Optimal Consumption and Asset Allocation with Unknown Income Growth *

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Abstract

Recent empirical evidence supports the view that the income process has an important individual-specific growth rate component (Baker (1997), Guvenen (2007b), and Huggett, Ventura, and Yaron (2007)). Moreover, the individual-specific growth component may be stochastic. Motivated by these empirical observations, I study an individual’s optimal consumption-saving and portfolio choice problem when he needs to learn about his income growth. As in standard income fluctuation problems, the individual cannot fully insure his income shocks. In addition to the standard income-risk-induced precautionary saving demand, the individual also has estimation-risk-induced precautionary saving, which is greater when belief is more uncertain. With constant unobserved income growth, changes of belief are not predictable. However, with stationary stochastic income growth, belief is no longer a martingale and mean reverts. Mean reversion of belief reduces the hedging demand and in turn mitigates the impact of estimation risk on the individual’s consumption-saving and portfolio decisions.

Key words: Incomplete markets, precautionary saving, learning, hedging, estimation risk.

JEL: E2, G11, G31

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1 Introduction

Consumption-saving and portfolio allocation decisions are at the center stage of modern economics. Two fundamental insights in the consumption-saving literature are consumption smoothing over time (Friedman’s permanent-income hypothesis (PIH) and Modigliani’s life-cycle hypothesis) and precautionary saving for future undesirable contingencies.\(^1\) Almost all existing research assumes that the individual has complete information about the structural parameters of his income process such as growth and volatility parameters. The complete-information assumption may be a sensible starting point, if one believes that an individual’s income process can be represented by a growth component common to all individuals plus an idiosyncratic shock process as argued by MaCurdy (1982) and Abowd and Card (1989). Intuitively, if all individuals share the same income growth rate (for example within the same education group), it is reasonable to argue that each individual can estimate the “common” income growth by using the time-series and cross-sectional data of income growth (within the same group).

However, Baker (1997), Guvenen (2007a), and Huggett, Ventura, and Yaron (2007) recently provide significant and convincing empirical evidence in support of a competing view that the individual’s income process has an important individual-specific growth component. If income growth is individual specific, it is then much more difficult for each individual to estimate his own income growth. An individual enters the labor market with a prior belief about his future income growth and updates his belief over time based on realized incomes. Learning about his income growth rate inevitably induces (current and future) belief uncertainty and estimation risk. By estimation risk, I refer to uncertainty which arises from estimating the individual’s income growth. Naturally, we expect that estimation risk affects the individual’s consumption-saving/portfolio allocation decisions and his welfare.

Learning has a significant effect on consumption-saving and portfolio allocation decisions. If the agent ignores the effect of his future belief updating on his consumption, he effectively behaves as an “anticipated utility” agent as in Kreps (1998) and Sargent (1999). Behaving in a “myopic” way, he has no hedging demand against estimation risk induced by belief changes. This suboptimal behavior distorts his consumption-saving decisions. The distortion is particularly costly in utility terms when the agent is more uncertain about his income growth.

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The effect of income volatility on precautionary saving is worth discussing in our learning model. The agent hedges both the standard income risk and also the unknown stochastic income growth. Naturally, the hedging demand against income risk is higher when the income stream is more volatile. More interestingly, unknown income growth also induces a hedging demand, which is stochastic and depends on the agent’s time-varying belief. For a given fixed spread between the two possible levels of income growth rates, perhaps surprisingly, estimation risk decreases with income volatility. This seemingly counter-intuitive result may be explained as follows. Past incomes from a more volatile income process provide less information about the unknown true income growth rate. Hence, the agent updates his belief less in response to unanticipated income innovations. As a result, estimation risk is smaller when the underlying income process is more volatile, ceteris paribus. Therefore, the net impact of income volatility on hedging demand depends on the relative magnitude of these two opposing effects.

Recent work also provides strong empirical evidence consistent with the hypothesis that income growth is stochastic. Haider (2001) and Guvenen and Kuruscu (2008) document that the cross-sectional dispersion in income growth have been rising since the 1970s. That income growth is stochastic further complicates the agent’s learning problem, which in turn makes his consumption-saving and portfolio decisions more difficult to solve. The potential empirical importance of stochastic income growth on decision rules and utility costs calls for models incorporating stochastic income growth. Incorporating stochastic income growth into the agent’s optimization problem not only makes the model more realistic, but also more importantly enriches the economics of learning. Intuitively, when income growth is stochastic and unknown, belief change is locally predictable (due to the expected change of income growth) and hence belief is no longer a martingale. For example, when the conditional probability of income growth being low is small, mean reversion (due to the stochastic transition of income growth from low to high) pulls the agent’s belief upward in expectation. Intuitively, this mean reversion of belief makes shocks driving the change of belief no longer permanent, unlike in settings with unknown constant income growth. The stationary belief updating process in turn lowers the impact of estimation risk on consumption by the notion of consumption smoothing and precautionary saving. Therefore, consumption responds less to belief change.

This paper contributes to the literature on incomplete-markets consumption, saving, and portfolio choice with learning. Earlier papers that explore the role of partially observed and uninsurable income on consumption include Goodfriend (1992), Pischke (1995), and a collection of papers in Hansen and Sargent (1991). All these studies postulate that the agent’s consumption is given by the certainty equivalence based PIH rule (Friedman (1957)), which precludes any possible effect of estimation risk on consumption. The most closely related
papers are Guvenen (2007b) and Wang (2004). Guvenen (2007b) solves for the consumption rule numerically for agents with constant relative risk averse utility. His work complements this one in terms of methodology and economic insights. Unlike Wang (2004), learning has implications not only on income volatility, but also on expected changes of income. More importantly, the conditional variance of belief updating is stochastic. As a result, learning induces stochastic belief-dependent precautionary saving demand. Unlike Guvenen (2007b) and Wang (2004), this paper also studies the effect of estimation risk and hedging and portfolio allocation.

2 Model Setup

Consider a consumption-saving and portfolio allocation problem. An infinitely-lived agent receives an exogenous perpetual stream of stochastic income. He chooses his consumption and portfolio allocation between a risk-free asset and a risky financial asset. As in standard self insurance (income fluctuations) problems, the agent cannot fully insure his stochastic income stream. That is, the optimization problem is one featuring an incomplete-markets setting. For technical convenience, I cast the model in continuous time.

Let $y(t)$ denote the level of the agent’s time-$t$ labor income. Assume that the dynamics of $\{y(t) : t \geq 0\}$ is given by

$$dy(t) = (\alpha(t) - \kappa y(t))dt + \sigma dZ(t),$$

(1)

where $Z$ is a standard Brownian motion. The parameter $\sigma$ measures the conditional volatility of the income change over an incremental unit of time. The income growth parameter $\{\alpha(t) : t \geq 0\}$ may change stochastically. The detailed specification for $\alpha$ is deferred to the next section. For convergence, assume $r + \kappa > 0$, i.e. income cannot grow too fast. When $\kappa = 0$, the income process (1) has a unit root (non-stationary). When $\kappa > 0$, (1) is stationary, known as an Ornstein-Uhlenbeck process. The parameter $\kappa$ measures the degree of mean reversion. The discrete-time counterpart of (1) when $\kappa > 0$ is

\footnote{Technically, our learning uses the filter by Wonham (1964), and Wang (2004) uses the Kalman filter, which implies deterministic conditional variance and also constant estimation risk induced precautionary saving. See Veronesi (2000) for an asset-pricing application of a learning model with stochastic income growth.}

\footnote{Otherwise, the agent’s incentive to borrow against future income is “too” high, which makes the problem economically uninteresting.}

\footnote{I thank the anonymous referee and Bob King (Editor) for the suggestion to extend the model analysis from the original unit-root process to allow for mean reversion of the income process.}

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represented by the following first-order autoregressive (AR1) process: \(^5\)

\[
y(t + 1) = a_0 + a_1 y(t) + \epsilon(t + 1), \tag{2}
\]

where \(a_1 = e^{-\kappa}, a_0 = \alpha (1 - e^{-\kappa}) / \kappa, \hat{\sigma} = \sigma \sqrt{(1 - e^{-2\kappa})/(2\kappa)}, \) and \(\epsilon(t + 1)\) a time-\((t + 1)\) innovation drawn from the standard normal distribution. The above AR1 process has been widely used to model income. \(^6\) In the precautionary saving literature, Caballero (1991) uses a discrete-time unit-root process \((\kappa = 0), a special case of (2) to model labor income and derives a closed-form consumption rule. Wang (2006) obtains the closed-form consumption rule and characterizes the stochastic precautionary saving demand for a class of the income process known as “affine” models nesting (1) as a special case.

