \psi, r^c)$ holds for all $a \in [\bar{a}(r),a^1(r)]$. This together with (29) implies that $F'(a,r) = F'(a + \psi, r^c)$ holds for all $a \in [\bar{a}(r),a^1(r)]$. Differentiating (30) with respect to $a \in [\bar{a}(r),a^1(r)]$ and using $F'(a,r) = F'(a + \psi, r^c)$ that holds for all $a \in [\bar{a}(r),a^1(r)]$ and the boundary conditions implies that
\[
\frac{dF''(a^1(r),r)}{da} = \frac{2(\gamma - r)}{\sigma^2} > 0.
\]
Note that as $F''(a^1(r),r) = 0$ and as we have that $\frac{dF''(a^1(r),r)}{da} > 0$ it implies that there exists $\varepsilon > 0$ such that $F''(a,r) < 0$ over the interval $(a^1(r) - \varepsilon, a^1(r))$. Also as $F'(a^1(r)) = 0$ and $F''(a,r) < 0$ over the interval $(a^1(r) - \varepsilon, a^1(r))$ it implies that $F'(a,r) > 0$ over the interval $(a^1(r) - \varepsilon, a^1(r))$. 

Step 2. Strict concavity of $b(a,r)$ over $[\bar{a}(r),a^1(r)]$, $\bar{a}(r) \geq \bar{a}(r)$, for which $\psi(a,r) \leq 0$: Let’s pick $r \in \{r_L,r_H\}$ and the smallest $\bar{a}(r) \geq \bar{a}(r)$ such that $\psi(a,r) \leq 0$ holds over the interval $[\bar{a}(r),a^1(r)]$. Such
\( \hat{a}(r) \) must exists given that over \([\hat{a}(r), a^1(r)]\) we have that
\[
\psi(a, r) = -\psi(a + \psi(a, r), r^c),
\]
which follows from the definition of function \( \psi \).

From (30) we have that
\[
F''(a, r) = \frac{K(a, r) - (\gamma a - \theta)F'(a, r)}{2\sigma^2},
\]
where
\[
K(a, r) = rF(a, r) + (\gamma - r)a + \psi(a, r)\delta(r)F'(a, r) - \delta(r) (F(a + \psi(a, r), r^c) - F(a, r)) - (\mu + \theta).
\]

Now we note that \( K(a^1(r), 0) = 0 \) and that
\[
K'(a, r) = rF'(a, r) + (\gamma - r) + \psi(a, r)\delta(r)F''(a, r)
\]
which holds over \([A, a^1(r)]\).

Then (32) and (33), and the fact that \( F''(a^1(r), r) < 0 \), imply that as long as \( F'(a, r) > 0 \) over \([\hat{a}(r), a^1(r)]\) we have that \( F''(a, r) < 0 \) in this interval. To see this note that (32) and (33) imply that we cannot have \( F'(a, r) > 0 \) over \([\hat{a}(r), a^1(r)]\) and at the same time \( F''(a, r) = 0 \) in this interval, as for the largest \( \hat{a} \in [\hat{a}(r), a^1(r)] \) for which \( F'' = 0 \) we would have that \( K(\hat{a}, r) < 0 \) as \( K \) would be increasing on \([\hat{a}, a^1(r)]\) and \( K(a^1(r), 0) = 0 \). But then given that \( F'(\hat{a}, r) > 0 \), \( K(\hat{a}, r) < 0 \), and \( (\gamma\hat{a} - \theta) > 0 \) (as by assumption \( A \geq \frac{\theta}{\gamma} \)) we would have by (33) that \( F''(\hat{a}, r) < 0 \), which is a contradiction. So as long as \( F'(a, r) > 0 \) over \([\hat{a}(r), a^1(r)]\) we would have that \( F''(a, r) < 0 \) over \([\hat{a}(r), a^1(r)]\).

