THE JOINT REPLENISHMENT PROBLEM WITH GENERAL JOINT COST STRUCTURES

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(Received July 1988; revisions received July 1989, July 1990; accepted October 1990)

We consider inventory systems with several distinct items. Demands occur at constant, item specific rates. The items are interdependent because of jointly incurred fixed procurement costs: The joint cost structure reflects general economies of scale, merely assuming a monotonicity and concavity (submodularity) property. Under a power-of-two policy each item is replenished with constant reorder intervals which are power-of-two multiples of some fixed or variable base planning period. Our main results include a proof that, depending upon whether the base planning period is fixed or variable, the best among all power-of-two policies has an average cost which comes within either 6% or 2% of an easily computable lower bound for the *minimum cost value*. We also derive two efficient algorithms to compute an optimal power-of-two policy. The proposed algorithms generate as a by-product, a specific *cost allocation* of the joint cost structure to the individual items. With this specific allocation, the problem with separable costs is in fact equivalent to the original problem with nonseparable joint costs in the sense that the two problems share the same sets of optimal power-of-two policies with identical associated long-run average costs.

One of the major complications in managing multi-item inventory systems stems from the fact that various cost components, in particular, setup costs, are often *jointly* incurred between several distinct items. The joint cost structure often reflects *economies of scale* which may be exploited by combining different items in the same production batch or delivery order. Such cost structures invoke the need for careful coordination of the items' replenishment strategies. The coordination problem and the resulting potential for efficiency improvements and cost savings is often ignored in the current practice of inventory management: Typically, a rather arbitrary *allocation* scheme is applied to allocate the joint cost structure to the individual items. The allocated costs are subsequently used for the determination of inventory rules for each of the items separately.

One of the most extensively studied multi-item inventory models with joint setup costs is the so-called "joint replenishment problem." In this model, demands are assumed to occur continuously, at item specific but time-homogeneous rates. Most of the literature addresses itself to one specific joint setup cost structure: The *first-order interaction* structure in which a major (uniform) setup cost is incurred for each order, regardless of which items are involved, in addition to item specific (minor) setup costs incurred for each specific item that is included in the replenishment batch.

Even under the first-order interaction cost structure, which is arguably the simplest of all joint cost structures, few structural properties have been identified that an *optimal* policy can be shown to satisfy. This explains why all existing approaches in the literature are based on an a priori restriction to a convenient class of inventory policies. In fact, all existing restriction approaches appear to be of one of the following two types (or hybrid combinations thereof):

Fixed Partitions. Each strategy in this class employs a fixed partition of the items into groups; each time the inventory of a given item is replenished, it is replenished jointly with the other members of its group and the setup cost associated with that group is incurred. No joint replenishment occurs between items that are assigned to different groups.

Integer Ratio Policies. All items are replenished at constant intervals which are integer multiples of some base planning period. A special subclass is the class of

Subject classifications: Inventory/production, aprpoximations: power-of-two policies. Inventory/production, deterministic models: joint replenishment. Mathematics, combinatorics: submodular function minimization.
Area of review: MANUFACTURING, PRODUCTION AND SCHEDULING.

0030-364X/92/4002-0384 $01.25

power-of-two rules under which these replenishment intervals are chosen as power-of-two multiples of the base period. (Such rules have often been implemented in practice because their simplicity facilitates the planning, scheduling and coordination problems.)

For the case of first-order interaction, it has been shown (Jackson, Maxwell and Muckstadt 1985, Roundy 1985) that the best among all power-of-two policies has an average cost which comes within either 6 or 2% of a lower bound for the minimum cost value (depending upon whether the base planning period is fixed in advance, or may be varied, respectively). (The minimum cost value is defined as the infimum over all possible policies, of their "long-run average cost values.")

The first contribution of this paper is to extend this result to a *general* joint setup cost structure, merely assuming a monotonicity and a concavity (or submodularity) property:

Monotonicity. The fixed cost of a joint replenishment does not decrease by the inclusion of additional items.

Submodularity. The incremental cost due to the addition of a new item to a given collection of items is no larger than if the same item were added to a subset of this collection.

(The monotonicity property may be assumed without loss of generality, as is argued below; the submodularity assumption reflects general economies of scale.) Joint cost structures described by submodular set functions were first considered by Queyranne (1985) in the context of general production/distribution networks. In a companion paper, Federgruen and Zheng (1988), we discuss several specific types of cost structures which satisfy these properties.

