Dynamic pricing when customers strategically time their purchase

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We study the dynamic pricing problem of a monopolist firm in presence of strategic customers that differ in their valuations and risk-preferences. We show that this problem can be formulated as a static mechanism design problem, which is more amenable to analysis. We highlight several structural properties of the optimal solution, and solve the problem for several special cases. Focusing on settings with low risk-aversion, we show through an asymptotic analysis that the "two-price point" strategy is near-optimal, offering partial validation for its wide use in practice, but also highlighting when it is indeed suitable to adopt it.

1 Introduction

The wide adoption of promotional and markdown pricing by major retailers has "trained" many consumers to anticipate these events and accordingly time their purchases. Given this observation, a natural question that arises is the following: How should the retailer price and allocate its inventory over time in presence of customers that strategize their purchasing decisions? Most pricing and revenue management models and associated commercial systems do not explicitly incorporate this level of consumer behavior. This paper is part of a growing literature that tries to model this effect and study its impact on the firm's controls and profitability.

In more detail, we consider a revenue-maximizing monopolist firm, the seller, that sells a homogeneous good over a time horizon to a market of heterogeneous strategic customers that differ in their valuations and risk-preferences. We allow for a discrete (but arbitrary) customer valuation distribution. The firm seeks to discriminate customers by selling the product at different points in time at different prices and fill-rates. This creates rationing risk, i.e., the risk of not being able to procure the product because its availability is limited, and introduces an incentive for customers with higher valuations, or that are more risk-averse, to pay more

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for the product offered during periods with higher availability. Customers observe or anticipate correctly the price and associated product availability at different points in time and decide when, if at all, to attempt to purchase the product in a way that maximizes their expected risk-adjusted, net utility from their purchase. The seller knows the market characteristics, i.e., the number of customer types, the market size of each type, which is assumed to be deterministic, its value and risk preferences, but cannot discriminate across types. The seller's problem is to choose its dynamic pricing and product availability strategies to maximize its profitability taking into account the strategic customer choice behavior.

The key contributions of this paper are two. First, it highlights a connection between the seller's dynamic pricing problem and a static product design problem, where the firm selects an optimal menu of (price, rationing probability) combinations to optimally segment the market. This connection allows one to use standard machinery from mechanism design to study the structural properties of the seller's policy; it also provides a novel way of addressing the dynamic pricing problem that can be extended in several ways. The second contribution of the paper is that it shows when is it optimal or near-optimal to adopt a two product heuristic, which if often used in the literature to simplify analysis and due to its practical appeal. To this point, it is optimal to offer a menu with one or two products when all customers are risk-neutral, and it is asymptotically optimal to follow a similar strategy even when customers have low risk aversion. The latter case asymptotically reduces to the hierarchical solution of two LPs:

1. The first LP solves the seller's problem treating all consumer types as being risk-neutral.
2. The second LP solves for price and rationing risk perturbations around the risk-neutral solution, taking into account the risk-aversion of the various customer types.

The above decomposition is justified asymptotically as the risk-aversion coefficients of all market participants approach one (i.e., the risk-neutral case). The optimal perturbation around the optimal risk neutral product menu retains the two product structure, but appropriately adjusts the price and rationing probabilities according to the "linearized" risk preferences of the various customer segments. In that sense the asymptotic optimality of the two-product solution is fairly robust. This lends credibility to a practical heuristic that would focus on identifying the optimal two product offering, which as we show can be solved very efficiently even without this asymptotic decomposition. Numerical results are used to benchmark this heuristic against the brute-force computational solution.

Our analysis partially extends prior work that has made restrictive assumptions with respect to customer heterogeneity, e.g., assuming two discrete types of customers, uniformly distributed valuations, no price control, and/or focus on two product variants (i.e., product offered at only two price points). The extension to multiple types and multiple products comes at the expense of the assumption that customers that are rationed out do not renter the market in future times. This assumption is not needed if the market has two types, or if the seller selects to offer just two products. In the general case, it can be justified if customers incur important fixed, transportation or other overhead and search costs in making subsequent store visits, or if they prefer outside opportunities upon being rationed out.

The remainder of this paper is organized as follows: this section concludes with a brief literature review. In §2, we pose the firm's dynamic pricing problem, and develop its reformulation as a product design problem. §3 characterizes the structure of the optimal solution, and §4 solves the two-product problem, the problem with risk-neutral customers, §5 addresses the problem with low risk-aversion, and §6 presents some numerical results.

Literature review: The economic literature that accounts for the effect of strategic consumer behavior in the context of revenue optimization for a monopolist dates back to Coase's [4] treatment of the durable good problem; see also Bulow [1]. We refer the reader to the recent papers by Liu and van Ryzin [8] and Shen and Su [13] for thorough reviews of the extensive economic literature in this area.

Our paper is more closely related to two sets of papers that focus on revenue optimization problems of perishable products with strategic consumers, which is in part motivated by the impact of this type of behavior in retailing coupled with the proliferation of revenue management (price markdown) systems in this industry. This literature is also reviewed in [8] and [13].

The first studies questions of dynamic pricing optimization in a variety of settings that are differentiated by the assumptions on customer heterogeneity. Liu and van Ryzin [8] studied the problem of offering two product variants at predetermined prices to a market of risk-averse consumers with uniformly distributed valuations. Other related papers include Su [14] looks at a problem with high-valuation and low-valuation customers (i.e with a two-point mass valuation distribution) that are either strategic or myopic (purchase at their time of arrival, if at all). Cachon and Swinney [2] study the problem of offering two product variants to a market of myopic, strategic and bargain-hunting customers (that only purchase if the price is low). Caldentey and Vulcano [3] also considered an essentially two period problem where consumers could choose whether to purchase in an auction or at a list price at a future time, and explicitly modeled the possibility of purchasing in the open market at a list price if their bid was not accepted in the auction. Zhang and Cooper [15] consider the two-period, two-product problem with strategic and myopic customers under a linear and multiplicative demand model. While we do not model myopic or bargain-hunting behavior as in Su [14], Cachon and Swinney [2] and Zhang and Cooper [15], we allow customer valuations to have any arbitrary discrete distribution, and model risk-aversion. Unlike some of the abovementioned papers, we assume that customers that are rationed out do not re-enter the market in future times, but on the hand extend our treatment to problems with multiple types of customers, multiple product variants, and, in general, time horizons with more than two periods. Specifically, this restrictive assumption is not needed if there are only two customer types, or if the number of offered products is two.

We extend the analysis in Liu and van Ryzin [8] to arbitrary discrete valuation distribution, allowing both prices and fill-rates to be optimization variables, and for consumers to also differ in terms of their risk preferences. We show that several of their insights carry through in this more general setting (such as the optimality of offering at most two products to a market of risk neutral customers). All of the above papers studied models with two periods, or equivalently where the seller offered the good in two (price, availability) variants. This paper lends credibility to this modeling choice by proving that this restriction is near-optimal in problems with low risk aversion. Specifically, the optimal product menu in settings with low risk aversion retains the two-product structure of the risk-neutral solution but perturbs it appropriately to take into account the risk preferences of the market. The solution of the LP that characterizes the optimal perturbation of the two product menu is quite different from the risk-neutral product design problem (which can also be reduced to an LP), it takes into account the "linearized" risk preferences of the market, and the set of feasible perturbations includes -or better yet is dominated- by price and rationing vectors that would lead to a menu with more products than just two. The fact that the optimal perturbation retains the two product structure of the risk-neutral solution supports the robustness of that policy. This insight is also robust to the form in which the risk preferences enter the utility calculation. Finally, this result relies on a form of asymptotic analysis that is novel in this literature.

The second set of papers that is related to our work is one that uses ideas from mechanism design in the study of this type of a problem. In this paper, the connection with mechanism design allows us to unify and extend several previously established results under a common and intuitive framework. While not considered in this paper, the mechanism design formulation could also be extended to include myopic customer behavior, discounted customer utility, or heterogenous outside opportunity. Our use of the direct revelation principle (Myerson [12]) towards solving the product design reformulation of the dynamic pricing problem is very similar to the approach adopted in Harris and Raviv [7] and Moorthy [10]. Our notion of fill-rate corresponds to their notion of quality. In both Harris and Raviv [7] and Moorthy [10], offered qualities affect cost and thus the revenues. However, customer utility is separable in product price and quality. In our case fill-rates and prices enter the objective multiplicatively, leading to a bilinear objective. This, in addition to the risk-averse behavior of customers, makes our problem different and more complicated. Another paper that uses the direct revealation principle to characterize the optimal mechanism for the seller is Gallien [6], where the revenue maximization problem of a monopolist firm operating in market of risk-neutral, time-sensitive customers with linear utility, and in which the customer arrival process is given by a renewal process is solved using dynamic programming. A recent preprint by Akan. et.al. [9] adopts a mechanism design for a market with a continuum of types that know their type at the beginning of the sales horizon but learn their true valuation at some future point that depends on their type. Customers are strategic and the goal is to design the optimal mechanism to maximize the sellers revenue, which ends up involving an appropriately designed set of options. [6] and [9] study models that differ than ours in many respects, but share in common a problem formulation that exploits a mechanism design structure.

Finally, we note that strategic rationing as a way to differentiate customers is also discussed in Dana [5], where, however, the primary motivation for rationing is demand uncertainty.

2 Dynamic pricing with strategic customers

In this section, we formulate the firm's dynamic pricing problem, and show how it can be reformulated as a static mechanism design problem, thereby making it more amenable to analysis.

2.1 Problem formulation

Seller: A monopolist firm seeks to sell a homogeneous good to a market of strategic customers that differ in their valuations and risk-aversion. In order to segment the market and maximize revenue, the firm sells the product over a period of time by varying product price and the associated fill-rate. Time is discrete, and the selling horizon is divided into T periods, indexed by $t = 1, ..., T$. The capacity, denoted by C, can be endogenous (an optimization variable) or exogenously given (fixed). The capacity cost is linear, and there is no inventory carrying cost. We denote by p_t and r_t the price and the fill-rate associated with the product offered by this monopolist in the t^{th} period. We also refer to it as the t^{th} product.

The firm's policy (p, r) is assumed to be known to the customers, either because it is announced to the market, or because customers have estimated it through repeated interactions with the firm. The firm sells to the market in every season, and as a consequence, the firm's strategy (p, r) over any season should be credible in the sense that the firm commits to it at the start of the selling horizon and cannot deviate from it at any point after that even if that would be optimal from that instant onwards; e.g., the firm cannot announce that the low price product variant will be offered with a significant rationing risk (i.e., at a low fill-rate), and then once the high valuation customers buy the high price variant, decide to offer the lower priced variant at full availability so as to capture more revenue.

Customers: We allow the customer valuation distribution to be arbitrary but discrete. We assume that there are N distinct customer valuations, $v_1, v_2, ..., v_N$. Without loss of generality, we assume that $T \geq N$, else the firm can further divide the time-horizon. The discrete valuations could be obtained as a result of some clustering analysis or by dividing the support of valuation distribution uniformly. Corresponding to each valuation v_i, there is an associated number π_i, denoting the size of customer segment with this valuation. The pair (v, π) defines an arbitrary discrete valuation distribution in a market with N types. We assume that the number of customers π_i with valuation v_i is deterministic and known to the firm. This can be viewed as a large market approximation of a stochastic model where the firm knows the stochastic demand primitives. [1]

For notational convenience, we will assume $v_1 > v_2 > ... > v_N > 0$ and refer to the customer segment that has valuation v_i as "type i". Type i customers, apart from their valuation, are also characterized by a risk-aversion parameter, γ_i, assumed to be rational, and are endowed with the power-utility function. Specifically, the net expected utility for a type i customer from product t is given by $(v_i - p_t)^{\gamma_i} r_t$. We also assume that higher valuation types are at least as risk-averse as the low valuation types, i.e., $0 < \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_N \leq 1$. The setting where all customers have the same degree of risk-aversion is a special case. Customers seek to purchase a product as long as their net expected utility is non-negative. If $v_i < p_t$, then we define the resulting utility to be 0^-. Customers choose the variant that maximizes their expected net utility according to:

$$\chi(i, p, r) = \begin{cases} \arg\max_{1 \leq t \leq T} \ (v_i - p_t)^{\gamma_i} \ r_t, & \text{if} (v_i - p_t)^{\gamma_i} \ r_t \geq 0 \ \text{ for some t}, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

If two or more products result in the same utility, we assume that customers prefer the one with the highest fill-rate amongst these products. In the following, we will often abbreviate $\chi(i, p, r)$ to $\chi(i)$. Note that given the discrete type space and the assumption that customers of each type are homogeneous, all customers of each particular type will make the same choice. We assume that each customer makes the decision to buy one of the offered products only once and buys only one unit of product. In particular, if a customer decides to enter the system in a particular period and does not obtain a unit of the product (because of being rationed out), then the customer leaves and does not contend to buy any other product offered by this firm. This is possible, for example, if customers incur transportation or overhead costs in making subsequent attempts to purchase, or if they prefer to purchase a substitute upon being rationed out. In a

[1] A more complex model could allow for the π_i's to be uncertain, perhaps driven by some aggregate source of uncertainty that would translate the distribution towards lower or higher valuations, and where the firm, and perhaps the consumers, would learn over time. A two-period version of this problem was analyzed in [2].

more general formulation, this customer could be expected to attempt to buy the product in a later period. However, we do not model this flexibility. We note that this restriction is not needed if the number of types is two, or if the seller restricts attention to offering at most two products. We also assume that there is no strategic interaction amongst the customers.