Now we remember that \( F'(a, r) > 0 \) in \([a^1(r) - \varepsilon, a^1(r)]\). We want to show that \( F'(a, r) > 0 \) over the entire \([\hat{a}(r), a^1(r)]\), and thus by the above discussion that \( F(a, r) \) is strictly concave over \([\hat{a}(r), a^1(r)]\). Now suppose by contradiction that \( F' \leq 0 \) for some \( \hat{a}(r) \leq a \leq a^1(r) - \varepsilon \), and let \( \hat{a} = \sup \{ a \leq a^1(r) - \varepsilon : F', r \leq 0 \} \). Then it follows that \( F''(\hat{a}, r) = 0 \), and that for all \( a \in (\hat{a}, a^1(r)) \) we have that \( F'' > 0 \). But this implies that \( F''(a, r) < 0 \) for \( a \in (\hat{a}, a^1(r)) \). From the Fundamental Theorem of Calculus it follows that:
\[
F'(a^1(r), r) = F'(\hat{a}, r) + \int_{\hat{a}}^{a^1(r)} F''(a, r)da,
\]
which given that \( F'(a^1(r), r) = 0 \) implies that
\[
F'(\hat{a}, r) = -\int_{\hat{a}(r)}^{a^1(r)} F''(a, r)da
\]
As \( F'' < 0 \) for \( a \in (\hat{a}, a^1(r)) \) the above implies that \( F'(\hat{a}, r) > 0 \), which is a contradiction. Hence we have that \( F' > 0 \) for \( a \in [\hat{a}(r), a^1(r)] \) and hence \( F''(a, r) < 0 \) for \( a \in [\hat{a}(r), a^1(r)] \). Also if \( \hat{a}(r) > \hat{a}(r) \), and noting
that $F$ is $C^2$ (in $a$) and $F''(\tilde{a}(r), r) < 0$, there exists $\varepsilon > 0$ such that $F(a, r)$ is strictly concave and increasing over the interval $(\tilde{a}(r) - \varepsilon, \tilde{a}(r))$.

**Step 3. Strict concavity of $b(a, r^c)$ over $[\tilde{a}(r^c), a^1(r^c)]$, $\tilde{a}(r^c) = \tilde{a}(r) + \psi(\tilde{a}(r), r)$, where $\psi(a, r^c) \geq 0$:**

From (31) it follows that if $\psi(a, r) \leq 0$ holds over the interval $[\tilde{a}(r), a^1(r)]$, then we have that $\psi(a, r^c) \geq 0$ holds over the interval $[\tilde{a}(r^c), a^1(r^c)]$. Now note that for $a \in [\tilde{a}(r^c), a^1(r^c))$ we have that

$$F'(a, r^c) = F'(a + \psi(a, r^c), r)$$

(34)

Now differentiating (34) we obtain

$$F''(a, r^c) = F''(a + \psi(a, r^c), r)(1 + \psi'(a, r^c)).$$

(35)

We note that (31) implies that for $a \in [\tilde{a}(r^c), a^1(r^c))$ we have that $(a + \psi(a, r^c)) \in [\tilde{a}(r), a^1(r))$ and so $F''(a + \psi(a, r^c), r) < 0$. Now suppose that by contradiction $F''(a, r^c) \geq 0$ for some $a \in [\tilde{a}(r^c), a^1(r^c))$. As in the neighborhood of reflecting barriers we have that $F''(a, r^c) < 0$, and as the function $F$ is continuous it would imply the existence of $\tilde{a} \in [\tilde{a}(r^c), a^1(r^c))$ such that $F''(\tilde{a}, r^c) = 0$. But then (35) and the fact that $F''(\tilde{a}, r) < 0$ would imply that

$$\psi'(\tilde{a}, r^c) = -1.$$  

But then differentiating

$$\psi(a, r^c) = -\psi(a + \psi(a, r^c), r)$$

with respect to $a$ and setting $a = \tilde{a}$ yields

$$\psi'(\tilde{a}, r^c) = -\psi'\left(\tilde{a} + \psi(\tilde{a}, r^c), r \right) \left[1 + \psi'(\tilde{a}, r^c)\right],$$

