On the Efficiency of Queueing in Dynamic Matching Markets

Laura Doval[*] Balázs Szentes[†]

July 18, 2019

Abstract

This paper considers a two-sided dynamic matching market where agents arrive at the market randomly. An arriving agent is immediately matched if there are agents waiting on the other side. Otherwise, the arriving agent has to decide whether to leave the market and take her outside option or to join a (possibly empty) queue and wait for a match. The equilibrium is characterized by a cutoff, k^*, so that an agent joins the queue if, and only if, the length of the queue is less than k^*. Our main result compares k^* with the socially optimal queue size, K^*. In particular, we show that if the arrival rate of the agents is (small) large then $k^* \geq (\leq) K^*$, that is, agents are too (im)patient. In addition, we characterize parameter values for which $K^* = \infty$.

1 Introduction

In many matching markets, agents arrive over time and matches are created sequentially. Examples for such dynamic markets include marriage, kidney exchange, public housing and child-adoption. A key determinant of the allocation in these markets is agents' patience. If agents are willing to wait more, the quality of matches might improve at the cost of increased waiting times. Despite the inherent dynamic nature of these markets, the majority of the literature on market design analyzes these environments by static models. This paper contributes to this problem in a small way: We analyze equilibria in a simple two-sided matching market and compare the allocation with the socially efficient one. In particular, we fully characterize the environment where the agents are more or less patient than efficiency would require them to be.

An agent's decision whether to join a queue or just leave in a two-sided market has the following opposing effects on social welfare. On the one hand, an agent who joins the queue

[*]Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, CA. E-mail: ldoval@caltech.edu

[†]Department of Economics, London School of Economics, London, UK. E-mail: b.szentes@lse.ac.uk

increases the waiting time of subsequently arriving agents on her side of the market. Hence, an agent's decision to wait decreases the welfare of others on the same side of the market. On the other hand, a waiting agent generates a positive externality on the agents arriving at the other side by reducing their waiting times and potentially increasing the number of matches created. When an agent makes a decision about waiting she ignores *all* the external effects of her decision. The goal of this paper is to understand what factors determine the relative magnitude of these two effects by comparing equilibrium queues with efficient ones.

In our model, agents arrive stochastically in each side of the market according to a Poisson process with arrival rate λ. If there is a queue in front of an arriving agent, she has to decide whether to join the queue or leave the market and take her outside option. The payoff of an agent from a match, h, exceeds her outside option l, but each agent discounts the future by the rate r. Throughout, we assume that the queueing discipline is first-come first-served.[1]

In order to analyze which of the two opposing effects dominate in our two-sided environment, we compare equilibrium queueing behavior with the socially optimal one. The latter maximizes the steady state payoff of an agent. We show that the equilibrium behaviour of an agent can be defined by a threshold, k^*. That is, an agent joins a queue if, and only if, the queue on her side of the market is shorter than k^*. Similarly, the socially optimal policy is characterized by the length of the longest queue the social planner wishes to maintain, denoted by K^*. In order to understand whether agents form inefficiently long queues in equilibrium or they are not willing to wait enough, we compare k^* and K^*.

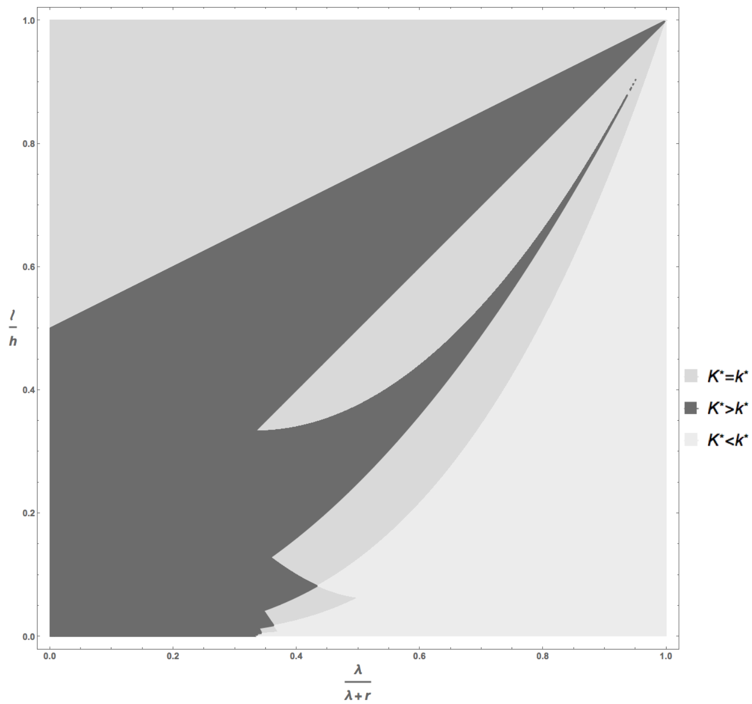
Our main results are as follows. First, when agents discount the future, we characterize sets of parameter values where the socially optimal queue length (K^*) is longer or shorter than the equilibrium queue length (k^*). In particular, we show that when $\lambda/(\lambda+r)$ is small, the socially optimal queue is longer than the equilibrium one. In contrast, when $\lambda/(\lambda+r)$ is large relative to l/h, the equilibrium queue is longer than the socially optimal one. Panel a. of Figure 1 illustrates the comparison between K^* and k^* based on computations.

We also show that for a certain set of parameters, the socially optimal queue is unbounded (i.e., $K^* = \infty$), while the agents equilibrium queue is always bounded. In fact, the socially optimal queue can be unbounded even if agents are not willing to queue at all.

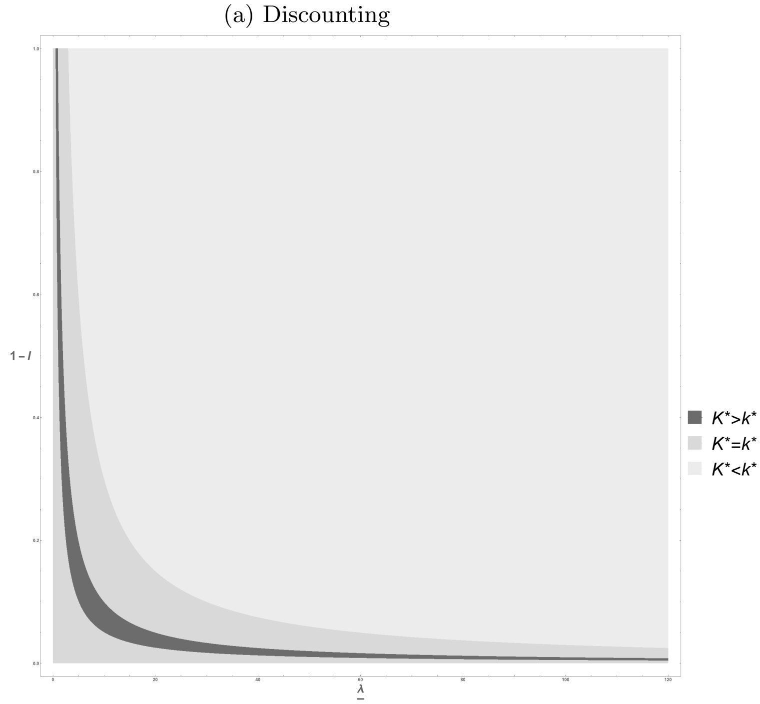
Finally, to compare our results to those in existence in the queueing literature, we also consider the case where agents do not discount the future but incur a flow cost c of waiting. We show that, when agents incur a flow cost while they wait, the equilibrium queue length is typically (weakly) longer than the socially optimal one.[2] In other words, when agents do not discount the future, the negative externality an agent imposes on her own side of the market generally dominates the positive externality she impose on the other side. Panel b. of Figure 1 depicts the comparison between K^* and k^* in the case of flow costs based on computations.

[1] We focus on the first-come first-served discipline because of its pervasiveness in practice. We do not intend to characterize the optimal queueing discipline.

[2] In fact, the only exception is when $K^* = 1$, in which case, it is possible that $k^* = 0$.



(a) Discounting



(b) Flow costs

Figure 1: Comparison between K^* and k^*: $K^* > k^*$ (dark grey), $K^* = k^*$ (gray), $K^* < k^*$ (light gray)

We now illustrate the possibility of unbounded queues in the case of discounting and leave the intuition for the rest of the results to the main text. To do so, consider the following example. Suppose that $h = 3$, $l = 1$, agents are very impatient, $r \approx \infty$ and agents arrive infrequently, $\lambda \approx 0$. Since agents are infinitely impatient, their payoff is zero whenever they wait. Therefore, in equilibrium, agents never wait ($k^* = 0$) and their payoff is just the outside option, 1. Also note that, even when a social planner decides whether an agent joins a queue, the average payoff of the agents cannot exceed 2. The reason is that, in each created match, one of the agents had to wait and her payoff is zero whereas the other was matched immediately upon arrival and received a payoff of zero. So, agents who are matched receive a payoff of 2 on average and the payoff of those agents who take their outside option is one. This argument also implies that the average payoff of the agents is maximized if each agent is matched. Therefore, the socially optimal policy is to make an agent join a queue irrespective of how long that queue is ($K^* = \infty$). In summary, when agents discount their future, the payoff of each agent is bounded by zero and hence, the opportunity cost of forcing an agent to join a queue never exceeds the outside option, l. Therefore, as long as the benefit from making an agent wait, h, is sufficiently large, the optimal queues can be long. In contrast, when agents pay a flow cost of waiting, their payoffs are unbounded from below. Therefore, it would never be optimal to force an agent to join a sufficiently long queue.