The agent can invest in both the risk-free asset (with a constant rate of return \(r\)) and the risky financial asset (e.g. the market portfolio). Investing in the risky asset offers the agent both the opportunity to earn a higher expected return than the risk-free rate \(r\) and the benefit of hedging labor-income related risk. Let \(\{ S(t) : t \geq 0 \}\) denote the cum-dividend value process for the risk asset, whose instantaneous return is given by:

\[
dS(t)/S(t) = (r + \zeta) dt + \nu dW(t), \tag{3}
\]

where \(\zeta\) is the market risk premium, \(\nu\) is the volatility of the market return, and \(W\) is a standard Brownian motion. Equation (3) specifies that the market return is independently and identically distributed (iid). Let \(\rho\) be the (instantaneous) correlation between the labor income process (1) and the return of the risky asset, i.e. the correlation between Brownian motions \(Z\) and \(W\) is \(\rho\). Let \(\eta = \zeta / \nu\) denote the Sharpe ratio of the market portfolio. Let \(\psi(t)\) denote the amount of wealth that the agent allocates to the market portfolio at time \(t\), and hence \(x(t) - \psi(t)\) corresponds to time-\(t\) wealth invested in the risk-free asset. The agent’s financial wealth dynamics is then given by

\[
dx(t) = (rx(t) + y(t) - c(t)) dt + \psi(t) (\zeta dt + \nu dW(t)), \tag{4}
\]

where the first term in (4) gives the sum of interest income \(rx\) from investing in the risk-free asset and labor income \(y\) minus consumption \(c\). That is, the first term gives the saving rate \(s = rx+y-c\) in standard self-insurance models, if the agent can only invest in the risk-free asset. The last term \(\psi(t) (\zeta dt + \nu dW(t))\) captures the “excess” return by borrowing at the risk-free rate and investing

\(^5\) For illustrative purposes, we set \(\alpha(t)\) in (1) to be constant in this discrete-time representation.

\(^6\) See Deaton (1992) for thorough discussions of the permanent income literature using the linear income models which include the AR1 process (2) as a special case. See Hansen and Sargent (2005) for an extensive treatment of linear economic models, which include the permanent-income models as an important class of applications.
in the risky asset. Next, I state the agent’s preference and his optimization problem.

Assume that the agent has a time-additive separable constant absolute risk aversion (CARA) utility function given by

\[ U(c) = E \left( \int_0^\infty e^{-\beta s} u(c(s)) \, ds \right), \]

where \( \beta > 0 \) is his subjective discount rate and \( \gamma > 0 \) is the coefficient of absolute risk aversion \( (u(c) = -e^{-\gamma c} / \gamma) \). It is well known that CARA utility gives much tractability in deriving the consumption rule because it ignores the wealth effect. Merton (1971), Kimball and Mankiw (1989), Caballero (1990), Svensson and Werner (1993), Davis and Willen (2000), and Wang (2006) have all adopted CARA utility in analyzing the agent’s consumption-saving decisions under incomplete markets with different income process specifications. The agent chooses his consumption \( c \) and wealth allocation to the risky asset \( \psi \) to maximize his utility given in (5) subject to his stochastic labor-income process (1), his wealth accumulation process (4), and the corresponding transversality condition specified in the online appendix.

3 Model Analysis

In standard consumption-saving models, the agent knows both his income process and the parameters governing his income process, such as the growth parameter \( \alpha \). However, much empirical evidence suggests that the agent’s income growth may be individual specific and hence the agent does not necessarily know his income growth parameter. Learning about income growth potentially has significant impact on the agent’s intertemporal consumption-saving and portfolio allocation rules. Moreover, income growth \( \alpha \) may change stochastically over time, further complicating the agent’s decision problem.

To understand the impact of learning and/or the stochastic feature of the income growth on consumption-saving and portfolio allocation in an intuitive and pedagogical way, we categorize our model into four special sub-models along two dimensions: Whether the agent knows the value of \( \{\alpha(t) : t \geq 0\} \), and whether the agent’s income growth \( \{\alpha(t) : t \geq 0\} \) is stochastic. The following table summarizes the structure of the model development. Each special

<table>
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case will provide new insights on the effect of learning on precautionary sav-
ing. I start with the models where the agent knows the value of his income growth parameter $\alpha$.

### 3.1 Models I and III: Known (but possibly stochastic) income growth

I proceed in two steps. First, describe the dynamics for the income growth $\{\alpha(t) : t \geq 0\}$ and then analyze the agent’s optimality. The agent’s income growth is often subject to both aggregate and idiosyncratic risks. One parsimonious way to capture the stochastic nature of income growth is to postulate that income growth $\{\alpha(t) : t \geq 0\}$ varies stochastically over time between $\alpha_1$ and $\alpha_2 < \alpha_1$, the two possible levels.\(^7\) Let $N(t)$ denote the regime for the agent’s income growth $\alpha(t)$ at time $t$. That is, income growth $\alpha(t)$ takes the value $\alpha_{N(t)}$ for $N(t) = 1, 2$. Fix a small time period $\Delta t$. If time-$t$ growth rate is high, i.e. $\alpha(t) = \alpha_1$, the growth rate remains high at time $(t + \Delta t)$ with probability $(1 - \lambda_1 \Delta t)$ and decreases to $\alpha_2$ at time $(t + \Delta t)$ with the remaining probability $\lambda_1 \Delta t$. Similarly, if time-$t$ growth rate is low, i.e. $\alpha(t) = \alpha_2$, $\lambda_2 \Delta t$ is the transition probability from low growth rate $\alpha_2$ to $\alpha_1$, the high value.\(^8\)

Now, I use dynamic programming to characterize the agent’s optimality.

When the agent observes his income growth, he not only takes wealth $x$ and income $y$, but also his current income growth rate as a state variable in his optimization problem. Let $V(x, y, 1)$ and $V(x, y, 2)$ denote the value function when his current income growth rate is high ($\alpha_1$) and low ($\alpha_2$), respectively. When the current income growth is high, the principle of optimality implies

\(^7\) Kimball and Mankiw (1989) model the “level” of income, rather than the growth rate (drift) of the income process stochastically switch between two states in their study of the impact of precautionary saving on Ricardian (tax) equivalence. Indeed, the income process in Kimball and Mankiw (1989) may be viewed as a special case of the income process (1).