which would imply that $-1 = 0$, which is a contradiction. Therefore we conclude that $F''(a, r^c) < 0$ and by (34) $F'(a, r^c) > 0$ for all $a \in [\tilde{a}(r^c), a^1(r^c))$. Also if $\tilde{a}(r^c) > \tilde{a}(r^c)$, and given that $F$ is $C^2$ (in $a$) and $F''(a^1(r^c), r^c) < 0$, there exists $\varepsilon > 0$ such that $F(a, r^c)$ is strictly concave and increasing over the interval $(\tilde{a}(r^c) - \varepsilon, \tilde{a}(r^c))$.

**Step 4: Strict concavity of $F(a, r)$ over $[\tilde{a}(r), a^1(r)]$:** If $\tilde{a}(r) = \tilde{a}(r)$ it follows by Steps 1, 2, and 3. Suppose that $\tilde{a}(r) > \tilde{a}(r)$. Then by Steps 1, 2, and 3 we know that the function $F(a, r)$ is strictly increasing and concave (in $a$) over the interval $(\tilde{a}(r) - \varepsilon, a^1(r))$. Then let’s pick $r' \in \{r_L, r_H\}$ and the smallest $\tilde{a}'(r) \geq \tilde{a}(r)$ such that $\psi(a, r^c) \leq 0$ holds over the interval $[\tilde{a}'(r'), \tilde{a}(r')]$. Then applying the same reasoning as in Steps 3 and 4 we get that the functions $F(a, r^c)$ and $F(a, r^c)$ are strictly increasing and concave over, respectively, $[\tilde{a}'(r'), a^1(r')]$ and $[\tilde{a}'(r^c), a^1(r^c)]$, where $\tilde{a}(r^c) = \tilde{a}' + \psi(\tilde{a}', r')$. Applying this argument over and over again we obtain that the function $F(a, r)$ is strictly increasing and concave over the interval $[\tilde{a}(r), a^1(r)]$.

**Step 5. Strict concavity of $F$ over $[A, \max(\tilde{a}(r), \tilde{a}(r^c))]$:** It follows from the definition of function
that \( \min(a(r), a(r^c)) = A \) and then Steps 1, 2, 3, and 4 imply that for \( r \in \{r_L, r_H \} \) for which \( a(r) = A \) we have that \( F(a, r) \) is strictly concave and increasing over \([A, a^1(r)]\). From the definition of function \( F \) it follows that \( \psi(a, r^c) = A - a \) over \([A, a(r^c)]\). By steps 1, 2, 3, 4 we know that \( F(a, r^c) \) is strictly increasing and concave over \([a(r^c), a(a^1(r^c))]\), and moreover because \( F \) is \( C^2 \) (in \( a \)) there exists \( \varepsilon > 0 \) such that \( F(a, r^c) \) is strictly increasing and concave over \((a(r^c) - \varepsilon, a(r^c))]\). But then because \( \psi(a, r^c) = A - a \leq 0 \) over \([A, a(r^c)]\), applying the same reasoning as in Step 2 yields us that \( F(a, r^c) \) is strictly increasing and concave over \([A, a(r^c)]\) and so is strictly increasing and concave over \([A, a^1(r^c)]\).

Steps 1, 2, 3, 4, and 5 imply that \( F \) is strictly increasing and concave over \([A, a^1(r)]\) and as \( b''(a, r) = F''(a, r) \) for any \( a \geq A \) and \( r \in \{r_L, r_H \} \) we have that \( b(a, r) \) is strictly concave (in \( a \)) over \([A, a^1(r)]\).

Now for any incentive compatible contract \((\tau, I, C, Y)\) we define:

\[
G_t = \int_0^t e^{-R_s}(dY_s - dI_s) + e^{-R_t}b(a_t, r_t),
\]

(36)

where \( a_t \) evolves according to (7). We note that the process \( G \) is such that \( G_t \) is \( \mathcal{F}_t \)-measurable.