Dynamic pricing problem formulation: Under the above assumptions, the revenue maximization problem faced by the monopolist is given by:

$$\max_{p,r} \quad \Sigma_{t=1}^{T} \; (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}} \pi_i) \; r_t \; p_t \tag{2}$$

$$s.t. \quad \Sigma_{t=1}^{T} \; (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}} \pi_i) \; r_t \leq C, \tag{3}$$

$$0 \leq r_t \leq 1, \quad 0 \leq p_t, \; t = 1, 2, ..., T. \tag{4}$$

The objective is the sum of the revenues from all product variants: $(\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}} \pi_i)$ is the number of customers that wish to purchase in period t, r_t is the fraction of customers that are served, and p_t is the price per unit sold. For each time period t, the price, fill-rate combination (p_t, r_t) can be interpreted as a "product" offered by the firm. The T products are sequenced in time, $t = 1, ..., T$, with $t = 1$ denoting the first and $t = T$ denoting the last product respectively. Capacity is consumed in this sequence as well, and hence defining $C_0 = C$, C_t to be the capacity at the end of period t, we observe that $C_t = C_{t-1} - (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}} \pi_i) \; r_t$. Equation (3) enforces the constraint that the cumulative sales over the sales horizon cannot exceed the available capacity, and hence $C_t \geq 0, \; \forall \; t = 1, ..., T$. If capacity is endogenous, i.e., an optimization variable, then constraint (3) can be dropped. The optimization variables are the price and fill-rate to offer in each of these T periods, and prices are non-negative, fill-rates are between 0 and 1, that the total sales cannot exceed the available capacity.

2.2 Reformulation as a mechanism design problem

This section develops a mechanism design formulation that is equivalent to the problem specified in (2)-(4), and enables the firm to incorporate strategic customer behavior within its revenue optimization problem.

Sufficiency of N products: As a starting observation we show that the firm needs to offer at most N distinct products, N being the number of customer types.

Lemma 1 *Let k^* be the optimal number of products for formulation (2)-(4). Then, $k^* \leq N$.*

The above result does not preclude the case where the firm may optimally choose less than N products, or even just one product. Hence the firm needs to segment the sales horizon of T periods into at most N intervals such that a distinct product (price, fill-rate) combination is offered in each interval. Since all customers are assumed to be strategic, the length (as long as it is non-zero) or the ordering of the intervals during which distinct products are offered does not matter.

Reformulation as a static, product design problem: Next note that we can rewrite the revenue in (2),

$$\Sigma_{t=1}^{T}\ (\Sigma_{i=1}^{N} 1_{\{\chi(i)=t\}}\pi_i)\ r_t\ p_t = \Sigma_{i=1}^{N}\ \pi_i\ p_{\chi(i)}\ r_{\chi(i)}, \tag{5}$$

where we define $p_0 := 0,\ r_0 := 0$. Hence, the T period optimization problem in (2)-(4) can be viewed as a single period problem, when we interpret the fill-rates associated with different time periods as quality attributes of the different product variants that the firm offers to this market of strategic customers. While customers choose the optimal time to enter the system and purchase a product (if at all), for the firm, time does not explicitly enter the problem. The firm needs to compute the optimal prices and fill-rates as if it were a single period problem and all the customers arrive, purchase (if at all), and depart in the same period. This mapping is illustrated in Figure 1.

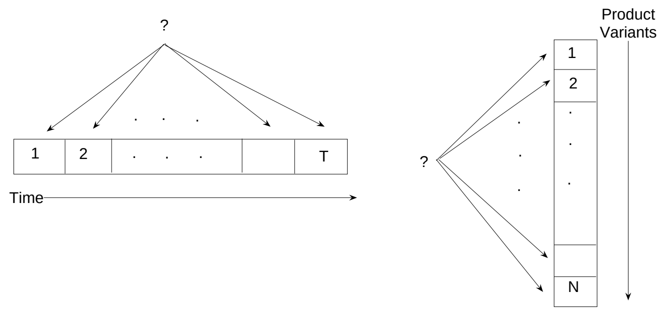


Figure 1: In model (a), customers strategize over the timing of their purchases. Model (b) interprets each time period as a product variant, and customers strategize over which variant to choose, if any. Also, a solution to model (a) can be mapped to a solution to model (b), and vice-versa.

The above observation allows us to reformulate the dynamic pricing problem as a static mechanism design problem. Customers arrive and observe the product menu offered by the firm, and make their choices accordingly. Each customer is characterized by its type designation, which is private information, i.e., not directly observed by the firm. The firm's problem is to design the optimal product menu so as to maximize its profitability.

To begin with, one can restrict the firm's optimization problem to so called "direct mechanisms", wherein the firm designs a payment and product allocation policy ("the mechanism"), under which the customers choose to truthfully self-report their type, as described in Myerson [11]. Essentially, in order to elicit this private type information, the mechanism is designed in a way such that the customer is allocated the product variant that she/he would have selected on her/his own. Following Lemma 1, which ensures that we need to offer at most N distinct products, the resulting problem can be formulated as follows:

$$\max \quad \Sigma_{i=1}^{N}\ p_i\ \pi_i\ r_i \tag{6}$$

$$s.t. \quad (v_i - p_i)^{\gamma_i}\ r_i \geq (v_i - p_j)^{\gamma_i}\ r_j, \quad \forall j \neq i, \tag{7}$$

$$(v_i - p_i)\ r_i \geq 0,\ \ i = 1, 2, ..., N, \tag{8}$$

$$\Sigma_{i=1}^{N}\ \pi_i\ r_i\ \ \leq C, \tag{9}$$

$$0 \leq p_i, \quad 0 \leq r_i \leq 1,\ \ i = 1, 2, ..., N. \tag{10}$$

Equations (6) assumes, without loss of generality, that customer type i buys product i, i.e., $\chi(i) = i$. Equations (7) are the Incentive Compatibility (IC) conditions, enforcing that customer type i (at least weakly) prefers product i over all other products offered by the firm. Equations (8) are the Individual Rationality (IR) conditions, enforcing that customer type i buys from the firm only if the resulting consumer surplus is non-negative. Equation (9) enforces the capacity constraint, and can be removed if capacity is endogenous. In what follows we will analyze (6)-(10) assuming that capacity C is exogenous, with the understanding that the endogenous capacity case can be solved by dropping the capacity constraint (or by setting $C = \infty$). In the optimal solution, some products can be the same, thereby allowing less that N distinct products to be offered. In our solution, $r_i = 0$ implies that customer type i is not offered a product. The above discussion leads to the following theorem, which we state without proof.

Theorem 1 *The problem (2)-(4) is equivalent to the product design problem (6)-(10) in the sense that both formulations lead to the same optimal solution.*

Translation of product design solution to a dynamic policy: Given a solution to (6)-(10), a solution to (2)-(4) can be obtained by assigning to each unique variant (price, fill-rate combination), an interval of time over which it will be applied. For example, if the solution (6)-(10) involves offering k distinct products, then one possible assignment is to offer the k variants in k disjoint intervals, each of length $\lfloor T/k \rfloor$. Since customers are strategic, arrive at the beginning of the time horizon, and demand is deterministic, the order in which different variants are offered, or the duration of time for which they are offered, does not matter in our stylized model. In a richer model, it might be optimal to offer products in a certain order, e.g., in increasing order of prices, as in Su [14]. The mechanism design formulation can incorporate other model attributes, such as time-discounting, myopic behavior, etc. that would force the solution to "define" the sequencing of product variants over time.

3 Analysis of the mechanism design problem

The mechanism design formulation (6) - (10) allows us to deduce several structural properties of the optimal solution. In what follows, without loss of generality, we will assume that $\chi(i) = i,\ i = 1, ..., N$, whenever we consider an N product solution.

Lemma 2 *(**Monotonicity of prices and fill-rates**) At the optimal solution for (6)-(10), $p_1 \geq p_2 \geq ... \geq p_N$ and $r_1 \geq r_2 \geq ... \geq r_N$. Moreover, if for some $i \neq j$, $p_i > p_j$ then $r_i > r_j$, and if $p_i = p_j$, then $r_i = r_j$.*

The next lemma shows that it suffices for type i customers (buying product i) to only check the IC constraints for products intended for types $i-1$ and $i+1$, if any, thus reducing the number of IC constraints from $N(N-1)$ to $2(N-1)$.

Lemma 3 *(**Transitivity of IC constraints**) The IC conditions in (7) are equivalent to:*

$$(v_i - p_i)^{\gamma_i}\ r_i \geq (v_i - p_{i+1})^{\gamma_i}\ r_{i+1},\ i = 1, 2, ...N-1, \tag{11}$$

$$(v_i - p_i)^{\gamma_i}\ r_i \geq (v_i - p_{i-1})^{\gamma_i}\ r_{i-1},\ i = 2, ...N. \tag{12}$$

We will refer to (11) and (12) as the downstream and upstream IC constraints, and from now on replace (6)-(10) by (6), (8)-(12). Lemma 4 shows that products offered by the firm partition the customer types into contiguous sets, so that if customer types $i-1$ and $i+1$ buy the same product l, then customer type i also buys product l. This property will be exploited in subsequent computational algorithms.

Lemma 4 *(**Contiguous Partitioning**) Suppose the firm offers $k \leq N$ distinct products with $p_1 > p_2 > ... > p_k$ and $r_1 > r_2 > ... > r_k$, and such that each generates non-zero demand. Then, these products partition the customer types into contiguous sets $\{1, ..., i_1\}$, $\{i_1+1, ..., i_2\}$, ..., $\{i_{k-1}+1, ..., i_k\}$, buying product 1, 2, ..., k, respectively, $1 \leq i_1 < i_2 < ... < i_k \leq N$, and customer types $\{i_k + 1, ..., N\}$, if any, not buying from the firm. In addition,*
a) $r_1 = \min\left(1, \frac{C}{\Sigma_{l=1}^{i_1}\pi_l}\right)$,
b) $p_j = v_{i_j}$, where $j = \max\{1 \leq l \leq k \mid r_l > 0\}$ in the optimal solution.

In what follows, we will assume that whenever $k \leq N$ products are offered, they partition the customer types as in Lemma 4, i.e., $\chi(1) = ... = \chi(i_1) = 1$, $\chi(i_1+1) = ... = \chi(i_2) = 2$, ..., $\chi(i_{k-1}+1) = ... = \chi(i_k) = k$, and $\chi(i_k+1) = \chi(i_k+2) = ... = 0$, if any.

Using Lemma 4, we can characterize the optimal one product solution.

Corollary 1 *(**One product solution**) The optimal one product solution is $p^* = v_i$, $r^* = \min\left(1, \frac{C}{\Sigma_{k=1}^{i}\pi_k}\right)$ where $i = \text{argmax} \min(C, \Sigma_{k=1}^{j}\pi_k)v_j$. Moreover, if $C \leq \pi_1$, then the globally optimal solution is to offer a single product to type 1 customers with $p_1 = v_1$ and $r_1 = \frac{C}{\pi_1}$.*

To avoid trivial solutions, hereafter we will assume that $C > \pi_1$. The next result shows that the upstream IC constraints for types 2, ..., N can be dropped.

