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AN INVENTORY MODEL WITH LIMITED PRODUCTION CAPACITY AND UNCERTAIN DEMANDS II. THE DISCOUNTED-COST CRITERION*

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This paper considers a single-item, periodic-review inventory model with uncertain demands. We assume a finite production capacity in each period. With stationary data, a convex one-period cost function and a continuous demand distribution, we show (under a few additional unrestrictive assumptions) that a modified basic-stock policy is optimal under the discounted cost criterion, both for finite and infinite planning horizons. In addition we characterize the optimal base-stock levels in several ways.

1. Introduction. This paper is a sequel to Federgruen and Zipkin [3]. We consider a single-item, periodic-review production (or inventory) model with linear production costs, a convex function representing expected one-period costs, and nonnegative i.i.d. demands. Stockouts are backordered. All data are stationary. Both finite- and infinite-horizon problems are treated. As in [3], the novel feature here is a finite production capacity in each period. Whereas [3] treats the discrete-demand, average-cost case, we assume here continuous demand and the expected-discounted-cost criterion.

Our goal, as in [3], is to prove that a stationary, modified base-stock policy, characterized by a single critical number, is optimal: when initial stock is below that number, produce enough to bring total stock up to that number, or as close to it as possible, given the limited capacity; otherwise, do not produce.

See [3] for a discussion of related prior work and related models. The proof in [3] is based on specific results of Federgruen, Schweitzer and Tijms [2] for denumerable-state, average-cost dynamic programs. Such results are not available (to date, at least) for the case treated here, so we adopt a different approach, based on the limiting behavior of the sequence of finite horizon problems. This is a relatively standard approach for uncapacitated problems (e.g., Iglehart [6]). This approach allows us to show also that the optimal base-stock level and optimal cost function are, respectively, the limits of their finite-horizon counterparts.

As in the average-cost case [3], if there is also a fixed cost for production, solutions to the infinite-horizon optimality equation continue to exist and any stationary policy satisfying this equation for certain such solutions is optimal. These results may be obtained from a simple adaptation of the analyses below. Whether our optimal policy has a simple (e.g. modified (s, S), cf. [3] and Heyman and Sobel [5]) structure remains an open question.

§2 sets forth the notation and a set of required assumptions under which the cost of *every* policy is finite. §3 examines finite-horizon problems, and demonstrates that a (nonstationary) base-stock policy is optimal in each period. The optimality proof for the infinite-horizon problem is presented in §4. §§3 and 4 also describe the dependence of the optimal critical number(s) on the production capacity in the finite- and

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infinite-horizon cases, respectively. §5, finally, provides a partial characterization of an optimal (modified) base-stock policy.

2. Notation and assumptions.

$\mathbb{R}$ = the real numbers.

D = generic random variable representing one-period demand.

D_t = demand in period t, $t = 0, 1, \ldots$.

$\bar{D} = \{D_t\}_{t=0}^{\infty}$.

$D^{(i)}$ = generic i-period demand, $i = 1, 2, \ldots$.

b = production capacity, or limit on order size, a finite positive number.

c = per-unit order (production) cost, a nonnegative number.

x_t = inventory at the beginning of period t.

y_t = inventory after ordering (production) but before demand in period t, $t = 0, 1, \ldots$.

$Y(x) = \{y : x \leqq y \leqq x + b\}$, the feasible values of y given $x \in \mathbb{R}$.

$G(y)$ = one-period expected cost function, exclusive of order costs.

α = discount rate, $0 < \alpha < 1$.

The D_t are assumed independent and distributed identically as D. D is nonnegative and possesses a density. We suppress the time index where possible, writing x for x_t for example. We now state additional assumptions:

ASSUMPTION 1. *(a)* $\lim_{|y|\to\infty} G(y) = \lim_{|y|\to\infty}[cy + G(y)] = \infty$; *(b) G is C^1 (continuously differentiable), nonnegative and convex.*

ASSUMPTION 2. $G(y) = 0(|y|^{\rho})$ *for some positive integer* ρ.