1.1 Related Literature

The paper contributes to the literatures on equilibrium behavior of customers in queueing systems, and dynamic matching. The literature on queuing is mainly concerned with server models, i.e., one-sided markets in which customers choose whether to exit and take and outside option, or to join a queue to get serviced by a server. If the queuing discipline is first-come-first-served, the only external effect of an agent's decision to join a queue is that she increases the waiting time of others. Consequently, equilibrium queues are inefficiently long. This was first noted by Naor [8]. Hassin [6] argues that a last-come-first-served queueing discipline implements the optimal allocation in this setting. Our paper extends the classic models in this literature to allow for two sides but we restrict attention to the commonly used first-come first-served discipline. This allows us to study the aforementioned positive externality not present in the previous papers on the equilibrium behavior: when a customer joins a queue, she exerts a positive externality on the customers arriving on the other side by creating matching opportunities.

Among the papers on the recent literature on dynamic matching, the closest to our work is the paper by Baccara et al. [5]. The authors analyze a two-sided dynamic matching market in which agents have a binary quality. The payoff of an agent depends on her own quality and the quality of her match and it is supermodular. The authors show that low-quality agents are willing to match with anybody, so the equilibrium queue length is determined by the high-quality agents' decision of whether to join queues or match with low-quality agents. Baccara et al. show that the negative externality caused by an agent on her side of the market always dominates the positive externality imposed on the other side and, as a consequence, equilibrium queues are inefficiently long. In Section 5, we argue that this model is similar to ours with flow costs of waiting, except that the low-quality agent in their model plays the role of the outside option in ours. As mentioned above, in that section, we confirm the result of Baccara et al. and explain that it is the artefact of modeling time preferences with flow cost of waiting instead of with discounting.

Ünver [10], Akbarpour et al. [1], Anderson et al. [2], and Ashlagi et al. [3] consider dynamic matching models in which preferences are compatibility based: agents get a payoff of 0 when unmatched, and 1 when matched. Thus, in their models, agents are always more impatient than the planner, and their focus is on understanding when it is optimal for the planner to force agents to wait to generate more matches in the future. Leshno [7] considers a model in which agents and objects, of one of two types each, arrive over time. The planner's objective is to minimize the number of agents who exit the system matched to an object of the opposite type, while agents may prefer to take this object if it takes too long to get the object that matches their type. As in the previous papers, agents in his model are always more impatient than the planner.

Some of the ideas we present have antecedents in the literature of matching with frictions. First, the idea that, in two sided economies, agents may wait too long to match (failing to internalize the externality on their own side), or too little (failing to internalize the positive externality they generate on the other side) already appears in Shimer and Smith [9]. In their model, ex ante heterogeneous agents have to (costly) search in order to find a partner on the other side. They show that in order to decentralize the efficient solution some agents have to be subsidized (because they search too little), while others have to be taxed (because they search too much). Second, the observation that the predictions under discounting and flow costs of waiting differ is also made, in the context of a model of search, by Atakan [4]. Since discounting alters the marginal value of waiting for a better match, while waiting flow costs do not, he finds that assortative matching holds in a frictional search model with waiting flow costs under weaker conditions than with discounting.

2 The Model

Time is continuous and there is a two-sided market. On each side, agents arrive according to a Poisson process with arrival rate λ. Agents are matched on a first-come, first-served basis. To be more specific, if there is a queue on one side of the market and an agent arrives on the other side then the agent who waited the longest in the queue is matched with the arriving agent and they leave the market. An arriving agent who is not matched immediately has to decide whether to join the queue[3] or leave the market. If an agent is matched, she receives a payoff of h, and if she leaves the market unmatched, she receives a payoff of l. Agents discount the future according to a common discount rate r.

In what follows, we show that both the equilibrium queue size as well as the size of the socially optimal queue size depend on l/h and $\lambda/(\lambda+r)$. Hence, we denote the latter quantity by x, that is, $x \equiv \lambda/(\lambda+r)$, and we normalize $h=1$.

2.1 Equilibrium

An agent's decision whether to join the queue or leave the market depends only on the length of the queue she faces. Let $u(k)$ denote the payoff of an agent who joins a queue of length $k-1$. The value function u is recursively defined by the following two equations:

$$ru(1) = \lambda(1-u(1)) \text{ and } ru(k) = \lambda(u(k-1)-u(k)) \text{ if } k>1.$$

The first equation says that the dividend from forming a queue of size one, $ru(1)$, is just the product of the arrival rate of a match, λ, and the increase in value from $u(1)$ to one. The second equation says that the dividend from joining a queue of size $k-1$ is again the product of the arrival rate λ and the increase in value from jumping ahead in the queue by one position, $u(k-1)-u(k)$. The equations can be rewritten as

$$u(1) = x \text{ and } u(k) = xu(k-1).$$

Solving for u by induction yields

$$u(k) = x^k. \tag{1}$$

So, when an agent arrives at the market and there is a queue of length k on her side she is willing to join the queue if, and only if,

$$x^k \geq l,$$

so the largest queue formed in equilibrium, k^*, is defined by

$$x^{k^*} \geq l > x^{k^*+1},$$

so

$$k^* = \left\lfloor \frac{\log l}{\log x} \right\rfloor. \tag{2}$$

Figure 2 below depicts the values of k^* as a function of x, l, where the white area corresponds to $k^*=0$, and darker shades of gray correspond to higher values of k^*:

From (2) we can conclude the following intuitive comparative static result.

Remark 1 *The largest equilibrium queue size, k^*, is decreasing in l and r and increasing in λ.*

[3] Or form a queue if there is no agent in the market.

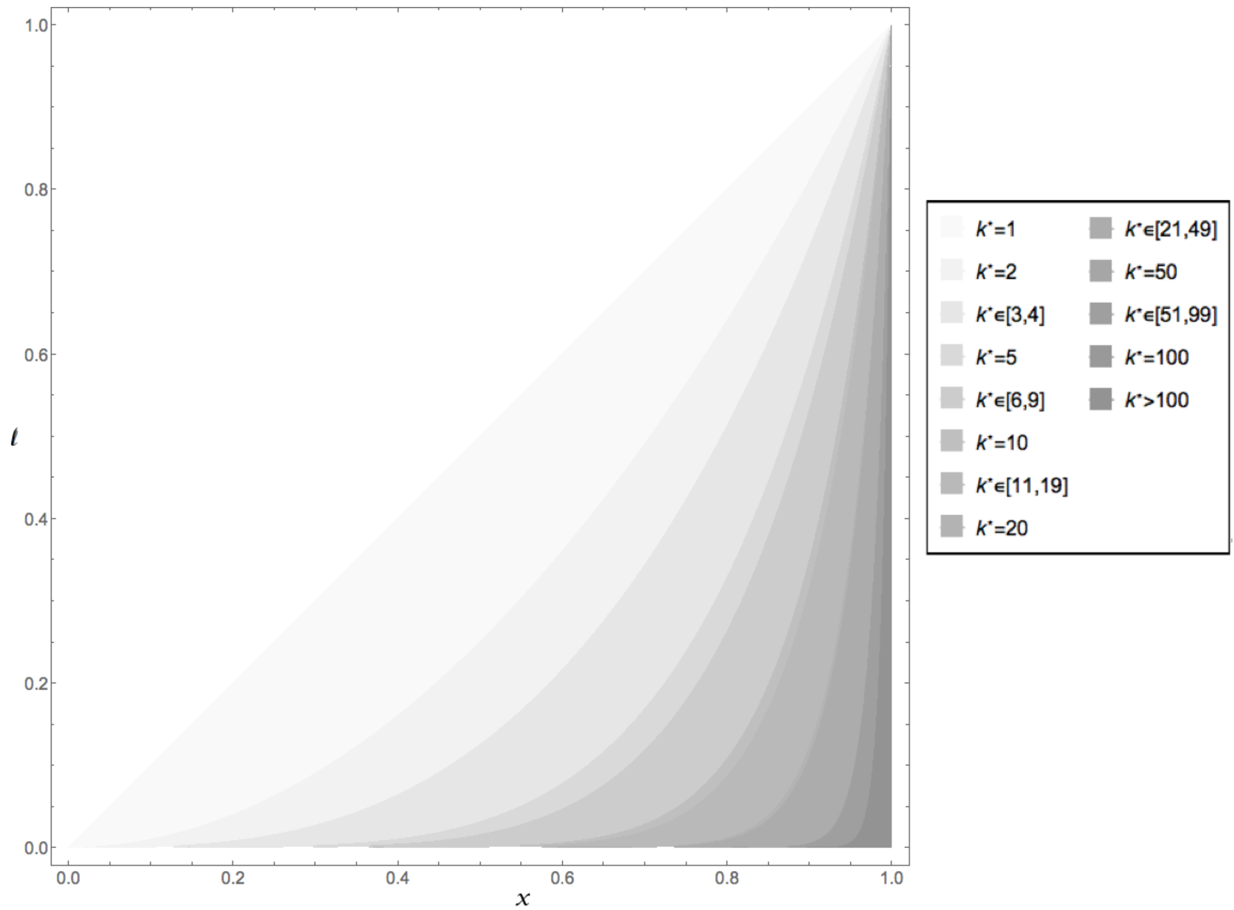


Figure 2: Equilibrium queue length, k^*.

Remark 1 says that an agent's willingness to join the queue is higher if her outside option is smaller (l is small), she is more patient (r is small), or if agents on the other side of the market arrive more frequently (λ is high).

3 Socially Optimal Queues

This section analyzes socially optimal queues and compares them with equilibrium queues. We consider the problem of a social planner who can force the agents to wait for other agents on the other side to arrive (queue), but can also prevent them from doing so, forcing them to take their outside option. We assume that the goal of the social planner is to maximize the steady state welfare of a randomly arriving agent.[4] We restrict attention to threshold policies, that is, a policy is characterized by a cutoff k such that the planner makes an agent queue if, and only if, there are less than k agents on the side of the arriving agent already in the market, and there is nobody on the other side. In what follows, we denote by $V(k)$ the value of welfare if the cutoff is k. In other words, $V(k)$ is the steady state payoff of an arriving agent if the planner's cutoff is k.