\(^8\) We may summarize this dynamics using the conditional transition probability matrix: $P(\Delta t) = \begin{pmatrix} 1 - \lambda_1 \Delta t & \lambda_1 \Delta t \\ \lambda_2 \Delta t & 1 - \lambda_2 \Delta t \end{pmatrix}$. This transition matrix is a discrete-time representation of the continuous-time Markov chain (regime-switching) model used in the paper. See Hamilton (1989) for the regime-switching model and its econometric analysis in discrete time. The implied stationary probabilities for high-income-growth and low-income-growth states are $\lambda_2/(\lambda_1 + \lambda_2)$, and $\lambda_1/(\lambda_1 + \lambda_2)$, respectively.
that the agent’s Hamilton-Jacobi-Bellman (HJB) equation is given by

\[
\beta V(x, y, 1) = \max_{c, \psi} \left\{ u(c) + (rx + \psi \zeta + y - c) V_x(x, y, 1) + (\alpha_1 - \kappa y) V_y(x, y, 1) \right. \\
\left. + \frac{\psi^2 \nu^2}{2} V_{xx}(x, y, 1) + \psi \rho \nu \sigma V_{xy}(x, y, 1) + \frac{\sigma^2}{2} V_{yy}(x, y, 1) \right\} + \lambda_1 (V(x, y, 2) - V(x, y, 1)).
\]  

(6)

The left side of (6) is the annuity value of his value function. The right side of (6) is the sum of his utility rate \( u(c) \) and the instantaneous expected changes of his value function. Optimality of consumption and portfolio rules implies that the two sides of (6) are equated. The \( V_x \) term describes the marginal increase of the agent’s value function from saving. The \( V_y \) term captures the marginal increase of the agent’s value function if income \( y \) increases by a unit. The \( V_{xx}, V_{xy}, \) and \( V_{yy} \) terms reflect the effects of stochastic return, income volatility and their correlation on the agent’s value function. The last term captures the effect of stochastic transition of his income growth rate on the expected change of his value function. Note that the value function changes discretely from \( V(x, y, 1) \) to \( V(x, y, 2) \) when the growth rate changes.

The first-order condition (FOC) with respect to consumption is as follows: \( u'(c) = V_x(x, y, 1) \). That is, marginal utility of consumption \( u'(c) \) is equal to the marginal value of wealth \( V_x \). The FOC with respect to the portfolio rule \( \psi \) gives

\[
\psi = -\frac{\zeta V_x(x, y, 1)}{\nu^2 V_{xx}(x, y, 1)} - \frac{\rho \sigma V_{xy}(x, y, 1)}{\nu V_{xx}(x, y, 1)},
\]

(7)

where the first term captures the risk-return piece from investing in the risky asset, and the second term reflects the agent’s motive to hedge his labor income shocks as in Merton (1971). We defer the details for consumption and portfolio rules to later sections. In the remaining part of this section, we incorporate the effect of learning on the agent’s optimality.

3.2 Models II and IV: Unknown (but possibly stochastic) income growth

First, turn to the agent’s belief updating process about his income growth parameter \( \alpha \). The agent’s time-\( t \) information set \( \mathcal{F}_t \) only contains the history of his income \( \{y(s) : s \leq t\} \), not the true (but possibly stochastic) value of \( \alpha(t) \). Let \( p(t) \) denote his time-\( t \) belief that the growth rate is of the higher value \( \alpha_1 \), in that \( p(t) = \text{Prob}(\alpha(t) = \alpha_1 | \mathcal{F}_t) \).

First use the observed historical incomes to construct the innovations process \( B \) which drives the dynamics of his posterior belief. Let \( \mu \) denote the expected growth rate of the income process. By definition, the expected growth rate \( \mu \)
is a weighted average of the two possible income growth rates, in that
\[ \mu(t) = E_t (\alpha(t)) = p(t) \alpha_1 + (1 - p(t)) \alpha_2 = \alpha_2 + \delta p(t), \]  
(8)
where \( \delta = \alpha_1 - \alpha_2 \) is the difference between the two possible values of \( \alpha \). Fix a small time period \((t, t + \Delta t)\). The change of income is \((y(t + \Delta t) - y(t))\). Out of this total change, \((\mu(t) - \kappa y(t)) \Delta t\) is the expected change. Thus, by definition, the unanticipated change is given by \((y(t + \Delta t) - y(t) - (\mu(t) - \kappa y(t)) \Delta t)\). Scaling by volatility \(\sigma \sqrt{\Delta t}\) and taking the limit \(\Delta t \to 0\), we may construct a “new” Brownian motion process \(B\) as follows:
\[ dB(t) = (dy(t) - (\mu(t) - \kappa y(t))) dt / \sigma, \]  
(9)
which will serve as the innovations process for belief updating.

Recall there is only one shock in the model. Re-writing the innovations process (9) gives the following equivalent representation of the income process (1):
\[ dy(t) = (\mu(t) - \kappa y(t)) dt + \sigma dB(t) = (\alpha_2 + \delta p(t) - \kappa y(t)) dt + \sigma dB(t), \]  
(10)
where the last equality uses (8) for \( \mu(t) \). The process (10) makes intuitive sense. The expected change of income over \( \Delta t \) is \((\mu(t) - \kappa y(t)) \Delta t\) and the volatility over \( \Delta t \) is \(\sigma \sqrt{\Delta t}\). The innovations representation (10) proves useful in deriving the agent’s optimal consumption and portfolio rules later. Using the results in Liptser and Shiryaev (1977), write the belief process as follows:
\[ dp(t) = (\lambda_2 - (\lambda_1 + \lambda_2) p(t)) dt + \sigma^{-1} \delta p(t) (1 - p(t)) dB(t), \]  
(11)
where \( B \) is given in (9). Note that belief \( p \) and income \( y \) are perfectly correlated (one shock model). We defer the economic interpretations of (11) to later sections when discussing model intuition.

When the income growth rate \( \alpha \) is unknown, the optimization problem is not Markovian with respect to the original information set \( \mathcal{F}_t \), which only contains the history of income \( y \). The belief updating process (11) and the innovations-representation process (10) for income \( y \) jointly convey the same information as the agent’s original income process (1) and his prior belief about his income growth do. We can transform the original non-Markovian optimization problem into a Markovian one. That is, the agent maximizes his utility function (5), subject to the innovations-based representation of his income process (10), his belief updating process (11), his wealth accumulation equation (4), and the standard transversality condition given in the online appendix.\(^9\) Importantly, the agent’s learning about his income growth implies

\(^9\) Gennette (1986) and Xia (2001) study the optimal asset allocation when the agent has incomplete information about his investment opportunities, such as the dividend growth rate or the expected stock return. See Detemple (1986), Wang (1993), and Veronesi (1999, 2000) for equilibrium asset pricing implications.
that belief $p$ is also a state variable in addition to wealth $x$ and income $y$.

There are three state variables for the agent’s optimization problem: wealth $x$, income $y$, and belief $p$ that income growth $\alpha$ is high. The HJB equation for the agent’s value function $J(x, y, p)$ is given as follows:

$$\beta J = \max_{c, \psi} u(c) + (rx + \psi \zeta + y - c) J_x + \frac{\psi^2 \nu^2}{2} J_{xx} + (\alpha_2 + \delta p - \kappa y) J_y + \psi \rho \sigma J_{xy} + (\lambda_2 - (\lambda_1 + \lambda_2) p) J_p + \frac{\delta^2}{2 \sigma^2} p^2 (1 - p)^2 J_{pp} + \psi \rho \sigma^{-1} \delta p (1 - p) J_{xp} + \delta p (1 - p) J_{yp} + \frac{1}{2} \sigma^2 J_{yy}. \quad (12)$$

The left side of (12) is the annuity (flow measure) of his value function. As for the HJB equation (6) when the income growth $\alpha$ is known, the left side includes standard terms such as $J_x$, $J_y$, $J_{xx}$, $J_{yy}$, and $J_{xy}$. Unlike the HJB equation (6) when the income growth $\alpha$ is known, the agent’s learning about his income induces additional terms in the HJB equation. For example, $J_p$ and $J_{pp}$ terms capture the effects of the agent’s belief about income growth on his value function. Since belief updating is solely driven by realized incomes, the agent’s income process is perfectly correlated with his belief updating, as reflected in the $J_{yp}$ term in (12). Finally, the $J_{xp}$ term captures the agent’s hedging demand induced by his estimation risk (associated with belief updating).