Proposition 1 *Formulation (6), (8)-(12) has the same (optimal) solution with (6), (8)-(11).*

The proof of Proposition 1 establishes that constraint (11) is tight in the optimal solution, and so we can use it to express the optimal fill-rates in terms of the optimal prices.

Corollary 2 *a) A price vector p defines a partitioning of the customer types, specifically, $p_1 = p_2 = ... = p_{i_1} > p_{i_1+1} = ... = p_{i_2} > ... > p_{i_{k-1}+1} = ... = p_{i_k},\ p_{i_j} \leq v_{i_j},\ j = 1, ..., k-1,\ p_k = v_{i_k}$, partitions the customer types as described in Lemma 4.*
b) Fixing the price vector as above, the optimal fill-rates for $j = 1, ..., k$, are given as follows:

$$r_j = \min\left(\max \left(\frac{C - \Sigma_{l=1}^{j-1} \left(\Sigma_{m=i_{l-1}+1}^{i_l} \pi_m\right) r_l}{\Sigma_{m=i_{j-1}+1}^{i_j} \pi_m},\ 0 \right),\ \Pi_{l=1}^{j-1} \left(\frac{v_{i_l} - p_l}{v_{i_l} - p_{l+1}} \right)^{\gamma_{i_l}} \right). \tag{13}$$

The above observation holds at the optimal solution. Also, given an exogenously fixed price vector p satisfying the monotonicity condition in Corollary 2, problem (6), (8)-(11) is solvable in closed form. Formulation (6), (8)-(11) also leads to the following corollary, which relates the optimal revenue with risk-neutral customers to optimal revenue with risk-averse customers.

Corollary 3 *Let $R(\gamma)$ be the optimal revenue achieved for (6), (8)-(11) with risk-aversion vector γ. Let $\mathbf{1}$ denote the N-vector of ones. Then $R(\gamma) \geq R(\mathbf{1})$, $\forall \gamma \leq \mathbf{1}$, where $R(\mathbf{1})$ denotes the optimal revenue achieved for (6), (8)-(11), for risk-neutral customers.*

4 Computations

Problem (6), (8)-(11) appears hard, in part due to the bilinear objective, but mostly due to the non-convex nature of the constraint (11) for $\gamma < 1$. We next discuss two special cases where it can be solved efficiently, and then relate them for our key computational and managerial insight regarding the near optimality of two-product strategies in low risk-aversion settings.

4.1 Risk-neutral case

When customers are risk-neutral, objective (6) and constraint (11) can be simplified through appropriate variable substitutions to lead into an equivalent LP formulation.

Proposition 2 *If $\gamma_i = 1$, $i = 1, 2, ..., N$, then (6), (8)-(11) can be reformulated as the following*

LP: choose $y_i,\ z_i,\ i = 1, ..., N$ *to*

$$\max\ \ \Sigma_{i=1}^{N}\ \pi_i\ z_i \tag{14}$$

$$s.t.\ \ z_i - z_{i+1} = v_i\ y_i,\quad i = 1, 2, ..., N-1, \tag{15}$$

$$\Sigma_{i=1}^{N}\ (\Sigma_{i=i}^{N}\ y_i)\ \pi_i \le C, \tag{16}$$

$$v_i\ \Sigma_{i=i}^{N}\ y_i \ge z_i, \tag{17}$$

$$0 \le z_i,\quad 0 \le \Sigma_{i=i}^{N}\ y_i \le 1, \tag{18}$$

where $z_i := p_i r_i$ *and* $y_i := r_i - r_{i+1}$, $i = 1, ..., N$, $y_{N+1} := 0$.

This LP gives us z_i and y_i as solution. However, $y_N = r_N$, and $y_i = r_i - r_{i+1}$, so we can obtain $r_i,\ i = 1, ..., N$. Next the relation $z_i = p_i r_i$ gives us the value of $p_i,\ i = 1, ..., N$. Proposition 2 implies that the firm's problem is easy to solve in the case of risk-neutral customers. Our next proposition shows that there exists a solution to (14)-(18) that involves offering at most two distinct products, irrespective of the customer valuation distribution and available capacity.

Proposition 3 *If* $\gamma_i = 1,\ i = 1, ..., N$, *then the optimal number of products to offer to risk-neutral customers,* $\overline{k}$, *is at most 2. In particular,* $\overline{k} = 1$ *if the capacity constraint is slack in the optimal solution, and* $\overline{k} = 2$ *if the capacity constraint is tight in the optimal solution.*

The proof of the above proposition also leads to the following corollary, which states that under our assumption that $C > \pi_1$, the highest fill-rate is always equal to 1 in the optimal risk-neutral solution irrespective of the available capacity.

Corollary 4 *For risk-neutral customers, under* $\pi_1 \le C$, *it is never optimal to set* $r_1 < 1$.

4.2 Two product case ($k = 2$)

For administrative reasons ("menu" costs associated with offering new products) or branding considerations (the firm may not want to ration in multiple periods, so that customers that are rationed our in an earlier period do not find the product to be available in a later period), the seller may want to offer a small number of products, typically two. This section solves that problem. The two products effectively partition the N customer types into 3 segments. The first segment of customers from types 1 to i_1 buys product 1, the second segment of customers from types $i_1 + 1$ to i_2 buys product 2, and the remaining customer types, if any, do not buy from the firm. Algorithm 1 outlines how to solve this problem efficiently in $O(N^2)$ time (the proof is presented in Appendix B).

From Lemma 1, we know that at most two distinct products need to be offered in a market with two customer types, and so it follows that we can solve the two customer type problem

Algorithm 1 To calculate two distinct product solution

$R^* = 0$
for $i_1 = 1$ to $N-1$ **do**
  for $i_2 = i_1+1$ to N **do**
    $R_{i_1,i_2} = 0$
    if $\gamma_{i_1} < 1$ **then**
      if $C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)(\frac{\Sigma_{l=i_1+1}^{i_2}\pi_l v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})})^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}$ && $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) > (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_1 + (\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)^{\gamma_{i_1}}(v_{i_1}-v_{i_2})^{\gamma_{i_1}}})^{\frac{1}{1-\gamma_{i_1}}}(\frac{1}{\gamma_{i_1}}-1)$
      else if $C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})})^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}$ && $C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l})^{\frac{1}{\gamma_{i_1}}} + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l})$
    else if $\gamma_{i_1} = 1$ **then**
      if $C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)$ && $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) == (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_1$
      else if $C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)$ && $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) < (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}$ **then**
        $R_{i_1,i_2} = (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l}) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}(\frac{C-\Sigma_{l=1}^{i_1}\pi_l}{\Sigma_{l=i_1+1}^{i_2}\pi_l})$
    if $R^* < R_{i_1,i_2}$ **then**
      $R^* = R_{i_1,i_2}$
if $R^* = 0$ **then**
  Not optimal to offer two distinct products

as a special case of the two product problem. The two product solution provides a lower bound to the optimal revenues attainable in problem (6), (8)-(11). Since the general problem is hard to solve, this provides a heuristic solution to the problem. The next section show that it is asymptotically optimal in settings with low risk-aversion, and its overall effectiveness is evaluated numerically in section 6.

5 Low risk-aversion: offering two products is near-optimal

We now focus on the setting where customers have low risk-aversion, i. e., γ close to 1. To that end, we will rewrite $\gamma_i = 1 - x_i$, where $x_i := \frac{1-\gamma_i}{1-\gamma_1}$ and $x_1 = 1 \geq x_2 \geq ... \geq x_N \geq 0$. We assume that $\gamma_1 < 1$, else the problem involves risk-neutral customers only. We will consider a sequence of problems indexed by n, where the n^{th} problem is characterized by the risk-aversion parameter vector γ^n, given by $\gamma_i^n = 1 - \frac{x_i}{n}$ for $i = 1, ..., N$. When $n = \frac{1}{1-\gamma_1}$, we "recover" the original model parameters, or in other words, the element in the above sequence that corresponds to $n = \frac{1}{1-\gamma_1}$ is exactly the one we started with. We are interested in the case where $\gamma_1 \uparrow 1$. [2]

[2] It is also possible to consider the case where $\gamma^n \downarrow 0$, wherein the asymptotically optimal solution can be characterized as follows. Let $i^* = \min\{i \mid \Sigma_{l=1}^{i}\pi_l > C\}$. Then, $p_i^n \to v_i,\ i < i^*$, $p_i^n = v_i,\ i \geq i^*$, $r_i^n \to 1,\ i < i^*$, $r_{i^*}^n = \frac{C-\Sigma_{l=1}^{i^*-1}\pi_l}{\pi_{i^*}}$, $r_i^n = 0,\ i > i^*$, $r_1^n > r_2^n > ... > r_{i^*}^n > 0$, and the optimal revenue converges to revenue achievable with myopic customers. This case corresponds to extremely high risk-aversion, and we do not analyze it in detail.

We will denote the optimal solution to the problem with γ^n as the risk-aversion parameter vector as (p^n, r^n), and the optimal solution to the risk-neutral problem as $(\overline{p}, \overline{r})$. Following Corollary 2, given price-vector p^n, the corresponding fill-rate vector r^n is uniquely determined. So, in what follows, we will often abbreviate (p^n, r^n) to p^n and $(\overline{p}, \overline{r})$ to $\overline{p}$. We will also refer to the problem with γ^n as the risk-aversion parameter vector as $\mathcal{P}^n$ and the risk-neutral problem as $\overline{\mathcal{P}}$. We will denote the feasible set, given by equations (8)-(12), for $\mathcal{P}^n$ as $\mathcal{S}^n$ and the feasible set for $\overline{\mathcal{P}}$ as $\overline{\mathcal{S}}$.

Asymptotic optimality of risk-neutral solution: We will make the following assumption in the subsequent analysis.
Assumption 1: The risk-neutral solution $(\overline{p}, \overline{r})$ is unique.

Proposition 4 *Under assumption 1, for any convergent subsequence* $\{p^{n_k}\}$, $p^{n_k} \rightarrow \overline{p}$, *and* $R(\gamma^{n_k}) \rightarrow R(\mathbf{1})$, *as* $n_k \uparrow \infty$.

The proof of Proposition 4 also shows that for n sufficiently large, the optimal risk-neutral solution $(\overline{p}, \overline{r})$ is feasible for the problem with risk-averse customers, $\mathcal{S}^n$, and that the optimal solution p^n is "close" to a feasible risk-neutral solution $p' \in \overline{\mathcal{S}}$.

Perturbations around $(\overline{p}, \overline{r})$: With this knowledge, we will look at the perturbed solution to $\overline{P}$ as a candidate optimal solution for P^n for n sufficiently large. Specifically, for $\mathcal{P}^n$, we will consider solutions of the form $(\overline{p} + \delta^n, \overline{r} + \rho^n)$ for the risk-averse problem, where $\delta^n := \frac{\delta}{n}$, and $\rho^n := \frac{\rho}{n}$, such that $\delta^n = p^n - \overline{p}$, and $\rho^n = r^n - \overline{r}$. Suppose the optimal risk-neutral solution partitions involves offering $k \leq N$ distinct products, where the products partition the customer types as in Lemma 4. If $i_k < N$, define $j := i_k + 1$, and set $r_i = 0,\ p_i = v_i,\ i \geq j$. Then, the revenue-maximization problem $\mathcal{P}^n$ can be written as follows.

$$\max \quad \Sigma_{i=1}^N \pi_i (\overline{p}_i + \delta_i^n)(\overline{r}_i + \rho_i^n) \tag{19}$$

$$s.t. \quad \Sigma_{i=1}^N \pi_i (\overline{r} + \rho_i^n) \leq C, \tag{20}$$

$$(v_i - \overline{p}_i - \delta_i^n)^{\gamma_i} (\overline{r}_i + \rho_i^n) \geq (v_i - \overline{p}_{i+1} - \delta_{i+1}^n)^{\gamma_i} (\overline{r}_{i+1} + \rho_{i+1}^n), \quad i = 1, ..., N-1, \tag{21}$$

$$(v_i - \overline{p}_i - \delta_i^n)(\overline{r}_i + \rho_i^n) \geq 0, \quad i = 1, ..., N-1, \tag{22}$$

$$\delta_i^n \geq \delta_{i+1}^n,\ i \notin \{i_1, i_2, ..., i_k\},\ i < i_k, \quad \rho_i^n \geq \rho_{i+1}^n, \quad i \notin \{i_1, i_2, ..., i_k\}, \tag{23}$$

$$\delta_{i_k}^n \leq 0, \quad \rho_i^n \leq 0, \quad i = 1, ..., i_1, \tag{24}$$

$$\delta_i^n \leq 0, \quad \rho_i^n \geq 0, \quad i \geq j. \tag{25}$$

Equations (19) and (20) represent the objective and the capacity constraint, respectively, for the problem $\mathcal{P}^n$. Equation (21) is the downstream IC constraint. Equation (22) ensures that the IR condition is satisfied. Equation (23) ensures that prices and fill-rates are monotonically non-increasing with respect to customer types. For n sufficiently large, we need to enforce this condition only for indices $i \notin \{i_1, i_2, ..., i_k\}$. Equation (24) ensures that $p_{i_k}^n$ cannot increase from the optimal price $\overline{p}_{i_k} = v_{i_k}$ for the risk-neutral case, and similarly that the fill-rate r_1^n cannot increase from the optimal fill-rate $\overline{r}_1 = 1$ for the risk-neutral case (Following Corollary 4 and our assumption that $\pi_1 \leq C$, the optimal solution involves setting $\overline{r}_1 = 1$). Finally, equation (25) ensures that prices and fill-rates for types that were not being sold a product in the risk-neutral case (indices $i \geq j$), if any, are all non-negative.