A key step in our analysis is to show that the marginal value of the planner from having an additional agent join the queue, $M(k) \equiv V(k+1) - V(k)$, is decreasing whenever it is positive, and can cross 0 at most once. A consequence of this observation is that if this marginal value is positive at the individually optimal queue size, i.e. $M(k^* - 1) > 0$, then the socially optimal cutoff is at least as large as k^*, that is, agents are (weakly) more impatient than the planner. Viceversa, if $M(k^* - 1) < 0$ then the socially optimal cutoff is smaller than k^*, that is, individuals are more patient than the planner.

The Markov Process.— The payoff of an arriving agent is determined by the distribution of queue sizes. Each planner's cutoff, k, and the arrival process induces a Markov process and an ergodic distribution over the possible queue sizes, $\{0, 1, ..., k\}$. Let p_s denote the ergodic probability that the size of the queue on a given side of the market is s. We do not introduce notation indicating the side of the market on which the queue is formed, but we note that, by symmetry, these probabilities are the same on the two sides. It is important to remember that agents arrive on either side with equal probabilities. Next, we argue that:

$$p_s = \frac{1}{2k+1} \tag{3}$$

for each $s \in \{0, ..., k\}$. Consider an interval of time of length Δ. The following three heuristic equations establish the relationship between the ergodic probabilities of different states:

$$\begin{aligned} p_0 &= (1 - 2\lambda\Delta)p_0 + 2\lambda\Delta p_1, \\ p_s &= (1 - 2\lambda\Delta)p_s + \lambda\Delta p_{s-1} + \lambda\Delta p_{s+1}, s \in \{1, ..., k-1\}, \\ p_k &= (1 - \lambda\Delta)p_k + \lambda\Delta p_{k-1}. \end{aligned} \tag{4}$$

[4] It is not hard to show that this social planner has limits-of-the-mean preferences. More formally, his policy maximizes

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N} u_i}{N}$$

where u_i denotes the expected payoff of the ith arriving agent.

The first equation states that the probability of the queue size being zero after time Δ is the sum of the probability that the initial queue size is zero and no one arrives on either side, and the probability that the initial queue size is one and an agent arrives on the opposite side.[5] The second equation states that the probability that the queue size is s after Δ time for $s \in \{1, ..., k-1\}$ is the sum of the probabilities of three events: (i) s agents are queueing initially and no one arrives on either side, (ii) $s-1$ agents are queueing on one side and an additional agent arrives on that side, and (iii) there are $s+1$ agents queueing and an agent arrives on the opposite side. Finally, the probability of transiting to state k is the sum of the probability that (i) there are k agents on one side and either no one arrives on either side, or someone arrives on that same side - and exits-, and (ii) there are initially $k-1$ agents queueing on one side, and an agent on that side arrives.

Note that setting $p_s = 1/(2k+1)$ for all s solves (4) and adds up to 1. It then follows that this solution is the unique stationary distribution for the Markov chain induced by the planner's policy. To see this, note first that the Markov chain induced by the planner's policy is irreducible. Since $p_s = 1/(2k+1)$ is a non-negative solution to (4) and adds up to 1, it follows that the Markov chain is positive recurrent. We then conclude that the identified solution is the unique stationary distribution of the Markov chain.

The Value Function.— Next, we provide an explicit characterization of the value function $V(k)$. We then use this characterization to describe some of the properties of the function V.

Lemma 1 *If the planner's cutoff is k, then the steady state payoff of an arriving agent is*

$$V(k) = \frac{1}{2k+1}\left[k + x\frac{1-x^k}{1-x} + l\right]. \tag{5}$$

Proof. We argue that the expected payoff of an arriving agent in steady state is

$$\frac{k}{2k+1} + \frac{l}{2k+1} + \sum_{i=0}^{k-1} \frac{x^{i+1}}{2k+1}. \tag{6}$$

The probability that there is at least one agent on the other side of the market is $\sum_1^k p_s = k/(2k+1)$ by (3). In this case, the arriving agent is immediately matched and receives a payoff of one, which explains the first term of the previous expression. With probability $p_k = 1/(2k+1)$, the arriving agent faces a queue of length k on her side of the market. Then, she leaves immediately and gets l, which explains the second term. Finally, the length of the queue on the side of the arriving agent is $i \in \{0, ..., k-1\}$ with probability $1/(2k+1)$, in which case, her expected payoff is $u(i) = x^{i+1}$ by (1). Summing over $i \in \{0, ..., k-1\}$ yields the last term. Finally, we note that the previous displayed expression is equivalent to (5). ■

[5] Since there are two sides, we count this event twice.

This lemma implies that the marginal value of increasing the size of the largest queue by one, $M(k) \equiv V(k+1) - V(k)$, can be expressed as

$$M(k) = \frac{1 + x^{k+1} - 2l + 2x \sum_{i=0}^{k-1} (x^k - x^i)}{(2k+1)(2k+3)} \tag{7}$$

Equation (7) illustrates the benefits and costs of increasing the length of the queue from k to $k+1$. On the one hand, increasing the length of the queue allows the planner to form an additional match by keeping an arriving agent when there are already k agents queueing, when he would have otherwise sent this agent away. On the other hand, a longer queue implies that a randomly arriving agent faces a higher probability of having to queue in order to be served than before. Since waiting is costly, this shows up as a cost in (7) (this is the term in the summation).

The next lemma characterizes some properties of the function M.

Lemma 2

(i) If $M(k) < 0$ then for all $k' > k$, $M(k') < 0$ and

(ii) If $M(k) \geq 0$ then $M(k+1) \leq M(k)$.

Proof. First, we prove that the numerator on the right-hand side of (7) is decreasing in k, that is,[6]

$$x^{k+2} + 2x \left[(k+1)\, x^{k+1} - \frac{1 - x^{k+1}}{1-x} \right] < x^{k+1} + 2x \left[k x^k - \frac{1 - x^k}{1-x} \right].$$

Since $x \in (0,1)$, this inequality would be implied by

$$(k+1)\, x^{k+1} - \frac{1 - x^{k+1}}{1-x} < k x^k - \frac{1 - x^k}{1-x},$$

or equivalently,

$$(1-x)\, x^k \left[(k+1)\, x - k \right] < (1-x)\, x^k.$$

Since $x \in (0,1)$, $(k+1)\, x - k < 1$ and hence, the inequality indeed holds.

To prove part (i), note that the sign of $M(k)$ is determined by the sign of its numerator. Since the numerator of $M(k)$ is decreasing, then if $M(k) < 0$, it follows that $M(k+1) < 0$.

To prove part (ii), note that:

$$\begin{aligned} M(k+1) &= \frac{1 - 2l + x^{k+2} + 2x \left[(k+1)\, x^{k+1} - \frac{1-x^{k+1}}{1-x} \right]}{(2k+5)(2k+3)} \\ &\leq \frac{1 - 2l + x^{k+1} + 2x \left[k x^k - \frac{1-x^k}{1-x} \right]}{(2k+5)(2k+3)} = \frac{2k+1}{2k+5} M(k), \end{aligned} \tag{8}$$

[6] Note that we rewrote equation (7) as:

$$\frac{1 + x^{k+1} - 2l + 2x \sum_{i=0}^{k-1} (x^k - x^i)}{(2k+1)(2k+3)} = \frac{1 - 2l + x^{k+1} + 2x \left[k x^k - \frac{1-x^k}{1-x} \right]}{(2k+1)(2k+3)}.$$

where the inequality follows from replacing the the numerator of $M(k+1)$ by that of $M(k)$. If $M(k) \geq 0$, the result follows from noting that $(2k+1)/(2k+5) < 1$. ■

Part (i) of Lemma 2 implies that the socially optimal queue length K^*, which solves $\max_{k \in \mathbb{N} \cup \{\infty\}} V(k)$, can be characterized as the threshold value at which the sign of the function M switches from positive to negative, that is,

$$M(k) \geq 0 \text{ if } k < K^* \text{ and } M(k) \leq 0 \text{ if } k \geq K^*. \tag{9}$$

Figure 3 below displays the values of K^* as a function of x, l, where the white area corresponds to $K^* = 0$, and darker shades of gray correspond to higher values of K^*:

