Optimal Dynamic Auctions for Revenue Management

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We analyze a dynamic auction, in which a seller with C units to sell faces a sequence of buyers separated into T time periods. Each group of buyers has independent, private values for a single unit. Buyers compete directly against each other within a period, as in a traditional auction, and indirectly with buyers in other periods through the opportunity cost of capacity assessed by the seller. The number of buyers in each period, as well as the individual buyers' valuations, are random. The model is a variation of the traditional single-leg, multiperiod revenue management problem, in which consumers act strategically and bid for units of a fixed capacity over time.

For this setting, we prove that dynamic variants of the first-price and second-price auction mechanisms maximize the seller's expected revenue. We also show explicitly how to compute and implement these optimal auctions. The optimal auctions are then compared to a traditional revenue management mechanism—in which list prices are used in each period together with capacity controls—and to a simple auction heuristic that consists of allocating units to each period and running a sequence of standard, multiunit auctions with fixed reserve prices in each period. The traditional revenue management mechanism is proven to be optimal in the limiting cases when there is at most one buyer per period, when capacity is not constraining, and asymptotically when the number of buyers and the capacity increases. The optimal auction significantly outperforms both suboptimal mechanisms when there are a moderate number of periods, capacity is constrained, and the total volume of sales is not too large. The benefit also increases when variability in the dispersion in buyers' valuations or in the number of buyers per period increases.

(*Optimal Auction; Strategic Behavior; Revenue Management; Dynamic Programming; Mechanism Design*)

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1. Introduction

Revenue management traditionally involves setting list prices and controlling a fixed capacity to maximize revenues. (See McGill and van Ryzin 1999 for a review.) Among the most well-studied problems this area is the single-leg revenue management problem. (See, for example, Belobaba 1987, Brumelle and McGill 1993, Curry 1990, Lee and Hersh 1993, Robinson 1995, and Wollmer 1992.) In this problem, a firm is assumed to sell a fixed capacity over T time periods indexed by t. In each period t, there is an exogenously specified list price p_t and a random demand for capacity (e.g., airline seats, hotel rooms) at this price. The firm cannot vary its prices, but it can control the number of units it sells in each period—i.e., it can ration its capacity. Its objective is maximizing total expected revenues.[1] An

[1] The reason for maximizing revenues is that in industries where revenue management is typically practiced, the capacity cost is sunk and the variable cost is negligible.

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important result for this problem is the optimality of critical threshold policies in which the firm sets limits on the number of units it is willing to sell in each period (called *booking limits*) (Brumelle and McGill 1993, Curry 1990, Robinson 1995, Wollmer 1992). Such list-price, capacity-controlled (LPCC) mechanisms are widely used in the airline, hotel, and car rental industry (see Belobaba 1987, Geraghty and Johnson 1996, and Kimes 1989)—and the single-leg problem for finding optimal capacity controls is central to both the theory and practice of revenue management in these industries.

However, with the rise of Internet commerce, many firms have begun experimenting with alternative pricing mechanisms such as auctions, guaranteed purchase contracts (Priceline.com's patented selling mechanism), group purchasing, etc. (See van Ryzin 2000.) Deregulation is also driving changes in pricing. For example, recently U.S. regulators have strongly encouraged the natural gas transmission industry—another user of revenue management techniques—to implement capacity auctions for allocating pipeline capacity. (See Valkov and Secomandi 2000.)

These trends raise some important theoretical and practical questions. In particular, exactly which mechanisms maximize revenues in any given context? How can such optimal selling mechanisms be designed and implemented? And how much benefit (if any) can be obtained from a different selling mechanism?

We make some initial progress in addressing these questions. Specifically, we consider a variant of the single-leg revenue management problem in which buyers are separated into T periods and in each period they bid for a limited capacity. Buyers bid only in their given periods. Our model follows the assumptions of the classical independent, private-value auction model. (See Vickrey 1961, Milgrom and Weber 1982, the recent survey by Klemperer 1999, and earlier surveys by McAfee and McMillan 1987, Milgrom 1989, Rothkopf and Harstad 1994, Matthews 1995, and Wolfstetter 1996.) Specifically, we assume that each buyer i has a private valuation v_i for a single unit of capacity. These values are independent and identically distributed according to a continuous distribution that is common knowledge. (This distribution may vary from one period to the next.) Buyers in each period act strategically to maximize their utility (i.e., their value minus the price they pay). As a result, a buyer's bidding behavior depends on both the pricing and allocation mechanism selected by the seller and on other buyers' bidding strategies, and game theory is required to determine the equilibrium behavior of buyers.

In contrast to the traditional auction problem, in our model the seller receives bids from T groups of buyers who are separated over time. In particular, in each period t we assume that a new set of buyers arrives and bids for the remaining capacity. The seller must determine winners in period t before observing the bids (or even the number of buyers) in future periods. This dynamic feature parallels the traditional revenue management model, in which the seller must determine the capacity to sell in a given period before observing demand in future periods.

Such separation of buyers over time is typical in many industries that practice revenue management. A canonical example is the airline industry. Airlines have two major customer segments, leisure travelers and business travelers. Leisure travelers typically make travel plans months in advance of departure, because they frequently must coordinate their vacation travel with other arrangements, like reserving resort accommodations, taking time off work, finding child care, etc. (i.e., they face "contingent decisions"). In contrast, business travelers—a salesperson following a hot lead or a lawyer meeting a client for an emergency consultation—may not even know of their need to travel until a few days in advance of departure. As a result, if an airline were to conduct a single auction months in advance of departure, they would likely lose many business travelers; and if they conducted a single auction a week before departure, they would likely lose many leisure travelers. This creates an incentive for them to conduct auctions at multiple points in time. Moreover, most leisure travelers (because of their contingent decisions) would not find the later auction attractive, and most business travelers (because they do not know of their need to travel early on), would not find the early auction attractive. These buyers are effectively separated in time.

Indeed, Dana (1998) shows that such differences in timing over the need to secure travel (and also differences in the valuation of customers for travel given their need to do so) help explain why advance reservations exist in the airline industry. Moreover, he shows that advance reservations can achieve more socially efficient outcomes than conducting a single auction. The multiple-auction solution achieves a similar advance-reservation benefit.

This airline example corresponds closely to the model we analyze. Moreover, other industries face similar situations, in which buyers' needs are realized at different points of time (e.g., the need to buy a gift for a birthday)—or are based on other contingent events (e.g., a new order to a manufacturer triggering a need for new supplies)—that effectively separate buyers in time. In such situations, a seller attempting to use a single auction at a single point in time would find herself eliminating many potential buyers. By conducting multiple auctions over time, the seller can reach a larger pool of buyers.

While the sequential decision-making feature of our model matches the traditional revenue management model, the assumption that buyers act strategically appears to be in sharp contrast. However, it is not hard to see that if the firm offers a fixed list price in each period, it is a dominant strategy for buyers in period t to attempt to buy if the price p_t is less than their valuation v_i—regardless of the capacity controls imposed by the seller. The number of such buyers is random, being determined by the total number of buyers in the period and their random valuations. Thus, our model is consistent with the assumptions of the traditional single-leg model when a LPCC mechanism is used.

1.1. Overview of the Main Results

The main focus of this paper is in embedding classical results from auction theory that describe the bidder and seller behavior in each period into a dynamic framework that is typical of the revenue management problem. Specifically, for our dynamic auction model we use classical optimal auction design results (Myerson 1981, Maskin and Riley 2000) to show that appropriate dynamic versions of the first-price and second-price auction mechanisms are optimal for the seller. We also provide an efficient method to compute the optimal auction parameters.

These optimal dynamic auctions are somewhat more complex than in the traditional, single-period setting. In particular, in the first-price auction, the seller solicits bids, sorts them, infers the bidders' valuations v from the bids (which we show can indeed be done), and then computes what Myerson (1981) calls the bidder's "virtual value", defined by $J(v) = v - 1/\rho(v)$, where v is the bidder's valuation and $\rho(v)$ is the hazard rate of the distribution of buyers' valuations. The seller then accepts a bid if its virtual value exceeds the expected marginal cost of capacity that depends on the number of units the seller chooses to award and the number of remaining periods (see §3 for a precise definition of this mechanism). In the second-price auction, if the seller chooses to award k units, the winners pay the maximum of the $(k+1)$th highest bid and a threshold price that also depends on k and the number of remaining periods. Under this mechanism, we show it is a dominant strategy for buyers to bid their values v—again, bids are accepted if their virtual value exceeds the expected marginal cost of capacity.

What is unusual from a revenue management perspective about these optimal auctions—but quite natural to auction theorists—is that, in the first-price auction, the seller may reject a bid even though its revenue strictly exceeds the expected marginal cost of capacity. That is, an optimal first-price auction mechanism in some cases will refuse bids that would be strictly profitable if accepted. The reason is that, just as in setting a reserve price in a classical auction, rejecting profitable bids ex post is sometimes necessary to induce buyers to submit higher equilibrium bids ex ante.

We then compare these optimal mechanisms to two suboptimal mechanisms. The first is a variation of a traditional revenue management LPCC mechanism, in which the seller optimally sets a fixed, take-it-or-leave-it price in each period together with a limit on the number of units of capacity that can be sold at the list price. This *dynamic list price, capacity control* (denoted DLPCC) mechanism is equivalent to the traditional single-leg revenue management model, but with prices as well as capacity controls optimized in each period. We show theoretically that this DLPCC mechanism is optimal when there is at most one buyer in each period, when capacity is not constrained, and asymptotically as the number of buyers and capacity increases. The second suboptimal mechanism is a simple auction heuristic in which the seller splits the capacity across the T periods. She then runs a multiunit standard auction with a fixed reserve price in each period. Units not sold in one period are carried over to the next period.