Characterization of the optimal perturbation around $(\overline{p}, \overline{r})$: Since $\gamma^n = 1 - \frac{x}{n}$, in what follows, we will use Taylor expansion and focus on the second-order terms. This will lead to a LP in terms of x, δ and ρ. This would imply that the price and rationing risk "corrections" needed because customers are not risk-neutral are captured through a solution to a LP. We proceed to derive this LP as follows.

The objective in (19) can be re-written as $\Sigma_{i=1}^{N}(\pi_i\overline{p}_i\overline{r}_i + \pi_i\overline{p}_i\rho_i^n + \pi_i\delta_i^n\overline{r}_i + \pi_i\rho_i^n\delta_i^n)$, wherein we note that first term is objective is a constant, while the last term is $O(\frac{1}{n^2})^3$. Similarly, equation (20) can be re-written as $\Sigma_{i=1}^{N}(\pi_i\overline{r}_i + \pi_i\rho_i^n) \leq C$. If the capacity constraint for $\overline{\mathcal{P}}$ is slack at the optimal solution $(\overline{p}, \overline{r})$, then this constraint can be dropped (since as n grows large, the first-order term will dominate). If not, since the optimal risk-neutral solution was capacitated, it can be re-written as $\Sigma_{i=1}^{N}\pi_i\rho_i^n \leq 0$. Finally, using Taylor expansion, the IC constraint (21) can be written as

$$\begin{aligned}&(v_i - \overline{p}_i)\rho_i^n - \delta_i^n\overline{r}_i - (v_i - \overline{p}_i)\overline{r}_i(1 - \gamma_i^n)\log(v_i - \overline{p}_i) + O\left(1/n^2\right) \geq \\ &(v_i - \overline{p}_{i+1})\rho_{i+1}^n - \delta_{i+1}^n\overline{r}_{i+1} - (v_i - \overline{p}_{i+1})\overline{r}_{i+1}(1 - \gamma_i^n)\log(v_i - \overline{p}_{i+1}) + O\left(1/n^2\right),\end{aligned} \tag{26}$$

where we used that $(v_i - \overline{p}_{i+1})\overline{r}_{i+1} = (v_i - \overline{p}_i)\overline{r}_i$, which follows from the tightness of the downstream IC condition in the optimal solution to $\overline{\mathcal{P}}$, as shown in Proposition 1. We will substitute this constraint by the following constraint.

$$\begin{aligned}&(v_i - \overline{p}_i)\rho_i^n - \delta_i^n\overline{r}_i - (v_i - \overline{p}_i)\overline{r}_i(1 - \gamma_i^n)\log(v_i - \overline{p}_i) + \geq \\ &(v_i - \overline{p}_{i+1})\rho_{i+1}^n - \delta_{i+1}^n\overline{r}_{i+1} - (v_i - \overline{p}_{i+1})\overline{r}_{i+1}(1 - \gamma_i^n)\log(v_i - \overline{p}_{i+1}) + \epsilon_i,\end{aligned} \tag{27}$$

where $\epsilon_i > 0$, if $i \in \{i_1, i_2, ..., i_{k-1}\}$, $\epsilon_i = 0$ otherwise. For n sufficiently large, feasibility of constraint (27) implies feasibility of constraint (26). In what follows we will use the following notation.

$$u_{i,i} = v_i - \overline{p}_i, \quad w_{i,i} = (v_i - \overline{p}_i)\overline{r}_i\log(v_i - \overline{p}_i), \tag{28}$$

$$u_{i,i+1} = v_i - \overline{p}_{i+1}, \quad w_{i,i+1} = (v_i - \overline{p}_i)\overline{r}_i\log(v_i - \overline{p}_{i+1}). \tag{29}$$

The above discussion leads to the following first-order optimization problem.

$$\max\ \Sigma_{i=1}^{N}(\pi_i\overline{p}_i\rho_i + \pi_i\overline{r}_i\delta_i) \tag{30}$$

$$s.t.\ \Sigma_{i=1}^{N}\pi_i\rho_i \leq 0, \tag{31}$$

[3] $g(x) = O(f(x))$ denotes that $\lim_{x\downarrow 0}\frac{g(x)}{f(x)} = c < \infty$

$$\rho_i \le 0, \quad i = 1, ..., i_1, \tag{32}$$

$$\delta_{i_k} \le 0, \tag{33}$$

$$\delta_i \ge \delta_{i+1}, \quad i \notin \{i_1, i_2, ..., i_k\}, \quad i < i_k, \quad \rho_i \ge \rho_{i+1}, \quad i \notin \{i_1, i_2, ..., i_k\}, \tag{34}$$

$$\delta_i \le 0, \quad \rho_i \ge 0, \quad i \ge j, \tag{35}$$

$$u_{i,i}\rho_i + \delta_{i+1}\overline{r}_{i+1} + w_{i,i+1}x_i \ge u_{i,i+1}\rho_{i+1} + \delta_i\overline{r}_i + w_{i,i}x_i + \epsilon_i, \quad i = 1, ..., N-1. \tag{36}$$

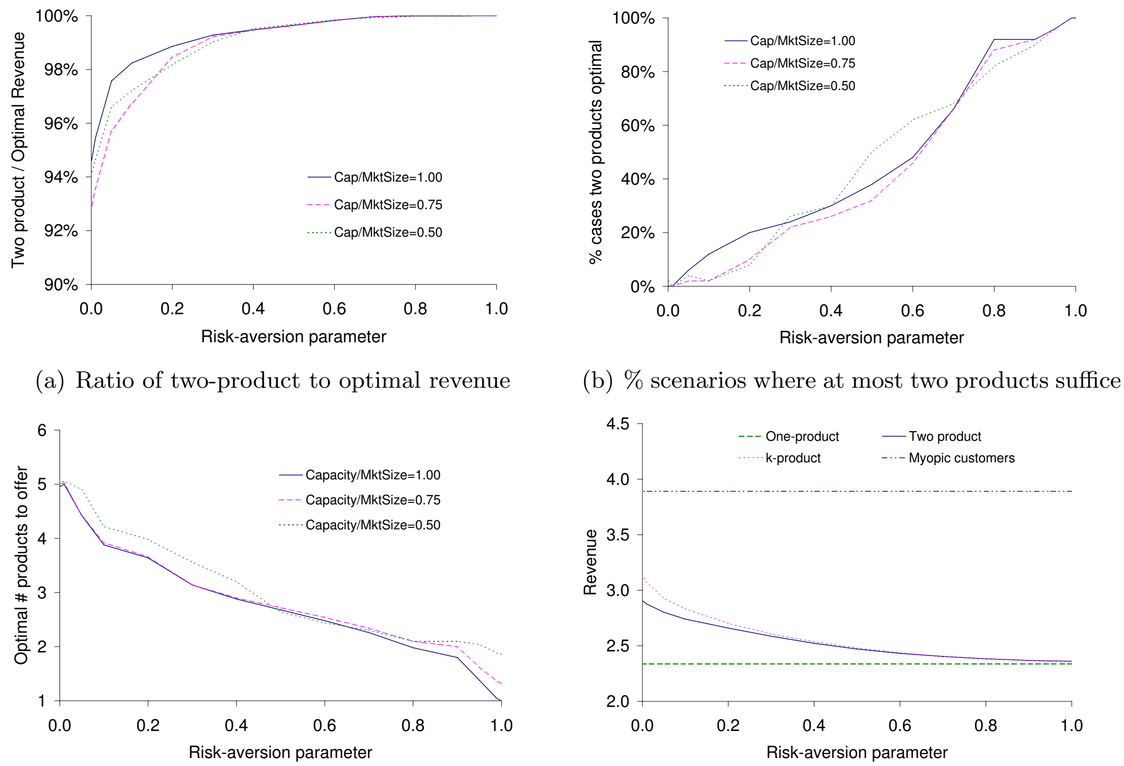
Analyzing the dual of problem (30)-(36), verifying its feasibility and using strong duality for LPs leads to the following proposition.

Proposition 5 *The problem (30)-(36) is feasible and has a finite solution. Let $\overline{k}$ denote the optimal number of products to offer for the risk-neutral problem $\overline{\mathcal{S}}$. Then,*
i) if $\overline{k} = 1$, $\rho_i = \delta_i = 0$, $i = 1, ..., N$,
ii) if $\overline{k} = 2$, $\rho_i = 0$, $i = 1, ..., N$, $\delta_1 = \delta_2 = ... = \delta_{i_1} = (w_{i_1,i_1+1} - w_{i_1,i_1})x_{i_1} - \epsilon_{i_1}$, $\delta_{i_1+1} = ... = \delta_N = 0$, for $\epsilon_{i_1} > 0$ sufficiently small.

Proposition 5 implies that as $\gamma \uparrow 1$, it becomes asymptotically optimal to offer at most two products. Following Lemma 1 and algorithm 1, the optimal one product and the optimal two distinct product solution can be computed efficiently, and together with Proposition 5, this implies that we can solve for the optimal prices and fill-rates for the low risk-aversion case. This section has methodically showed when and why is a two product solution, i. e., offering the product at two price points at different fill-rates, near optimal. This allows justification for this practical heuristic, and allows one to circumvent the intractability of the general formulation (6), (8)-(11). It also lends credibility to numerous papers that have restricted attention to two product models but without any theoretical justification, highlighting the conditions under which it is suitable to do so.

6 Numerical Results

We conclude with some numerical results that study the performance of the two-product heuristic. We consider a market with seven customer types ($N = 7$), with uniform valuations $v_i = 8 - i$, $i = 1, ..., 7$. Type i population-size is sampled from a normal distribution with mean μ_i, and variance σ_i^2, where $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 2$, $\mu_4 = 1$, $\mu_5 = 3.5$, $\mu_6 = 5$, $\mu_7 = 3$, and $\sigma_i = 0.2\mu_i$. Correlation is assumed to be 0, though it can be easily added. For $\sigma_i = 0$, $i = 1, ..., 7$, this corresponds to a bimodal distribution of customer valuations. Other valuation distributions, e.g., uniform, geometric, lead to similar results and are therefore not included. Capacity is fixed at 1, while the capacity to market-size ratio $\frac{C}{\Sigma_{i=1}^N \pi_i}$ varies between $(0, 1]$ and is also a simulation input. The risk-aversion parameter varies between $(0, 1]$ and is a simulation



(a) Ratio of two-product to optimal revenue

(b) % scenarios where at most two products suffice

(c) Optimal number of products to offer

(d) Revenue at Capacity/Market-size = 0.75

Figure 2: This figure shows how the two-product solution compares to the k-product solution as a function of risk-aversion for different capacity to market-size ratios. Figure (a) shows that the two-product solution approaches the optimal solution as risk-aversion parameter approaches 1. Figures (b) and (c) show how the proportion of cases where it is optimal to offer two products, and the optimal number of products to offer, respectively, varies with the risk-aversion parameter. These results are averaged over 50 demand scenarios. Figure (d) examines one such demand scenario in detail and shows that the k-product revenue decreases monotonically and approaches the two-product revenue as risk-aversion approaches one. The maximum revenue is obtained with myopic customers, followed by the k-product, two-product and one-product revenue.

input. $\gamma_2, ..., \gamma_N$ are assumed to be order-statistics of random samples drawn uniformly from the interval $[\gamma_1, 1]$, so that $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_N \leq 1$. Given γ_1 and a fixed capacity to market-size ratio, 50 scenarios are generated, wherein for each scenario, the risk-aversion parameters $\gamma_2, ..., \gamma_N$ are generated randomly as described above, and the customer population size π is sampled from a normal distribution with the parameters given above. Negative demand, if any, is truncated to zero, and the resulting customer population sizes are scaled proportionately to achieve the given capacity to market-size ratio. For each scenario, the optimal one-product, two-product and k-product solutions are computed. These are averaged over scenarios to obtain results corresponding to a data point in Figure 2.