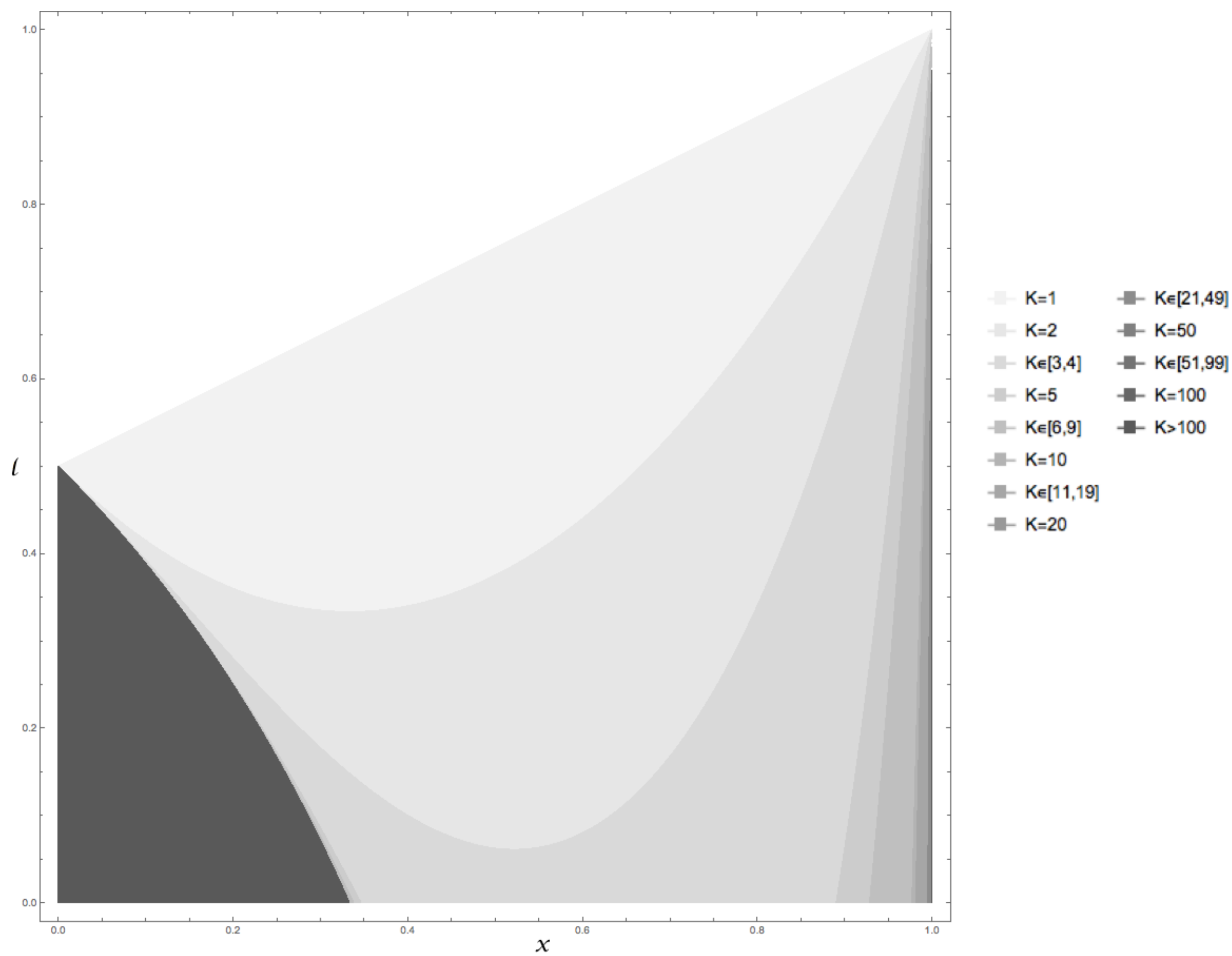


Figure 3: Socially optimal queue length, K^*.

Next, we establish a comparative static result regarding the socially optimal queue size. Note that the numerator of $M(k)$ in (7) is decreasing in l. Hence, if $M(k) \geq 0$ for a given l, this inequality continues to hold for any $l' \leq l$. By (9), this implies the following:

Remark 2 *The socially optimal queue length, K^*, is decreasing in l.*

Remark 2 formalizes the intuition that if agents have better outside options, the planner is less willing to have them join the queue.

Figure 3 shows that K^* is not necessarily monotonic in x. This is in contrast to the observation in Remark 1 for k^*. To explain this observation, recall from (9) that the change in the optimal queue seize in response to an increase in x depends on the effect on $M(k)$. By (6), $M(k)$ can be expressed as

$$\frac{x^{k+1}}{2k+3} - 2\left[\sum_{i=0}^{k-1} \frac{x^{i+1}}{(2k+3)(2k+1)}\right] + \left[\frac{k}{2k+3} - \frac{k}{2k+1}\right] + \left[\frac{l+1}{2k+3} - \frac{l}{2k+1}\right].$$

An increase in x has the following two countervailing effects on this expression. The first term, corresponding to the payoff of the agent who is last in line if the maximum queue-length is $k+1$, is increasing in x. The second term is also increasing in x; however, it has a negative sign. This term corresponds to the change in payoffs of those agents who are not matched immediately and arrive at states $0, ..., k$. This term has a negative sign because the probability of each of these states goes down from $1/(2k+1)$ to $1/(2k+3)$. The final effect on K^* of an increase in x depends on the relative impact of x on the first and second terms. Figure 3 shows that, depending on the parameter values, either of these effects can dominate the other. The next remark shows that the net effect largely depends on l.

Remark 3 *If $l \geq 1/2$, then K^* is non-decreasing in x, whenever $K^* \geq 1$; if $l < 1/2$, there exists $\overline{x}(l)$ such that for all $x \geq \overline{x}(l)$, K^* is non-decreasing in x.*

The proof of Remark 3 is in Claim 1 in the appendix. It follows from Remark 3 that K^* can only be non-increasing in x when $l \leq 1/2$; this can already be seen in Figure 3.

To understand the statement in Remark 3, it is useful to first understand why $l \lesseqgtr 1/2$ is the relevant cutoff. Note that from each pair of arriving agents, one on each side of the economy, the planner can always guarantee a payoff of $2l$ by forcing each of them to take their outside option upon arrival. However, the planner can also guarantee that he obtains a payoff of (at least) 1 by forcing the first to arrive member of the pair to wait and match with the second member of the pair, who obtains a payoff of 1. Note that this may entail the first agent waiting for an arbitrarily long period of time before being matched, making her payoff close to 0. Whenever $l < 1/2$, there is x low enough so that such a scheme may be profitable. However, when $l > 1/2$, it better be the case that the agent who waits does not wait for an arbitrarily long period of time; matching agents with their outside option upon arrival dominates the above scheme.[7]

The above discussion implies that, when $l \geq 1/2$, the planner chooses to keep a finite (possibly empty) queue.[8] Claim 1 shows that in this case the length of the queue is non-decreasing in x, and hence the first of the two aforementioned effects dominate: the benefit from increased matching opportunities is greater than the increase in waiting costs. This is because given a certain queue length the planner always benefits from the increased arrivals, and can keep the waiting costs at bay by not increasing the queue length. When $l < 1/2$, the planner may choose to keep an infinite length queue when x is small enough. Then, as x increases, the increase in waiting costs generated by this policy is of first order, and thus, the socially optimal queue length initially diminishes. However, as x increases, the planner no longer keeps long queues, and the increase in matching opportunities eventually becomes first order in the planner's welfare calculations.

4 Main Results

We compare the equilibrium queue size, k^*, with the socially optimal one, K^*, using the statement of Lemma 2. Lemma 2 implies that if $M(k) \geq 0$ for $k \in \mathbb{N}$ then $K^* \geq k$. The reason is that, by part (ii) of Lemma 2, the value of the social welfare function is increasing on $[0, k]$. On the flip side, if $M(k) \leq 0$ then part (i) of Lemma 2 implies that the function V is decreasing on $[k, \infty)$ and hence, $K^* \leq k$. The proof of the next proposition uses this argument with $k = k^*$, that is, we evaluate the sign of the function M at the equilibrium queue size k^*.

Proposition 1

(i) If $x \leq 1/3$, then $K^* \geq k^*$; this inequality is strict whenever $x \geq l$.

(ii) If $x^2 < l \leq x$, then $K^* \geq 1 = k^*$.

(iii) If $x^5 \geq l$, and $x \geq 1/2$, then $k^* > K^*$.

In what follows, we prove statements (i) and (ii) of Proposition 1. The proof of statement (iii) is carried out in Appendix A.2.

Proof. To show (i), we consider two cases. If $x < l$, then, by (2), $k^* = 0$ so $k^* \leq K^*$. Next, we consider the case of $x \geq l$. We show that if $x \leq 1/3$ then $M(\log l / \log x) > 0$ and hence, by (2), $M(k^*) \geq M(\log l / \log x) > 0$. By (9), this inequality implies that the planner wants to form a queue of length at least $k^* + 1$, that is, $K^* > k^*$. Note that, by (7), the numerator of

[7] This discussion highlights how the planner takes into account both sides of the market in his calculations (he compares $1 + x^K$ to $2l$), while the agents only compare their own benefit (they compare 1 to l).

[8] See Proposition 2.

$M(\log l/\log x)$

$$1-2l+x^{\frac{\log l}{\log x}+1}+2x\left[\frac{\log l}{\log x}x^{\frac{\log l}{\log x}}-\frac{1-x^{\frac{\log l}{\log x}}}{1-x}\right]=1-\frac{2x}{1-x}+l\left(2x\frac{\log l}{\log x}+\frac{2x}{1-x}+x-2\right)$$
$$\geq 1-\frac{2x}{1-x}+l\left(2x+\frac{2x}{1-x}+x-2\right)\geq 1-\frac{2x}{1-x}+\frac{1}{3}\left(2x+\frac{2x}{1-x}+x-2\right),$$

where the equality follows from $x^{\frac{\log l}{\log x}}=l$, the first inequality follows from $x\geq l$ and the last one from the observation that the term in brackets in the third expression is negative for $x\leq 1/3$ and $x\in[l,1/3]$. This inequality chain implies that for $x\in[l,1/3]$

$$M\left(\frac{\log l}{\log x}\right)\geq\frac{1-\frac{2x}{1-x}+\frac{1}{3}\left(2x+\frac{2x}{1-x}+x-2\right)}{\left(2\frac{\log l}{\log x}+1\right)\left(2\frac{\log l}{\log x}+3\right)}>0,$$

where the latter step follows from algebra.

To show part (ii), note that equation (2) implies that $x^2<l\leq x$ if, and only if, $k^*=1$. By (7), the numerator of $M(0)$ is $1+x-2l\geq 1-l>0$ so $M(0)>0$. Then (9) implies that $K^*\geq 1=k^*$. ■

Recall that $x=\lambda/(\lambda+r)$, and note that it is increasing in λ and decreasing in r. Therefore, statement (i) of Proposition 1 implies that if the arrival rate of the agents, λ, is small or the discount rate of the agents, r, is high, then equilibrium queues are too short relative to the efficient ones. In this case, it is too costly for agents to wait to be matched, and thus they leave the market too early, ignoring that by doing so they reduce the matching opportunities encountered by future agents who arrive on the opposite side.

Proposition 2 below identifies parameter values for which the social planner always asks the agents to join the queue irrespective of its length, that is, $K^*=\infty$.

Proposition 2 *If* $1-2l-2x/(1-x)>0$ *then* $K^*=\infty$.