ASSUMPTION 3. *D has finite moments of all orders up to ρ.*

Assumption 1 is required (even in the uncapacitated case) to guarantee that a (modified) base-stock policy is optimal. Assumption 2 seems satisfied in all cases of practical interest. Given Assumption 2, Assumption 3 is required to guarantee finite expected costs even in finite-horizon problems and even in the uncapacitated case. Note, we do *not* require $b > E(D)$, a crucial condition in the undiscounted case [3]. (Only in Corollary 1 of §5 is $b \geqq E(D)$ assumed.)

Let Δ denote the set of pure, stationary, measurable policies; $y = \delta(x) \in Y(x)$ denotes the action prescribed by $\delta \in \Delta$ in state $x \in \mathbb{R}$. We shall also use $\delta \in \Delta$ to denote a one-period policy, and $\bar{\delta}$ to denote a sequence of one-period policies; the meaning will be clear from the context. Define

$$B_t(x \mid \bar{\delta}, \bar{D}) = \left\{ \sum_{i=0}^{t} \alpha^i [c(y_i - x_i) + G(y)] \mid x_0 = x, \bar{\delta}, \bar{D} \right\},$$

$$B(x \mid \bar{\delta}, \bar{D}) = \limsup_{t\to\infty} B_t(x \mid \bar{\delta}, \bar{D}), \qquad B(x \mid \bar{\delta}) = EB(x \mid \bar{\delta}, \bar{D}),$$

for $x \in \mathbb{R}$. $B(x \mid \bar{\delta})$ is the expected discounted cost of the policy sequence $\bar{\delta}$ starting in state x. Observe that $\{B_t(x \mid \bar{\delta}, \bar{D})\}$ is nondecreasing, so we may replace the lim sup by $B(x \mid \bar{\delta}, \bar{D}) = \lim_{t\to\infty} B_t(x \mid \bar{\delta}, \bar{D})$. Also by the monotone convergence theorem (Royden [7, p. 227]) we have $B(x \mid \bar{\delta}) = \lim_{t\to\infty} EB_t(x \mid \bar{\delta}, \bar{D})$. Specifically, let $\delta[\bar{y}]$ denote either the one-period base-stock policy with critical number $\bar{y}$, or the corresponding stationary, infinite-horizon policy, $\bar{y} \in \mathbb{R}$.

We now show that *every* feasible policy has finite expected cost under the assumptions above.

LEMMA 1. *For a fixed integer q, $0 \leqq q \leqq \rho$,* $E[D^{(i)}]^q = 0(i^q)$.

PROOF. $E[D^{(i)}]^q = E(\sum_{t=1}^{i} D_t)^q \leqslant i^q C_q$, where

$$C_q = \max\left\{ E\left(D_1^{l_1}\right)E\left(D_2^{l_2}\right) \dots E\left(D_q^{l_q}\right) : \sum_{j=1}^{q} l_j = q, l \geqslant 0, \text{ integer} \right\}. \quad ∎$$

THEOREM 1. *$B(x \mid \bar{\delta}) = (|x|)^p)$ for all policy sequences $\bar{\delta}$.*

PROOF. Define $\hat{D} = \max(D, b)$, and $\hat{D}^{(i)}$ the i-fold convolution of $\hat{D}$. In view of Lemma 1 applied to $\hat{D}$ there exists a constant $\hat{C}_p$ such that $E(\hat{D}^{(t)})^p \leqslant \hat{C}_p t^p$. Note that $x - D^{(t)} \leqslant y_t \leqslant x + tb$ for all policies and demands. Now, by the (quasi-) convexity of G and Assumption 1,

$$G(y_t) \leqslant \max\left\{ G(x - D^{(t)}), G(x + tb) \right\}$$
$$\leqslant A + B\max\left\{ |x - D^{(t)}|^p, |x + tb|^p \right\},$$

for some positive constants A and B. From the convexity of the function $|z|^p$, $|x - D^{(t)}|^p \leqslant \frac{1}{2}(2x)^p + \frac{1}{2}| - 2D^{(t)}|^p$. Hence,

$$G(y_t) \leqslant A + 2^{p-1}B\left(|x|^p + \max\left\{ \left(D^{(t)}\right)^p, (tb)^p \right\}\right)$$
$$\leqslant A + 2^{p-1}B\left(|x|^p + \left(\hat{D}^{(t)}\right)^p\right), \quad \text{and}$$
$$EG(y_t) \leqslant A + 2^{p-1}B\left(|x|^p + \hat{C}_p t^p\right).$$

The theorem now follows from

$$B(x \mid \bar{\delta}) = \sum_{t=0}^{\infty} \alpha^t E\left[c(y_t - x_t) + G(y_t) \right]$$
$$\leqslant (1 - \alpha)^{-1} cb + \sum_{t=0}^{\infty} \alpha^t EG(y_t). \quad ∎$$

3. Finite-horizon problems. In this section we characterize the optimal policies and value functions in finite-horizon problems.