Figure 2 (a) shows how the two-product revenue compares with the k-product revenue for different capacity to market-size ratios as γ_1 varies between zero and one. We observe that as γ_1

approaches one, the two-product revenue approaches the k-product revenue. Even when γ_1 is small and not close to 1, the two-product revenue achieves within 8% of the k-product revenues on average, therefore serving as a useful approximation and lower bound. (As described in Footnote 1 in section 5, as $\gamma \downarrow 0$, the optimal number of products may grow large.) Figure 2 (b) shows how the fraction of scenarios where it suffices to offer at most two products varies with γ_1. We observe that this fraction is non-monotonic, but the overall trend suggests that it increases as γ_1 increases. It equals one in the limit of risk-neutral customers, but for other risk-aversion values, it can be much less than one. From Figure 2 (a), we know that the two product revenue is close to the k-product revenue, and together this implies that even though the two-product solution might be suboptimal in a large fraction of cases, the two-product solution is close to the k-product solution and hence the suboptimality gap is small. Figure 2 (c) shows the optimal number of products to offer as a function of γ_1. Again, while non-monotonic, the overall trend suggests that this number decreases as γ_1 increases. For the risk-neutral case, this number lies between one and two, consistent with our result that at most two products need to be offered in this case. Figure 2 (d) shows the one-product, two-product and k-product revenue as a function of γ_1 for the case where capacity to market-size ratio is fixed at 0.75. It also plots the optimal achievable revenue if the customers were all myopic. We observe that the highest revenue is achieved when customers are myopic, followed by the k-product solution, the two-product solution, and the one-product solution, respectively. Both the revenue with myopic customers and the one-product revenue do not depend on customer risk-aversion. The k-product revenue dominates the two-product revenue, and approaches it as γ_1 approaches one. Also, the k-product revenue decreases as γ_1 increases, and as expected, exceeds the revenue with risk-neutral customers.

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Appendix A

Proof of Lemma 1 From (1), all type i customers select the same product variant, $\chi(i, p, r)$. Since there are N types, there can be at most N distinct products that generate non-zero demand. Hence, it suffices to offer at most N distinct products. $\square$

Proof of Lemma 2 We first show that higher prices are associated with higher fill-rates. IC conditions for types i and j (that prefer product i and j, respectively) imply that, $\left(\frac{v_i - p_i}{v_i - p_j}\right)^{\gamma_i} \geq \frac{r_j}{r_i}$

and $\left(\frac{v_j - p_j}{v_j - p_i}\right)^{\gamma_j} \geq \frac{r_i}{r_j}$, respectively. There are three cases to consider. 1) Suppose $p_i = p_j$: then, from the IC condition we obtain $r_i = r_j$. 2) $p_i > p_j$: then, the IC condition for type i implies that $\frac{r_j}{r_i} \leq \left(\frac{v_i - p_i}{v_i - p_j}\right)^{\gamma_i} < 1$. So, $r_j < r_i$. 3) $p_i < p_j$: this case is symmetric to 2). Similarly, one can verify that higher fill-rates are associated with higher prices.

Next we show that "higher" types prefer higher priced products. Suppose that there exist $i < j$, such that $p_i < p_j$. IC conditions for customer types i and j imply that $\left(\frac{v_i - p_i}{v_i - p_j}\right)^{\gamma_i} \geq \frac{r_j}{r_i} \geq \left(\frac{v_j - p_i}{v_j - p_j}\right)^{\gamma_j}$. Also, $p_i < p_j$ and $v_i > v_j$ imply that $1 < \frac{v_i - p_i}{v_i - p_j} < \frac{v_j - p_i}{v_j - p_j}$, and since $\gamma_i \leq \gamma_j$, $\left(\frac{v_i - p_i}{v_i - p_j}\right)^{\gamma_i} < \left(\frac{v_j - p_i}{v_j - p_j}\right)^{\gamma_j}$. The latter implies that the IC conditions cannot hold, and this leads to a contradiction. □

Proof of Lemma 3 The proof comprises of two parts. First we show that customer type $i-1$ would rather make the choice made by type i than by $i+1$. Then we show that customer type $i+1$ would rather make the choice made by type i than by $i-1$. Together they imply that IC conditions are transitive.

Step 1: We will show that if type i chooses product i over product $i+1$, then type $i-1$ will also choose product i over product $i+1$. Following Lemma 2, $p_{i-1} \geq p_i \geq p_{i+1}$. Consequently, $1 \geq \frac{v_{i-1} - p_i}{v_{i-1} - p_{i+1}} \geq \frac{v_i - p_i}{v_i - p_{i+1}}$, and so $\gamma_{i-1} \leq \gamma_i$ implies that $\left(\frac{v_{i-1} - p_i}{v_{i-1} - p_{i+1}}\right)^{\gamma_{i-1}} \geq \left(\frac{v_i - p_i}{v_i - p_{i+1}}\right)^{\gamma_i}$. The IC condition that guarantees that type i customers choose product i over product $i+1$ implies that $\left(\frac{v_i - p_i}{v_i - p_{i+1}}\right)^{\gamma_i} \geq \frac{r_{i+1}}{r_i}$, thereby leading to the desired inequality.

Step 2: Similarly, one can show that if type i chooses product i over product $i-1$, then type $i+1$ will also choose product i over product $i-1$ (details are omitted). □

Proof of Lemma 4 Suppose the seller decides to offer only $k \leq N$ products. Then, we will show that if customer types $i-1$ and $i+1$ choose product l, then customer type i chooses product l, this would imply that offered products partition the customer types into contiguous sets. IC conditions for type $i-1$ and type $i+1$ customers imply that, for any $m \neq l$, $(\frac{v_{i-1} - p_l}{v_{i-1} - p_m})^{\gamma_{i-1}} \geq \frac{r_m}{r_l}$ and $\left(\frac{v_{i+1} - p_l}{v_{i+1} - p_m}\right)^{\gamma_{i+1}} \geq \frac{r_m}{r_l}$. Consider all products m s.t. $v_i > p_m > p_l$. Then, $1 < \frac{v_{i-1} - p_l}{v_{i-1} - p_m} < \frac{v_i - p_l}{v_i - p_m}$. Next $\gamma_{i-1} \leq \gamma_i$ implies that $\left(\frac{v_i - p_l}{v_i - p_m}\right)^{\gamma_i} > \frac{r_m}{r_l}$, and so type i customers also prefer product l over all products m with $v_i > p_m > p_l$. Now consider all products m s.t. $p_m < p_l$. Then, $1 > \frac{v_i - p_l}{v_i - p_m} > \frac{v_{i+1} - p_l}{v_{i+1} - p_m}$. Next $\gamma_i \leq \gamma_{i+1}$ implies that $1 > \left(\frac{v_i - p_l}{v_i - p_m}\right)^{\gamma_i} > \frac{r_m}{r_l}$, so that type i customers also prefer product l over all products m with $p_m < p_l$.

Parts a) and b) of the Lemma proceed as follows. a) Since $p_1 \geq p_2 \geq ... \geq p_k$ following Lemma 2, setting r_1 to the highest possible value is optimal. This is $\min\left(1, \frac{C}{\Sigma_{l=i_0+1}^{i_1} \pi_l}\right)$. b) Suppose in a given optimal solution (p, r), $p_j < v_{i_j}$ where $j = \max\{1 \leq l \leq k | r_l > 0\}$. Define a new price vector as follows: $p'_j = v_{i_j}$, $p'_l = v_{i_l} - (v_{i_l} - p'_{l+1})(\frac{r_{l+1}}{r_l})^{\frac{1}{\gamma_{i_l}}}$, $1 \leq l < j$. Observe that

$v_{i_l} \geq p'_l \geq p_l$, $1 \leq l \leq j$, and none of the customer types that were buying a product switch classes or discontinue to buy the product. The solution (p', r) is also incentive-compatible and feasible, and results in a greater revenue. Hence $p_j < v_{i_j}$ cannot hold in an optimal solution. $\square$

Proof of Corollary 1 From Lemma 4, we know that $r^* = r_1 = \min\left(1, \frac{C}{\Sigma_{k=1}^{i}\pi_i}\right)$, where customer types $1, .., i$ buy product 1. Also the optimal price to offer the product at when up to type i are being served the first product is v_i, and hence we can search for the optimal value of i by evaluating the revenue at each of the N possible price points $v_1, v_2, ..., v_N$. $\square$

Proof of Proposition 1 Suppose the optimal solution to formulation (6), (8)-(12) involves offering $k \leq N$ distinct products, where the products partition the customer types as in Lemma 4. Given this partitioning, it suffices to impose downstream IC constraints for types $i_1, i_2, ..., i_{k-1}$ and upstream IC constraints for types $i_1 + 1, i_2 + 1, ..., i_{k-1} + 1$. Just as in the case of individual customer classes, transitivity across groups also holds.

We next determine the necessary conditions that the optimal prices and the fill-rates must satisfy. Since k distinct products are offered, $0 < v_{i_k} < p_j < v_{i_j}$, $j = 1, ..., k-1$, $r_j > 0$, $j = 1, ..., k$, and the Lagrange multipliers with the associated bounding constraints $v_i \geq p_i$, $i = 1, ..., i_k - 1$, $p_i \geq 0$, $i = 1, ..., i_k$, $1 \geq r_i$, $i = i_1 + 1, ..., i_k$, $r_i > 0$, $i = 1, ..., i_k$ are zero. Moreover, since the solution is optimal, $p_k = v_{i_k}$ and $r_1 = 1$, so that we have $2k$ - 2 optimization variables. We can write the Lagrangian as follows (fulfilment of the constraint qualification condition is shown in Appendix B):

$$\begin{aligned} L = \Sigma_{j=1}^{k}(\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l)p_j r_j + \Sigma_{j=1}^{k-1}\mu_j\left((v_{i_j} - p_j)^{\gamma_{i_j}} r_j - (v_{i_j} - p_{j+1})^{\gamma_{i_j}} r_{j+1}\right) \\ + \Sigma_{j=1}^{k-1}\zeta_j\left((v_{i_j+1} - p_{j+1})^{\gamma_{i_j+1}} r_{j+1} - (v_{i_j+1} - p_j)^{\gamma_{i_j+1}} r_j\right) + \lambda(C - \Sigma_{j=1}^{k}\Sigma_{l=i_{j-1}}^{i_j}\pi_l r_j). \end{aligned} \quad (37)$$

Note that $\mu_j\zeta_j = 0$, since exactly one of the respective constraints is tight; otherwise we can increase revenues by changing the price or the fill-rate. Differentiating with respect to p_1, we obtain

$$\frac{\partial L}{\partial p_1} = \Sigma_{l=1}^{i_1}\pi_l - \mu_1\gamma_{i_1}(v_{i_1} - p_1)^{\gamma_{i_1}-1} + \zeta_1\gamma_{i_1+1}(v_{i_1+1} - p_1)^{\gamma_{i_1+1}-1} = 0,$$

implying that $\mu_1 > 0, \zeta_1 = 0$. Differentiating with respect to p_u, we get that

$$\begin{aligned} \frac{\partial L}{\partial p_u} = (\Sigma_{l=i_{u-1}+1}^{i_u}\pi_l)r_u - \mu_u\gamma_{i_u}(v_{i_u} - p_u)^{\gamma_{i_u}-1}r_u + \mu_{u-1}\gamma_{i_{u-1}}(v_{i_{u-1}} - p_u)^{\gamma_{i_{u-1}}-1}r_u \\ -\zeta_{u-1}\gamma_{i_{u-1}+1}(v_{i_{u-1}+1} - p_u)^{\gamma_{i_{u-1}+1}-1}r_u + \zeta_u\gamma_{i_u+1}(v_{i_u+1} - p_u)^{\gamma_{i_u+1}-1}r_u = 0. \end{aligned}$$

Now using the induction hypothesis that $\mu_{u-1} > 0, \zeta_{u-1} = 0$, we find that $\mu_u > 0, \zeta_u = 0$. This implies that all the downstream constraints are tight, while all the upstream constraints are slack. Hence, given any partitioning, we can drop the upstream constraints. Moreover, we can set the downstream constraints to be tight. Since, the choice of partitioning does not matter, this holds for all partitions, and in particular the optimal partition, and hence formulation (6), (8)-(11) leads to the same optimal solution as formulation (6), (8)-(12).