The fact that the social planner finds it optimal to form arbitrarily long queues if l and x are small might appear surprising at first glance. Let us explain this observation. To this end, let us assume that the agents are very impatient, that is, $r\approx\infty$ and hence, $x\approx 0$. In this case, whenever an agent has to wait she receives a payoff close to zero. If the maximum length of the queue is K then, by (5), a newly arriving agent's payoff is roughly $(K+l)/(2K+1)$. The reason is that the probability of having an agent on the other side of the market is $K/(2K+1)$, in which case, the agent is matched immediately and gets a payoff of one. With probability $1/(2K+1)$, there is already a K-long queue on the agent's side, in which case she takes her outside option and receives l. In any other case, the agent joins a queue and, since she is very impatient, receives a payoff close to zero. The expression $(K+l)/(2K+1)$ is strictly increasing in K if and only if $l>1/2$ and converges to one half. So, by forming arbitrarily long queues, the planner maximizes the probability that a newly arriving agent does not have to wait. This probability cannot exceed 1/2, so a newly arriving agent's payoff cannot be larger than one. Hence, the planner can find it optimal to set $K^* = \infty$ only if $1/2 > l$. Note that when $x \approx 0$ the requirement $l < 1/2$ is exactly the hypothesis of Proposition 2.

We emphasize that the possibility of unbounded queues is a consequence of the two-sidedness of the market we analyse. To see this, consider a planner who only determines the queue on one side of the market and the welfare function which only takes into account the payoffs of the agents on that side. The planner's problem then becomes similar to the one arising in a *server model*, see for example *[8],* where customers join a queue and they are served as service opportunities arrive randomly. Then the optimal queue size is always bounded, in fact, it is smaller than the largest equilibrium queue size. The reason is that, just like an individual agent in equilibrium, this planner ignores the positive externality that joining the queue imposes on the other side of the market. On the other hand, unlike an agent in equilibrium, this planner takes into account the negative externality that joining the queue imposes on the same side of the market. Therefore, unbounded queues can be socially optimal in our two-sided market because a queueing agent contributes to welfare not only through her payoff upon being matched but also through the increase in speed at which agents are matched on the other side of the market.

Proof. Recall that the sign of $M(k)$ is the same as the sign of its numerator, which is given by:

$$1 - 2l + x^{k+1} + 2x \sum_{i=0}^{k-1} (x^k - x^i).$$

Note that:

$$\lim_{k \to \infty} 1 - 2l + x^{k+1} + 2x \sum_{i=0}^{k-1} (x^k - x^i) = 1 - 2l - 2\frac{x}{1-x}.$$

Hence, if the above condition holds, $M(k) > 0$ for all k, and therefore $K^* = \infty$. ■

Propositions 1 and 2 characterize sets of parameter values (x, l) on which the socially optimal queue size, K^*, is larger or smaller than the equilibrium queue size, k^*. This characterization is, however, incomplete as the regions corresponding to the inequalities in the statements of these propositions do not cover all the possible pairs of parameters, $[0, 1]^2$. Figures 4 and 5 below display our computational results regarding the comparison of k^* and K^*. They also illustrate the regions on which we have analytical results.

Figure 4 depicts the regions identified by Propositions 1 and 2 where $K^* \geq k^*$. The hatched area in red corresponds to $K^* \geq k^*$, the solid line corresponds to $x = 1/3$ (statement (i) in Proposition 1), the blue shaded area corresponds to statement (ii), and the shaded gray area close to the origin corresponds to $K^* = \infty$.

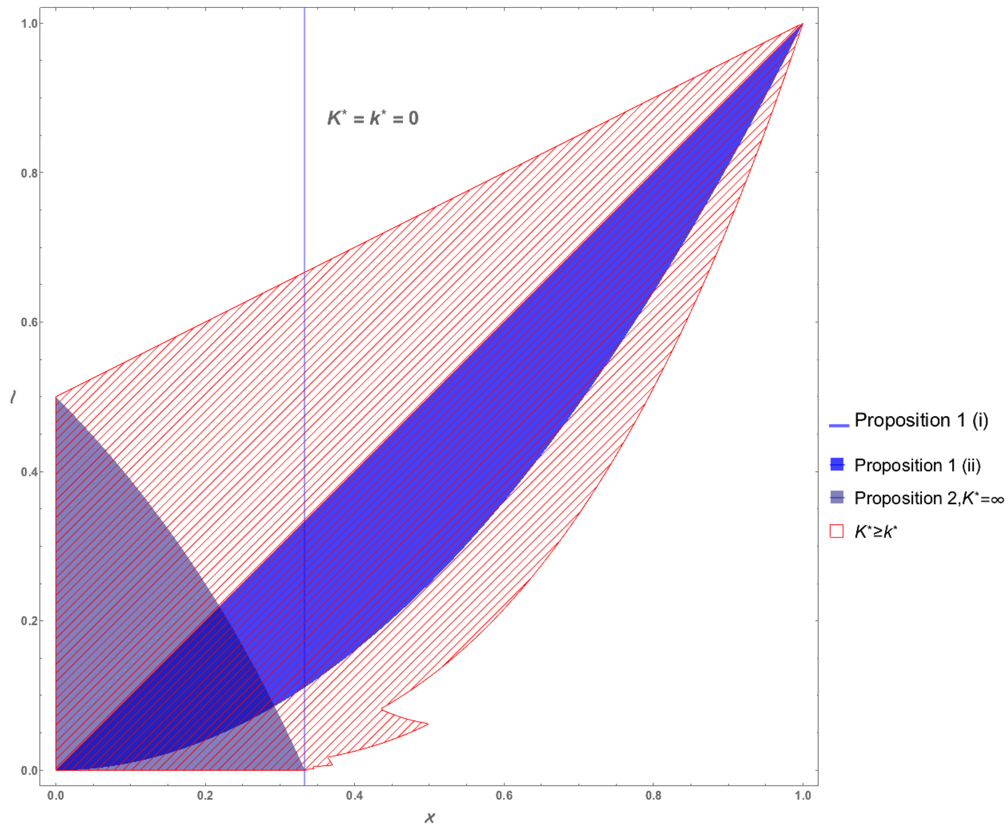


Figure 4: $K^* \geq k^*$; Proposition 1, (i)-(ii), and Proposition 2

Figure 5 depicts the region identified by Proposition 1 where $K^* < k^*$: the hatched area in red corresponds to $K^* < k^*$, while the shaded subset corresponds to statement (iii) in Proposition 1.

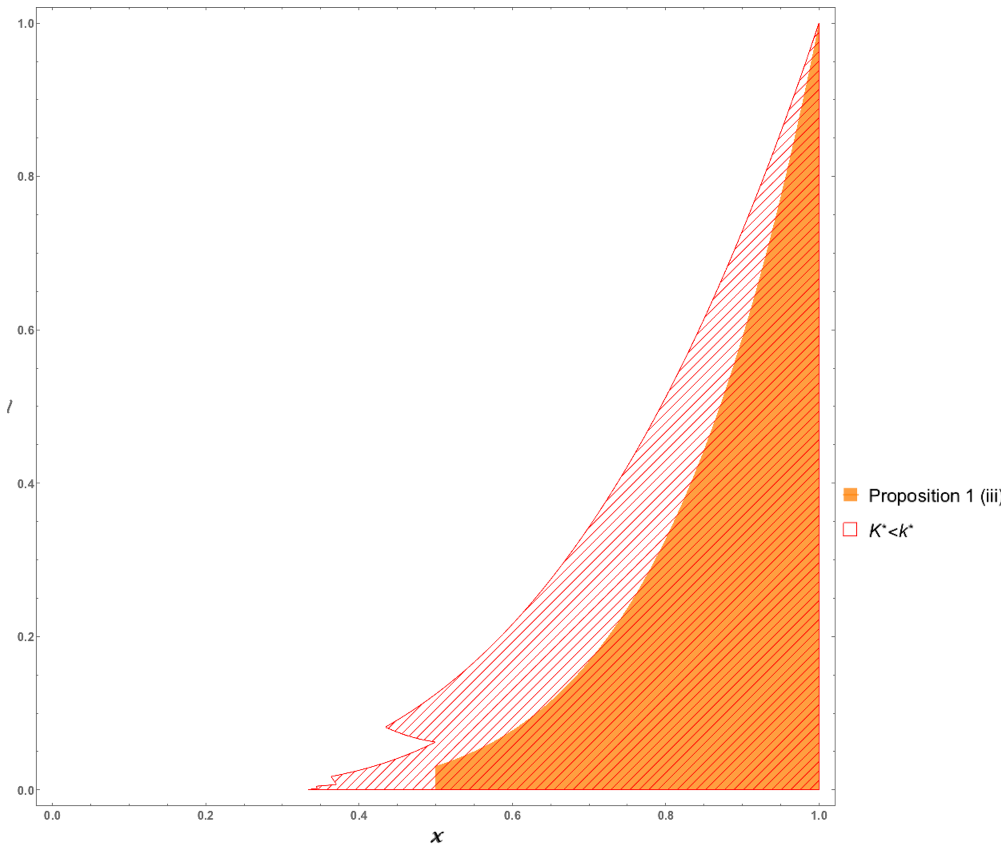


Figure 5: $K^* < k^*$; Proposition 1,(iii)

It follows from our results and the computations illustrated in the above figures that, fixing a value of l and increasing x, the planner starts off being more patient than the agents, i.e., $K^* \geq k^*$, and then becomes more impatient, i.e. $K^* < k^*$ (recall also Figure 1 in the introduction). In order to explain this observation, we argue that if x is small, the planner benefits from increasing the maximum queue size at k^*, that is, $M(k^*) \geq 0$. Similarly, if x is large, welfare is increased if the maximum queue length is decreased at k^*, that is, $M(k^*) \leq 0$. To this end, observe that if x is small then k^* is also small and the last term in the numerator of $M(k^*)$, (7), is close to zero. Since $x^{k^*+1} \approx l$ (by 2), this numerator can be approximated by $1 - l$, which is indeed positive and hence, $M(k^*) \geq 0$. In contrast, if x is close to one, k^* is large and the last term in the numerator becomes a small negative number. In fact, this term is decreasing in x and can be arbitrarily small. Again, since $x^{k^*+1} \approx l$ (by 2), the remaining terms in this numerator is around $1 - l$ which is dominated by the last term if x is big, showing that $M(k^*) \leq 0$. Indeed, Claim 2 in Appendix A.3 shows that, for each l, $M(k^* - 1) < 0$ for x large enough.