Our numerical experiments show that the DLPCC revenues decrease relative to the optimal auction revenues when the buyers are concentrated into fewer periods. In contrast, the precommitting auction heuristic exploits the increasing bidding competition caused by aggregating buyers and performs relatively better under the same conditions; however, because it does not account for the opportunity cost of capacity, it performs poorly when buyers are disaggregated into a large number of periods. Scenarios that seem to hurt both suboptimal mechanisms are when buyers are concentrated into a moderate number of periods, the product is scarce, and the total capacity is moderate. The optimal auction also generates relatively larger benefits than either suboptimal mechanism when variability increases—either the variability in buyers' valuations or the variability in the number of buyers per period. Overall, our results point to relatively specific conditions under which an auction significantly outperforms the DLPCC mechanism.

1.2. Literature Review

Several papers have addressed the link between revenue management and auctions. Cooper and Menich (1998) proposed a Vickrey-Clarke-Groves (Vickrey 1961, Clarke 1971, Groves 1973) mechanism to auction airline tickets on a network of flights. However, this work does not capture the dynamic decision-making feature of our problem. Eso (2001) analyzes an iterative sealed-bid auction for excess seat capacity for an airline, where buyers get instant feedback, including minimum bid suggestions for declined bids. She models every iteration as a multiunit combinatorial auction (see de Vries and Vohra 2001 for a survey on this topic).

Motivated by Internet auctions, Segev et al. (2001) deal with a problem quite similar to ours in which an auctioneer tries to sell multiple units of a product using a multiperiod auction. However, the key difference is that customer bids in their model are exogenous, based on a Markov chain model of buyer behavior. Thus, their analysis does not endogenize the strategic behavior of buyers and they do not employ game theory to analyze equilibrium bidding strategies, as we do in our work. Another difference is that Segev et al. (2001) assume the seller precommits to the number of units to award in each period. We do not impose this restriction, and indeed show that precommitting is suboptimal; the seller is better off observing the bids first and then deciding how many units to award based on the realized bid values she receives. Indeed, the optimality of not precommitting the number of units to award is observed in other auction contexts. For example, Lengwiler (1999) studies a variable-supply auction motivated by the problem of a firm that issues new securities to finance its operations. In this setting, the firm has an incentive to adjust the total number of securities issued based on both the volume and value of the bids it receives.

Pinker et al. (2001) also analyze a similar problem. They study how to run a sequence of second-price auctions, determining the lot size and duration of each auction, and the number of auctions to run. In their model, however, the seller again precommits to the number of units to award and also does not use reserve prices. Again, we show that it is not optimal to precommit to the number of units to sell, and moreover the seller must use dynamic reserve prices that depend on the remaining capacity and time to achieve the optimal revenue.

A related problem is the one analyzed by Lavi and Nisan (2000). They study an *online* auction for a fixed inventory when the seller does not know the distribution of buyer valuations. In their model, each buyer arrives separately and bids for units. When receiving a bid, the auctioneer must decide how many units to allocate to the arriving bidder and at what price. The lack of any distributional assumption, the game-theoretic approach, and the competitive-ratio analysis with respect to an offline Vickrey auction are the most remarkable features of this paper. Other papers that also treat the problem of auction design under no distributional assumption are Segal (2002), Baliga and Vohra (2001), and Fiat et al. (2002).

1.3. Organization of this Paper

The remainder of this paper is organized as follows: In §2 we introduce notation and describe our model. We also review key results from the theory of optimal auctions. In §3, we formulate and analyze a dynamic program for the seller's decision problem and derive optimal first-price and second-price auction mechanisms. Some extensions of the basic model are discussed in §4. In §5, we describe the DLPCC mechanism and the precommitting auction heuristic, and present some theoretical and numerical comparisons of these mechanisms to the optimal auction mechanism. Finally, our conclusions are given in §6.

2. Model Formulation

2.1. Notation

All vectors are assumed to be in $\mathbf{R}^n_+$. v_j denotes the jth component of vector v, and $v_{-j} \equiv (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$ is the vector of components other than j. Subscripts between parentheses stand for reverse-order statistics; that is, for any vector v, $v_{(1)} \geq v_{(2)} \geq \cdots \geq v_{(n)}$.

$\mathbf{I}\{\cdot\}$ denotes the indicator function, and $\mathcal{A} \setminus \mathcal{B}$ the set difference between $\mathcal{A}$ and $\mathcal{B}$. *LHS* and *RHS* are shorthand for *left-hand side* and *right-hand side*, respectively. A function is said to be *increasing (decreasing)* when it is nondecreasing (nonincreasing).

2.2. Description of the Model

A seller has an initial capacity of C units of a good that she wants to sell over a finite time horizon T. She does this by conducting a sequence of auctions, indexed by $t = T, T-1, \ldots, 1$. The time index is assumed to run backwards, and smaller values of t represent later points in time.

Buyers are separated in time. In period t, N_t risk-neutral potential buyers arrive. N_t is a nonnegative, discrete-valued random variable, distributed according to a known probability mass function $g(\cdot)$ with support $\{0, \ldots, M\}$ for some $M > 0$, and strictly positive first moment. We assume buyers do not select their time of arrival, and they participate in only one auction period. That is, we do not model the fact that buyers may adjust their time of arrival in response to the seller's behavior or that they may choose to participate in multiple auction periods (e.g., rebid in later periods if they are unsuccessful). As discussed above, this assumption is reasonable in situations where buyers find it inconvenient (or impossible) to participate in auctions at different points in time. This issue is discussed further in §4.3.

Each buyer wishes to purchase at most one unit and has a reservation value v_i^t, $1 \leq i \leq N_t$, which represents the maximum amount buyer i is willing to pay for a unit. When the context is clear, we will drop the time index and write v_i. Reservation values are private information, independent and identically distributed samples from a distribution $F(\cdot)$, which is strictly increasing with a continuous density function $f(\cdot)$ on the support $[\underline{v}, \bar{v}]$, with $F(\underline{v}) = 0$ and $F(\bar{v}) = 1$. Without loss of generality, assume $\underline{v} = 0$ throughout. We will use v both for the random vector of valuations (from the seller's perspective), and for its realization, where the meaning should be clear from the context. To simplify notation and subsequent analysis, we restrict attention to distribution functions g and F that do not depend on the time t. However, the extension to time-dependent distributions is straightforward as long as the realizations of (N_t, v^t) remain independent over time.[2]

The distributions F and g are assumed to be common knowledge to the seller and all potential buyers (although this assumption can be relaxed for the second-price mechanism as discussed in §3.3.1). In addition, each buyer i knows his own (private) valuation v_i. Without loss of generality, we assume that the unit salvage value for the seller at time $t = 0$ is $v_0 = 0$.

The seller's problem is to design an auction mechanism that maximizes her expected revenue. The auctioneer will specify a set of rules (the mechanism)

[2] Note this independence over time would not be valid if the firm could "learn" about the valuations of customers from one period to the next by observing their bidding behavior. However, if buyers in different periods represent different segments (e.g., the leisure and business travelers of our canonical airline example), then it is not likely that bids received in one period provide much information about the valuations of customers in subsequent periods. However, one can well imagine cases where buyers' values in different periods may be highly correlated, so the seller could learn over time and adjust her strategy accordingly. This would be an worthwhile extension to consider.

according to which the auction will be conducted. These rules may depend on the time t and the remaining capacity at the beginning of each period, denoted by x. Each buyer, based on his private valuation, his knowledge of the distribution functions F, g, and the set of rules established by the auctioneer, chooses his bid (or strategy) to maximize his expected utility. Then, the auctioneer observes the set of submitted bids and applies the rules specified earlier to decide the number of units to award in period t and the payments to be made by the various bidders.

2.3. Results from the Theory of Optimal Auctions

Our analysis relies on some basic results on optimal auctions due to Myerson (1981), Riley and Samuelson (1981), and Maskin and Riley (2000). We briefly review these results here.

Consider an auction in which we are selling one or more homogeneous objects to n buyers. As above, each buyer i wants at most one of the objects, which he values at v_i; the values v_i are private information, but it is common knowledge that v_is are i.i.d. with distribution F. Define the *allocation* to buyer i, denoted $q_i(v_i, v_{-i})$, to be 1 if buyer i is awarded a unit, and 0 otherwise. We let $q(v) = (q_1(v_1, v_{-1}), \dots, q_n(v_n, v_{-n}))$.[3] We restrict attention to only symmetric equilibria, in which all buyers adopt the same bidding strategy; this is reasonable since buyers are "similar" through their common valuation distribution F.

Maskin and Riley (2000), extending Myerson's (1981) results, show the rather remarkable fact that the seller's expected revenue can be expressed *only* in terms of the allocations $q_i(v_i, v_{-i})$—independent of the buyers' payments. Specifically, the expected revenue for the seller is given by

$$E_{v_i, v_{-i}}\left[\sum_{i=1}^{n} J(v_i) q_i(v_i, v_{-i})\right], \qquad (1)$$

where $J(v) = v - 1/\rho(v)$, and $\rho(v) = f(v)/[1-F(v)]$ is the hazard rate function associated with the distribution F. The requirements for this result are rather general: It holds provided: (i) F is continuous and strictly increasing, (ii) the allocations $q_i(\cdot, v_{-i})$ are increasing in v_i, and (iii) buyers with value $v_i = 0$ have zero expected surplus in equilibrium (see Maskin and Riley 2000, Proposition 2). From this fact, it follows that all mechanisms that result in the same allocations $q(v)$ for each realization of v yield the same expected revenue. This is the so-called *Revenue Equivalence Theorem*.