We remember that under an arbitrary incentive compatible contract, \((\tau, I, C, Y)\), \( a_t \) evolves as

\[
da_t(r_t) = (\gamma a_t - \theta - \psi_t \delta(r_t)) dt - dB_t + \beta_t dN_t + \psi_t dN_t.
\]

where \( \beta_t \geq \sigma \) m.a.s. for any \( 0 \leq t \leq \tau \). From Ito’s lemma we get that

\[
db(a_t, r_t) = [(\gamma a_t - \theta - \psi_t \delta(r_t))b'(a_t, r_t) + \frac{1}{2}\beta_t^2 b''(a_t, r_t)] dt - b'(a_t, r_t) dt + \beta_t b'(a_t, r_t) dN_t + [b(a_t + \psi_t, r_t) - b(a_t, r_t)] dN_t.
\]

Then combining the above with (36) yields

\[
e^{R_t} dG_t = \left[ \mu + (\gamma a_t - \theta - \psi_t \delta(r_t))b'(a_t, r_t) + \frac{1}{2}\beta_t^2 b''(a_t, r_t) - r_t b'(a_t, r_t) \right] dt
\]

\[
- \left[ 1 + b'(a_t, r_t) \right] dI_t + \left[ \sigma + \beta_t b'(a_t, r_t) \right] dN_t + [b(a_t + \psi_t, r_t) - b(a_t, r_t)] dN_t.
\]

Combining the above with (28) yields

\[
e^{R_t} dG_t \leq \left[ \frac{1}{2}(\beta_t^2 - \sigma^2) b''(a_t, r_t) + \delta(r_t)b'(a_t, r_t)[\psi(a_t, r_t) - \psi_t] \right] dt - \left[ 1 + b'(a_t, r_t) \right] dI_t
\]

\[
+ \left[ \sigma + \beta_t b'(a_t, r_t) \right] dN_t + [b(a_t + \psi_t, r_t) - b(a_t + \psi(a_t, r_t), r_t)] dN_t.
\]

36
with equality whenever \( a \in [A, a^1(r_t)] \). From the above we have that for any \( 0 \leq t < \tau \):

\[
e^{R_t} dG_t \leq \left[ \frac{1}{2} (\beta_t^2 - \sigma_t^2) b'(a_t, r_t) \right] dt - (1 + b'(a_t, r_t)) dI_t
\]

\[
\leq 0
\]

\[
+ \delta(r_t) \left( [b(a_t + \psi_t, r_t^e) - \psi_t b'(a_t, r_t)] - [b(a_t + \psi_t, r_t^e) - \psi(a_t, r_t)b'(a_t, r_t)] \right) dt
\]

\[
\leq 0
\]

\[
+ (\sigma + \beta_t b'(a_t, r_t)) dZ_t + [b(a_t + \psi_t, r_t^e) - b(a_t + \psi(a_t, r_t), r_t^e)] dM_t,
\]

(37)

with equality whenever \( a \in [A, a^1(r_t)] \). The first component of the RHS of the above inequality is less or equal to zero because the function \( b \) is concave (Lemma 4) and \( \beta_t \geq \sigma \) for any \( t \leq \tau \). The second component is less or equal to zero because \( b' \geq -1 \) and \( dI_t \geq 0 \). The third component is less or equal to zero because, by definition, the function \( \psi \) is a solution to

\[
\max_{\psi \geq A-a} [b(a + \psi, r^e) - \psi b'(a, r)]
\]

The condition (37) implies that the process \( G \) is an \( F_t \)-supermartingale up to time \( t = \tau \), where we recall that \( Z \) and \( M \) are martingales. It will be an \( F_t \)-martingale if and only if, for \( t > 0 \), \( a_t \leq a^1(r_t) \), \( \beta_t = \sigma \) m.a.s., \( \psi_t = \psi(a_t, r_t) \), and \( I_t \) is increasing only when \( a_t \geq a^1(r_t) \).