Define

$v_n(x)$ = minimal expected discounted cost with $n \geqslant 0$ periods remaining in the problem starting with inventory $x \in \mathbb{R}$.

Then the v_n satisfy the following standard functional equations, expressed in terms of auxiliary functions J_n and I_n:

$v_0(\cdot) = J_0(\cdot) = I_0(\cdot) = 0$;
$J_n(y) = cy + G(y) + \alpha E v_{n-1}(y - D)$, $y \in \mathbb{R}$,
$I_n(x) = \min\{J_n(y) : y \in Y(x)\}$, $x \in \mathbb{R}$,
$v_n(x) = -cx + I_n(x)$, $x \in \mathbb{R}$, $n \geqslant 1$.

(A simple induction shows that each $v_n(x)$ is $0(|x|^p)$, so $Ev_{n-1}(y - D)$ exists and is finite, hence J_n is well defined.)

THEOREM 2. *For all $n \geqslant 1$*

(a) *J_n is C^1 and convex; there exists a finite number which achieves the global minimum of J_n. Let y_n^* be the smallest value of y that minimizes J_n.*

(b) *The optimal policy in period n is $\delta[y_n^*]$.*

(c) *I_n and v_n are C^1 and convex.*

(d) *$I_n'(x) \leqslant I_{n-1}'(x)$, $x \leqslant y_n^*$.*

(e) *$J_{n+1}'(x) \leqslant J_n'(x)$, $x \leqslant y_n^*$.*

(f) $y_{n+1}^* \geqslant y_n^*$.
(g) $\{v_n(x)\}$ *is nonnegative and nondecreasing in* n, $x \in \mathbb{R}$.

PROOF. A simple induction using $c(y-x)+G(y) \geqslant 0$, $y \in Y(x)$, $x \in \mathbb{R}$ verifies (g). We shall prove (a)–(f) by induction.

For $n = 1$, (a) and (b) are obvious.

$$I_1(x) = \begin{cases} J_1(x+b), & x \leqslant y_1^* - b, \\ J_1(y_1^*), & y_1^* - b \leqslant x \leqslant y_1^*, \\ J_1(x), & y_1^* \leqslant x, \end{cases}$$

from which (c) follows immediately. $I_1'(x) \leqslant 0 = I_0'(x)$, $x \leqslant y_1^*$, which is (d). Thus, $EI_1'(y-D) \leqslant 0$, $y \leqslant y_1^*$. Also, the integrals $EI_1(y-D)$ and $EI_1'(y-D)$ converge uniformly over y in any closed interval, so $dEI_1(y-D)/dy = EI_1'(y-D)$. This, together with (c) establishes part (a) for $n = 2$. (Note, $\lim_{|y|\to\infty} J_2(y) = \infty$ in view of Assumption 1, and part (g).) Thus,

$$J_2'(y) = (1-\alpha)c + G'(y) + \alpha EI_1'(y-D) \leqslant c + G'(y) = J_1'(y), \qquad y \leqslant y_1^*,$$

which is (e). In particular $J_2'(y_1^*) \leqslant J_1'(y_1^*) = 0$, yielding (f).