More importantly, expression (1) can be used to design an optimal mechanism by simply choosing the allocation rule $q^*(v)$ that maximizes $\sum_{i=1}^{n} J(v_i)$ $q_i(v_i, v_{-i})$ subject to any constraints one might have on the allocation (e.g., we have k units to sell so we may require that the allocation q satisfies $\sum_i q_i \le k$). In such cases, it is convenient to make the following regularity assumption on the distribution function F, which we will assume holds in our case as well:

Assumption 1. *$J(v)$ is strictly increasing in v.*

Assumption 1 is not overly restrictive, and is satisfied by many standard distributions.[4]

If we define

$$v^* = \max\{v\colon J(v) = 0\} \qquad (2)$$

(and by convention, $v^* = \infty$ if $J(v) < 0$, $\forall v$), then from (1) it follows that it is never optimal to allocate a unit to a buyer with valuation $v_i < v^*$. Indeed, this simple observation is the basis for using (2) as an optimal reserve price in a standard, k-unit auction.

3. Optimal Dynamic Allocations and Mechanisms

The trick in applying the above approach to auction design is to find an implementable mechanism that produces the optimal allocation $q^*(v)$. This requires a

[3] The fact that the allocation can be expressed as a function of v follows from the Revelation Principle of Myerson (1981); namely, that for every bidding mechanism which induces a symmetric equilibrium, there exists a corresponding *direct revelation mechanism*, in which a buyer's optimal strategy is to bid their value. Thus, the allocation variables can be associated with the corresponding direct revelation mechanism.

[4] In particular, the assumption holds when the hazard rate is increasing—or, more generally, satisfies $\rho'(v) > -\rho(v)^2$, for all $v \in [0, \bar{v}]$. Distributions that have increasing hazard rate include the uniform, normal, logistic, exponential, and extreme value (double exponential) distributions, etc. (See Bagnoli and Bergstrom 1989.)

separate analysis. Thus, the design process proceeds in two steps: (1) Find an optimal allocation $q^*(v)$; then, (2) find an implementable mechanism that produces $q^*(v)$ for each realization v. We apply this approach to our dynamic auction problem next.

3.1. Optimal Allocations

Our first objective is finding an optimal allocation $q(v)$ in each period. Define the value function $V_t(x)$ as the maximum expected revenue obtainable from periods $t, t-1, \ldots, 1$ given that there are x units remaining at time t. Using (1) for the expected revenue in each period, the Bellman equation for $V_t(x)$ in terms of the allocation variables $q(v)$ can be written as

$$V_t(x) = E_{N_t,v}\left[\max_q \left\{\sum_{i=1}^{N_t} J(v_i)\, q_i + V_{t-1}(x-k)\colon q_i \in \{0,1\},\ k = \sum_{i=1}^{N_t} q_i,\ k \le x\right\}\right], \quad (3)$$

where k is the total number of units awarded in period t. The boundary conditions are

$$V_0(x) = 0,\ x = 1, \ldots, C, \quad (4)$$

and recall that C denotes the initial capacity. An allocation $q(\cdot)$ that achieves the maximum above given x, t, and v will be an optimal dynamic allocation policy. (See Bertsekas 1995.)

The solution of the dynamic program (3)–(4) is greatly simplified by the fact that the marginal value of capacity, defined by $\Delta V_t(x) \equiv V_t(x) - V_t(x-1)$, is decreasing in x. Indeed, we have (the proof can be found in an online appendix that accompanies this paper at ⟨mansci.pubs.informs.org/ecompanion.html⟩):

Lemma 1. *$\Delta V_t(x)$ is decreasing in x for any fixed t, and is increasing in t for any fixed x.*

These are quite natural economic properties. At any point in time, the marginal benefit of each additional unit declines because the future number of buyers is limited; therefore, the chance of selling the marginal unit—and/or the expected revenue if we sell it—decreases. Similarly, for any given remaining quantity x, the marginal benefit of an additional unit increases with t, because the more time remaining, the greater the number of future buyers; therefore, the chance of selling the marginal unit—and/or the expected revenue if we sell it—goes up.

Lemma 1 can be used to characterize the optimal allocation policy in each period. First, note that from (3) and the monotonicity of $J(\cdot)$, it is clear that if the seller allocates k total units, these units should be awarded to those buyers with the k highest values v_i. Therefore, define

$$R(k) = \begin{cases} 0 & \text{if } k = 0, \\ \sum_{i=1}^{\min\{k,N_t\}} J(v_{(i)}) & \text{if } k > 0, \end{cases} \quad (5)$$

and note that

$$R(k) = \max_q \left\{\sum_{i=1}^{N_t} J(v_i) q_i\colon q_i \in \{0,1\},\ \sum_i q_i = \min\{k, N_t\}\right\}. \quad (6)$$

(In fact, the integrality constraints above can be relaxed to $0 \le q_i \le 1$.)

Formulation (3) can therefore be rewritten in terms of $R(k)$ as follows:

$$V_t(x) = E_{N_t,v}\left[\max_{0\le k\le x} \{R(k) + V_{t-1}(x-k)\}\right], \quad (7)$$

subject to (4). Let $k_t^*(x)$ denote the optimal solution above; this is the optimal number of bids to accept at time t given remaining capacity x. Clearly, $k_t^*(x) \le N_t$. Letting $\Delta R(i) \equiv R(i) - R(i-1)$, we can rewrite $V_t(x)$ as

$$V_t(x) = E_{N_t,v}\left[\max_{0\le k\le x}\left\{\sum_{i=1}^{k} [\Delta R(i) - \Delta V_{t-1}(x-i+1)]\right\}\right] + V_{t-1}(x), \quad (8)$$

where the sum is defined to be 0 if $k = 0$.

Let n_t denote any realization of the random variable N_t, and v be a realization of buyers' types. The following theorem characterizes the optimal allocation:

Theorem 1. *For any realization (n_t, v), the optimal number of units to allocate in state (x, t) is given by*

$$k_t^*(x) = \begin{cases} \max\{1 \le k \le \min\{x, n_t\}\colon \\ \quad \Delta R(k) > \Delta V_{t-1}(x-k+1)\} \\ \quad \text{if } R(1) > \Delta V_{t-1}(x), \\ 0 \quad \text{otherwise}. \end{cases}$$

Moreover, it is optimal to award these $k_t^*(x)$ units to those buyers with the $k_t^*(x)$ highest values v_i.

PROOF. Given that k units are awarded, that the units go to the buyers with the k highest values v_i is clear from (6). To find $k_t^*(x)$, from Assumption 1 it follows that $\Delta R(k)$ is decreasing in k and by Lemma 1, $\Delta V_{t-1}(x)$ is decreasing in x. Therefore, $\Delta R(k) - \Delta V_{t-1}(x-k+1)$ in (8) is also decreasing in k; hence, $k_t^*(x)$ is the largest k for which this difference is positive. □

Theorem 1 shows how the seller should allocate units—provided she can infer the values v_i of the buyers. In particular, note that $\Delta R(i) = J(v_{(i)})$ for $i = 1, \ldots, \min\{x, N_t\}$, so the optimal number of bids to accept is simply based on sorting the values v_i and progressively awarding units to the highest-value buyers until the value $J(v_{(i)})$ drops below the marginal opportunity cost $\Delta V_{t-1}(x-i+1)$. (Note that this means a buyer's allocation is increasing in his valuation, as required by the Revenue Equivalence Theorem.) Thus, given the buyer's values v_i and the value function $V_{t-1}(x)$, the optimal allocation rule is quite simple.

Indeed, the seller's problem in each period is equivalent to that of a monopolist who seeks to price discriminate amongst N_t privately informed buyers and has a variable supply governed by a convex cost function. In our case, however, the cost function is endogenously determined by the opportunity cost of capacity. For example, Segal (2002) shows the same optimality of comparing virtual values with marginal costs in his variable-supply auction problem. Lengwiler (1999) studies a subgame perfect equilibria of a variation of this problem.[5]

Finally, note that $V_t(x)$ is increasing in x (this can be shown easily) and $\Delta V_{t-1}(x) \geq 0$. Thus, if $k_t^*(x) \geq 1$, then $\Delta R(k_t^*(x)) = J(v_{(k_t^*(x))}) > 0$, and so $v_{(k_t^*(x))} > v^*$, where v^* is the optimal reserve price in a single-period auction defined by (2). Therefore, the seller never awards a unit in any period to buyers with values below v^*.

3.2. Computing the Optimal Solution

We next briefly consider the computation of the optimal auction solution. To compute the function $V_t(x)$, one can use simulation. The following lemma is useful in this regard (it is also useful theoretically; see the proof of Lemma 1 in the online appendix):

LEMMA 2. $k_t^*(x) \leq k_t^*(x+1) \leq k_t^*(x)+1$, *for all* $x \geq 0$.

That is, if we have one more unit available to sell, we will allocate at most one more unit to the buyers (see the online appendix for the proof).

The simulation method proceeds as follows: For each period t, generate m samples of the number of buyers and their valuations, where m is a parameter of the simulation. For each instance and for each pair of values (x, t), calculate $k_t^*(x)$, the optimal number of units to allocate using Theorem 1. Taking advantage of Lemma 2, we know that $k_t^*(x) = k_t^*(x-1)$ or $k_t^*(x) = k_t^*(x-1)+1$. So, we need $O(m\,C)$ operations to compute all values $V_t(x)$ of x for a single t and, therefore, $O(m\,C\,T)$ operations for the whole algorithm. For example, a problem with $C = 100$, $T = 10$, and $m = 100$ samples per cell requires order 10^5 operations to compute the entire value function. Our experience is that this simulation method is both fast and accurate.