The next contribution of the paper is to derive two efficient algorithms to compute an optimal power-of-two policy. The problem of determining an optimal power-of-two policy can be formulated as a nonlinear mixed integer program with a special type of integrality constraint. We derive a characterization theorem exhibiting necessary and sufficient conditions for an optimal solution of its continuous relaxation. This continuous relaxation generates the above mentioned *lower bound* for the *minimum* achievable cost value among *all* possible policies. A similar characterization theorem can, interestingly enough, be derived for the original mixed integer program. Our algorithms are based directly on these two characterization theorems.

Our first algorithm is a *two-stage procedure* which computes a solution to the model's continuous relaxation in *stage one* and applies a rounding procedure to transform the obtained vector of replenishment intervals into an optimal power-of-two vector in *stage two.* The second algorithm solves the integer program *directly.* The complexity of the direct algorithm is an order of magnitude lower than that of the two-stage procedure. An advantage of the latter is, however, that it generates the above mentioned lower bound for the minimum cost value. In this paper, we confine ourselves to a brief discussion of the algorithms' general complexity. Our companion paper, Federgruen and Zheng, contains a detailed discussion of efficient implementations for several types of cost structures.

The proposed algorithms generate, as a by-product, a specific *cost allocation* of the joint cost structure to the individual items. With this specific allocation, the problem with separable costs is, in fact, equivalent to the problem with joint costs in the sense that the two systems share the same sets of optimal power-of-two policies with identical associated long-run average costs.

In Section 1, we introduce notation, discuss structural properties of globally optimal policies, and derive our nonlinear (mixed integer) programming formulation for the problem of determining an optimal power-of-two policy. In Section 2, we derive the characterization theorem for the model's continuous relaxation as well as the two-stage procedure. In Section 3, we prove that the continuous relaxation results in a lower bound for the minimum achievable cost-value and derive the above discussed worst case bound for the performance of power-of-two policies. In Section 4, we discuss the faster, single-stage algorithm which results in an optimal power-of-two vector *directly* rather than via the solution of the model's continuous relaxation. In Section 5, we illustrate the two algorithms by solving a numerical example. In Section 6, we give a discussion of the relative performance of fixed partition strategies versus power-of-two (or integer ratio) strategies, thus establishing a connection between two distinct bodies of literature.

We conclude our introduction with a review of the recent literature on the joint replenishment problem. Chakravarty, Orlin and Rothblum (1982) apply the fixed partition approach to "symmetric" cost structures, where the setup cost is a concave function of the *number* of items included in the replenishment batch. Chakravarty, Orlin and Rothblum (1985) use the same approach for generalized symmetric structures of order 1; see Federgruen and Zheng for a precise definition. The results of Barnes, Hoffman and

Rothblum (1989) may be applied to general symmetric structures of higher order; see Federgruen and Zheng. (Instead of the assumption $\nabla^2 f \geqslant 0$ ibid, the authors require f to be (quasi) concave.) The first two papers show that an optimal partition may be found by a simple $O(N^2)$ shortest path computation. Barnes, Hoffman and Rothblum provide interesting characterizations for generalized symmetric cost structures of a higher order but these do not, as of yet, translate into efficient solution methods. Aggarwal (1984a, b, c) and Chakravarty (1982) give heuristic solution methods for the determination of a good fixed partition.

Rosenblatt and Kaspi (1985) propose a dynamic programming algorithm to compute an optimal partition for fully general joint cost structures (which are not even required to be submodular). Queyranne (1987a) shows that the Rosenblatt-Kaspi algorithm may *fail* to find optimal partitions and proposes an alternative dynamic programming algorithm with complexity $O(3^n)$. Schwarz (1987) points out that the Rosenblatt-Kaspi algorithm may fail even when the joint cost structure represents first-order interaction only. This observation is due to Quernhein and Bastian.

For the special case of first-order interaction cost structures, Jackson, Maxwell and Muckstadt (1985) derive an $O(n \log n)$ algorithm which generates an optimal power-of-two policy. We refer the reader to this paper for a review of heuristic search methods for this problem. Queyranne (1987b) shows that a modification of the Jackson, Maxwell and Muckstadt (1985) algorithm, based on a *linear* medium finding procedure, has linear ($O(n)$) complexity. Roundy (1985) obtains similar results for a *more general* one-warehouse multiretailer model. Roundy (1986) applies the same restriction to the class of power-of-two policies, but his general model allows for joint cost structures generated by an arbitrary *family model.* The family model is a special case of the general submodular cost structures considered here see (example V in Federgruen and Zheng. His proposed algorithm grows as the cube of the number of "families" and the latter may grow exponentially with the number of items considered.