Now, assume the result for $n-1$. Part (a) for n follows from (c) for $n-1$ (using $dEv_{n-1}(y-D)/dy = Ev_{n-1}'(y-D)$ and $\lim_{|y|\to\infty} J_n(y) = \infty$, as above), and this yields (b) immediately. Thus,

$$I_n(x) = \begin{cases} J_n(x+b), & x \leqslant y_n^* - b, \\ J_n(y_n^*), & y_n^* - b \leqslant y_n^*, \\ J_n(x), & y_n^* \leqslant x, \end{cases}$$

which yields (c). For (d) we consider two cases; in each case x falls in one (or more) of four intervals:

(A) $y_n^* - b \leqslant y_{n-1}^*$:

$$I_n'(x) = J_n'(x+b) \leqslant J_{n-1}'(x+b) = I_{n-1}'(x), \qquad x \leqslant y_{n-1}^* - b;$$

$$I_n'(x) = J_n'(x+b) \leqslant 0 = I_{n-1}'(x), \qquad y_{n-1}^* - b \leqslant x \leqslant y_n^* - b;$$

$$I_n'(x) = 0 \leqslant I_{n-1}'(x), \qquad y_n^* - b \leqslant x \leqslant y_{n-1}^*;$$

$$I_n'(x) = 0 \leqslant J_{n-1}'(x) = I_{n-1}'(x), \qquad y_{n-1}^* \leqslant x \leqslant y_n^*.$$

(B) $y_n^* - b \geqslant y_{n-1}^*$: same as (A), $x \leqslant y_{n-1}^* - b$;

$$I_n'(x) = J_n'(x+b) \leqslant 0 = I_{n-1}'(x), \qquad y_{n-1}^* - b \leqslant x \leqslant y_{n-1}^*;$$

$$I_n'(x) = J_n'(x+b) \leqslant 0 \leqslant J_{n-1}'(x) = I_{n-1}'(x), \qquad y_{n-1}^* \leqslant x \leqslant y_n^* - b;$$

$$I_n'(x) = 0 \leqslant J_{n-1}'(x) = I_{n-1}'(x), \qquad y_n^* - b \leqslant x \leqslant y^*.$$

Parts (e) and (f) follow immediately from (d), as above for the case $n = 1$. ∎

Let $\bar{\delta}^*$ denote the sequence of one-period policies $\{\delta[y_n^*]\}$; thus $\bar{\delta}^*$ specifies an optimal n-period policy for all $n \geqslant 1$.

We now show how the optimal policy depends on b. Let $v_n(x;b)$, $y_n^*(b)$, etc. indicate the quantities above parameterized on b.

THEOREM 3. *If* $0 < b_1 < b_2$, *then for all* $n \geqslant 1$
(a) $v_n(x;b_1) \geqslant v_n(x;b_2)$, $x \in \mathbb{R}$;
(b) $y_n^*(b_1) \geqslant y_n^*(b_2)$.

PROOF. (a) $\bar{\delta}^*(b_1)$ is feasible for $b = b_2$.

(b) We show $J_n'(y; b_1) \leqslant J_n'(y; b_2)$, $y \in \mathbb{R}$, and hence $y_n^*(b_1) \geqslant y_n^*(b_2)$ by induction on n. For $n = 1$ we have equality. Assuming the result for n, we show $I_n'(x; b_1) \leqslant I_n'(x; b_2)$, $x \in \mathbb{R}$. There are two cases to consider, and several intervals for x in each case:

(A) $y_n^*(b_1) - b_1 \leqslant y_n^*(b_2)$:

$$I_n'(x; b_1) = J_n'(x + b_1; b_1) \leqslant J_n'(x + b_1; b_2)$$
$$\leqslant J_n'(x + b_2; b_2) = I_n'(x; b_2), \qquad x \leqslant y_n^*(b_2) - b_2;$$
$$I_n'(x; b_1) = J_n'(x + b_1; b_1) \leqslant 0 = I_n'(x; b_2), \qquad y_n^*(b_2) - b_2 \leqslant x \leqslant y_n^*(b_1) - b_1;$$
$$I_n'(x; b_1) = 0 = I_n'(x' b_2), \qquad y_n^*(b_1) - b_1 \leqslant x \leqslant y_n^*(b_2);$$
$$I_n'(x; b_1) = 0 \leqslant J_n'(x; b_2) = I_n'(x; b_2), \qquad y_n^*(b_2) \leqslant x \leqslant y_n^*(b_1);$$
$$I_n'(x; b_1) = J_n'(x; b_1) \leqslant J_n'(x; b_2) = I_n'(x; b_2), \qquad y_n^*(b_1) \leqslant x.$$