3.3. Mechanism Design

The next step in our analysis is to find auction mechanisms that achieve the optimal allocation policy derived above. The main result of this section is to demonstrate that appropriately modified versions of two standard procedures—the first- and the second-price auctions—achieve this objective.

3.3.1. Second-Price Auction. It is well known that in a traditional k-unit second-price auction, where all k winners pay the $(k+1)$th highest bid (i.e., the first losing bid), it is a dominant strategy for the buyers to bid their own values (Vickrey 1961). However, if one uses a straightforward application of the second-price mechanism in our setting, it is no longer optimal for buyers to bid their valuation.

The following informal reasoning shows why. Suppose it is optimal to bid truthfully under the

[5] In Lengwiler's model, the seller produces quantities of the commodity at a unit cost β, which is private information, but is drawn from a common knowledge distribution. The proposed multiunit auction is a two-stage game, with buyers submitting their bids in the first stage of the game, and the seller picking a price from a grid—or even canceling the auction—in the second stage. He proves the existence of a subgame perfect equilibrium for both first- and second-price auctions in this setting.

second-price mechanism and let

$$\hat{v}_i \equiv J^{-1}(\Delta V_{t-1}(x-i+1)), \quad i \geq 1. \tag{9}$$

The thresholds $\hat{v}_i$ are directly computable from the solution of (3) described in the previous section, which uses common-knowledge information, and are in principle known to all buyers and the seller. Following Theorem 1, the seller will accept bid $v_{(i)}$ as long as $v_{(i)} > \hat{v}_i$. Now suppose the seller decides to award k units: That means $v_i > \hat{v}_i$, $i = 1, \dots, k$; and $v_{(j)} \leq \hat{v}_j$, $j = k+1, \dots, n_t$. However, if the first loser, $v_{(k+1)}$, had bid $\hat{v}_{k+1} + \varepsilon$ instead (which in fact verifies $v_{(k+1)} < \hat{v}_{k+1} + \varepsilon$), the seller would include him among the winners, award $k+1$ units, and the buyer would only pay $v_{(k+2)}$ and make a positive profit. Hence, buyers have some incentive to bid above their own values (i.e., a pure second-price mechanism fails to elicit truthful bids).

However, the following modification to the second-price mechanism avoids this pitfall: In each period t, the seller first computes the thresholds $\hat{v}_i$ using the current capacity x. Given the vector of submitted bids, b, the seller will award k units, where

$$k = \max\{i \geq 1\colon b_{(i)} > \hat{v}_i\}, \tag{10}$$

and $k = 0$ if $b_{(1)} \leq \hat{v}_1$; and all winners will pay

$$b^{(2\text{nd})}_{(k+1)} = \max\left\{b_{(k+1)}, \hat{v}_k\right\}, \tag{11}$$

where $b_{(k+1)}$ is the $(k+1)$th highest bid and $\hat{v}_k$ is the threshold to award the kth unit. Ties between bids are broken by randomization. For simplicity we will refer to (10)–(11) as the *modified second-price* mechanism.

We then have the following result (the proof is in the appendix):

THEOREM 2. *For the modified second-price auction with allocation and payments given by* (10)–(11), *a buyer's dominant strategy is to bid their own values. Moreover, under this dominant strategy equilibrium, the mechanism is optimal.*

This same result was again shown by Segal (2002) for his variable-supply auction problem. Indeed, the above modified second-price mechanism can be viewed as a type of Vickrey-Clarke-Groves (Vickrey 1961, Clarke 1971, Groves 1973) mechanism, in which the price paid by a winning bidder equals the minimum bid that guarantees him a unit. In our case, this minimum bid is given by (11), because to be one of the k winners, a buyer's bid must exceed both the $(k+1)$th highest bid and the seller's threshold, $\hat{v}_k$, for awarding the kth unit.

Note that because bidding one's own value is a dominant strategy for buyers, then under this modified second-price mechanism we can relax the assumption that the capacity C and the distributions F and g are known to the buyers. This leads to a more realistic assumption about what information the buyers have and the their level of sophistication in bidding.

Also, observe that when $T = 1$ our problem reduces to a single-period, multiunit auction, which was reviewed in §2.3. Because the salvage value of unsold units at $t = 0$ is zero, the thresholds $\hat{v}_i$ are all equal to v^* defined in (2) in this case, and the mechanism indeed reduces to the standard second-price auction with optimal reserve price v^*.

3.3.2. First-Price Auction. In a first-price auction, units are awarded to the buyers with the highest bids and winners pay their bids. This type of mechanism may be more natural in certain applications.

To establish that the first-price auction achieves the optimal expected revenue, one needs to show that units are again awarded according to the optimal allocation rule of Theorem 1. To do this, it suffices to show that there exists a symmetric equilibrium bidding strategy $B(\cdot)$ that is strictly increasing in the buyer's value. In this case, the seller can use this bid function to invert a bid and infer the bidder's value, which she can then use to correctly compute the number of units to award.

The setup of the first-price auction is the following. The buyers are informed of the time t, the remaining capacity x, and the following allocation rule used by the seller: Given a vector of bids b, the seller will award k units, where $k = \max\{i \geq 1\colon B^{-1}(b_{(i)}) > \hat{v}_i\}$, and $k = 0$ if $B^{-1}(b_{(1)}) \leq \hat{v}_1$. Here, $B(\cdot)$ is the equilibrium bid function, which we show below can be computed by the seller. The units are awarded to the highest bidders, and winners pay their bids.

Our main result is the following (the proof is in the Appendix):

THEOREM 3. *Under the first-price auction, there exists a symmetric, equilibrium, strictly increasing, bidding strategy $B(v_i)$. The strategy B depends on the current values of t and x as given by*

$$\hat{B}(v_i) = v_i - \frac{\int_{\hat{v}_1}^{v_i} P(v)\,dv}{P(v_i)} \quad \text{and} \quad B(v_i) \equiv \lim_{\varepsilon\to 0^+} \hat{B}(v_i-\varepsilon), \tag{12}$$

where $P(v)$ is the probability that a bidder with value v is among the winners,

$$P(v) = \begin{cases} 0 \quad \text{if } k^*(v) = 0, \\ \sum_{n=1}^{M} \left\{ \sum_{k=0}^{k^*(v)-1} \binom{n-1}{k} [1-F(v)]^k \right. \\ \qquad \left. \times [F(v)]^{n-1-k} \right\} g(n) \quad \text{if } k^*(v) \geq 1, \end{cases} \tag{13}$$

and $k^*(v) = \max\{0 \le i \le \min\{x, N_t\}: v > \hat{v}_i\}$, and by convention, $\hat{v}_0 < 0$. Moreover, under this symmetric equilibrium, the first-price auction is optimal.

Since (12) shows that $B(v) < v$, under the first-price mechanism buyers shade their values to make a positive surplus. This is, of course, expected because winners are required to pay what they bid. The fact that $B(\cdot)$ is strictly increasing means the mechanism is implementable, because once the seller observes bids $b_1, \ldots, b_{n_t}$, she can calculate the values $v_1, \ldots, v_{n_t}$ through the well-defined inverse bidding function, $v_i = B^{-1}(b_i)$.[6]

Note that this first-price mechanism is not greedy, in the sense that it does not maximize the sum of the current bid revenues plus the expected revenue-to-go. This is true because the seller compares the virtual values $J(v_i)$s to the marginal value $\Delta V_t(\cdot)$ rather than comparing the bids themselves. As a result, the seller may (a) accept bids below the marginal value when $J(v_{(k)}) > \Delta V_{t-1}(x-k+1) \ge B(v_{(k)})$, and (b) reject bids that are above the marginal value when $B(v_{(k)}) > \Delta V_{t-1}(x-k+1) \ge J(v_{(k)})$. Our numerical experiments show that both cases may occur.

This feature of the optimal policy, admittedly, raises some concerns from a practical point of view. Indeed, it might be hard to explain to managers exactly why profitable options ought to be turned down—even though the logic for doing so is quite familiar to auction theorists.

On a theoretical level, it shows that—just as with the use of reserve prices in a standard auction—our optimal auction mechanism is not subgame perfect.[7] Equivalently, we assume the seller can credibly commit to the rules she uses in each period.

This assumption, however, is no more restrictive than a traditional auction, and in a certain sense is necessary to analyze any auction mechanism. Indeed, as McAfee and McMillan (1987) point out, in most auctions the seller, upon observing the bids, can infer the buyers' values. Having done so, she then has an incentive to renege on her previously announced rules and simply offer the high-value buyers a price marginally less than their value. At that point, it is in the buyers' interests to accept her offers. However, knowing the seller would behave in this way, the buyers would not bid as hypothesized, which invalidates the equilibrium analysis.

Moreover, in our problem the usual repeated-interaction arguments in support of the commitment assumption are quite plausible. (See McAfee and McMillan (1987).) In particular, if our seller is conducting her dynamic auctions repeatedly over time, then if she reneges on her commitment in one auction, she will be unable to maintain credibility in future auctions. The resulting lost future stream of revenues creates an incentive for her to hold to her commitments. For example, in the airline case, an airline using auctions as a pricing mechanism would need to conduct dynamic auctions for a large number of departures extending far into the foreseeable future. Given the potential threat to their future revenues, it would be foolhardy for them to myopically renege on the auction rules in a particular period.