Proof of Corollary 2 a) This follows directly from Lemma 4. b) The expression for r follows from the tightness of constraint (11) in formulation (6), (8)-(11), the second part of Lemma 4, and the non-negativity of fill-rates. □

Proof of Corollary 3 Note that the risk-aversion parameter enters formulation (6), (8)-(11) only via constraints (11). Next, denote the optimal solution to formulation (6), (8)-(11) when customers are risk-neutral by $(\overline{p}, \overline{r})$. Then $\left(\frac{v_i - \overline{p}_i}{v_i - \overline{p}_{i+1}}\right)^{\gamma_i} \geq \left(\frac{v_i - \overline{p}_i}{v_i - \overline{p}_{i+1}}\right) \geq \frac{\overline{r}_{i+1}}{\overline{r}_i}$, and hence $(\overline{p}, \overline{r})$ is feasible for formulation (6), (8)-(11) with γ as risk-aversion parameter vector. Hence the revenue with $(\overline{p}, \overline{r})$ serves as a lower bound for the optimal revenue to the risk-averse problem. (Note that $(\overline{p}, \overline{r})$ might not be feasible for formulation (6), (8)-(12) with γ as risk-aversion parameter vector.) Similarly, one can also show that the optimal revenue with risk-aversion parameter γ serves as a lower bound for revenue with risk-aversion parameter γ', if $\gamma' \leq \gamma$. □

Proof of Proposition 2 Define $z_i := p_i r_i,\ i = 1, ..., N$, and $y_i := r_i - r_{i+1},\ i = 1, ..., N-1,\ y_N = r_N$. Then the IR condition (equation (8)), the IC condition (equation (11)), and the capacity constraint (equation (9)) can be written as $v_i \Sigma_{l=i}^N y_l \geq z_i$, $z_i - z_{i+1} \leq v_i y_i$ and $\Sigma_{i=1}^N (\Sigma_{l=i}^N y_l) \pi_i \leq C$, respectively, while the objective (equation (6)) becomes $\Sigma_{i=1}^N \pi_i z_i$, which are all linear in the variables z_i, y_i, thereby leading to an LP. □

Proof of Proposition 3 Suppose customers are risk-neutral and the firm decides to offer $k > 1$ distinct products such that they partition customer types as in Lemma 4. Using Proposition 1, we can write the Lagrangian as

$$\begin{aligned} L = {} & \Sigma_{j=1}^k (\Sigma_{l=i_{j-1}+1}^{i_j} \pi_l) p_j r_j + \Sigma_{j=1}^{k-1} \mu_j \left((v_{i_j} - p_j) r_j - (v_{i_j} - p_{j+1}) r_{j+1} \right) \\ & + \lambda (C - \Sigma_{j=1}^k \Sigma_{l=i_{j-1}+1}^{i_j} \pi_l r_j). \end{aligned} \tag{38}$$

Using Lemma 4 and differentiating with respect to p_1 yields $\Sigma_{l=1}^{i_1} \pi_l - \mu_1 = 0$. Differentiating with respect to p_j gives $\Sigma_{l=i_{j-1}+1}^{i_j} \pi_l - \mu_j + \mu_{j-1} = 0,\ j = 2, ..., k-1$. Together these imply that $\mu_j = \Sigma_{l=1}^{i_j} \pi_l$. Differentiating with respect to r_j and using the above we get $\mu_j v_{i_j} - \mu_{j-1} v_{i_{j-1}} - \lambda \Sigma_{l=i_{j-1}+1}^{i_j} \pi_l = 0, j = 2, ..., k-1$. Using $\mu_j = \Sigma_{l=1}^{i_j} \pi_l$ gives $(\Sigma_{l=i_{j-1}+1}^{i_j} \pi_l) v_{i_j} - (\Sigma_{l=1}^{i_{j-1}} \pi_l)(v_{i_{j-1}} - v_{i_j}) - \lambda \Sigma_{l=i_{j-1}+1}^{i_j} \pi_l = 0$. There are two cases to consider.

a) $\Sigma_{l=1}^{i_1} \pi_l v_{i_1} = \Sigma_{l=1}^{i_2} \pi_l v_{i_2} = ... = \Sigma_{l=1}^{i_k} \pi_l v_{i_k}$, when capacity is unconstrained and $k > 1$,

b) $\lambda = \frac{\Sigma_{l=1}^{i_2} \pi_l v_{i_2} - \Sigma_{l=1}^{i_1} \pi_l v_{i_1}}{\Sigma_{l=i_1+1}^{i_2} \pi_l} = \frac{\Sigma_{l=1}^{i_3} \pi_l v_{i_3} - \Sigma_{l=1}^{i_2} \pi_l v_{i_2}}{\Sigma_{l=i_2+1}^{i_3} \pi_l} = ... = \frac{\Sigma_{l=1}^{i_k} \pi_l v_{i_k} - \Sigma_{l=1}^{i_{k-1}} \pi_l v_{i_{k-1}}}{\Sigma_{l=i_{k-1}+1}^{i_k} \pi_l} \geq 0$, if capacity is scarce and $k \geq 2$.

The remainder of this proof verifies (details are omitted) that the revenue in a) is given by $\Sigma_{l=1}^{i_1} \pi_l v_{i_1}$, and is achieved by offering a single product at price $p_1 = v_{i_1}$, $r_1 = 1$, and that the revenue in case b) is given by $\Sigma_{l=1}^{i_1}\pi_l v_{i_1} + \lambda(C - \Sigma_{l=1}^{i_1}\pi_l)$, and is achieved by offering two distinct products at the following prices and fill-rates. Define $u := \max\{j \mid \Sigma_{l=1}^{i_j}\pi_l < C\}$, then $p_1 = v_{i_u} - (v_{i_u} - v_{i_{u+1}})\frac{C-\Sigma_{l=1}^{i_u}\pi_l}{\Sigma_{l=i_u+1}^{i_{u+1}}\pi_l}$, $r_1 = 1$, $p_2 = v_{i_{u+1}}$ and $r_2 = \frac{C-\Sigma_{l=1}^{i_u}\pi_l}{\Sigma_{l=i_u+1}^{i_{u+1}}\pi_l}$. □

Proof of Corollary 4 Suppose $r_1 < 1$. There are two cases to consider.

a) $k > 1$: Consider setting $r_1' = \min\left(1, \frac{\Sigma_{l=1}^{i_1}\pi_l}{C}\right)$, $r_j' = \min\left(r_j, \frac{C-\Sigma_{u=1}^{j-1}(\Sigma_{l=i_{u-1}+1}^{i_u}\pi_l)r_i'}{\Sigma_{l=i_{j-1}+1}^{i_j}\pi_l}\right)$, $j > 1$, and increasing p_1' s.t. the downstream IC constraint $(v_{i_1} - p_1')r_1' \geq (v_{i_1} - p_2')r_2'$ for type i_1 customers is tight. Then, no type chooses to buy a different product, but the revenues strictly increase. This leads to a contradiction. If $r_1' < 1$, then it implies that only one product is being offered and hence this reduces to case b).

b) $k = 1$: Suppose $r_1 < 1$. This implies that $\Sigma_{l=1}^{i_1}\pi_l > C$ and $\pi_1 v_1 < Cv_{i_1} < (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1}$. Consider the following two-product offering: $p_1' = v_1 - (v_1 - p_2)r_2$, $r_1' = 1$, $p_2' = v_{i_1}$, $r_2' = \frac{C-\pi_1}{\Sigma_{l=i_1+1}^{i_1}\pi_l}$. Then, the new revenue equals $\pi_1 v_1 + \frac{(\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - \pi_1 v_1}{(\Sigma_{l=2}^{i_1}\pi_l)}(C - \pi_1) > Cv_{i_1}$, again implying that the original one product revenue was suboptimal, thereby leading to a contradiction. □

Proof of Proposition 4: We will first prove the following:

i) There exists $M \in \mathbb{N}$ sufficiently large s. t. $(\overline{p}, \overline{r}) \in \mathcal{S}^n, \forall n \geq M$

ii) There exist $p^1 \in \overline{\mathcal{S}}$, $M \in \mathbb{N}$ s. t. $|p_i^n - p_i^1| < c(1 - \gamma_1^n), \forall n \geq M$, where c is a constant

i) Suppose the optimal risk-neutral solution involves offering $k \leq N$ distinct products, where the products partition the customer types as in Lemma 4. Then, following Proposition 1, $(\overline{p}, \overline{r})$ satisfies the following constraints.

$$(v_{i_j} - \overline{p}_j)\overline{r}_j = (v_{i_j} - \overline{p}_{j+1})\overline{r}_{j+1}, \quad j = 1, ..., k-2,$$
$$(v_{i_j+1} - \overline{p}_{j+1})\overline{r}_{j+1} > (v_{i_j+1} - \overline{p}_j)\overline{r}_j, \quad j = 2, ..., k-1.$$

The feasible sets $\mathcal{S}^n$ and $\overline{\mathcal{S}}$ of problems $\mathcal{P}^n$ and $\overline{\mathcal{P}}$, respectively, only differ in their IC constraints. Hence, in order to establish claim i), it suffices to show that $(\overline{p}, \overline{r})$ satisfies the IC constraints of $\mathcal{P}^n$, for n sufficiently large. Note that

$$1 > (\frac{v_{i_j} - \overline{p}_j}{v_{i_j} - \overline{p}_{j+1}})^{\gamma_{i_j}^n} > \frac{v_{i_j} - \overline{p}_j}{v_{i_j} - \overline{p}_{j+1}} = \frac{\overline{r}_{j+1}}{\overline{r}_j},$$

implying that the downstream IC condition for problem $\mathcal{P}^n$ is satisfied by $(\overline{p}, \overline{r})$. If $(v_{i_j+1} - \overline{p}_j) \leq 0$, the upstream IC condition is satisfied as well, otherwise, consider the following. Define ϵ_j, $j = 2, ..., k-1$ such that

$$\epsilon_j \overline{r}_{j+1}(v_{i_j+1} - \overline{p}_j) = (v_{i_j+1} - \overline{p}_{j+1})\overline{r}_{j+1} - (v_{i_j+1} - \overline{p}_j)\overline{r}_j > 0.$$

We want to show that for sufficiently large n,

$$(v_{i_j+1} - \overline{p}_{j+1})^{\gamma_{i_j+1}^n}\overline{r}_{j+1} \geq (v_{i_j+1} - \overline{p}_j)^{\gamma_{i_j+1}^n}\overline{r}_j,$$

$$\Leftrightarrow \quad \left(\frac{v_{i_j+1}-\overline{p}_{j+1}}{v_{i_j+1}-\overline{p}_j}\right)^{\gamma^n_{i_j+1}} \geq \frac{\overline{r}_j}{\overline{r}_{j+1}} = \left(\frac{v_{i_j+1}-\overline{p}_{j+1}}{v_{i_j+1}-\overline{p}_j}\right) - \epsilon_j.$$