4 Model I: Known & constant growth parameter $\alpha$

First, consider the benchmark setting when $\alpha(t)$ is known and is constant over time. This is the standard consumption-saving and portfolio allocation problem for an agent endowed with uninsurable stochastic labor income. There is no learning for the agent in this setting. It is a special case of the model presented in Section 3, where the transition probability from the current $\alpha$ is zero. That is, let $\alpha(t) = \alpha$ for all $t$ and set the transition probabilities to zero, i.e. $\lambda_1 = \lambda_2 = 0$. The following proposition summarizes the main results on consumption and portfolio rules for this setting, dubbed as Model I:

**Proposition 1** If the agent knows his constant income growth $\alpha$, his consumption $c^*$ and wealth allocation to the risky asset $\psi^*$ are given by

$$c^*(t) = r (x(t) + g(y(t); \alpha)), \quad (13)$$

$$\psi^*(t) = \frac{\zeta}{\gamma r \nu^2} - \frac{\rho \sigma}{\nu} \frac{1}{r + \kappa}, \quad (14)$$
where the risk-adjusted certainty equivalent human wealth \( g(y; \alpha) \) is given by

\[
g(y; \alpha) = \frac{1}{r + \kappa} \left( y + \frac{\alpha - \rho \sigma \eta}{r} \right) - \frac{\gamma (1 - \rho^2) \sigma^2}{2 (r + \kappa)^2} + \frac{\beta - r}{\gamma r^2} + \frac{\eta^2}{2 \gamma r^2}.
\]  

(15)

First turn to the effect of the agent’s degree of impatience (discount rate \( \beta \)). The agent’s subjective discount rate \( \beta \) has no impact on his portfolio allocation rule \( \psi \). If the agent’s discount rate \( \beta \) is higher than the risk-free rate \( r \), his consumption is higher by a constant amount \( (\beta - r) / (\gamma r) \) in perpetuity due to the CARA utility specification.\(^1\)

Now consider the effect of the agent’s opportunity to invest in the risky asset. There are two effects. First, investing in the risky asset earns a higher expected return and hence shall raise the agent’s current consumption (Merton (1971)). This effect is captured by the first term in the agent’s portfolio rule (14), and also by the constant positive term \( \eta^2 / (2 \gamma r^2) \) in the agent’s risk-adjusted certainty equivalent wealth \( g(y) \) given in (15). Second, investing in the risky asset allows the agent to partially hedge against his labor income risk (i.e. the second term in the portfolio allocation rule (14)). A higher systematic volatility \( \rho \sigma \), the greater the agent’s hedging demand is, \textit{ceteris paribus}. The more persistent the income shock is (a lower \( \kappa \)), the stronger this hedging demand is. Intuitively, the more persistent labor income is, the greater impact of a unit innovation to income on the certainty equivalent wealth \( g(y) \).

Intuitively, hedging not only “changes” the agent’s labor income growth from \( \alpha \) to \( (\alpha - \rho \sigma \eta) \), but also reduces the agent’s undiversifiable component of his idiosyncratic labor income volatility from \( \sigma \) to \( \sigma \sqrt{1 - \rho^2} \). Since precautionary saving demand arises from the agent’s uninsurable idiosyncratic risk, hedging lowers the agent’s precautionary saving demand. Using (13), we see that the agent’s precautionary saving demand is given by

\[
\pi(t) = \frac{\gamma (1 - \rho^2) \sigma^2}{2 (r + \kappa)^2}.
\]  

(16)

Consistent with our analysis on hedging demand \( \psi \), the more persistent the income process is, the riskier the income process is in the long run, and hence a greater demand for precautionary saving. If labor income is perfectly correlated with the risky asset return, the agent can fully hedge his income risk. As a result, his precautionary saving demand is zero (i.e. complete markets setting). Finally, Model I nests Caballero (1991) and Wang (2006), settings

\(^{10}\)Note that impatience affects the agent’s level of consumption, not his marginal propensity to consume. This specific prediction is present in effectively all CARA-based utility models. Throughout the remainder of the model, we will not further discuss this “obvious” effect of discounting on consumption.
where the agent cannot invest in the risky asset \((\psi(t) = 0)\), as special cases.\(^{11}\)

5 Model II: Unknown & constant growth parameter \(\alpha\)

Now consider Model II, the case where the agent has a constant growth rate \(\alpha\), but he does not know the value of \(\alpha\). First, I analyze the agent’s learning problem. Then, I use dynamic programming to solve the decision rules. Finally, I highlight the model intuition on learning-induced precautionary saving.

5.1 Model Solution

When the agent does not know his income growth, he needs to use his past realized incomes to estimate the likelihood that his income growth \(\alpha\) is high. Note that Model II is a special case of the general learning model of Section 3.2 with \(\lambda_1 = \lambda_2 = 0\). Write the updating process (11) as follows:

\[
dp(t) = \sigma^{-1} \delta p(t) (1 - p(t)) dB(t),
\]

where \(B\) is the Brownian motion process under the innovations representation, given in (9). The intuition behind the belief updating process is as follows. Because the unknown growth rate \(\alpha\) is constant, the change of the agent’s belief is unpredictable, i.e. the agent’s expected income growth is a martingale, in that \(\mu(t) = E_t (\alpha(t)) = E_t (E_s (\alpha(t))) = E_t (\mu(s))\), for any \(t < s\). As a result, the belief process \(p\) is also a martingale because \(\mu(t) = \alpha_2 + p(t)\delta\).

The instantaneous volatility of belief updating is symmetric in \(p\) and \((1 - p)\) because the unobserved growth rate can only take two possible values: \(\alpha_1\) and \(\alpha_2\). The greater the wedge \(\delta = \alpha_1 - \alpha_2 > 0\) is, the more volatile belief updating is. Moreover, a higher income volatility \(\sigma\) implies a less volatile belief updating. Intuitively, a higher realized value of income is more informative about the unknown income growth if the income process is less volatile (lower \(\sigma\)). The following proposition summarizes the results on consumption and portfolio rules when the agent learns about his (constant) income growth.

\(^{11}\)Wang (2006) extends the discrete-time CARA-Gaussian formulation of Caballero (1991) in a continuous-time setting to allow for conditionally heteroskedastic labor income process. The key advantage of introducing conditional heteroskedasticity of labor income process is that the agent’s marginal propensity to consume (MPC) out of labor income may be less than the MPC out of financial wealth, a desirable feature argued in Friedman (1957), Hall (1978) and Zeldes (1989).
Proposition 2 If income growth $\alpha$ is constant but unknown to the agent, his consumption $c^*$ and wealth allocation to the risky asset $\psi^*$ are given by

$$c^*(t) = r \left( x(t) + g(y(t); \alpha_2) + f(p(t)) \right),$$

$$\psi^*(t) = \frac{\zeta}{\gamma r \nu^2} - \xi(t),$$

where $g(y; \alpha_2)$ is given by (15), hedging demand $\xi(t)$ is given by

$$\xi(t) = \frac{\rho}{\nu} \left[ \frac{\sigma}{r + \kappa} + \frac{\delta p(t) (1 - p(t)) f'(p(t))}{\sigma} \right],$$

and $\{f(p) : 0 \leq p \leq 1\}$ solves the following non-linear ODE:

$$rf(p) = \frac{\delta p}{r + \kappa} - \frac{\rho \eta}{\sigma} \delta p (1 - p) f'(p) + \frac{\delta^2}{2\sigma^2 p^2} (1 - p)^2 f''(p)$$

$$- \gamma r \left( 1 - \rho^2 \right) \left[ \frac{1}{r + \kappa} \delta p (1 - p) f'(p) + \frac{\delta^2}{2\sigma^2 p^2} (1 - p)^2 f'(p)^2 \right],$$

subject to $f(0) = 0$ and $f(1) = \delta / (r(r + \kappa))$.

Unlike the baseline model without learning (Model I), the agent now also consumes out of the “risk-adjusted” certainty equivalent wealth $f(p)$ in addition to financial wealth $x$ and the risk-adjusted certainty equivalent “human” wealth $g(y; \alpha_2)$. The certainty equivalent wealth $f(p)$ arises from the agent’s learning about his income growth $\{\alpha(t) : t \geq 0\}$. We now turn to the determination of the boundary conditions at $p = 0$ and $p = 1$. If income growth is always low (i.e. $\alpha(t) = \alpha_2$), we are back to Model I with constant known income growth $\alpha_2$. Using the implied consumption rule given in Proposition 1 and matching the one given here, we have $f(0) = 0$. Similarly, if income growth $\alpha$ is always high (i.e. $\alpha(t) = \alpha_1$), to make the consumption rules implied by Propositions 1 and 2 compatible, we need to have $f(1) = \delta / (r(r + \kappa))$.