[6] Keeping the assumption of symmetric equilibria, alternatively the seller could announce that she will use the allocation rule described in (10) and that winners will pay $B(b_i)$, where the b_is are the bids. In this case, buyers will truthfully reveal their values. See the online appendix for a proof.

[7] Note, however, that since we formulate the problem as a dynamic program, the seller's decision in period t still reflects the fact that she will behave optimally given their remaining capacity in the next period.

Finally, note that the implementation of the allocation rule for the first-price auction is much more complex than in the second-price auction, because the seller must compute the equilibrium bidding strategy and invert it. Moreover, the bidding problem faced by buyers is considerably more complex as well, because they have to solve the same equilibrium as well as the seller's dynamic program. As a result, it is questionable whether the parties would really behave as predicted under this mechanism. However, the general characteristics of the mechanism—that bidders will shade their values and that the seller should not necessarily accept all bids that exceed the marginal cost of capacity to induce buyers to bid higher in equilibrium—are likely to be more robust.

4. Some Extensions

We next briefly discuss some extensions to our basic model.

4.1. Varying the Number of Periods

How do the seller's revenues vary with respect to the number of periods? Suppose we fix the total number of buyers across all periods. Intuitively, the more the seller aggregates the buyers, the higher her revenues, because with more buyers per period the seller is able to make allocation decisions with better information about the actual values v_i of each buyer. For example, in the limiting case where all buyers are concentrated in just one period (i.e., a standard multiunit auction), the seller can allocate based on perfect information about all values v_i. Dividing the buyers across more periods forces the seller to allocate without knowing the future values v_i, hence the need for the dynamic program to assess the opportunity cost of capacity.

The next proposition formalizes the fact that the seller is worse off having more periods provided the total demand remains the same (see the online appendix for a proof):

Proposition 1. *Let N (a random variable) denote the total number of buyers over time, and consider splitting time into T and $2T$ periods, where $T \geq 1$ so that $N = \sum_{t=1}^{T} N_t = \sum_{t=1}^{2T} N_t$. This is done by arbitrarily allocating buyers in period t of the original, T-period problem to periods $2t-1$ and $2t$ in the expanded, $2T$-period problem. Let $V^T(x)$ and $V^{2T}(x)$ denote, respectively, the resulting optimal expected revenues in each case. Then, $V^T(x) \geq V^{2T}(x)$, $\forall x$.*

This proposition shows that the seller is better off with fewer periods. All else being equal, therefore, the seller would prefer only a single auction period.

Of course, this result is true only if no buyers "drop out" when the number of periods is reduced. But as we've argued above, a primary motivation for conducting a sequence of auctions in the first place is precisely to increase the number of buyers who can participate. So assuming that the number of buyers will remain fixed independent of the number of periods the seller uses is not realistic. In general, then, the seller faces a trade-off between the convenience of offering many auction opportunities over time—leading to a potential increase in the number of buyers who participate—and the inefficiency of having to allocate units to buyers with increased uncertainty about both the number and valuations of future buyers. How this trade-off can be resolved optimally is likely to be context specific, but in principle could be evaluated using our model combined with a model of how the population of buyers is affected by changes in the number of auction periods.

4.2. Mixed LPCC and Auction Periods

Another extension is obtained by noting that one can easily combine auction periods with periods in which a traditional list-price, capacity-control mechanism is used, in which prices are fixed ex ante and capacity is optimally rationed. For example, one might use list prices and capacity controls in the initial periods, but run an optimal auction in the last period; or run an optimal auction early on and then follow this by a sequence of periods in which list price and capacity controls are used. Or, these mechanisms could be used in arbitrary, mixed order from one period to the next. Such mixed use of both types of mechanisms is quite common, for example, in airlines, where auctions are typically used in supplementary "channels" (e.g., online sales) along with traditional pricing mechanisms.

The fact that these mechanisms can be combined follows simply by noting that the optimal allocations from Theorem 1 in period t (and hence the optimal mechanism of Theorems 2 and 3) remain optimal as long as the marginal value of capacity in the next period, $\Delta V_{t-1}(x)$ is decreasing in x. Moreover, Lemma 1 shows that if $\Delta V_{t-1}(x)$ is decreasing in x and we run an optimal auction in period t, then $\Delta V_t(x)$, is decreasing in x as well. Similarly, the traditional single-leg model has this same property (see Brumelle and McGill 1993, Robinson 1995, Wollmer 1992); critical-threshold capacity controls are optimal in period t as long as the marginal value of capacity is decreasing, and under optimal capacity controls, the value function in period t has decreasing marginal values $\Delta V_t(x)$. As a result, one can easily combine our results with the traditional single-leg model results to analyze combinations of auction and list-price, capacity-control mechanisms.

4.3. Buyers Who Strategize Over the Time of Purchase

A key assumption of our model is that buyers are constrained to participate in only one auction period. If we assume that buyers can choose the period to bid in, then their strategic behavior becomes more complex. How does this affect the outcome?

At first glance, this strategic-timing problem appears more difficult. However, it turns out that if one assumes that buyers have complete flexibility over timing (i.e., are able to participate in all T auctions) and their values v_i are independent of the period in which they buy (e.g., there is no discounting of their utility), then one can show that the strategic-timing problem is not much more complex than a classical single-period auction.

To see this, let N denote the total number of buyers ($N = \sum_{t=1}^{T} N_t$), and v_i denote the value of buyer i, $i = 1, \ldots, N$, $v = (v_1, \ldots, v_N)$ with common distribution $F(\cdot)$. (Here, we do require the distribution of values to be the same for all periods.) Let $q_i(v_i, v_{-i})$ denote the allocation to buyer i under some dynamic auction scheme. Then by the Revenue Equivalence Theorem, the seller's revenue is

$$E\left[\sum_{i=1}^{N} J(v_i) q_i(v_i, v_{-i})\right] \tag{14}$$

provided the usual assumptions are satisfied; namely, that $q_i(v_i, v_{-i})$ is increasing in v_i and buyers with value zero have zero expected surplus. The allocations must also satisfy the capacity constraint $\sum_{i=1}^{N} q_i(v_i, v_{-i}) \leq C$. Note that this claim holds for our original dynamic auction problem as well. However, because in our case buyers are separated over time, the seller cannot make her allocation decisions $q_i(v_i, v_{-i})$ based on perfect knowledge of the vector of values v for all N buyers. Rather, she needs to make a *sequence* of allocation decisions *under uncertainty* that maximize (14) given only probabilistic information about future demand; this is precisely what the dynamic program (3) accomplishes. Hence, her allocations (in hindsight) do not necessarily maximize $\sum_{i=1}^{N} J(v_i) q_i(v_i, v_{-i})$ for each realization of v.

However, when buyers act strategically and choose the time of purchase, the problem in a certain sense becomes easier. This is because the seller can exploit the buyers' flexibility in timing to construct mechanisms that allow her to obtain perfect information about v and therefore achieve the same optimal allocation as when all the N buyers participate in a single-period, optimal auction.

One such mechanism is simply to post an arbitrarily high reserve price (e.g., a reserve price of $\bar{v}$) in all but the last period and then run a standard, optimal C-unit auction with reserve price v^* in the last period. All buyers with values greater than v^* will then choose to bid in the last period, and thus this mechanism trivially achieves the single-period, optimal auction revenue.

Many other mechanisms achieve this same result. Indeed, this strategic-timing problem is equivalent to one analyzed by Bulow and Klemperer (1994). (See also the related price-skimming problem of Bensanko and Winston 1990.) Bulow and Klemperer define a mechanism in which the seller uses dynamic list prices, which are lowered and raised over time in such a way that the C highest-value buyers (with values in excess of v^*) are awarded units. Thus, by revenue equivalence, (14) is maximized. Their argument is quite general and only requires that high-value buyers have an incentive to "bid" first under the seller's mechanism. By bidding first, high-value buyers are the first ones awarded units, which produces the optimal allocation.

For example, one can show that the same optimal allocation is achieved if the seller runs a sequence of second-price auctions with a sequence of reserve prices that are higher than v^* in all but the last period, where the reserve price is exactly v^*. Under this mechanism, buyers have a higher probability of winning by bidding early, because they can always bid again later if unsuccessful. Since bidding only affects a buyer's probability of winning (and not his payment) in a second-price auction, high-value buyers prefer strategies that increase their probability of winning, and bid early. Because they bid first, the highest-value buyers are awarded units first, which again produces the optimal allocation.

5. Comparisons to Suboptimal Pricing Mechanisms

We next compare the optimal dynamic auction to two suboptimal mechanisms. The comparisons provide insights into the benefit provided by an optimal auction.

The first suboptimal mechanism we consider is the traditional revenue management LPCC mechanism, but with both prices and capacity controls set optimally in each period. This is a refinement of the traditional single-leg model, in which prices are exogenous and only capacity controls are optimized. We call this the dynamic LPCC—or DLPCC—mechanism. Buyers who are interested in acquiring one unit at that list price submit "acceptances" (i.e., an offer to buy). If the number of acceptances exceeds the capacity limit set by the seller, the units are randomly rationed to the buyers. As mentioned above, it is easy to see that a dominant strategy for a buyer is to submit an "acceptance" if and only if their value v_i exceeds the seller's list price.

The second suboptimal mechanism uses a simple heuristic to design a sequence of auctions in each period. Specifically, the seller first evenly allocates the C units over the T periods (which is reasonable when F and g are the same in each period). Then, she runs a standard, multiunit auction in each period, with a fixed reserve price v^* given by (2). Units that are left over in one period are carried on to the next period.