(B) $y_n^*(b_1) - b_1 \geqslant y_n^*(b_2)$: same as (A), $x \leqslant y_n^*(b_2) - b_2$;

$$I_n'(x; b_1) = J_n'(x + b_1; b_1) \leqslant 0 = I_n'(x; b_2), \qquad y_n^*(b_2) - b_2 \leqslant x \leqslant y_n^*(b_2);$$
$$I_n'(x; b_1) = J_n'(x + b_1; b_1) \leqslant 0 \leqslant J_n'(x; b_2)$$
$$= I_n'(x; b_2), \qquad y_n^*(b_2) \leqslant x \leqslant y_n^*(b_1) - b_1;$$
$$I_n'(x; b_1) = 0 \leqslant J_n'(x; b_2) = I_n'(x; b_2), \qquad y_n^*(b_1) - b_1 \leqslant x \leqslant y_n^*(b_1);$$

same as (A), $y_n^*(b_1) \leqslant x$.

Therefore, $v_n'(x; b_1) \leqslant v_n'(x; b_2)$, $x \in \mathbb{R}$, which implies $J_{n+1}'(y; b_1) \leqslant J_{n+1}'(y; b_2)$, $y \in \mathbb{R}$, completing the induction. ∎

4. Optimality proof for the infinite-horizon problem. In this section we show that a stationary (modified) base-stock policy is optimal for the infinite-horizon problem. In addition we show that the infinite-horizon minimal-cost function and the corresponding optimal base-stock level arise as limits of the sequences of their finite-horizon counterparts.

THEOREM 4. (a) *The sequence* $\{v_n\}$ *converges pointwise to a limit* v_∞; v_∞ *is convex, and* $\lim_{|x| \to \infty} v_\infty(x) = \infty$.

(b) *The function* $J_\infty(y) = cy + G(y) + \alpha E v_\infty(y - D)$ *is well defined*; J_∞ *is convex, and some finite number achieves its global minimum. Let* y_∞^* *denote the smallest such number.*

(c) $\lim_{n \to \infty}\{y_n^*\} = y_\infty^*$.

(d) *The function* v_∞ *satisfies the optimality equation*

$$v(x) = \min\{c(y - x) + G(y) + \alpha E v(y - D) : y \in Y(x)\},$$

and the minimum is achieved by the policy $\delta[y_\infty^*]$, *for all* $x \in \mathbb{R}$.

PROOF. (a) Choose any $\bar{y} \in \mathbb{R}$. By the optimality of $\bar{\delta}^*$ for the n-period problem, $v_n(x) = EB_n(x \mid \bar{\delta}^*, \bar{D}) \leqslant EB_n(x \mid \delta[\bar{y}], \bar{D}) \leqslant B(x \mid \delta[\bar{y}])$, for all $n \geqslant 1$ and $x \in \mathbb{R}$. Using Theorems 2(g) and 1, $\{v_n\}$ is nondecreasing and bounded above, so it is convergent. Each v_n is convex, so v_∞ is also, and $\lim_{|x| \to \infty} v_\infty(x) \geqslant \lim_{|x| \to \infty} v_1(x) = \infty$, by Assumption 1.

(b) $v_\infty(x) = 0(|x|^p)$ by Theorem 1, so $EV_\infty(y - D) < \infty$ for all y, and J_∞ is well defined. The convexity of J_∞ follows from that of G and v_∞. Also, $\lim_{|y| \to \infty} J_\infty(y) \geqslant \lim_{|y| \to \infty} J_1(y) = \infty$, again by Assumption 1, so y_∞^* is finite.

(c) Since each $v_n \leqslant v_\infty$, by the Lebesgue Convergence Theorem, $\{Ev_n(y - D)\}$

$\to \{Ev_\infty(y - D)\}$, so $\{J_n(y)\} \to \{J_\infty(y)\}$. Define $\hat{y}_\infty = \sup\{y_n^*\}$. If $y_\infty^* < \hat{y}_\infty$, then choose N such that $y_\infty^* < y_N^* < \hat{y}_\infty$. Let $\epsilon = J_N(y_\infty^*) - J_N(y_N^*)$; by the definition of y_N^*, $\epsilon > 0$. By Theorem 2(e), (f) for $n \geqslant N$

$$J_n(y_\infty^*) - J_n(y_N^*) = -\int_{y_\infty^*}^{y_N^*} J_n'(y)\,dy \geqslant -\int_{y_\infty^*}^{y_N^*} J_N'(y)\,dy = \epsilon,$$