We now evaluate the lender’s expected utility for an arbitrary incentive compatible contract \((\tau, I, C, Y)\), which equals

\[
E \left[ \int_0^\tau e^{-R_t} (dY_s - dI_s) + e^{-R_t} L \right].
\]

We note that \( b(a_t, r_t) = L \) as, from the definition of \( a, a_\tau = A \). Using this, and the definition of process \( G \), we have that under any arbitrary incentive compatible contract \((\tau, I, C, Y)\) and any \( t \in [0, \infty) \):

\[
E \left[ \int_0^\tau e^{-R_t} (dY_s - dI_s) + e^{-R_t} L \right] =
\]

\[
E [G_{t \wedge \tau}] + E \left[ 1_{t \leq \tau} \left( \int_t^\tau e^{-R_s} (dY_s - dI_s) + e^{-R_s} L - e^{-R_t} b(a_t, r_t) \right) \right] \\
\]

\[
= \]

\[
= \]

\[
= \]

\[
= b(a_0, r_0) + E \left[ 1_{t \leq \tau} \left( e^{-R_t} L \right) \int_t^\tau e^{-R_s} (dY_s - dI_s) + e^{-R_s} L \right] + e^{-R_t} L | F_t \right] - b(a_t, r_t) \right) \right],
\]

(38)

where, the inequality follows from the fact that \( G_{t \wedge \tau} \) is supermartingale and \( G_0 = b(a_0, r_0) \). We note that in the above

\[
E \left[ \int_t^\tau e^{R_{t-s}} (dY_s - dI_s) + e^{R_{t-s}} L \right] | F_t < \frac{\mu}{r_L} + \frac{\theta}{\gamma} - a_t,
\]

37
as the RHS of the above inequality is the upper bound on the lender’s expected profit under the first-best (public information) contract. Using the above inequality in (38) we have that

$$E\left[ \int_0^\tau e^{-R_s (dY_s - dI_s)} + e^{-R_t L} \right] \leq b(a_0, r_0) + e^{-R_t} E\left[ 1_{t \leq \tau} \left( \frac{\mu}{r_L} + \frac{\theta}{\gamma} - a_t - b(a_t, r_t) \right) \right].$$

Using $b'(a, r) \geq -1$, we have that, for any $a \geq A, -a - b(a, r) \leq -A - L$. Applying this to the above inequality yields

$$E\left[ \int_0^\tau e^{-R_s (dY_s - dI_s)} + e^{-R_t L} \right] \leq b(a_0, r_0) + e^{-R_t} E\left[ 1_{t \leq \tau} \left( \frac{\mu}{r_L} + \frac{\theta}{\gamma} - A - L \right) \right].$$

Taking $t \to \infty$ yields

$$E\left[ \int_0^\tau e^{-R_s (dY_s - dI_s)} + e^{-R_t L} \right] \leq b(a_0, r_0).$$

Let $(\tau^*, I^*, C^*, Y)$ be a contract satisfying the conditions of the proposition. We remember that this contract is incentive compatible as it is feasible and $\beta_t = \sigma \geq \sigma$ for any $t \leq \tau$. Also under this contract the process $G_t$ is a martingale until time $\tau$ (note that $b'(a, r)$ is bounded). So we have that

$$E\left[ \int_0^{\tau^*} e^{-R_s (dY_s - dI_s^*)} + e^{-R_t^* L} \right] = b(a_0, r_0) + e^{-R_t^*} E\left[ 1_{t \leq \tau^*} \left( E\left[ \int_0^{\tau^*} e^{R_t - R_s (dY_s - dI_s^*)} + e^{R_t^* - R_t^* L} \left| \mathcal{F}_t \right] \right] - b(a_t, r_t) \right].$$