We first make some simple theoretical observations about the optimality of the DLPCC mechanism. Then, we describe some numerical experiments comparing the two suboptimal mechanisms to the optimal dynamic auction.

5.1. Theoretical Analysis of DLPCC

Let $V_t^{\text{DLPCC}}(x)$ denote the seller's expected revenue starting with x units of capacity in period t under DLPCC. Let s denote the reserve price set by the seller in period t, and

$$N_t(s) \equiv \left| \{v_i \colon v_i > s,\ \forall\ 1 \le i \le N_t\} \right|, \quad \forall\ 1 \le t \le T,$$

be the random variable representing the number of buyers with valuations exceeding the reserve price s. The seller solves the recursion,

$$V_t^{\text{DLPCC}}(x) = \max_{\substack{s \ge 0 \\ 0 \le k \le x}} E_{N_t, v}\Big[s \min\{N_t(s), k\} + V_{t-1}^{\text{DLPCC}}(x - \min\{N_t(s), k\}) \Big],$$

with boundary condition $V_0^{\text{DLPCC}}(x) = 0$ for all x.

In general, the DP associated with the DLPCC mechanism is solved using simulation, although in same cases (e.g., uniform valuations) it can be solved exactly by using standard optimization techniques. Our next proposition summarizes cases when the DLPCC mechanism is optimal (see the online appendix for proofs).

Proposition 2. *The DLPCC mechanism maximizes the seller's expected revenue when either*:

(a) $N_t \le 1, \forall 1 \le t \le T$;

(b) $C \ge \sum_t N_t$ *(with probability 1);*

(c) *Asymptotically as both the number of buyers and the number of units to sell becomes large* ($N_t, C \uparrow \infty$ *and* $N_t/C \to \alpha_t$ *w.p.1, where* α_t *are deterministic limits).*

Part (a) shows that having isolated buyers submit offers to buy has no inherent advantage over dynamic list pricing.[8] One needs to aggregate multiple buyers in each period to generate a strict benefit from

[8] The one-bidder-per-period problem is equivalent to dynamic pricing problems of the type analyzed by das Varma and Vettas (2001), Gallego and van Ryzin (1994), Kincaid and Darling (1963), Stadje (1990), and Zhao and Zheng (2000).

an optimal dynamic auction. In particular, this result suggests that individualized bidding schemes, like the one used by Priceline.com,[9] have no inherent advantage over dynamic list pricing.

Part (b) states that an optimal mechanism has no advantage over list pricing if capacity is always unconstrained. Thus, some scarcity of the product or service is necessary to realize a gain over list pricing.

Finally, part (c) shows that DLPCC is asymptotically optimal as both the number of buyers and number of units sold increase in the same proportion. That is, when the overall volume of sales becomes large (holding the ratio fixed), list pricing is near optimal.[10] This suggests in particular that sellers of high-volume, mass-market products are not likely to gain much benefit from the use of a dynamic auction mechanism.

To see why (c) holds intuitively, consider a k-unit, single-period, second-price auction. Roughly speaking, as the number of buyers N increases, the bid realization becomes an accurate and dense sample of the value distribution F. As a result, the order statistics, which determine the winning buyers and price they pay, will correspond closely to fractiles of this distribution. By the law of large numbers, the seller's revenue therefore approaches a deterministic function of C/N and of the minimum reserve price v^*.[11] In the dynamic setting, this gives rise to a limiting deterministic control problem, in which the auctioneer chooses the allocation of units per auction period to optimize overall revenue. This resulting limiting optimal solution is a DLPCC rule.

Finally, we note that (b) and (c) above hold even for a LPCC mechanism, in which the seller precommits to the prices used in each period. This is because the optimal unconstrained prices in (b) and the asymptotically optimal prices in (c) are deterministic and not state dependent. Therefore, the seller can precommit to an optimal sequence of prices.

5.2. Numerical Comparisons

We next consider several numerical examples comparing the optimal auction to the two suboptimal heuristics. In every experiment, the optimal mechanism and the heuristics were computed and then simulated over 20 runs to estimate the mean revenue produced by each method.[12] Buyers' valuations were assumed to be uniformly distributed. This assumption simplified the numerical analysis and did not appear to affect the nature of the results.[13] In the first two subsections the valuation distribution was uniform $U(0, 1)$.

5.2.1. The Effect of the Concentration of Buyers. In the first experiment, we studied how the revenue changes as we varied the *concentration of buyers* by clustering the same number of buyers into varying numbers of periods. Specifically, we assume the seller has $C = 16$ units, and the total number of buyers in all periods is constant at 64. Buyers are evenly distributed across the periods. We then varied the number of periods from 1 to 64, producing different numbers of buyers per period. Thus, the examples run from 64 buyers in one period (high concentration of buyers) to one buyer in each of 64 consecutive periods (low concentration of buyers).

The results are summarized in Table 1.[14]

A few observations are apparent from Table 1. The first is that the optimal auction revenue increases as the concentration of buyers increases. This is consis-

[9] In Priceline.com's system, a buyer submits an isolated bid for a product (most popularly, airline tickets) and is notified of acceptance or rejection in 10 to 15 minutes. Priceline.com therefore does not have enough time to aggregate bids, and so each one is evaluated in isolation.

[10] A result related to a combination of Proposition 2, parts (a) and (c), is the *adaptive monopoly pricing* procedure under no distributional assumption for valuations studied by Baliga and Vohra (2001).

[11] A similar asymptotic analysis is provided in Segal (2002, §3.2).

[12] The optimal mechanism and heuristics were computed using the simulation method outlined in §3.2, using 1,000 samples in (N_t, v) space in each period to estimate the value function. To generate the descending uniform order statistics, we used the method described in Tadikamalla and Balakrishnan (1998).

[13] We tested the methods using other distributions, such as the truncated normal and exponential distributions, and there was no significant difference in the qualitative results.

[14] The experiments for the precommitting auction heuristic are reported only up to the case of 1 unit per period, i.e., $T = 16$, because after this point the number of periods exceeds the number of units.

Table 1 Revenue for Different Concentration of Buyers

Buyers per period	Number of periods	Optimal auction rev.		DLPCC rev.		Precommitting auction rev.		
		Mean	95% CI	Mean	Gap	Mean	95% CI	Gap
1	64	11.410	(11.390, 11.430)	11.412	0.16%	—	—	—
2	32	11.434	(11.420, 11.448)	11.401	0.41%	—	—	—
4	16	11.480	(11.466, 11.495)	11.382	0.98%	10.162	(10.146, 10.177)	11.49%
8	8	11.534	(11.511, 11.556)	11.348	1.79%	10.822	(10.803, 10.841)	6.17%
16	4	11.621	(11.602, 11.639)	11.292	2.99%	11.311	(11.296, 11.327)	2.66%
32	2	11.722	(11.704, 11.740)	11.201	4.59%	11.639	(11.624, 11.654)	0.71%
64	1	11.796	(11.780, 11.812)	11.060	6.36%	11.796	(11.780, 11.812)	0.00%

tent with Proposition 1; the seller is better off having the same number of buyers in a smaller number of periods. The gap between the two extreme cases is 3.27%, which by revenue management standards is a significant difference.

Second, observe that as buyers are concentrated into a smaller number of periods, the DLPCC revenues decrease. This is true because the multiperiod DLPCC revenue cannot be worse than the single-period DLPCC revenue, because posting the same fixed price in each period is always a feasible policy in the multiperiod case. However, with multiple periods the seller can exploit the realized demand information to dynamically adjust her prices based on the remaining time and capacity, producing higher revenues. The opposite pattern holds for the optimality gap of the DLPCC policy; it increases as buyers are concentrated into fewer periods.

This behavior is intuitive. Indeed, as shown theoretically in Proposition 2, part (a), the DLPCC mechanism is optimal when there is only one buyer per period. But as more buyers are concentrated into fewer periods, the DLPCC mechanism will "leave money on the table" so to speak, because many of the winners would have been willing to pay more than the list price to get their units. Like all list-price mechanisms, it fails to achieve the competition benefits of an auction, and this disadvantage shows up most acutely when the concentration of buyers is high. Therefore, a lower concentration of buyers per period benefits the DLPCC mechanism—both in absolute terms and relative to the optimal auction.

In contrast, the precommitting auction heuristic revenues increase as more buyers are concentrated into fewer periods. This is because this heuristic—like all auction mechanisms—exploits bidding competition. However, it fails to take into account the "option value" of reserving capacity for the future (the opportunity cost of capacity). Rather, recall that the auction heuristic uses only a simple, static reserve price of v^* in every period. As buyers are separated into more periods, this opportunity cost disadvantage dominates the bidding competition advantage, and the auction heuristic suffers. Indeed, with only one period, the opportunity cost doesn't matter at all and the auction heuristic is optimal. Thus, a high concentration of bidders benefits the auction heuristic.

The optimal mechanism, of course, considers both factors. It exploits the within-period bidding competition of an auction mechanism, but does so using a set of reserve prices that are based on the opportunity cost of capacity. As seen from Table 1, both factors are necessary to achieve the optimal revenues in the intermediate cases.

5.2.2. The Effect of Demand and Capacity Values. The second experiment compares the suboptimality of the heuristics under various levels of capacity and demand. The number of periods was kept constant at $T = 5$. The number of buyers per period was varied, with $N_t = 10$, 30, 50, and 100; and for each of these four values we considered three choices of capacity, $C = 0.1\,T\,N_t$, $C = 0.3\,T\,N_t$, and $C = 0.5\,T\,N_t$. The optimality gaps are shown in Tables 2 and 3. (For completeness, Table 6 in the Appendix gives the optimal auction revenues for this example.)