Substituting $\gamma^n_{i_j+1} = 1 - x_{i_j+1}$ and using the Taylor expansion, this condition can be written as

$$\left(\frac{v_{i_j+1}-\overline{p}_{j+1}}{v_{i_j+1}-\overline{p}_j}\right) - c_1 x_{i_j+1} + O((x_{i_j+1})^2) \geq \left(\frac{v_{i_j+1}-\overline{p}_{j+1}}{v_{i_j+1}-\overline{p}_j}\right) - \epsilon_j,$$

where c_1 is a constant. The latter is satisfied if $c_1 x_{i_j+1} \leq \epsilon_j + O((x_{i_j+1})^2)$, from which it follows that for any $\epsilon_j > 0$, there exists a M sufficiently large such that for all $n \geq M$, $(\overline{p}, \overline{r})$ satisfies both upstream and downstream IC constraints, and this completes the proof of claim i).

ii) Suppose the optimal solution (p^n, r^n) involves offering $k \leq N$ distinct products, where the products partition the customer types as in Lemma 4. Since (p^n, r^n) is the optimal solution for $\mathcal{P}^n$, it satisfies the following constraints

$$(v_{i_j} - p^n_j)^{\gamma^n_{i_j}} r^n_j = (v_{i_j} - p^n_{j+1})^{\gamma^n_{i_j}} r^n_{j+1}, \quad j = 1, ..., k-1, \tag{39}$$

$$(v_{i_j+1} - p^n_{j+1})^{\gamma^n_{i_j+1}} r^n_{j+1} > (v_{i_j+1} - p^n_j)^{\gamma^n_{i_j+1}} r^n_j, \quad j = 2, ..., k-1. \tag{40}$$

We will construct a new solution $(p', r') \in \overline{\mathcal{S}}$, where $r' = r^n$, and

$$(v_{i_j} - p'_j) r'_j = (v_{i_j} - p'_{j+1}) r'_{j+1} \quad [\Longrightarrow (v_{i_j+1} - p'_{j+1}) r'_{j+1} > (v_{i_j+1} - p'_j) r'_j],$$

such that $|p^n_i - p_i| < c(1-\gamma^n_i)$, for n sufficiently large and some constant c. Consider the $k-1^{th}$ downstream IC constraint. From (39), it follows that

$$(v_{i_{k-1}} - p^n_{k-1}) r^n_{k-1} < (v_{i_{k-1}} - p^n_k) r^n_k.$$

Set $p'_k = p^n_k$, $p'_{k-1} = p^n_{k-1} - \epsilon_{k-1}$, $\epsilon_{k-1} > 0$, s.t.

$$(v_{i_{k-1}} - p'_{k-1}) r'_{k-1} = (v_{i_{k-1}} - p'_k) r'_k.$$

This requires that

$$\begin{aligned}
\epsilon_{k-1} &= (v_{i_{k-1}} - p^n_k) \frac{r'_k}{r'_{k-1}} - (v_{i_{k-1}} - p^n_{k-1}), \\
&= (v_{i_{k-1}} - p^n_k) \left(\frac{v_{i_{k-1}} - p^n_{k-1}}{v_{i_{k-1}} - p^n_k}\right)^{\gamma^n_{i_{k-1}}} - (v_{i_{k-1}} - p^n_{k-1}), \\
&= c_{k-1}(1 - \gamma^n_{i_{k-1}}) + O((1 - \gamma^n_{i_{k-1}})^2),
\end{aligned}$$

following a Taylor expansion, where c_{k-1} is a constant. Note that $\epsilon_{k-1} > 0$ since $p_{k-1} > p_k$. Next consider the $k-2^{th}$ downstream IC constraint. Set $p'_{k-2} = p^n_{k-2} - \epsilon_{k-2}, \epsilon_{k-2} > 0$, s.t.

$$(v_{i_{k-1}} - p'_{k-2}) r'_{k-2} = (v_{i_{k-1}} - p'_{k-1}) r'_{k-1}.$$

This requires that

$$\epsilon_{k-2} = (v_{i_{k-2}} - p'_{k-1}) \frac{r'_{k-1}}{r'_{k-2}} - (v_{i_{k-2}} - p^n_{k-2}),$$

$$
\begin{aligned}
&= (v_{i_{k-2}} - p'_{k-1}) \left(\frac{v_{i_{k-2}} - p^n_{k-2}}{v_{i_{k-2}} - p^n_{k-1}} \right)^{\gamma^n_{i_{k-2}}} - (v_{i_{k-2}} - p^n_{k-2}), \\
&= (v_{i_{k-2}} - p^n_{k-1} + \epsilon_{k-1}) \left(\frac{v_{i_{k-2}} - p^n_{k-2}}{v_{i_{k-2}} - p^n_{k-1}} \right)^{\gamma^n_{i_{k-2}}} - (v_{i_{k-2}} - p^n_{k-2}), \\
&= c^1_{k-2}(1 - \gamma^n_{i_{k-2}}) + c^2_{k-2}(1 - \gamma_{i_{k-1}}) + O((1 - \gamma^n_{i_{k-1}})^2) + O((1 - \gamma^n_{i_{k-2}})^2), \\
&\leq c_{k-2}(1 - \gamma^n_{i_{k-2}}) + O((1 - \gamma^n_{i_{k-1}})^2),
\end{aligned}
$$

following a Taylor expansion, where c_{k-2}, c^1_{k-2} and c^2_{k-2} are constants. Note that $\epsilon_{k-2} > 0$. Proceeding in a similar fashion, one can construct $p'_1, ..., p'_{k-3}$ as well, wherein $p'_i - p^n_i \leq c_i(1 - \gamma^n_{i_1}) + O((1 - \gamma_{i_1})^2)$, c_i constant. Finally $p'_1 > p'_2 > ... > p'_k$ is ensured if $\epsilon_i < p^n_i - p^n_{i-1}$, $i = 1, .., k-1$, which is guaranteed for n sufficiently large. Hence $|p^n_i - p'_i| < c(1 - \gamma^n_{i_1})$, $i = 1, .., k$, and this completes the proof of claim ii).

We will now prove the statement of the proposition. Denote the feasible set in equations (7), (8)-(11) for the problem with risk-aversion parameter γ_n as $\mathcal{T}^n$, and for the risk-neutral case as $\overline{\mathcal{T}}$. Following Proposition 1, $p^n \in \mathcal{T}^n$ and $(\overline{p}, \overline{r}) \in \overline{\mathcal{T}}$. Consider $n > m$ so that $\gamma^m < \gamma^n$. We will show that $\mathcal{T}^n \subset \mathcal{T}^m$. Consider any $p \in \mathcal{T}^n$. Then it satisfies constraint (11). However, $\gamma^m < \gamma^n$ implies that $1 \geq \left(\frac{v_i - p_i}{v_i - p_{i+1}}\right)^{\gamma^m_i} \geq \left(\frac{v_i - p_i}{v_i - p_{i+1}}\right)^{\gamma^n_i} \geq \frac{r_{i+1}}{r_i}$ implying that $p \in \mathcal{T}^m$. This implies $\{p^n\}_{n \geq m} \subset \mathcal{T}^m$. $\mathcal{T}^m$ is a compact set, implying that there exists a subsequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}, m \leq n_1 < n_2 < ...$ s.t. $p^{n_k} \to \widetilde{p}$, $\widetilde{p} \in \mathcal{T}^m$.

Suppose $\widetilde{p} \in \overline{\mathcal{S}}$ and $\widetilde{p} \neq \overline{p}$. Then $\forall \epsilon > 0$, $\exists M$ s.t. $\forall k \geq M$, $|p^{n_k}_i - \widetilde{p}_i| < \epsilon$. This implies that the optimal revenue for problem $\mathcal{P}^{n_k}$, $R^{n_k}(p^{n_k}) \leq \overline{R}(\widetilde{p}) + c\epsilon$, where $R^n(\cdot)$ denotes the revenue with risk-aversion parameter γ^n and $\overline{R}(\cdot)$ denotes the revenue for the risk-neutral case (under feasible price vectors). Now since the optimal solution to the risk-neutral problem P is unique, $\overline{R}(\overline{p}) > \overline{R}(\widetilde{p})$ and for ϵ small enough, $\overline{R}(\overline{p}) > R^{n_k}(p^{n_k})$. However, this would violate the optimality of p^{n_k}, since for n_k large enough, $\overline{p}$ is feasible for $\mathcal{S}^{n_k}$. Then, from assumption 1, it follows that $\widetilde{p} \in \overline{\mathcal{S}} \implies \widetilde{p} = \overline{p}$.

Now suppose $\widetilde{p} \notin \overline{\mathcal{S}}$. Then let $\delta = \min_{p' \in \overline{\mathcal{S}}} ||\widetilde{p} - p'||_2 > 0$. Note the for k large enough, $||p^{n_k} - \widetilde{p}||_2 < \epsilon_1$, and $\exists p' \in \overline{\mathcal{S}}$ s.t. $||p^{n_k} - p'||_2 < \epsilon_2$. Now using the triangle inequality, $||p^{n_k} - \widetilde{p}||_2 \geq ||\widetilde{p} - p'||_2 - ||p^{n_k} - p'||_2 \geq \delta - \epsilon_2$. Hence choosing ϵ_1, ϵ_2 to be such that $\epsilon_1 + \epsilon_2 < \delta$, we will achieve a contradiction. Hence $\widetilde{p} \in \overline{\mathcal{S}}$ and consequently $\widetilde{p} = \overline{p}$. In a similar fashion, it is also possible to show that $\{p^n\}$ has a unique limit point. Moreover, this directly implies that $R^{n_k}(p^{n_k}) \to \overline{R}(\overline{p})$ (Actually we know that $R^{n_k}(p^{n_k}) \geq \overline{R}(\overline{p})$ since $\overline{p} \in \mathcal{S}^{n_k}$, for k large, or alternatively, by using Corollary 3 directly). $\square$

Proof of Proposition 5 It is easy to verify that the following assignment,

$$\rho_i = 0, i = 1, 2, ..., N,$$

$$\delta_{i_{l-1}+1} = \delta_{i_{l-1}+2} = ... = \delta_{i_l}, l = 1, 2, ..., k,$$
$$\delta_j = \delta_{j+1} = ... = \delta_N = 0,$$
$$\delta_{i_l} = \frac{\delta_{i_l+1}\overline{r}_{l+1} + w_{i_l,i_l+1}x_{i_l} - w_{i_l,i_l}x_{i_l} - \epsilon_{i_l}}{\overline{r}_l}, l = 1, 2, ..., k,$$

is feasible for the LP (30)-(36). Constraints (31)-(32) and (34)-(36) are tight, while $\delta_{i_k} = -\frac{\epsilon_{i_k}}{\overline{r}_k}$ implies that constraint (33) is also satisfied.

We will next show that (30)-(36) has a finite optimal solution by establishing that its dual is itself feasible. The dual to (30)-(36) can be written as follows.

$$\min \quad \Sigma_{i=1}^{i_k}\mu_i[(w_{i,i+1} - w_{i,i})x_i - \epsilon_i] + \Sigma_{i=j}^{N-1}\mu_i(-\epsilon_i) \tag{41}$$
$$s.t. \quad \mu_i \geq 0,\ i = 1, 2, ..., N-1,\ \eta_i \geq 0,\ i < i_k,\ \theta_i \geq 0,\ i \leq i_k, \tag{42}$$
$$\lambda \geq 0,\ \alpha_1, ..., \alpha_{i_1} \geq 0,\ \beta_{i_k} \geq 0,\ \phi_j, ..., \phi_N \geq 0,\ \nu_j, ...\nu_N \geq 0, \tag{43}$$
$$\pi_1\lambda - u_{1,1}\mu_1 - \eta_1 1_{1\notin I} + \alpha_1 = \pi_1 p_1, \tag{44}$$
$$\pi_i\lambda - u_{i,i}\mu_i + u_{i-1,i}\mu_{i-1} - \eta_i 1_{i\notin I} + \eta_{i-1}1_{i-1\notin I} + \alpha_i 1_{i\leq i_1} = \pi_i p_i, \quad i = 2, 3, ..., i_k - 1, \tag{45}$$
$$\pi_{i_k}\lambda - u_{i_k,i_k}\mu_{i_k} + u_{i_k-1,i_k}\mu_{i_k-1} + \eta_{i_k-1}1_{i_k-1\notin I} + \alpha_{i_k}1_{i_k\leq i_1} = \pi_{i_k}p_{i_k}, \tag{46}$$
$$\pi_j\lambda - u_{j,j}\mu_j + u_{j-1,j}\mu_{j-1} - \phi_j = 0, \tag{47}$$
$$\pi_i\lambda - u_{i,i}\mu_j + u_{i-1,i}\mu_{i-1} - \phi_i = 0, \quad i = j+1, ..., N-1, \tag{48}$$
$$\pi_N\lambda + u_{N-1,N}\mu_{N-1} - \phi_N = 0, \tag{49}$$
$$\mu_1 r_1 - \theta_1 1_{1\notin I} = \pi_1 r_1, \tag{50}$$
$$-\mu_{i-1}r_i + \mu_i r_i + \theta_{i-1}1_{i-1\notin I} - \theta_i 1_{i\notin I} = \pi_i r_i, \quad i = 2, ..., i_k - 1, \tag{51}$$
$$-\mu_{i_k-1}r_{i_k} + \mu_{i_k}r_{i_k} + \theta_{i_k-1}1_{i_k-1\notin I} - \theta_{i_k}1_{i_k\notin I} + \beta = \pi_{i_k}r_{i_k}, \tag{52}$$
$$\theta_{j-1}1_{j-1\notin I, j-1\leq i_k} + \nu_j = 0, \tag{53}$$
$$\nu_i = 0, \quad i = j+1, ..., N. \tag{54}$$