Now focus on the effect of belief ($0 < p < 1$) on consumption. First construct a “reference” model where the agent is myopic. By myopia, I mean that the agent does not anticipate the evolution of his belief about income growth. This “myopia” assumption is dubbed as “anticipated utility” by Kreps (1998) and Sargent (1999). For a given $p$, the agent’s expected income growth is $\mu(p) = p \alpha_1 + (1 - p) \alpha_2 = \alpha_2 + \delta p$. The myopic agent ignores his learning problem and uses Model I to form his consumption rule by evaluating his expected income growth at $\mu(p)$, in that

$$c = r \left( x + g(y; \mu(p)) \right) = r \left( x + g(y; \alpha_2) + \bar{f}(p) \right),$$

(22)
where \( g(y; \alpha_2) \) is given by (15) and \( \bar{f}(p) \) is given by

\[
\bar{f}(p) = \frac{\delta p}{r(r + \kappa)}.
\]  

(23)

Note that \( \bar{f}(p) \) is linear in \( p \). In this reference (anticipated utility) model, there is no learning effect on the agent’s consumption. Intuitively, the only precautionary saving demand comes from the standard income risk, which is given by (16). The “anticipated utility” model of Kreps (1998) and Sargent (1999) is an approximation of the underlying Bayesian optimization problem stated earlier. It is sometimes used because the underlying learning/optimization problem is computationally too complicated. However, the joint learning/optimization problem posed here (e.g. Models II and IV) is analytically tractable, and reasonably realistic. Indeed, an important message of our analysis is while the approximation (via anticipated utility models) is sometimes useful, it may overly simplify the economic analysis and hence may not potentially capture the key underlying economic mechanism of learning.\(^{12}\)

Let \( l(p) = \bar{f}(p) - f(p) \). The wedge \( l(p) \) proves useful in understanding the economics of learning.

5.2 Model implications: Learning, precautionary saving, and hedging

First consider the case where the agent cannot invest in the risky asset. He faces the standard self-insurance (income fluctuation) problem with learning.

Figure 1 plots the “risk-adjusted” certainty equivalent wealth \( f(p) \) and the wedge \( l(p) = \bar{f}(p) - f(p) \) for various levels of the coefficient of absolute risk aversion \( \gamma \) in self insurance models, where the agent can only invest in the risk-free asset to buffer against his income shocks. The left panel shows that \( f(p) \) is increasing and convex in belief \( p \). The higher the coefficient of absolute risk aversion \( \gamma \), the lower the certainty equivalent wealth \( f(p) \) is. The straight line for \( \bar{f}(p) \) given in (23) is the certainty equivalent wealth \( f(p) \) for a “myopic” agent as in the “anticipated utility” model of Kreps (1998) and Sargent (1993). The wedge \( l(p) = \bar{f}(p) - f(p) \) measures learning induced precautionary saving in the self insurance model. The right panel of Figure 1 plots \( l(p) \) as a function of belief \( p \) for \( \gamma = 1, 2 \). Because the unknown income growth is constant, there is no learning involved at either end of the belief (\( p = 0, 1 \)), and hence no

\(^{12}\) Kreps and Sargent are certainly fully aware of the implications of using anticipated utility models to approximate the true model solutions. They suggest the approximation for convenience reasons. Therefore, we agree with the philosophies of Kreps and Sargent. Readers may view our model as a laboratory to evaluate the accuracy of using the anticipated utility model to approximate the exact model solution in an incomplete-markets consumption setting.
learning induced precautionary saving demand \( l(0) = l(1) = 0 \). Note that learning induced precautionary saving \( l(p) \) is concave (which follows from the convexity of \( f(p) \)). Intuitively, when the agent is more uncertain about his income growth (i.e. in the interior region of \( p \)), learning induced precautionary saving \( l(p) \) is higher \textit{ceteris paribus}. However, note that \( l(p) \) is not symmetric around \( p = 1/2 \), and is rather skewed. This is due to the fact that \( f(p) \) is convex, (i.e. \( f'(1-p) > f'(p) \) for \( 0 < p < 1/2 \)), and the fact that \( l(p) \) depends both on \( p(1-p) \) and \( f'(p) \). The nonlinear term in ODE (21) and the right panel of Figure 1 capture this asymmetry.\(^{13}\)

Now turn to the more general case where the agent invests in the risky asset. As in our Model I and Merton (1971), the agent invests in the risky asset to earn a higher expected return, (i.e. the first term \( \zeta/(\gamma r \nu^2) \) in (19)), and to hedge risks (i.e. the second term \( \xi(t) \) given in (20)). Note that positive \( \xi(t) \) is associated with adding a “negative” position in the risky asset. There are two hedging components. The first component in (20) is the standard hedging demand for the labor income risk as in our Model I, Svensson and Werner (1993), and Davis and Willen (2000). While the hedging demand against labor income risk is constant, the hedging demand with respect to the estimation risk is stochastic, and depends on both time-varying volatility \( \sigma^{-1}(1-p) \) of the belief updating process (17) and \( f'(p) \), which measures the sensitivity of \( f(p) \) with respect to belief \( p \).

Income volatility \( \sigma \) has two \textit{opposite effects} on the total hedging demand. On the one hand, a higher income volatility \( \sigma \) increases the hedging demand of labor income risk. On the other hand, incomes from a more volatile income process provide less precise information about the unknown income growth \( \alpha \) given a fixed dispersion \( \delta = \alpha_1 - \alpha_2 \). Hence, the agent updates his belief less in response to “unexpected” income news. Therefore, a higher income volatility \( \sigma \) maps to a lower estimation risk and a lower hedging demand against estimation risk, \textit{ceteris paribus}. The key assumption underlying this argument is that the conditional volatility of income changes (in terms of levels) is constant in our model given the income process specification (1). However, the income volatility may increase with the income level. If so, then estimation risk will also increase with income volatility. In that case, the estimation risk induced precautionary saving and the standard income risk effect on precautionary saving may move towards the same direction if the income process is specified in logarithm as in Guvenen (2007b).\(^{14}\)

Now consider the effect of correlation \( \rho \) and hedging on the certainty equivalent wealth \( f(p) \). There are three terms in the ODE (21). First, hedging estimation risk tilts the portfolio position to exploit the covariation between the belief

\(^{13}\) Formally, we can show that the implied ODE for \( l(p) \) depends on \( f'(p) \) and \( p(1-p) \).

\(^{14}\) I thank the referee for pointing this out.
updating process (17) and the risky asset return (3). This is captured by the second term in ODE (21). Second, the portion of estimation risk that cannot be hedged induces an uninsurable idiosyncratic (estimation) risk premium. The last term (with the multiplier $\gamma(1 - \rho^2)$ in (21) captures the two components of this estimation risk premium: one from the pure estimation risk and the other from the perfect (instantaneous) correlation between income risk and estimation risk. Figure 2 plots the effect of correlation on $f(p)$.

On the left panel of Figure 2, the solid and the dashed lines correspond to the certainty equivalent wealth $f(p)$ with correlation $\rho = 0.5$ and $\rho = -0.5$, respectively. For the comparison purpose, the dotted line depicts the case with $\rho = 0$, holding all other parameters the same. Unlike the self insurance case, $f(p)$ may be concave when $\rho \neq 0$. When $\rho < 0$, hedging tilts the agent’s portfolio position towards the risky asset. This on average raises the agent’s certainty equivalent wealth $f(p)$ since the risky asset on average offers a higher expected return, ceteris paribus. Therefore, when $\rho < 0$, an increase in the absolute value $|\rho|$ increases $f(p)$ and consumption (holding wealth $x$ fixed) for two reasons: reduced precautionary motive and higher expected wealth going forward (i.e. the last term in (21)). When $\rho > 0$, an increase in $\rho$ lowers estimation risk induced precautionary saving and hence increases consumption, but also tilts the agent’s position towards the risk-free asset, and hence lowers both the expected wealth and hence current consumption, ceteris paribus. Therefore, there are two opposing effects on $f(p)$ when $\rho > 0$.