hence $J_\infty(y_\infty^*) \geqslant J_\infty(y_N^*) + \epsilon$, contradicting the definition of y_∞^*. If $\hat{y}_\infty < y_\infty^*$, $J_n(y)$ is nondecreasing for $y \geqslant \hat{y}_\infty \geqslant y_n^*$, $n \geqslant 1$, so $J_n(\hat{y}_\infty) \leqslant J_n(y_\infty^*) \leqslant J_\infty(y_\infty^*)$. Therefore, $J_\infty(\hat{y}_\infty) - J_n(\hat{y}_\infty) \geqslant J_\infty(\hat{y}_\infty) - J_\infty(y_\infty^*) > 0$, by the definition of y_∞^*. But this contradicts $\{J_n(\hat{y}_\infty)\} \to J_\infty(\hat{y}_\infty)$.

(d) We shall take limits of both sides of the equation

$$v_n(x) = -cx + \min\{J_n(y) : y \in Y(x)\} = -cx + J_n(\delta[y_n^*](x)).$$

The left-hand side converges to $v_\infty(x)$. By (c) the policies $\delta[y_n^*]$ converge pointwise to $\delta[y_\infty^*]$, so the measures induced by the variables $\delta[y_n^*](x) - D$ converge setwise, in the sense of Royden [7, pp. 231–232]. Applying his Proposition 18, p. 232, therefore,

$$\{J_n(\delta[y_n^*](x))\} \to J_\infty(\delta[y_\infty^*](x)),$$

which yields the result, by the convexity of J_∞. ∎

THEOREM 5. *The policy $\delta[y_\infty^*]$ is optimal.*

PROOF. Follows immediately from Theorem 4(a) and (d) and Bertsekas and Shreve [1, Propositions 9.16 and 9.12]. ∎

THEOREM 6. *As a function of b, $y_\infty^*(b)$ is nondecreasing.*

PROOF. Follows from Theorems 3(b) and 4(c). ∎

5. The expected cost of (modified) base-stock policies. In this section we derive expressions for the expected costs of (modified) base-stock policies. These expressions are used to provide a (partial) characterization of the optimal policy.

For a given such policy $\delta(\bar{y})$ define T as the length of a cycle, i.e. T is the first time period $t \geqslant 1$ such that $y_t = \bar{y}$ conditional on $y_0 = \bar{y}$. (If $y_t < \bar{y}$ for all $t \geqslant 1$, set $T = \infty$.) Define $Q_t(w) = \Pr\{D^{(i)} - ib > 0,\ i = 1, \ldots, t-1,\ D^{(t)} - tb \leqslant w\}$. This is the probability that $T \geqslant t$ and $x_t \geqslant \bar{y} - (b + w)$. Note that $\int_{-b}^{0} dQ_t(w)$ is the probability that the cycle length is t and $\int_0^\infty dQ_t(w)$ is the probability that the cycle length is greater than t. Let $H(\bar{y})$ denote the expected discounted cost during such a cycle, including the order cost in period T, but not $G(y_T) = G(\bar{y})$. Letting C denote the discounted final expected order cost,

$$C = \sum_{t=1}^{\infty} \alpha^t \int_{-b}^{0} c(b + w)\,dQ_t(w).$$

Then

$$H(\bar{y}) = G(\bar{y}) + \sum_{t=1}^{\infty} \alpha^t \int_0^\infty [cb + G(\bar{y} - w)]\,dQ_t(w) + C$$

which is finite by Theorem 1. Also, let $\beta = E(\alpha^T \mid x_0 = \bar{y}, \delta(\bar{y})) < \alpha$ which is independent of $\bar{y}$.

The following theorem provides a (partial) characterization of the optimal critical number.

THEOREM 7. *A necessary condition for $\delta[\bar{y}]$ to be an optimal policy is that $\bar{y}$ minimize $(1 - \beta)cy + H(y)$ over $y \in \mathbb{R}$.*

PROOF. Observe that $H(y)$ is convex in y. Let y^- denote the smallest y realizing the minimum of $(1-\beta)cy + H(y)$ and y^+ the largest. First, suppose $\bar{y} < y^-$. For any ϵ, $0 < \epsilon < \min\{b, y^- - \bar{y}\}$, $(1-\beta)c(\bar{y}+\epsilon) + H(\bar{y}+\epsilon) < (1-\beta)c\bar{y} + H(\bar{y})$, so $c\epsilon + (1-\beta)^{-1}H(\bar{y}+\epsilon) < (1-\beta)^{-1}H(\bar{y})$.