5 Flow Costs

This section considers the case in which an agent incurs a flow cost of $c > 0$ every period she stays in the queue. To be more precise, if the waiting time of an agent is t before being matched, then her payoff is $h - tc$. This analysis allows us to provide further intuition about the results of the previous section and facilitates the comparison with Baccara et al. [5]. We first solve a single agent's problem when the queueing discipline is first-come-first-served and derive the equilibrium queue length. We then solve for the socially optimal queue length and compare it to the equilibrium one.

5.1 Equilibrium

If an agent joins a queue of length $k-1$, her expected waiting time is k/λ because k agents must arrive on the other side of the market before this agent is matched. So, if she joins the queue, the payoff of the agent is $1 - (ck)/\lambda$. Therefore, an agent is willing to join a queue of size $k-1$ if, and only if

$$1 - l \geq \frac{ck}{\lambda}.$$

That is, the agent joins the queue of size $k-1$ if the benefit from queueing, $1-l$, is larger than her expected waiting cost $(ck)/\lambda$. Thus, the equilibrium queue length is determined by

$$\widetilde{k}^* = \left\lfloor \frac{(1-l)\lambda}{c} \right\rfloor$$

Figure 6 below depicts the equilibrium cutoff for $(\lambda/c, 1-l) \in [0,100] \times [0,1]$, where the white area corresponds to $\widetilde{k}^* = 0$ and darker shades of gray correspond to higher values of k^*:

5.2 Socially Optimal Queue Size

We denote the value function of the planner by $\widetilde{V}$ and the marginal value of social welfare by $\widetilde{M}$ (as a function of the largest queue size). Using arguments similar to the derivations of V and M, it can be shown that

$$\widetilde{V}(k) = \frac{1}{2k+1}\left[l + 2k - \frac{c}{\lambda}\frac{k(k+1)}{2}\right],$$

and

$$\begin{aligned}\widetilde{M}(k) &\equiv \widetilde{V}(k+1) - \widetilde{V}(k) \\ &= \frac{1}{(2k+1)(2k+3)}\left[2(1-l) - \frac{c}{\lambda}(k+1)^2\right].\end{aligned}$$

The socially optimal queue length, $\widetilde{K}^*$ satisfies:

$$\widetilde{M}(\widetilde{K}^* - 1) > 0 \text{ and } \widetilde{M}(\widetilde{K}^*) \leq 0,$$

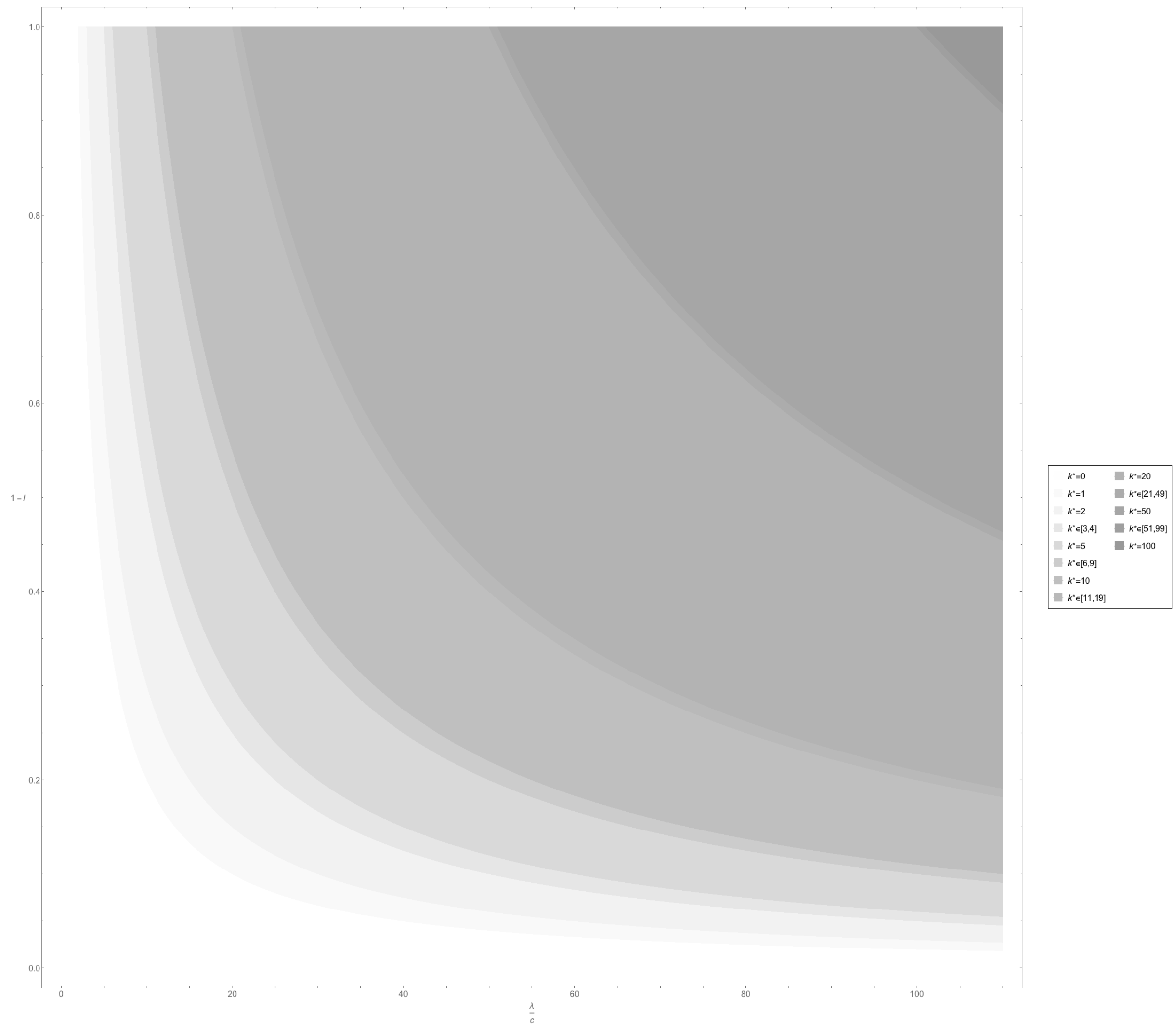


Figure 6: Equilibrium queue length, k^*.

or,

$$\widetilde{M}(\widetilde{K}^* - 1) \geq 0 \text{ and } \widetilde{M}(\widetilde{K}^*) < 0.$$

Thus, $\widetilde{K}^*$ satisfies

$$\sqrt{\frac{2(1-l)\lambda}{c}} - 1 \leq K^* \leq \sqrt{\frac{2(1-l)\lambda}{c}}.$$

Figure 7 below depicts the planner's cutoff for $(\lambda/c, 1-l) \in [0,100] \times [0,1]$, where the white area corresponds to $\widetilde{K}^* = 0$ and darker shades of gray correspond to higher values of $\widetilde{K}^*$:

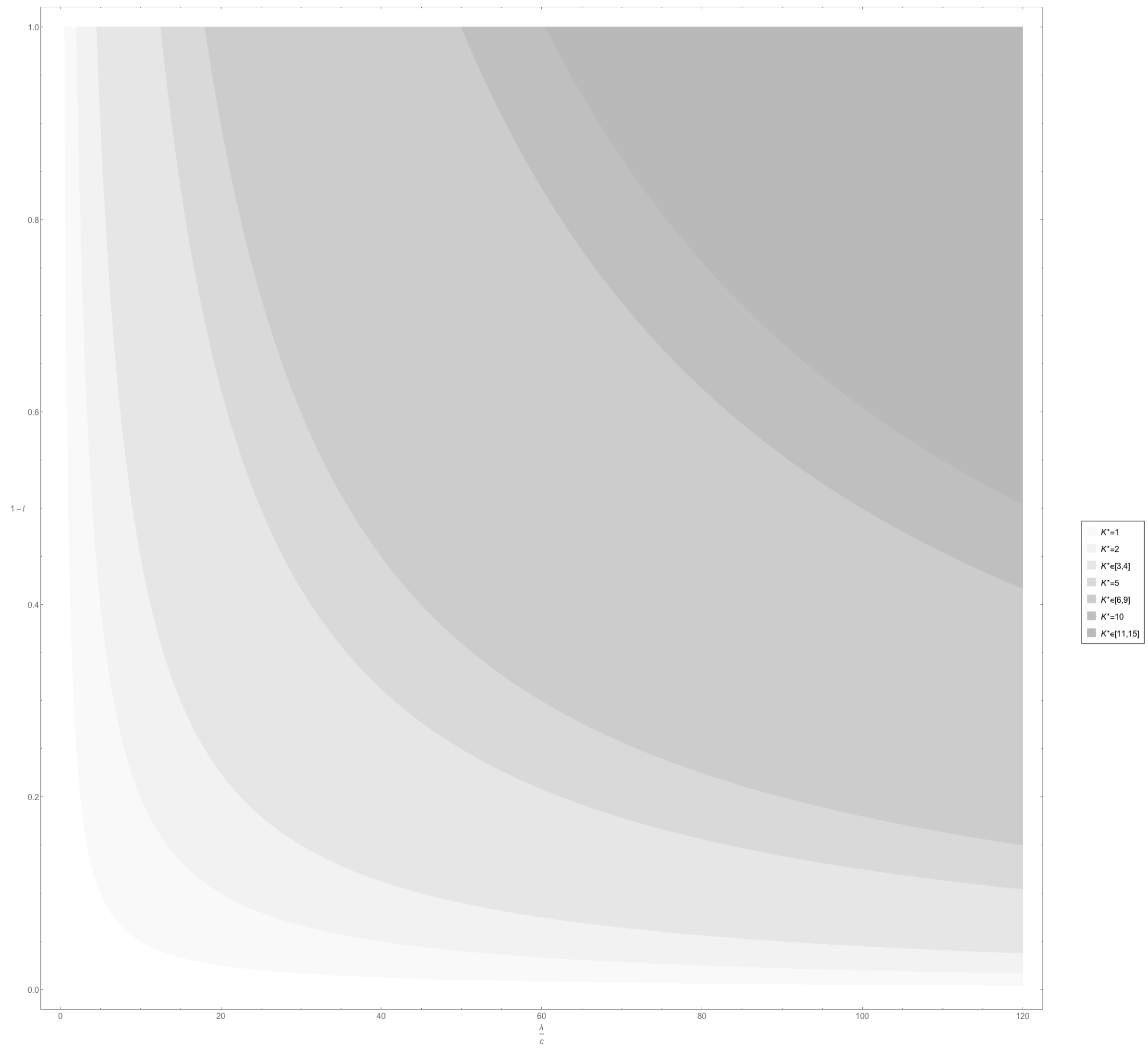


Figure 7: Socially optimal queue length, K^*.