Note that the optimality gaps for both suboptimal mechanisms decrease from left to right in Tables 2 and 3 (which corresponds to increasing the capac-

Table 2 Optimality Gaps for DLPCC

N_t	$C = 0.1\,T\,N_t$	$C = 0.3\,T\,N_t$	$C = 0.5\,T\,N_t$
10	2.37%	2.32%	0.58%
30	1.77%	1.77%	0.38%
50	1.43%	1.43%	0.21%
100	1.06%	1.13%	0.14%

Table 3 Optimality Gaps for Precommitting Auction Heuristic

N_t	$C = 0.1\,T\,N_t$	$C = 0.3\,T\,N_t$	$C = 0.5\,T\,N_t$
10	4.92%	3.99%	1.16%
30	1.81%	1.70%	0.34%
50	1.07%	0.96%	0.27%
100	0.55%	0.55%	0.14%

ity/demand ratio), and from top to bottom (which corresponds to increasing, proportionally, the number of buyers and capacity). In particular, the former observation is consistent with Proposition 2, part (b) for DLPCC. For the latter observation, going from the top to the bottom in each column corresponds to approaching the asymptotic regime of "large sales volume," which is again provably optimal in the limiting case for DLPCC by Proposition 2, part (c). The optimal DLPCC price approaches a fixed fractile of the distribution F, which is roughly the price paid under a standard second-price auction.

The precommitting heuristic follows the DLPCC behavior pattern in these same limiting cases (see Table 3). The intuition is similar in this case. First, that the auction heuristic improves as we approach the case where capacity is unconstrained (moving left to right in Table 3) is because the marginal opportunity cost of each unit becomes negligible. Hence, the optimal auction mechanism reduces to accepting all bids above the fixed reserve price v^* in (2) in each period, which is precisely what the precommitting auction heuristic does. In the large-sales-volume regime (going top to bottom in Table 3), the distribution of buyers' values in each period is becoming (relatively) more predictable; hence, the revenue in each period approaches a deterministic function of the number of units auctioned, so allocating a fixed number of units to each period becomes close to optimal.

In summary, both heuristics do well when capacity is not constrained or when the overall volume of sales is large. The optimal auction is most beneficial when capacity is tightly constrained and the volume of sales is moderate.

5.2.3. The Effect of Different Levels of Variability in Buyers' Valuations. Next, we looked at the effect of different levels of variability in buyers' valuations for units. Buyers' valuations were assumed to be uniformly distributed with a mean of 10. The variance of the valuations was changed by adjusting the range of the distribution (the "Min" and "Max" values in Table 4). There are $T = 5$ periods, $N_t = 10$ buyers per period, and $C = 10$ units to sell. Results are shown in Table 4.

The main observation is that the seller benefits from increased variability in buyers' valuations under all three mechanisms, as one might expect because the amount that high-value buyers are willing to pay increases. The performance improvement can be dramatic (on the order of 50%).

Also, the optimality gap of both the DLPCC and precommitting heuristic increases as the variance in

Table 4 Revenue vs. Variance in Buyers' Distributions

Range of types		Optimal auction rev.		DLPCC rev.		Precommitting auction rev.		
Min	Max	Mean	95% CI	Mean	Gap	Mean	95% CI	Gap
9.5	10.5	102.656	(102.640, 102.671)	102.185	0.46%	102.289	(102.276, 102.301)	0.36%
9	11	105.312	(105.294, 105.330)	104.456	0.81%	104.565	(104.541, 104.589)	0.71%
8	12	110.593	(110.547, 110.639)	109.127	1.33%	109.112	(109.061, 109.163)	1.34%
6	14	121.181	(121.103, 121.260)	118.788	1.98%	118.171	(118.077, 118.266)	2.48%
4	16	131.771	(131.650, 131.892)	128.728	2.31%	127.233	(127.096, 127.370)	3.44%
2	18	142.449	(142.286, 142.613)	138.858	2.52%	136.489	(136.274, 136.703)	4.18%
0	20	153.128	(152.947, 153.309)	149.126	2.61%	145.954	(145.739, 146.169)	4.69%

Table 5 Revenue vs. Variance in Number of Buyers

Range of buyers		Optimal auction rev.		DLPCC rev.		Precommitting auction rev.		
Min	Max	Mean	95% CI	Mean	Gap	Mean	95% CI	Gap
50	50	9.514	(9.511, 9.516)	9.413	1.06%	9.412	(9.410, 9.415)	1.06%
40	60	9.509	(9.506, 9.512)	9.406	1.08%	9.406	(9.402, 9.409)	1.08%
30	70	9.500	(9.498, 9.503)	9.388	1.18%	9.376	(9.373, 9.380)	1.31%
20	80	9.483	(9.480, 9.485)	9.354	1.35%	9.321	(9.315, 9.327)	1.71%
10	90	9.447	(9.444, 9.450)	9.300	1.56%	9.198	(9.191, 9.206)	2.64%

buyers' valuations increases as well. This suggests that an optimal mechanism may be more beneficial in cases where there is a lot of heterogeneity in buyer valuations for a product. Furthermore, note that DLPCC tends to outperform the precommitting auction. This is consistent with the intuition above: Higher volatility implies a higher option value for capacity, which is captured by DLPCC but not by the precommitting heuristic.

5.2.4. The Effect of Different Levels of Variability in the Number of Buyers. Our last experiment shows how the seller's revenues are affected by the level of variability in the number of buyers. There are $T = 5$ periods and $C = 10$ units to sell. The valuation distribution is uniform $U(0, 1)$. The number of buyers per period is uniformly distributed with a mean of 50. The variance of the distribution was then adjusted by changing the range of this discrete uniform distribution. Results are shown in Table 5.

The main observation here is that as the variance in the number of buyers per period increases, the seller's revenues decrease under all three studied mechanisms; however, this increased uncertainty seems to favor the use of an optimal mechanism. The relative deviation from optimality of DLPCC is smaller than the deviation of the precommitting heuristic as the variability increases. This is again explained intuitively by the fact that the option value of capacity becomes more significant as variability increases, and the DLPCC accounts for the option value while the precommitting heuristic does not.

6. Conclusions

The optimal dynamic auction provides an important theoretical benchmark for evaluating traditional revenue management mechanisms and simple auctioning heuristic schemes. Our results also provide some important insights into how to conduct an optimal dynamic auction; in particular, that (1) the number of units to award in each period should be varied depending on the quality of the bids received, (2) the opportunity cost of capacity should form the basis for evaluating how many units to award, and (3) that the seller should not be greedy in the first-price auction about accepting bids in each period, but rather should refuse bids that may be profitable to increase the equilibrium bids.

On a practical level, our optimal auction may provide a feasible alternative to traditional pricing mechanisms. It is reasonably simple, at least in the second-price case, and optimal policies can be computed relatively efficiently. However, we also showed that in some important cases, DLPCC is optimal or near optimal. These include the cases where either at most one buyer bids at a time, capacity is unconstrained, or where the sales volumes are large. However, in some specialized cases, our numerical results indicate that the optimal mechanism can indeed produce significant revenue improvements over both DLPCC and the precommitting auction heuristic. These include cases where: (1) the concentration of buyers per period is moderate (not all buyers in one period and not one buyer per period), (2) capacity is tightly constrained, (3) the total volume of sales is moderate, and (4) variability—either in buyers' valuations or in the number of buyers—is high.

As for additional work, we see several topics worthy of further study. One is to extend our analysis to the case where $J(\cdot)$ is not necessarily monotone; this was done in Myerson (1981) and Maskin and Riley (2000) in a single-period setting, and it is likely that similar techniques would work for our setting.

Another extension is to apply the same sort of analysis to a production-inventory setting where the seller controls how many units he or she is willing to produce, stock for the next period, or whether to offer through an auction in the current one. This is the topic of a recent paper by van Ryzin and Vulcano (2002).

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Appendix

Proof of Theorem 2. When $t = 1$, $\Delta V_{t-1}(x-k+1) = 0$ and $J^{-1}(0) = v^*$. In other words, in the last period a standard second-price auction takes place and therefore, bidding one's own value is a dominant strategy for buyers.

For $t \geq 2$, the seller will observe the vector of bids and compute the number of units to award through (10). Take a player i with value v_i. We will show that he has no incentive to bid $\tilde{b}_i \neq v_i$. There are two possible cases.

Case 1. Buyer i bids his value and is in the winning set. This implies that $v_i > \max\{b_{(k+1)}, \hat{v}_k\}$, and buyer i makes some positive surplus. Bidding $\tilde{b}_i > v_i$, does not change this outcome: There are still k bids above $\hat{v}_k$ and fewer than $k+1$ bids greater than $\hat{v}_{k+1}$, and buyer i is still in the winning set paying $\max\{b_{(k+1)}, \hat{v}_k\}$ for the same profit. If he bids $\tilde{b}_i < v_i$, three things can happen: (a) $\tilde{b}_i > \max\{b_{(k+1)}, \hat{v}_k\}$, he is still in the winning set, pays the same amount, and makes the same profit; (b) $\tilde{b}_i < \max\{b_{(k+1)}, \hat{v}_k\}$, he is no longer in the winning set, makes zero surplus, and is worse off; and (c) $\tilde{b}_i = \max\{b_{(k+1)}, \hat{v}_k\}$, in which case either $\tilde{b}_i = \hat{v}_k$, he is not a winner any more, and is worse off; or $\tilde{b}_i = b_{(k)} > \hat{v}_k$, a winner will be selected at random between the two equal bids, and he is worse off in expectation.