Since under policy $\delta[\bar{y}+\epsilon]$, the process $\{y_t\}$ is regenerative at epochs t with $y_t = \bar{y}+\epsilon$ we have

$$B(\bar{y}+\epsilon \mid \delta[\bar{y}+\epsilon]) = \sum_{t=0}^{\infty} \beta^t H(\bar{y}+\epsilon) = (1-\beta)^{-1}H(\bar{y}+\epsilon) \quad \text{and}$$

$$B(\bar{y} \mid \delta[\bar{y}+\epsilon]) = c\epsilon + (1-\beta)^{-1}H(\bar{y}+\epsilon).$$

Thus $B(\bar{y} \mid \delta[\bar{y}+\epsilon]) < B(\bar{y} \mid \delta[\bar{y}])$. For $x = \bar{y}$, the policy $\delta[\bar{y}+\epsilon]$ thus yields a lower cost than $\delta[\bar{y}]$, so $\delta[\bar{y}]$ cannot be optimal. Second, if $\bar{y} > y^+$, choose ϵ with $0 < \epsilon < \min\{b, \bar{y} - y^+\}$ and use a similar argument to show $B(\bar{y}-\epsilon \mid \delta[\bar{y}-\epsilon]) < B(\bar{y}-\epsilon \mid \delta[\bar{y}])$ so $\delta[\bar{y}]$ is not optimal for $x = \bar{y}-\epsilon$. ∎

Recall that in the uncapacitated problem, the optimal critical number minimizes $(1-\alpha)cy + G(y)$. Theorem 7 provides the analogous result in the capacitated case. Under the additional assumption that $b \geqslant ED$, Corollary 1 provides an alternative way to view the connection.

COROLLARY 1. *Assume $b \geqslant ED$. A necessary condition for $\delta[\bar{y}]$ to be an optimal policy is that $\bar{y}$ minimizes $\int_0^\infty[(1-\alpha)c(y-w) + G(y-w)]\,d\{\sum_{t=0}^{\infty}\alpha^t Q_t(w)\}$, where Q_0 is the unit step function at zero.*

PROOF. Note that in the first $(T-1)$ periods under policy $\delta[\bar{y}]$, the process $\{y_t\}$ follows a random walk with increments distributed as $b - D$. Since $b \geqslant ED$, T has a proper distribution (see Feller [4, p. 396]). Thus,

$$\begin{aligned} 1-\beta &= 1 - \sum_{t=1}^{\infty} \alpha^t \Pr\{T = t \mid x_0 = \bar{y}, \delta[\bar{y}]\} \\ &= (1-\alpha)\left[1 + \sum_{t=1}^{\infty} \alpha^t \Pr\{T > t \mid x_0 = \bar{y}, \delta[\bar{y}]\}\right] \\ &= (1-\alpha)\left[1 + \sum_{t=1}^{\infty} \alpha^t \int_0^\infty dQ_t(w)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} (1-\beta)cy + H(y) &= (1-\alpha)\left[1 + \sum_{t=1}^{\infty} \alpha^t \int_0^\infty dQ_t(w)\right]cy + G(y) \\ &\quad + \sum_{t=1}^{\infty} \alpha^t \int_0^\infty [cb + G(y-w)]\,dQ_t(w) + C \\ &= [(1-\alpha)cy + G(y)] + \sum_{t=1}^{\infty} \alpha^t \\ &\quad \times \int_0^\infty [(1-\alpha)c(y-w) + G(y-w)]\,dQ_t(w) \\ &\quad + \sum_{t=1}^{\infty} \alpha^t \int_0^\infty [(1-\alpha)w + b]c\,dQ_t(w) + C \\ &= \text{constant} + \int_0^\infty [(1-\alpha)c(y-w) + G(y-w)]\,d\left\{\sum_{t=0}^{\infty} \alpha^t Q_t(w)\right\}. \end{aligned}$$

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