Proposition 3 is the main result of this section:

Proposition 3 *When agents incur flow costs while they wait in the queue, the following holds:*

(i) If $\widetilde{k}^* \geq 1$ *then* $\widetilde{K}^* \leq \widetilde{k}^*$ *and the inequality is strict whenever* $\widetilde{k}^* \geq 3$.

(ii) There exist parameter values such that $\widetilde{K}^* = 1$ *and* $\widetilde{k}^* = 0$.

The proof is in Appendix A.4. To see why the result holds, ignore the integer constraints on $\widetilde{k}^*$ and $\widetilde{K}^*$ that is, take $\widetilde{k}^* = \lambda(1-l)/c$ and $\widetilde{K}^* = \sqrt{2\lambda(1-l)/c}$. Then $\widetilde{k}^* = {\widetilde{K}^*}^2/2$ and therefore,

$$\widetilde{k}^{*}-\widetilde{K}^{*}\geq 0\Leftrightarrow \widetilde{K}^{*}(\widetilde{K}^{*}-2)\geq 0\Leftrightarrow \widetilde{K}^{*}\geq 2,$$

Thus, once the socially optimal queue entails at least two agents then it is always the case that the equilibrium queue length is inefficiently high. Moreover, when the individual queue length is exactly one, i.e. $\lambda(1-l)/c\in[1,2)$, then $\widetilde{K}^{*}=1$.[9] However, when the socially optimal queue has length one then it is possible to find parameter values such that every agent takes the outside option upon arrival.

According to Proposition 3, when agents incur flow costs of waiting, the planner generally keeps shorter queues than those that arise in equilibrium. This is in sharp contrast with our results regarding discounting agents in Section 3. To explain this observation, note that there are two ways in which the model with discounting differs from the one with waiting costs, both of which suggest the planner will hold shorter queues in the case of flow costs. First, in the model with flow costs, the planner is reluctant to maintain long queues, since an agent's payoff goes to minus infinity as her waiting time converges to infinity. As a consequence, the cost of forcing an additional agent to join the queue in order to create another match is prohibitively high if the queue is already long. In the case of discounting, an agent's payoff is at least zero irrespective of her waiting time. Therefore, the cost of forcing an additional agent to join the queue is bounded which, in turn, might result in arbitrarily long queues (see Proposition 2). The second difference relates to how discounting and explicit waiting costs shape agents' decisions to join the queue. To see this, let us define the net value of matching of an agent as the difference between the payoff of matching evaluated at the moment of joining the queue and the outside option, l. Discounting introduces a wedge between the net value of matching of an agent who finds a queue of size $k-1$ on her side, $x^{k}-l$, and that of an agent who finds a queue of size $k-1$ on the opposite side, $1-l$. This is different from the net value of matching of a randomly arriving agent when there are $k-1$ agents in the system, $(1+x^{k}-l)/2$. Note that the latter corresponds to the first part of the numerator in $M(k)$, which is the positive term in the marginal social value of increasing the queue from $k-1$ to k. In contrast, when agents incur flow costs of waiting, the net value of matching is the same in both cases and coincides with that of a randomly arriving agent in the system, $2(1-l)/2=1-l$, which is the same as the positive term in the numerator in $\widetilde{M}(k)$. Thus, while in both models, the agents and the planner "disagree" on how to compare the negative effects generated by joining the queue (captured by the negative terms in $M(k)$ and $\widetilde{M}(k)$, respectively), in the case of flow costs, they "agree" on how to compare the positive effects generated by joining the queue, conditional on the agents joining the queue in equilibrium.

To compare our results to those in Baccara et al. [5], let us briefly describe their model. Baccara et al. [5] consider a discrete-time, dynamic matching environment with two sides. Unmatched agents on the market pay a flow cost of c. Each agent has a binary quality, either

[9] To see this, note that $\lambda(1-l)/c\in[1,2)$ implies $\sqrt{2\lambda(1-l)/c}\in[\sqrt{2},2)$, and hence $K^{*}=1$.

high or low. Low-quality agents in their model play the role of our outside option: all agents prefer to match with a high-quality agent and match with a low-quality agent only when the waiting time for a high-quality agent is excessive. Every period, a new agent arrives on each side of the market. Baccara et al. [5] show that, in equilibrium, agents *always* maintain longer queues than the socially optimal ones. Proposition 3 is similar to their result, except that, in our model, agents may choose not to join any queue, when it is socially optimal to have a queue of length one, $\widetilde{K}^* = 1$. If the equilibrium queue size is zero in our model and the planner forces an agent to form a queue of length 1, this agent generates a positive externality for the other side because the next agent arriving on the other side obtains a payoff of 1 instead of l. In addition, this agent does not increase the waiting time of agents arriving on her side because they take their outside options.[10] Thus, the planner may set $\widetilde{K}^* = 1$, despite $\widetilde{k}^* = 0$. In the model of Baccara et al. [5], making a high-quality agent to wait to be matched with a high-quality agent on the other side instead of matching her with the newly arrived low-quality agent immediately has two effects. On the one hand, the waiting agent generates a future benefit for an arriving high-quality agent on the other side, who would have been matched with a low-quality agent otherwise. On the other hand, she generates a cost for the low-quality agent on the other side who is now forced to wait but would have been matched immediately otherwise. Thus, even though this queueing agent does not increase waiting costs on her side, she does so on the other side of the market. This cost overturns the benefit of keeping a queue of length one, whenever it is not an equilibrium for agents to form a queue of length one. Finally, notice that as long as the socially optimal queue is at least two then the equilibrium in our model entails inefficiently long queues, as in Baccara et al. [5]. Now, when the planner is evaluating whether to add an agent to an existing queue of length K, he not only considers the benefit for agents on the other side, but also he considers the increased waiting times for agents on the same side, product of the longer queues.

References

[1] M. Akbarpour, S. Li, and S. Oveis Gharan. Thickness and information in dynamic matching markets. *Available at SSRN 2394319*, 2016.

[2] R. Anderson, I. Ashlagi, D. Gamarnik, and Y. Kanoria. A dynamic model of barter exchange. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1925–1933. SIAM, 2015.

[3] I. Ashlagi, M. Burq, P. Jaillet, and V. Manshadi. On matching and thickness in heterogeneous dynamic markets. 2016.

[10] The planner compares $2(1-l)$ (benefit) to c/λ (cost). On the contrary, an individual agent only compares her benefit from waiting, $1-l$, with her waiting cost, c/λ, failing to internalize the benefit she generates for the other side.

[4] A. E. Atakan. Assortative matching with explicit search costs. *Econometrica*, 74(3):667–680, 2006.

[5] M. Baccara, S. Lee, and L. Yariv. Optimal dynamic matching. *Available at SSRN 2641670*, 2015.

[6] R. Hassin et al. On the optimality of first come last served queues. *Econometrica*, 53(1):201–02, 1985.

[7] J. D. Leshno. Dynamic matching in overloaded waiting lists. Technical report, Columbia University Working Paper, 2015.

[8] P. Naor. The regulation of queue size by levying tolls. *Econometrica: journal of the Econometric Society*, pages 15–24, 1969.

[9] R. Shimer and L. Smith. Matching, search, and heterogeneity. *Advances in Macroeconomics*, 1(1), 2001.

[10] M. U. Ünver. Dynamic kidney exchange. *The Review of Economic Studies*, 77(1):372–414, 2010.

A Discounting

A.1 Comparative statics results for K^*.

In this section, we prove the claims made in Remark 3 concerning the comparative statics of K^* as a function of x (i.e., fixing l).

Claim 1 *The following hold:*

(i) If $K^* \geq 1$ and $l \geq 1/2$, then K^* is increasing in x.

(ii) If $l < 1/2$, then there exists $\overline{x}(l)$ such that for $x \geq \overline{x}(l)$, K^* is increasing in x.

In order to prove the claim, we characterize in Lemma 3 below properties of the function $j(x,k)$, defined as follows:

$$j(x,k) = x^k(2k - 1 + \frac{2}{1-x}) - \frac{2x}{1-x}.$$

That $j(x,k)$ is the object of interest follows from noting that K^* solves $j(x, K^*) \geq 2l - 1$ and $j(x, K^* + 1) < 2l - 1$.

Lemma 3

(i) $j(x,k)$ is increasing whenever it is positive,

(ii) For all l, k, $\lim_{x\to 1} \frac{\partial j(x,k)}{\partial x} > 0$, and

(iii) $j(0,k) = 0$, and $j(x,k)$ is decreasing around $x = 0$ for $k \geq 2$.