Case 2. Buyer i bids his value and is in the losing set. If he lowers his bid $\tilde{b}_i < v_i$, nothing changes. If he increases his bid $\tilde{b}_i > v_i$, four things can happen: (a) $\min\{b_{(k)}, \tilde{b}_i\} > \hat{v}_{k+1}$, in which case there are now $k+1$ winners that pay $\max\{b_{(k+2)}, \hat{v}_{k+1}\}$, and buyer i makes a surplus of $v_i - \max\{b_{(k+2)}, \hat{v}_{k+1}\} \leq 0$ (since i was in the losing set, and thus $v_i \leq \hat{v}_{k+1}$); (b) $\tilde{b}_i > b_{(k)}$, there are k winners, buyer i displaces the lowest winner, makes a payment larger than $\hat{v}_k$ for a negative surplus, and is worse off; (c) $\tilde{b}_i = b_{(k)}$, there are k winners, the last winner will be selected at random between the equal bids, and if he wins he makes a negative surplus and is worse off; and (d) $b_{(k)} > \tilde{b}_i$ and nothing changes.

Therefore, independently of the bidding strategies of all other players, buyer i has no incentive to change his bid from v_i in both Cases 1 and 2.

That the mechanism is optimal under this dominant strategy equilibrium follows immediately from Theorem 1. □

Proof of Theorem 3. The proof assumes that the seller can correctly invert the bids into values to implement the optimal allocation rule, and solve for the symmetric, equilibrium strategy B. Redefine the function $k^*(\cdot)$ from Theorem 1 in terms of a value v:

$$k^*(v) = \max\{0 \leq i \leq \min\{x, N_t\}: v > \hat{v}_i\},$$
$$\text{and by convention, } \hat{v}_0 < 0.$$

Let $P(v)$ be the probability that a buyer with valuation v is among the winners. For $k^*(v) \geq 1$ and $N_t = n$,

$$\begin{aligned} P(v \mid N_t = n) &= \sum_{k=1}^{\min\{n,x\}} \text{Prob}(v = v_{(k)} \text{ out of } n \text{ buyers}) \\ &\quad \times \mathbf{I}\{J(v) > \Delta V_{t-1}(x-k+1)\} \\ &= \sum_{k=1}^{k^*(v)} \text{Prob}(v = v_{(k)} \text{ out of } n \text{ buyers}) \\ &= \sum_{k=0}^{k^*(v)-1} \text{Prob}(v = v_{(k+1)} \text{ out of } n \text{ buyers}) \\ &= \sum_{k=0}^{k^*(v)-1} \binom{n-1}{k} [1-F(v)]^k [F(v)]^{n-1-k}. \end{aligned} \tag{A1}$$

Unconditioning on N_t gives (13).

Let $\hat{v} \equiv \hat{v}_1$ (i.e., $\hat{v}$ is the threshold for the seller to accept the first bid). It follows that $P(v) = 0$, $\forall\ 0 \leq v \leq \hat{v}$. It is also easy to check that $P(v)$ is strictly increasing in v, $\forall\ \hat{v} \leq v \leq \bar{v}$. This is done by rewriting (A1) as

$$P(v \mid N_t = n) = \sum_{k=1}^{\min\{n,x\}} \text{Prob}(\text{there are } k \text{ winners} \mid v)\, \text{Prob}(v \geq v_{(k)}),$$

and noting that the first term is nondecreasing, and the second is strictly increasing. The expectation over N_t preserves the monotonicity property. Finally, $P(\cdot)$ is not continuous in v. The probability of winning jumps up at points where the allocation rule $k^*(v)$ increases by one more unit; this adds one more term in the sum in (13). The set of discontinuity points is $\mathcal{D}_t(x) = \{v: J(v) = \Delta V_{t-1}(x-k+1), 1 \leq k \leq x\}$, which is finite and of Lebesgue measure zero, and $P(v)$ is continuously differentiable almost everywhere (a.e.), and left continuous at discontinuity points $\mathcal{D}_t(x)$.

The proof is completed using standard arguments from Maskin and Riley (2000, Proposition 3), or Riley and Samuelson (1981, Proposition 1). Assume that there exists a symmetric, equilibrium

Table 6 Optimal Auction Revenues Used in §5.2.2

	$C = 0.1\,T\,N_t$		$C = 0.3\,T\,N_t$		$C = 0.5\,T\,N_t$	
N_t	Mean	95% CI	Mean	95% CI	Mean	95% CI
10	4.307	(4.301, 4.312)	10.066	(10.052, 10.080)	12.272	(12.241, 12.303)
30	13.301	(13.296, 13.305)	31.031	(31.003, 31.058)	37.281	(37.233, 37.330)
50	22.295	(22.285, 22.306)	52.003	(51.977, 52.029)	62.247	(62.178, 62.316)
100	44.795	(44.782, 44.809)	104.548	(104.501, 104.595)	124.748	(124.656, 124.841)

bidding strategy $\hat{B}(\cdot)$. Suppose buyer i bids $b_i = \hat{B}(v)$ for some v not necessarily equal to v_i. (Note that from the Revelation Principle of Myerson (1981), it is sufficient to consider perturbing the value v alone and not the bid itself.) His expected surplus is

$$\Pi_i(v, v_i) = P(v)[v_i - \hat{B}(v)], \quad v > \hat{v}. \tag{A2}$$

Differentiating with respect to v, we get that

$$\frac{\partial \Pi_i}{\partial v}(v, v_i) = v_i\,\frac{dP(v)}{dv} - \frac{d}{dv}[P(v)\hat{B}(v)], \quad \forall v \in [\hat{v}, \bar{v}] \setminus \mathcal{D}_t(x).$$

Assume that the equilibrium strategy $\hat{B}(v_i)$ satisfies the first-order condition

$$\frac{\partial \Pi_i}{\partial v}(v_i, v_i) = 0.$$

This simplifies the previous expression to the differential equation

$$\frac{d}{dv}[P(v)\,\hat{B}(v)] = v\,\frac{dP(v)}{dv}, \quad \forall v \in [\hat{v}, \bar{v}] \setminus \mathcal{D}_t(x); \;\; \hat{B}(\hat{v}) = \hat{v}. \tag{A3}$$

The next step is to solve the differential equation for $\hat{B}$. Integrating between $\hat{v}$ and v_i, we obtain[15]

$$P(v_i)\hat{B}(v_i) = \int_{\hat{v}}^{v_i} v\,\frac{dP(v)}{dv}. \tag{A4}$$

Since $dP(v)/dv > 0\ \forall v \in (\hat{v}, \bar{v}) \setminus \mathcal{D}_t(x)$,

$$\int_{\hat{v}}^{v_i} v\,\frac{dP(v)}{dv} < \int_{\hat{v}}^{v_i} v_i\,\frac{dP(v)}{dv} = v_i P(v_i), \quad v_i > \hat{v},$$

which implies together with (A4) that $\hat{B}(v_i) < v_i$ for all $v_i > \hat{v}$. Evaluating (A3) at $v = v_i$, and using the fact that $\text{Prob}(v_i \in \mathcal{D}_t(x)) = 0$, we get that

$$P(v_i)\,\frac{d\hat{B}(v_i)}{dv} = \frac{dP(v_i)}{dv}[v_i - \hat{B}(v_i)].$$

Since $\hat{B}(v_i) < v_i$, it follows that $\hat{B}(v_i)$ is strictly increasing. Finally, integrating (A4) by parts and using the fact that $P(\hat{v}) = 0$, we get (12).

[15] Because $P(\cdot)$ is monotonic, it is of bounded variation and hence the Riemann-Stieltjes integral exists; see, for example, Apostol (1974, Theorems 6.5 and 7.27).

To complete the definition of the bid function at points of discontinuity, we define $B(v_i)$ as the left limit of $\hat{B}(v_i)$ according to (12).

The last step is to verify that $\hat{B}(\cdot)$ is indeed an equilibrium strategy. This is done by checking the condition that $\Pi_i(v_i, v_i) > \Pi_i(v, v_i)$, $\forall v > \hat{v}$, $v \neq v_i$. We first consider points of continuity. Substituting (12) into (A2), this is equivalent to

$$\int_{\hat{v}}^{v_i} P(z)\,dz > P(v)(v_i - v) + \int_{\hat{v}}^{v} P(z)\,dz.$$

If $v > v_i$, this implies that

$$P(v)(v_i - v) + \int_{v_i}^{v} P(z)\,dz < 0,$$

which holds true because $P(\cdot)$ is positive and strictly increasing. The analogous condition is true if $v < v_i$.

It remains to check the condition at points of discontinuity, $v_i \in \mathcal{D}_t(x)$. Since $P(\cdot)$ is left continuous and

$$\underline{b} \equiv \limsup_{\varepsilon \to 0^+} B(v_i - \varepsilon) \; < \; \bar{b} \equiv \liminf_{\varepsilon \to 0^+} B(v_i + \varepsilon), \tag{A5}$$

$P(v_i)(v_i - B(v_i)) = P(v_i)(v_i - \underline{b}) > P(v_i)(v_i - b)$, $\forall b \in (\underline{b}, \bar{b}]$. Note that if $B(v_i) < \underline{b}$ or $B(v_i) > \bar{b}$, then looking respectively at $v = v_i - \varepsilon$ or $v = v_i + \varepsilon$, for $\varepsilon > 0$, $B(v)$ would not be an equilibrium strategy. □

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