The right panel plots the difference $f(p; \rho) - f(p; \rho = 0)$ with respect to belief $p$. Using parameters given in the caption of Figure 2, we have $f(p; -0.5) > f(p; \rho = 0)$, but $f(p; 0.5) < f(p; \rho = 0)$. The wedges between $f(p; \rho)$ and $f(p; 0)$ plotted in the right panel help identify the impact of hedging estimation risk on $f(p)$. Note that the wedge between $f(p; \rho)$ and $f(p; 0)$ disappears when $p = 0$ or $p = 1$, both of which are absorbing states (i.e. income growth while unknown is constant in Model II depicted here.) While the wedge $f(p; \rho = 0.5) - f(p; 0) \leq 0$, two points are worth clarifying. First, for sufficiently low positive correlation $\rho$, the agent may have $f(p; \rho) > f(p; 0)$. Recall the two competing effects between reducing uninsurable idiosyncratic estimation risk and lowering the expected return in order to hedge the estimation risk when $\rho > 0$. Second, even if $f(p; \rho) < f(p; 0)$, we cannot state that the agent’s welfare is lower when investing in a positively correlated asset. Here, our analysis holds the agent’s wealth $x$, income $y$ and belief $p$ fixed and analyze the impact of correlation $\rho$ on $f(p)$. We cannot draw welfare implications simply from $f(p; \rho)$ without taking into account endogenous wealth accumulation.
Now turn to the setting with stochastic income growth. First consider the case where the agent knows his stochastic income growth at all times. That is, the agent’s information set $\mathcal{F}_t$ includes $\{N(s) : s \leq t\}$, where $N(t) = 1, 2$ correspond to high and low income growth rates, respectively. Recall that the agent’s income growth is given by a two-state Markov chain detailed in Section 5. The following proposition summarizes the main results of Model III.

**Proposition 3** When the agent knows income growth $\{\alpha(t) : t \geq 0\}$, his portfolio allocation is given by (14), and his consumption is given by

$$c^*(t) = r \left( x(t) + g(y(t); \alpha_2) + \phi_{N(t)} \right),$$

(24)

where $g(y; \alpha_2)$ is given in (15) and $\{\phi_1, \phi_2\}$ jointly solve

$$r \phi_1 = -\frac{\lambda_1}{\gamma r} \left( e^{-\gamma r (\phi_2 - \phi_1)} - 1 \right) + \frac{\delta}{r + \kappa},$$

(25)

$$r \phi_2 = -\frac{\lambda_2}{\gamma r} \left( e^{-\gamma r (\phi_1 - \phi_2)} - 1 \right).$$

(26)

The portfolio rule (14) is the same as in Model I. The new economic insight of Model III is the effect of stochastic income growth on consumption and saving. Stochastic income growth induces both persistence and volatility effects on consumption and saving, beyond the effect due to stochastic income shocks as in Model I. First, consider the impact of income persistence on consumption. Taking the first-order expansion on the right sides of (25) and (26), we have the following approximate expressions for $\phi_1$ and $\phi_2$:

$$\phi_1 \approx \frac{r + \lambda_2}{r + \lambda_1 + \lambda_2} \frac{\delta}{r (r + \kappa)}, \quad \phi_2 \approx \frac{\lambda_2}{r + \lambda_1 + \lambda_2} \frac{\delta}{r (r + \kappa)}.$$ 

(27)

Up to the first-order approximation for $\phi_1$ and $\phi_2$ as given in (27), stochastic income growth affects consumption decisions via the standard permanent income channel. For given wealth $x$ and income $y$, consumption is higher if income growth is larger (i.e. $\phi_1 > \phi_2$). Finally, note that the approximation analysis for the special case ($\lambda_1 = \lambda_2 = 0$) yields identical results as Model I does, as we expect. (It is immediate to see that $\phi_1 = \delta / (r (r + \kappa))$ and $\phi_2 = 0$.)

Perhaps more interestingly, the agent not only smooths his consumption with respect to the fluctuation of his income growth due to the *expected* changes

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15 The heterogeneity between the two regimes is income growth $\alpha_{N(t)}$. The degree of mean reversion is the same across the two regimes, and hence the agent’s portfolio allocation is the same in both regimes.
of income, but also has precautionary saving motive against his stochastic income growth. The difference between the values of \( \phi_1 \) and \( \phi_2 \) given in (25) and (26), and their respective first-order approximations given in (27) captures the precautionary saving demand induced by stochastic income growth. We will return to analyze this precautionary saving channel in the next section.

7 Model IV: Unknown & stochastic growth parameter \( \alpha \)

Now consider the effects of learning on consumption and portfolio decisions when income growth is stochastic and unknown. First, I review the agent’s belief updating about his stochastic income growth \( \{\alpha(t) : t \geq 0\} \), and describe his optimal consumption and portfolio allocation rules. Then, I highlight the new economic insights that arise from the agent’s learning about his unknown stochastic income growth, not captured by Models I-III.

Without observing his income growth \( \alpha \), the agent uses his past incomes to estimate the likelihood that his income growth is high. Equation (11) gives the belief updating process. The intuition for the volatility specification in (11) is the same as the one for the belief process (17) in Model II, a special case of Model IV. See discussions on volatility in Section 5 for Model II. The intuition for the drift specification in (11) is richer than the one for Model II. Because the underlying unknown income growth \( \alpha \) is “stochastic”, the expected change of the agent’s belief is no longer zero, unlike Model II in Section 5. Consider a small time period \( (t, t + \Delta t) \). Suppose the current income growth is high (i.e. \( \alpha(t) = \alpha_1 \)). The conditional probability that income growth changes from \( \alpha_1 \) to \( \alpha_2 \) is \( \lambda_1 \Delta t \). The size of this change is \( \alpha_2 - \alpha_1 = -\delta \). Therefore, the expected change of income growth (conditional on \( \alpha(t) = \alpha_1 \)) is \( -\delta \lambda_1 \Delta t \). The time-\( t \) probability that \( \alpha(t) = \alpha_1 \) is \( p(t) = \text{Prob}_t(\alpha(t) = \alpha_1) \). The unconditional expected change of income growth is thus given by

\[
E_t (\mu(t + \Delta t) - \mu(t)) = \delta (\lambda_2 - (\lambda_1 + \lambda_2) p(t)) \Delta t. \tag{28}
\]

Equation (28) indicates that \( (\lambda_2 - (\lambda_1 + \lambda_2) p(t)) \) is the instantaneous expected change of belief \( p \), because \( \mu(t) \) is linear in \( p(t) \), i.e. \( \mu(t) = \alpha_2 + \delta p(t) \). Intuitively, when the current posterior \( p(t) \) is larger than the long-run probability \( \lambda_2/\lambda_1 + \lambda_2 \), belief \( p(t) \) is likely to move downward on average. This reflects the mean reversion property of the belief process \( \{p(t) : t \geq 0\} \). The next proposition summarizes main results of the agent’s optimality.

**Proposition 4** When the agent does not know his stochastic income growth \( \alpha \), his consumption \( c^* \) and wealth allocation \( \psi^* \) are given by (18) and (19),
respectively, where \( g(y; \alpha_2) \) is given in (15), and \( \{f(p) : 0 \leq p \leq 1\} \) solves

\[
rf(p) = \frac{\delta p}{r + \kappa} + \left[ (\lambda_2 - (\lambda_1 + \lambda_2)p) - \left( \frac{\rho \eta}{\sigma} + \gamma \frac{r(1 - \rho^2)}{r + \kappa} \right) \delta p (1 - p) \right] f'(p)
\]

\[+ \frac{\delta^2}{2\sigma^2} p^2 (1 - p)^2 f''(p) - \frac{\gamma r(1 - \rho^2)}{2\sigma^2} \delta^2 p^2 (1 - p)^2 f'(p)^2, \tag{29}\]

subject to the following boundary conditions:

\[
rf(0) = \lambda_2 f'(0), \tag{30}\]

\[
rf(1) = \frac{\delta}{r + \kappa} - \lambda_1 f'(1). \tag{31}\]

Unlike Model II, the belief updating process is mean reverting, which is reflected by the drift term in the belief updating process and the corresponding term \((\lambda_2 - (\lambda_1 + \lambda_2)p)\) in the ODE (29). Since neither \( p = 0 \) nor \( p = 1 \) is an absorbing state, boundary conditions (30)-(31) are different from those for Model II.