Here λ is the dual variable associated with the capacity constraint (31), μ_i is the dual variable associated with constraint (36), $i = 1, ..., N-1$, η_i is the dual variable associated with constraint $\rho_i \geq \rho_{i+1}$, $i = 1, ..., N-1$, θ_i is the dual variable associated with constraint $\delta_i \geq \delta_{i+1}$, $i = 1, ..., i_k$. α_i is the dual variable associated with the constraint $\rho_i \leq 0$, $i = 1, ..., i_1$, β is the dual variable associated with the constraint $\delta_{i_k} \leq 0$, ϕ_i is the dual variable associated with the constraint $\rho_i \geq 0$, $i = j, j+1, ..., N$, and ν_i is the dual variable associated with the constraint $\delta_i \leq 0$, $i = j, j+1, ..., N$.

Following Proposition 3, without loss of generality, we restrict attention to the case where the optimal number of products to offer to risk-neutral customers $\overline{k} \leq 2$. By brute-force once can verify that the following assignment of variables is feasible for (41)-(54), and therefore that

(30)-(36) has a finite feasible solution (details are omitted):

$$\mu_i = \Sigma_{l=1}^{i}\pi_l, \quad i = 1, ..., N, \tag{55}$$

$$\theta_i = 0, \quad i = 1, ..., N-1, \tag{56}$$

$$\nu_i = 0, \quad i = j, j+1, ..., N, \ \beta = 0, \tag{57}$$

$$\eta_1 = 0, \quad \alpha_1 = \pi_1(v_1 - \lambda), \tag{58}$$

$$\alpha_i - \eta_i = -\lambda\pi_i + \mu_i v_i - \mu_{i-1}v_{i-1} - \eta_{i-1}, \quad \alpha_i \geq 0, \quad \eta_i \geq 0, \quad \alpha_i\eta_i = 0, \quad 1 < i < i_1, \tag{59}$$

$$\eta_{i_1} = 0, \quad \alpha_{i_1} = -\lambda\pi_{i_1} + \mu_{i_1}v_{i_1} - \mu_{i_1-1}v_{i_1-1} - \eta_{i_1-1}, \tag{60}$$

$$\eta_i = \lambda(\Sigma_{l=i_1+1}^{i}\pi_l) + \mu_{i_i}v_{i_1} - \mu_i v_i, \quad i_1 < i < i_2, \tag{61}$$

$$\phi_i = \lambda\pi_i + \mu_{i-1}(v_{i-1} - v_i), \quad i = j, j+1, ..., N, \tag{62}$$

$$\lambda = \begin{cases} 0, & \text{if } \overline{k} = 1, \\ \frac{(\Sigma_{l=1}^{i_2}\pi_l)v_{i_2} - (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1}}{\Sigma_{l=i_1+1}^{i_2}\pi_l}, & \text{if } \overline{k} = 2. \end{cases} \tag{63}$$

We next compute the optimal solution to (30)-(36). We consider the one-product and the two-product case separately.

i) $\overline{k} = 1$: In this case, $w_{i,i+1} - w_{i,i} = 0, \ i = 1, ..., N-1$. Also, $\epsilon_i = 0, \ i = 1, ..., N-1$ would ensure that the IC conditions are not violated. Together these imply that zero is feasible revenue for the dual problem (41)-(54). However, this is also attained by setting $\delta_i = \rho_i = 0, \ i = 1, ..., N$ in the primal problem. Hence by strong duality, this must be the optimal solution.

ii) $\overline{k} = 2$: In this case, $w_{i,i+1} - w_{i,i} = 0, \ i = 1, ..., i_1 - 1, i_1 + 1, ..., N-1$. Also, we can set $\epsilon_i = 0, \ i = 1, ..., i_1 - 1, i_1 + 1, ..., N-1$. This implies that a feasible dual revenue is given by $(\Sigma_{l=1}^{i_1}\pi_l)[(w_{i_1,i_1+1} - w_{i_1,i_1})x_{i_1} - \epsilon_{i_1}]$. However, this is also attained by setting $\delta_1 = \delta_2 = ... = \delta_{i_1} = (w_{i_1,i_1+1} - w_{i_1,i_1})x_{i_1} - \epsilon_{i_1}, \ \delta_i = 0, \ i > i_1, \ \rho_i = 0, \ i = 1, ..., N$, in the primal solution. Hence, by strong duality, this must be the optimal solution. □

Appendix B

Existence of optimal solution and constraint qualification: For completeness, we justify the use of the Lagrangian approach. We observe that the objective (6) is continuous, and the feasible sets (7), (8)-(12) and (7), (8)-(11) are compact. Hence, following Weierstrass theorem, a maxima exists. Suppose the optimal solution to formulation (6), (8)-(12) involves offering $k \leq N$ distinct products, where the products partition the customer types as in Lemma 4. Define $a_l = \gamma_{i_l}(v_{i_l} - p_l)^{\gamma_{i_l}-1}r_l$, $b_l = \gamma_{i_l}(v_{i_l} - p_{l+1})^{\gamma_{i_l}-1}r_{l+1}$, $c_l = (v_{i_l} - p_l)^{\gamma_{i_l}}$, $d_l = (v_{i_l} - p_{l+1})^{\gamma_{i_l}}$, $l = 1, ..., k-1$. Also define $e_l = \Sigma_{l=i_{l-1}+1}^{i_l}\pi_l, \ l = 1, ..., k$. Then the matrix obtained by differentiating the $k-1$ downstream IC conditions and the capacity constraint is given as follows.

$$\begin{pmatrix} -a_1 & b_1 & 0 & 0 & \dots & 0 & -d_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -a_2 & b_2 & 0 & \dots & 0 & c_2 & -d_2 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots & & & \ddots & & & \vdots \\ & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -a_{k-1} & 0 & 0 & 0 & \dots & c_{k-1} & -d_{k-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & -e_2 & -e_3 & & \dots & & -e_k \end{pmatrix}$$

The first $k-1$ rows in this matrix correspond to the $k-1$ downstream constraints, and the k^{th} corresponds to the capacity constraint. The first $k-1$ columns correspond to the variables $p_1, ..., p_{k-1}$, while the next $k-1$ columns correspond to variables $r_2, ..., r_k$. Note that this matrix has rank k (since there are k linearly independent rows (the matrix is in row-echelon form, and can easily be converted into reduced row-echelon form with k non-zero rows)), and hence the constraint qualification condition is met for the Lagrangian in equation (38). In case we also add the upstream constraints to the Lagrangian, as in equation (37), we observe that either the upstream or the downstream constraint can be tight, but not both, and since the derivative of any upstream constraint would lead to the same non-zero entries as the derivative of the corresponding downstream constraint, the constraint qualification condition would be met.

Proof of algorithm 1: For risk-neutral customers, the required conditions are obtained from the proof of Proposition 3. For risk-averse customers, the Lagrangian can be written as follows:

$$L = (\Sigma_{l=1}^{i_1}\pi_l)p_1 + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}r_2 + \mu\left((v_{i_1}-p_1)^{\gamma_{i_1}} - (v_{i_1}-v_{i_2})^{\gamma_{i_1}}r_2\right) + \lambda\left(C - \Sigma_{l=1}^{i_1}\pi_l - (\Sigma_{l=i_1+1}^{i_2}\pi_l)r_2\right).$$

Differentiating with respect to p_1 and r_2 respectively, yields the following:

$$(\Sigma_{l=1}^{i_1}\pi_l) - \mu\gamma_{i_1}(v_{i_1}-p_1)^{\gamma_{i_1}-1} = 0,$$
$$(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2} - \lambda(\Sigma_{l=i_1+1}^{i_2}\pi_l) - \mu(v_{i_1}-v_{i_2})^{\gamma_{i_1}} = 0.$$

There are two cases to consider: $\lambda = 0$ and $\lambda > 0$. $\lambda = 0$ implies that

$$p_1 = v_{i_1} - \left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})^{\gamma_{i_1}}}\right)^{\frac{1}{1-\gamma_{i_1}}} \quad r_2 = \left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.$$

The conditions $p_1 > v_{i_2}$ and $C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)r_2$ require that

$$(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) > \gamma_{i_1}(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}, \quad C \geq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.$$

under which the optimal revenue is given by

$$R = (\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})^{\gamma_{i_1}}}\right)^{\frac{1}{1-\gamma_{i_1}}} + v_{i_2}(\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{\pi_1(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.$$

and which exceeds the one product revenue at price v_{i_1} and v_{i_2} under the sufficient condition $(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2}) > (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}$. $\lambda > 0$ implies that

$$p_1 = v_{i_1} - (v_{i_1}-v_{i_2})\left(\frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}\right)^{\frac{1}{\gamma_{i_1}}}, \quad r_2 = \frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}.$$

The conditions $p_1 > v_{i_2}$ and $C \leq (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)r_2$ require that

$$C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l), \quad C < (\Sigma_{l=1}^{i_1}\pi_l) + (\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}}{(\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})}\right)^{\frac{\gamma_{i_1}}{1-\gamma_{i_1}}}.$$

under which the optimal revenue is given by

$$(\Sigma_{l=1}^{i_1}\pi_l)v_{i_1} - (\Sigma_{l=1}^{i_1}\pi_l)(v_{i_1}-v_{i_2})\left(\frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}\right)^{\frac{1}{\gamma_{i_1}}} + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\left(\frac{C-(\Sigma_{l=1}^{i_1}\pi_l)}{(\Sigma_{l=i_1+1}^{i_2}\pi_l)}\right),$$

and which exceeds the one product revenue at price v_{i_1} and v_{i_2}.

That the constraint qualification condition is met follows from 6. To see that the proposed solution is indeed a maxima, write the Lagrangian as

$$L = (\Sigma_{l=1}^{i_1}\pi_l)p_1 + (\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\left(\frac{v_{i_1}-p_1}{v_{i_1}-v_{i_2}}\right)^{\gamma_{i_1}} + \lambda\left(C - \Sigma_{l=1}^{i_1}\pi_l - (\Sigma_{l=i_1+1}^{i_2}\pi_l)\left(\frac{v_{i_1}-p_1}{v_{i_1}-v_{i_2}}\right)^{\gamma_{i_1}}\right).$$

Differentiating with respect to p_1, we obtain that

$$(\Sigma_{l=1}^{i_1}\pi_l) - \frac{(\Sigma_{l=i_1+1}^{i_2}\pi_l)v_{i_2}\gamma_{i_1}(v_{i_1}-p_1)^{\gamma_{i_1}-1}}{(v_1-v_{i_2})^{\gamma_{i_1}-1}} + \frac{\lambda(\Sigma_{l=1}^{i_1}\pi_l)\gamma_{i_1}(v_{i_1}-p_1)^{\gamma_{i_1}-1}}{(v_1-v_{i_2})^{\gamma_{i_1}-1}} = 0.$$

If $\lambda = 0$, then it is easy to verify that p_1 is the same as obtained earlier and the second derivative with respect to p_1 is negative. If $\lambda > 0$, the tightness of the IC condition implies that we obtain the same solution as before. Solving for λ and substituting to calculate the second derivative with respect to p_1, we find it to be negative, implying that the method does yield revenue-maximizing solution.