Note that when $l \geq \frac{1}{2}$, $K^* \geq 1$ requires that $j(x, K^*) > 0$. It follows from part (i) that for $l \geq 1/2$, $j(x, K^*) \geq 2l - 1$ implies that $j(x', K^*) \geq 2l - 1$ for $x' \geq x$. Thus, the socially optimal queue length at x' cannot be less than K^*, and part (i) of Claim 1 follows. When $l < 1/2$, the same logic does not apply. However, part (ii) implies that for x large enough K^* is non-decreasing in x, which proves part (ii) of Claim 1. Finally, statement (iii) implies that for low values of x the increase in waiting costs is of first order in the planner calculations.

In what follows, we prove Lemma 3:

Proof. In what follows, we focus on the case in which $k \geq 2$. To see the reason for this, note that $j(x, 1) = x$ which is increasing in x. Hence, whenever the planner wants to form a queue of 1 at some (x, l), then as x increases, the socially optimal queue length will never fall below 1.

To prove part (i) in Lemma 3, we start by noting that $j(x,k) \geq 0$ implies that:

$$x^k(1 + 2(k-1)) \geq 2x\frac{1 - x^{k-1}}{1 - x}. \tag{10}$$

Now, differentiating $j(x, k)$ with respect to x, we obtain:

$$\begin{aligned}
\frac{\partial}{\partial x} j(x,k) &= kx^{k-1} + 2((k-1)x^{k-1} - \frac{1 - x^{k-1}}{1 - x}) + 2x((k-1)^2 x^{k-2} - \frac{-(k-1)x^{k-2}(1-x) + 1 - x^{k-1}}{(1-x)^2}) \\
&= x^{k-1} + 2((k-1)x^{k-1} - \frac{1 - x^{k-1}}{1 - x}) + (k-1)x^{k-1} + 2x((k-1)x^{k-2}((k-1) + \frac{1}{1-x}) - \frac{1 - x^{k-1}}{(1-x)^2}) \\
&= \frac{j(x,k)}{x} + x^{k-1}((k-1) + 2(k-1)^2 + 2\frac{(k-1)}{1-x}) - 2x\frac{(1 - x^{k-1})}{(1-x)^2}
\end{aligned}$$

Equation (10) implies that

$$x^{k-1}((k-1) + 2(k-1)^2 + 2\frac{(k-1)}{1-x}) \geq 2\frac{(k-1)}{1-x}. \tag{11}$$

Moreover, $(k-1) \geq x(1 - x^{k-1})/(1-x)$ since

$$k - 1 - x\frac{1 - x^{k-1}}{1 - x} = k - 1 - x\sum_{i=0}^{k-2} x^i \geq k - 1 - x\sum_{i=0}^{k-2} 1 = (k-1)(1-x). \tag{12}$$

Inequalities (11) and (12) implies that $\partial j(x,k)/\partial x > 0$ whenever $j(x,k) > 0$.

To prove part (ii), note that rewriting $j(x,k) = x^k(1 + 2(k-1) + 2/(1-x)) - 2x/1 - x$, we get:

$$\lim_{x\to 1} \frac{\partial}{\partial x} j(x,k) = \lim_{x\to 1} kx^{k-1}(1 + 2(k-1)) + \frac{kx^{k-1} - \frac{1-x^k}{1-x}}{\frac{1-x}{2}}.$$

Applying L'Hôpital's rule, it follows that $\lim_{x\to1} kx^{k-1} - \frac{1-x^k}{1-x} = 0$, and also note that $\lim_{x\to1}(1-x)/2 = 0$. Thus, we can apply L'Hôpital to determine $\lim_{x\to1} \frac{kx^{k-1}-\frac{1-x^k}{1-x}}{\frac{1-x}{2}}$. Note that:

$$\lim_{x\to1} \frac{kx^{k-1} - \frac{1-x^k}{1-x}}{\frac{1-x}{2}} = \lim_{x\to1} \frac{k(k-1)x^{k-2} - \frac{\partial}{\partial x}\frac{1-x^k}{1-x}}{-\frac{1}{2}}$$

Now, by L'Hôpital's rule $\lim_{x\to1} \partial((1-x^k)/(1-x))/\partial x = \lim_{x\to1}(1+(k-1)x^k - kx^{k-1})/(1-x)^2 = k(k-1)/2$. Thus,

$$\lim_{x\to1} \frac{kx^k - \frac{1-x^k}{1-x}}{\frac{1-x}{2}} = -k(k-1).$$

We then conclude that

$$\lim_{x\to1} \frac{\partial}{\partial x} j(x,k) = k(1+2(k-1)) - k(k-1) = k^2 > 0.$$

This completes the proof of part (ii).

Finally, to show part (iii), note that $j(0,k) = 0$, and $\frac{\partial}{\partial x} j(x,k)|_{x=0} = -2$ for $k \geq 2$. ■

A.2 Proposition 1

We complete in this section the proof of Proposition 1 by showing statement (iii) holds, i.e., for $x^5 \geq l$ and $x \geq 1/2$, $k^* > K^*$.

To see why this is true, note that by (9) to show that $k^* > K^*$, it suffices to argue that $M(k^*-1) < 0$. Now, if $k^* = k$, for some $k \geq 2$, then $x^k \geq l \geq x^{k+1}$. The sign of $M(k-1)$ is the same as the sign of the numerator of (7) for $k-1$

$$1 - 2l + x^k + 2x\left[(k-1)x^{k-1} - \frac{1-x^{k-1}}{1-x}\right].$$

Since $l \geq x^{k+1}$, the above expression is weakly smaller than

$$f(x,k) = 1 - 2x^{k+1} + x^k + 2x[(k-1)x^{k-1} - \frac{1-x^{k-1}}{1-x}] = \frac{1-3x}{1-x} + x^k\left[2k - 1 + \frac{2-2x+2x^2}{1-x}\right].$$

We now show that f is decreasing in k for each x. To see this, note that

$$\frac{\partial f(x,k)}{\partial k} = x^k\left[2 - \log x\left(2k-1+\frac{2-2x+2x^2}{1-x}\right)\right].$$

The sign of $\partial f(x,k)/\partial k$ is the same as the sign of $2 - \log x\left(2k-1+\left(2-2x+2x^2\right)/(1-x)\right)$. For it to be negative, it has to be that

$$2 \leq \log\frac{1}{x}\left(2k-1+\frac{2-2x+2x^2}{1-x}\right).$$

Since $x^5 \geq l$, it follows that $k \geq 5$, and hence it suffices to show that

$$2 \leq \log\frac{1}{x}\left(9+\frac{2-2x+2x^2}{1-x}\right).$$

Note that the right hand side of the above expression is decreasing in x. Hence, we only need to show that evaluated at $x = 1$, it is greater or equal than 2. The result follows from noticing that:

$$\begin{aligned}\lim_{x\to 1^-} \log \frac{1}{x}\frac{2-2x+2x^2}{1-x} &= \lim_{x\to 1^-} -\frac{\log x}{\frac{1-x}{2-2x+2x^2}} \\ &= \lim_{x\to 1^-} -\frac{\frac{1}{x}}{\frac{-2(-2+x)x^2}{(2-2x+2x^2)^2}} = 2,\end{aligned}$$

where the second equality follows from applying L'Hôpital's rule. Hence, we only need to check that $f(x,5) \le 0$ for $x \ge \frac{1}{2}$. $f(x,5) \le 0$ is equivalent to

$$g(x) = \frac{3x-1}{x^5} - (2x^2 - 11x + 11) \ge 0,$$

for $x \in [1/2, 1]$. Note that $g(1/2) > 0 = g(1)$. We next show that g is decreasing on $[1/2, 1]$. To see this, note that:

$$\begin{aligned}g'(x) &= 11 + \frac{5}{x^6} - \frac{12}{x^5} - 4x \\ g''(x) &= -4 - \frac{30}{x^7} + \frac{60}{x^6}.\end{aligned}$$

Note that $g'(1/2) < g'(1) = 0$. On the other hand, $g''(x) < 0$ for $.5 \le x \le .51$ and $g''(x) \ge 0$ for $x \ge 0.51$. Hence, g' is decreasing on $[.5, .51]$ and increasing on $[.51, 1]$, which means that $g'(x) \le 0$ on $[1/2, 1]$. Thus, $g(x) \ge 0$ as desired.

A.3 Comparison between k^* and K^*

Claim 2 *Fix l. Then, there exists $\hat{x}(l)$ such that for all $x \ge \hat{x}(l)$, $M(k^* - 1) < 0$.*

Proof. It suffices to show that $\lim_{x\to 1} M(\log l/\log x - 2) < 0$. To see this is indeed the case, notice that the sign of $M(\log l/\log x - 2)$ is equal to the sign of its numerator, which is given by:

$$1 - 2l + \frac{l}{x} + 2((\frac{\log l}{\log x} - 2)\frac{l}{x} - \frac{x - \frac{l}{x}}{1-x}).$$

Noting that $\lim_{x\to 1} \frac{\log l}{\log x} = -\infty$ yields the result. ■

A.4 Comparison between $\widetilde{k}^*$ and $\widetilde{K}^*$

Assume that $n \le \widetilde{K}^* \le n+1$; that is

$$n \le \sqrt{\frac{2(1-l)}{c}} \le n+1.$$

Note that this implies that:

$$
\begin{aligned}
n^2 \leq \frac{2(1-l)}{c} \leq (n+1)^2 \\
\frac{n^2}{2} \leq \widetilde{k}^* \leq \frac{(n+1)^2}{2}.
\end{aligned}
$$

When $n \geq 3$, it follows that $n+1 < n^2/2$ and hence $\widetilde{K}^* < \widetilde{k}^*$. Moreover, suppose that $K^* = 2$, i.e., $2 \leq \sqrt{2(1-l)/c} < 3$. Then, $\widetilde{k}^* \geq 2$. The argument in the main text implies that when $\widetilde{k}^* = 1$, then $\widetilde{K}^* = 1$ as well.