First, compare Model IV with Model III where the agent knows his stochastic income growth. Figure 3 plots the certainty equivalent wealth \( f(p) \) as a function of belief \( p \) for Model IV and compares with the corresponding complete-information setting (Model III).

[Insert Figure 3 here.]

The left panel of Figure 3 corresponds to the setting where income growth \( \alpha(t) \) will permanently be at \( \alpha_1 \), once income growth \( \alpha(t) \) becomes \( \alpha_1 \) (i.e. \( \lambda_1 = 0 \)). The solid line gives the certainty equivalent wealth \( f(p) \) in Model IV (i.e. learning model with stochastic growth). For the comparison purpose, the dotted and dash-dotted lines depict \( \phi_1 \) and \( \phi_2 \) given in (25) and (26) for Model III, the corresponding complete-information setting with stochastic income growth. Note that \( f(1) = \phi_1, \) in that Models IV and III are the same once the absorbing high-income-growth state is reached. More interestingly, even when \( p(t) = 0 \), the belief \( p(t + \Delta t) \) in the next instant \( (t + \Delta t) \) will be uncertain \((\lambda_2 > 0)\). The forward-looking agent incorporates the future belief uncertainty into his current decisions. Therefore, the effect of estimation risk continues to exist, implying \( f(0) < \phi_2 \). At very low values of \( p \), \( f(p) \) is lower than the complete-information counterpart \( \phi_2 \) due to the estimation risk. When \( p \) is sufficiently high, the expected growth effect may dominate the belief uncertainty effect, giving \( f(p) > \phi_2 \). The right panel plots the certainty equivalent wealth \( f(p) \) with respect to belief \( p \) when the low-income-growth state is absorbing (i.e. \( \lambda_2 = 0 \)). The solid line gives \( f(p) \) in Model IV, and the dotted and dash-dotted lines depict \( \phi_1 \) and \( \phi_2 \) given in (25) and (26) for the corresponding Model III under the complete-information setting. By essentially the same argument, \( f(0) = \phi_2 = 0 \) because belief uncertainty is
permanently resolved once \( p = 0 \). However, \( f(1) < \phi_1 \) because of estimation risk (i.e. \( \lambda_2 > 0 \)) even when the current belief is (locally) certain \( p = 1 \).

Next we analyze the impact of income growth transition intensities \((\lambda_1, \lambda_2)\) on the certainty equivalent wealth \( f(p) \). Figure 4 compares the “risk-adjusted” certainty equivalent wealth \( f(p) \) for various pairs of transition intensities growth \((\lambda_1, \lambda_2)\) for the unobserved stochastic income growth.

[Insert Figure 4 here.]

On the left panel of Figure 4, the dashed line gives the certainty equivalent wealth \( f(p) \) with \((\lambda_1 = 0, \lambda_2 > 0)\). Because the high-income-growth state is absorbing \((\lambda_1 = 0)\), we have \( f(1) = \delta/(r(r + \kappa)) \). However, \( f(0) > 0 \) due to \( \lambda_2 > 0 \). By essentially the same argument but towards the opposite direction, when \( \lambda_2 = 0 \) (the low-income-growth state being absorbing), we have \( f(0) = 0 \). However, \( f(1) < \delta/(r(r + \kappa)) \) due to \( \lambda_1 > 0 \). The solid line illustrates the corresponding \( f(p) \). Now turn to the general case where neither state is absorbing. The corresponding dash-dotted line for \( f(p) \) lies between the lines for the previous two cases. With \( \lambda_1, \lambda_2 > 0 \), we have \( f(0) > 0 \) and \( f(1) < \delta/(r(r + \kappa)) \). Note that the orderings of \( f(p) \) for various \((\lambda_1, \lambda_2)\). The dashed line (with the high-income-growth state being absorbing) is the highest, and the solid line (with the low-income-growth state being absorbing) is the lowest. The dash-dotted line with neither state being absorbing lies between the two. That is, mean reversion of income growth flattens \( f(p) \) across \( p \). For the comparison purpose, we plot the dotted line for Model II with \((\lambda_1 = 0, \lambda_2 = 0)\). Intuitively, \( f(p) \) for Model II has the steepest slope, but its level is lower than that for \((\lambda_1 = 0, \lambda_2 > 0)\) and higher than the one for \((\lambda_1 > 0, \lambda_2 = 0)\). Note the line for Model II intersects with the line for the case with neither state absorbing \((\lambda_1 > 0, \lambda_2 > 0)\).

The right panel reinforces the message from the left panel. We plot \( f(p; \lambda_1, \lambda_2) - f(p; 0, 0) \) for the three pairs of \((\lambda_1, \lambda_2)\). If the high-income-growth state is absorbing, \( f(p; 0, \lambda_2) > f(p; 0, 0) \) (the dashed line). If the low-income-growth state is absorbing, \( f(p; \lambda_1, 0) < f(p; 0, 0) \) (the solid line). Finally, if neither state is absorbing, \( f(p; \lambda_1, \lambda_2) > f(p; 0, 0) \) for sufficiently low \( p \), and \( f(p; \lambda_1, \lambda_2) < f(p; 0, 0) \) for sufficiently high \( p \). The last result is interesting. Intuitively, the mean reversion effect of income growth is stronger when belief \( p \) is low (or \( p \) is high), and the effect of estimation risk is greater for beliefs in the intermediate range, where belief uncertainty is larger.
This paper studies the effect of learning about income growth on consumption-saving and portfolio choice decisions when the agent cannot fully insure his labor-income shocks. When the agent does not know his income growth, he rationally updates his belief. Estimation risk about income growth naturally arises. This estimation risk generates additional precautionary savings demand beyond the standard income-risk-induced precautionary savings. The more uncertain the agent’s belief, the stronger his precautionary savings demand against estimation risk. By investing in the risky asset, the agent partially hedges against both income risk and estimation risk. On the one hand, a higher income volatility induces a greater hedging demand against income shocks. On the other hand, a higher income volatility (for a fixed income growth wedge) induces a less volatile updating because income is less informative about unknown income growth. Hence a higher income volatility implies a lower estimation risk, which in turn suggests that the agent’s hedging demand with respect to the estimation risk may decrease with income volatility. Therefore, the total hedging demand depends on the relative magnitude of the two forces.

When income growth is stochastic and unknown, the agent’s learning about income growth becomes less volatile. Intuitively, the change of beliefs is locally predictable and hence beliefs are no longer a martingale. Mean reversion of beliefs makes shocks driving the change of beliefs no longer permanent, unlike in settings with unknown \textit{constant} income growth. The stationary belief updating process in turn lowers the impact of estimation risk on consumption. Therefore, consumption responds less to belief change.

The main objective of this paper is to study the effects of incomplete information about the income growth rate on his consumption and portfolio allocations, when the agent’s income shocks are not insurable. In order to deliver this intuition in a simplest possible way, I have intentionally chosen the CARA utility for technical convenience. While analytically convenient, this utility specification ignores the wealth effect on consumption and portfolio allocation rules. The natural next step is to extend our analysis to settings with iso-elastic utility, which captures the wealth effect and hence allows us to make quantitative assessments on the role of learning about income growth.