Taking $t \to \infty$ and using

$$\lim_{t \to \infty} e^{-R_t^*} E\left[ 1_{t \leq \tau^*} \left( E\left[ \int_0^{\tau^*} e^{R_t - R_s (dY_s - dI_s^*)} + e^{R_t^* - R_t^* L} \left| \mathcal{F}_t \right] \right] - b(a_t, r_t) \right] = 0,$$

yields

$$E\left[ \int_0^{\tau^*} e^{-R_s (dY_s - dI_s^*)} + e^{-R_t^* L} \right] = b(a_0, r_0). \quad \Box$$

**Proof of Proposition 4**

Let $(C, \tilde{Y})$ be any borrower’s feasible strategy given the contract $(\tau, I)$. The borrower’s private saving’s account balance, $S$, under the strategy $(C, \tilde{Y})$ and the contract $(\tau, I)$ grows, for $t \in [0, \tau]$, according to

$$dS_t = \rho_t S_t dt + (dY_t - d\tilde{Y}_t) + dI_t - dC_t, \quad (39)$$
where we remember that \( \rho_t \leq r_t \). Define the process \( \hat{V} \) as

\[
\hat{V}_t = \int_0^t e^{-\gamma s} dC_s + \int_0^t e^{-\gamma s} \theta ds + e^{-\gamma t} (S_t + a_t),
\]

From the above it follows that

\[
e^\gamma t \hat{V}_t = dC_t + \theta dt + dS_t - \gamma S_t dt + da_t - \gamma a_t dt.
\]

Using (17) and (39) yields

\[
e^\gamma t \hat{V}_t = (\rho_t - \gamma) S_t dt + (dY_t - \mu dt) dt + \psi_t dM_t = (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi_t dM_t.
\] (40)

Noting that \( e^{\gamma t} \geq 1 \) for any \( t \geq 0 \), we have that

\[
d\hat{V}_t \leq (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi_t dM_t.
\]

Recall that \( Z \) and \( M \) are martingales, \( \rho_t < \gamma \), and that the process \( S \) is nonnegative. So it follows from the above that the process \( \hat{V} \) is supermartingale up to time \( \tau \) (note that \( a \) is bounded from below). Using this and the fact that by definition \( a_\tau = A \), we have that for any feasible strategy of the borrower,

\[
a_0 = \hat{V}_0 \geq E \left[ \hat{V}_\tau \right] = E \left[ \int_0^\tau e^{-\gamma s} dC_s + \int_0^\tau e^{-\gamma s} \theta ds + e^{-\gamma \tau} (S_\tau + A) \right],
\] (41)

The right-hand-side of (41) represents the expected utility for the borrower under any feasible \( (C, \hat{Y}, S) \). This utility is bounded by \( a_0 \). If the borrower maintains zero savings, \( S_t = 0 \), and reports cash flows truthfully, \( d\hat{Y}_t = dY_t \), then \( \hat{V} \) is a martingale up to time \( \tau \), which means that (41) holds with equality and the borrower’s expected utility is \( a_0 \). Thus, this is the optimal strategy for the borrower. \( \square \)

**Proof of Proposition 5**

Define \( \tilde{a}_t \) as follows:

\[
\tilde{a}_t = A + C_t^L(r_t) - B_t = p_t + a_1^L(r_t) - B_t.
\] (42)

Under the candidate mortgage contract the debt balance evolves according to

\[
dB_t = \left( \bar{r}_t B_t + (\bar{r}_t - \bar{r}_t) (B_t - p_t) + \right) dt - d\hat{Y}_t + dI_t,
\] (44)
when \( B_t \leq C_t^T \), where \( I_t \) represents cumulative withdrawal of money by the borrower. In addition,

\[
dC_t^L = dp_t + da^1(r_t)
= \psi(p_t + a^1(r_t) - B_t, r_t)dN_t
\] (45)