References


Kimball, M. S. and N. G. Mankiw (1989). Precautionary saving and the


Fig. 1. The “risk-adjusted” certainty equivalent wealth \( f(p) \) and learning induced precautionary saving \( l(p) = \bar{f}(p) - f(p) \) in “self-insurance” models with constant unknown income growth. In self-insurance (income fluctuation) models, there is no investment opportunity in the risky asset \( (\psi^* = 0) \). The parameters are set as follows. The annual (continuously compounded) interest rate \( r = 4\% \), the dispersion of income growth \( \delta = \alpha_1 - \alpha_2 = 3\% \), income volatility \( \sigma = 40\% \), and the degree of income mean reversion \( \kappa = 5\% \). (Since there is no risky asset, we effectively assume correlation coefficient \( \rho = 0 \), and market risk premium \( \zeta = 0 \)). On the left panel, the solid and the dashed lines correspond to the certainty equivalent wealth \( f(p) \) with \( \gamma = 2 \) and \( \gamma = 1 \), respectively. For the comparison purpose, the dotted “straight” line depicts the certainty equivalent wealth \( \bar{f}(p) = \delta p/(r(r + \kappa)) \) given in (23), as if the agent’s income growth is held fixed at \( \mu = \alpha_2 + \delta p \) permanently. The certainty equivalent wealth \( f(p) \) is increasing and convex in belief \( p \), and lies below the straight line \( \bar{f}(p) \). Moreover, \( f(p) \) decreases in \( \gamma \), ceteris paribus. On the right panel, we plot the learning induced precautionary saving \( l(p) = \bar{f}(p) - f(p) \), where \( \bar{f}(p) \) is the dashed line in the left panel. Note that \( l(0) = l(1) = 0 \). Learning induced precautionary saving \( l(p) \) is concave in \( p \) in self insurance models. While concave, precautionary saving \( l(p) \) does not reach peak at \( p = 1/2 \).
Fig. 2. The effect of the correlation coefficient $\rho$ between the income shock and the risk asset return on the “risk-adjusted” certainty equivalent wealth $f(p)$ in a model where the agent learns about his unknown constant income growth (Model II). The parameters are set as follows. The coefficient of absolute risk aversion $\gamma = 1$. The annual (continuously compounded) interest rate $r = 4\%$, the dispersion of income growth $\delta = \alpha_1 - \alpha_2 = 3\%$, income volatility $\sigma = 40\%$, the degree of income mean reversion $\kappa = 5\%$, market (portfolio) risk premium $\zeta = 6\%$, and market portfolio return volatility $\nu = 20\%$ (with an implied Sharpe ratio $\eta = 30\%$). On the left panel, the solid and the dashed lines correspond to the certainty equivalent wealth $f(p; \rho)$ with correlation $\rho = 0.5$ and $\rho = -0.5$, respectively. For the comparison purpose, we plot the dotted line for the case with $\rho = 0$ and the same risk aversion ($\gamma = 1$). Note that $f(p)$ is no longer necessarily convex. The right panel plots the difference $f(p; \rho) - f(p; \rho = 0)$ with respect to belief $p$. Note that $f(p; 0.5) < f(p; 0) < f(p; -0.5)$. The wedges $(f(p; 0.5) - f(p; 0))$ and $(f(p; 0) - f(p; -0.5))$ in the right panel identify the impact of correlation and hedging effects on $f(p)$. Note that the wedge disappears when $p = 0$ or $p = 1$, both of which are absorbing states (i.e. income growth while unknown is constant in Model II depicted here.)
Fig. 3. The effect of learning on certainty equivalent wealth $f(p)$ in a model with stochastic income growth (comparison between Model IV and Model III). The parameters are set as follows. The coefficient of absolute risk aversion $\gamma = 1$. The annual (continuously compounded) interest rate $r = 4\%$, the dispersion of income growth $\delta = \alpha_1 - \alpha_2 = 3\%$, income volatility $\sigma = 40\%$, the degree of income mean reversion $\kappa = 5\%$, market (portfolio) risk premium $\zeta = 6\%$, market portfolio return volatility $\nu = 20\%$ (with an implied Sharpe ratio $\eta = 30\%$), and the correlation coefficient between income shocks and risky asset return $\rho = 0.5$. On the left panel, the income transition intensity parameters are $\lambda_1 = 0\%$ and $\lambda_2 = 3\%$. That is, the high-income-growth state is absorbing. The solid line corresponds to the certainty equivalent wealth $f(p)$ in the learning model (with stochastic growth). For the comparison purpose, the dotted and dash-dotted lines depict the corresponding $\phi_1$ and $\phi_2$ given in (25) and (26) for Model III, where the agent knows his stochastic income growth. Note that $f(1) = \phi_1$ due to the absorbing high-income growth state. More interestingly, $f(0) < \phi_2$ due to $\lambda_2 > 0$ and learning. The right panel plots certainty equivalent wealth $f(p)$ with respect to belief $p$ where income transition intensity parameters are $\lambda_1 = 0\%$ and $\lambda_2 = 3\%$. That is, the low-income-growth state is absorbing. The solid line corresponds to the certainty equivalent wealth $f(p)$ in the learning model (with stochastic growth). For the comparison purpose, the dotted and dash-dotted lines depict the corresponding $\phi_1$ and $\phi_2$ given in (25) and (26) for the corresponding complete-information setting (Model III). Note that $f(0) = \phi_2 = 0$ because the low-income growth state is absorbing. More interestingly, $f(1) > \phi_2$ due to $\lambda_1 > 0$ and learning.
Fig. 4. The “risk-adjusted” certainty equivalent wealth $f(p)$ for various income growth transition parameters $(\lambda_1, \lambda_2)$ in a stochastic income growth model with learning (Model IV). The parameters are set as follows. The coefficient of absolute risk aversion $\gamma = 1$. The annual (continuously compounded) interest rate $r = 4\%$, the dispersion of income growth $\delta = \alpha_1 - \alpha_2 = 3\%$, income volatility $\sigma = 40\%$, the correlation coefficient between income shock and risk asset return $\rho = 0.5$, the degree of income mean reversion $\kappa = 5\%$, market (portfolio) risk premium $\zeta = 6\%$, and market (portfolio) return volatility $\nu = 20\%$. On the left panel, the dashed line corresponds to the setting where income growth may change from low to high (i.e. $\lambda_2 > 0$), but the high income growth is an absorbing state (i.e. $\lambda_1 = 0$). Therefore, $f(1) = \delta/(r(r + \kappa))$ as in Model II, however, unlike Model II, $f(0) > 0$. The solid line corresponds to the setting where income growth may change from high to low (i.e. $\lambda_1 > 0$), but the low income growth state is absorbing (i.e. $\lambda_2 = 0$). Therefore, we have $f(0) = 0$ and $f(1) < \delta/(r(r + \kappa))$. Third, the dash-dotted line corresponds to the general case where neither income growth state is absorbing, i.e. $\lambda_1 > 0$ and $\lambda_2 > 0$. Naturally, $f(0) > 0$ and $f(1) < \delta/(r(r + \kappa))$. Neither boundary conditions resemble the corresponding ones in Model II. For the comparison purpose, the dotted line depicts the setting with (unknown) constant income growth, i.e. $(\lambda_1, \lambda_2) = (0, 0)$. Note $f(0) = 0$ and $f(1) = \delta/(r(r + \kappa))$ for Model II. On the right panel, we plot the certainty equivalent wealth difference $f(p; \lambda_1, \lambda_2) - f(p; 0, 0)$. The dashed line shows $f(p; 0, \lambda_2) - f(p; 0, 0) \geq 0$, if the high-income-growth state is absorbing (i.e. $\lambda_1 = 0$, but $\lambda_2 > 0$). The solid line shows $f(p; \lambda_1, 0) - f(p; 0, 0) \leq 0$, if the low-income-growth state is absorbing (i.e. $\lambda_2 = 0$, but $\lambda_1 > 0$). Finally, when neither high- nor low-income-growth state is absorbing, $f(p; \lambda_1, 0) - f(p; 0, 0) \geq 0$, for low $p$ and $f(p; \lambda_1, 0) - f(p; 0, 0) \leq 0$ for high $p$. See the dash-dotted line.