Using (18)-(20), (43)-(45), for \( B_t \geq p_t \) we can write

\[
d\tilde{a}_t = dC_t^L(r_t) - dB_t
= \psi(p_t + a^1(r_t) - B_t, r_t)dN_t - (\tilde{r}^p_t B_t + (\tilde{r}_t - \tilde{r}^p_t) (B_t - p_t)) dt + d\tilde{Y}_t - dI_t
\]

\[
= -(\tilde{r}^p_t p_t + \tilde{r}_t (B_t - p_t) - \delta\psi(p_t + a^1(r_t) - B_t, r_t)) dt + d\tilde{Y}_t - dI_t
+ \psi(p_t + a^1(r_t) - B_t, r_t)dB_t
\]

\[
= - (\theta + \mu - \gamma a^1(r_t) + \gamma (B_t - p_t)) dt + d\tilde{Y}_t - dI_t + \psi(p_t + a^1(r_t) - B_t, r_t)dB_t
\]

\[
= \gamma \tilde{a}_t dt - \mu dt - \theta dt + d\tilde{Y}_t - dI_t + \psi(\tilde{a}_t, r_t)dB_t\] (46)

The borrower’s savings evolve according to

\[
dS_t = \rho_t S_t dt + dI_t + (d\tilde{Y}_t - d\tilde{Y}_t) - dC_t.\] (47)

Consider

\[
\tilde{V}_t = \int_0^t e^{-\gamma s} (\theta dt + dC_s) + e^{-\gamma t} (\Omega_t + S_t)
\]

where

\[
\Omega_t = \begin{cases} 
  a^1(r_t) + (p_t - B_t), & \text{if } B_t < p_t \\
  \tilde{a}_t, & \text{if } B_t \geq p_t 
\end{cases}\] (48)

We will show that for any feasible strategy \((C, \tilde{Y}, S)\) of the borrower, \(\tilde{V}_t\) is a supermartingale. Note that

\[
d\tilde{\Omega}_t = \begin{cases} 
  \frac{[a^1(r_t^e) - a^1(r_t)]}{\psi(a^1(r_t^e), r_t)}dN_t - dB_t, & \text{if } B_t < p_t \\
  d\tilde{a}_t, & \text{if } B_t \geq p_t 
\end{cases}\] (49)

Using (47),

\[
e^{\gamma t}d\tilde{V}_t
= \theta dt + dC_t + dS_t - \gamma S_t dt + d\tilde{\Omega}_t - \gamma \tilde{\Omega}_t dt
\]

\[
= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + (d\tilde{Y}_t - d\tilde{Y}_t) + d\tilde{\Omega}_t - \gamma \tilde{\Omega}_t dt.
\]

First, we consider the case with \( B_t \geq p_t \). Using (1), (46), and (48)-(49),

\[
e^{\gamma t}d\tilde{V}_t
= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + (d\tilde{Y}_t - d\tilde{Y}_t) + d\tilde{a}_t - \gamma \tilde{a}_t dt
\]

\[
= (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi(\tilde{a}_t, r_t)dB_t.\] (50)
Now, let $B_t < p_t$. Using (1), (19), (44)-(49) yields

$$e^{\tau B_t} = \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left( dY_t - d\tilde{Y}_t \right) + d\Omega_t - \gamma \Omega_t dt$$

$$= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left( dY_t - d\tilde{Y}_t \right) + \psi(a^1(r_t), r_t) dN_t - dB_t - \gamma \left( a^1(r_t) + (p_t - B_t) \right) dt$$

$$= - (\tilde{r}_t^0 B_t + \gamma (p_t - B_t) + \gamma a^1(r_t) - \theta - \mu + (\gamma - \rho_t) S_t) dt$$

$$+ \psi(a^1(r_t), r_t) dN_t + \sigma dZ_t$$

$$= - (\tilde{r}_t^0 B_t + \gamma (p_t - B_t) - \tilde{r}_t^0 p_t + \delta \psi(a^1(r_t), r_t) + (\gamma - \rho_t) S_t) dt$$

$$+ \psi(a^1(r_t), r_t) dN_t + \sigma dZ_t$$

$$= - (\gamma - \tilde{r}_t^0) (p_t - B_t) dt - (\gamma - \rho_t) S_t dt + \psi(a^1(r_t), r_t) dM_t + \sigma dZ_t$$  \hspace{1cm} (51)

Recall that $Z$ and $M$ are martingales, $\tilde{r}_t^0 \leq \gamma$, $\rho_t < \gamma$. Thus, it follows from (50), (51), and the fact that $\tilde{a}_t$ is bounded from below, that for any feasible strategy $(C, \tilde{Y}, S)$ of the borrower $\tilde{V}_t$ is a supermartingale until default time $\tau(C, \tilde{Y}, S) = \inf \{ t : B_t = C_t^L \}$. Since $\Omega_t = A$,

$$A + C_0^L(r_0) - B_0 = \tilde{a}_0 = \tilde{V}_0 \geq E \left[ \tilde{V}_{\tau(C, \tilde{Y}, S)} \right]$$

$$= E \left[ \int_0^{\tau(C, \tilde{Y}, S)} e^{-\gamma s} (\theta dt + dC_s) + e^{-\gamma t(C, \tilde{Y}, S)} \left( A + S_t^0(C, \tilde{Y}, S) \right) \right]$$  \hspace{1cm} (52)

where $B_0$ is the time-zero draw on the credit line.

The right-hand side of (52) represents the expected utility for the borrower under strategy $(C, \tilde{Y}, S)$, given the terms of the mortgage. This utility is bounded by $A + C_0^L(r_0) - B_0$. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\tilde{Y}_t = dY_t$, and consumes all excess cash flows once the balance on the credit line reaches $p_t(r_t)$, so that $B_t \geq p_t$ and $C_t = I_t^* = \max(0, p_t - B_t) = \max(0, p_t - A_t)$, then $\tilde{V}_t$ is a martingale, which means that (52) holds with equality and the borrower’s expected utility is $A + C_0^L(r_0) - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower’s optimal strategy, $(C, \tilde{Y}, S) = (I^*, Y, 0)$, and the process $\tilde{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined as

$$\tilde{V}_{t', t} = \int_{t'}^t e^{-\gamma(s-t')} (dC_s + \theta ds) + e^{-\gamma(t-t')} \tilde{a}_t, \hspace{1cm} t \geq t'$$  \hspace{1cm} (53)

yields that, for any $0 \leq t \leq \tau(C, \tilde{Y}, 0)$, $\tilde{a}_t$ is equal to the borrower’s continuation utility under the proposed mortgage contract with the initial expected utility for the borrower given by $a_0 = A + C_0^L(r_0) - B_0$, which establishes (22).

Under the proposed mortgage contract and the borrower’s optimal strategy $(C, \tilde{Y}, S) = (I^*, Y, 0)$, the
lender’s expected utility equals

\[
E \left[ \int_0^{\tau(I^*, Y, 0)} e^{-R_t (dY_t - dI^*_t)} + e^{-R_{\tau(I^*, Y, 0)} \tau(I^*, Y, 0)} L | \mathcal{F}_t} \right],
\]

where

\[
\tau(I^*, Y, 0) = \inf \{ t : B_t = C_t^L \} = \inf \{ t : \tilde{a}_t = A \} = \tau^*(Y),
\]

as the borrower’s continuation utility, \( \tilde{a} \), evolves according to the equation (46), e.g. as in the optimal contract. Therefore, we conclude that the proposed mortgage contract implements the optimal contract. \( \square \)
References


Sannikov, Yuliy, 2006a, A continuous-time version of the principal-agent problem, working paper, University of California at Berkeley.

Sannikov, Yuliy, 2006b, Agency problems, screening and increasing credit lines, working paper, University of California at Berkeley.