Goyal and Soni (1984) (see Goyal 1987) consider a hybrid combination of the two restriction approaches: first a fixed partition is constructed with no more than three sets; next, the common reorder intervals for each of the (at most three) groups are modified to exploit additional cost savings by combining items into groups at some or all replenishment epochs. Chakravarty (1983, 1984b) and Chakravarty and Goyal (1986) consider policies which partition the items into a number of groups and apply arbitrary integer ratio policies within each of these groups. The paper by Chakravarty (1984a) differs from the previous two papers in that a common reorder interval is assigned to all items in the same group; these common intervals are chosen as integer multiples of some base period. See also Akroy and Erenguk (1988) for a recent survey of the above joint replenishment models.

Balintfy (1964), Ignall (1969), Silver (1965, 1974, 1981), Peterson and Silver (1979), Naddor (1975), Thompstone and Silver (1975), Federgruen, Groenevelt and Tijms (1984), and Atkins and Iyogun (1987, 1988) consider versions of the joint replenishment problem with stochastic demands and the first-order interaction cost structure.

Anily and Federgruen (1990) consider a class of joint replenishment problems in which an item is stored at n distinct locations and distributed by a fleet of vehicles. These vehicles combine deliveries to multiple locations into efficient routes. The resulting cost structure fails to be submodular. The authors restrict themselves to a class of strategies where each employs a fixed collection of regions (sets of locations); a class of $O(n \log n)$ heuristics is designed to generate solutions that, under mild probabilistic assumptions, are shown to be *asymptotically optimal* as n tends to infinity, within the specified class of strategies.

We refer to Zheng (1987) for a discussion of inventory models with joint cost structures in more general production/distribution networks.

1. THE GENERAL MODEL: A MATHEMATICAL PROGRAM FOR THE DETERMINATION OF OPTIMAL POWER-OF-TWO POLICIES

We consider a system with n distinct items $N = \{1, \ldots, n\}$. Demands for these items are assumed to occur continuously, at item specific but constant rates. We assume that no backlogging is allowed and that inventory carrying costs are proportional to the inventory sizes. Thus, for $i \in N$ let:

d_i = the rate at which demands for item i occur;
h_i = the per unit holding cost rate for item i.

We assume without loss of generality that $h_i > 0$, $i = 1, \ldots, n$. (An item with $h_i = 0$ does not need to be considered; the cost of managing this item can be made arbitrarily small by ordering sufficiently large quantities.)

An order for product i arrives after a fixed lead time of L_i time units ($i \in N$). We assume that at time 0 the starting inventory for product i equals $d_i L_i$. This implies that at time 0 orders need to be placed for *all* products. As in the single-item EOQ model, the problem is easily seen to be equivalent to the special case where all lead times are zero. (For different combinations of starting inventories, one minimizes long-run average costs by first achieving the above inventory balance with an appropriately constructed finite horizon policy.) We therefore assume zero lead times henceforth.

The general joint setup cost structure discussed in the Introduction is represented by a general set function $K\colon 2^N \to R^+$ which specifies, for any collection of items $S \subseteq N$, a setup cost $K(S)$ to be incurred whenever this specific collection is replenished together (e.g., when the corresponding production operations are combined in the same production run). The monotonicity and submodularity assumptions discussed above, may be expressed as follows.

Monotonicity

$$K(S) \leq K(T) \qquad S \subset T.$$

Submodularity

$$K(S \cup \{i\}) - K(S) \geq K(T \cup \{i\}) - K(T) \quad S \subset T \quad i \notin S.$$

The monotonicity property is made without loss of generality: If $K(\cdot)$ fails to be monotone, replace K by $\bar{K}$ defined by $\bar{K}(S) = \min\{K(T)\colon S \subseteq T\}$. ($\bar{K}(S)$ is clearly monotone; if $K(S) > K(T)$ for some $S \subset T$, we may replace a setup for the collection S by one for the larger collection T even though generating zero units of the items in $T \backslash S$.) We also assume without loss of generality that $K(\{i\}) > 0$ for all $i \in N$ and $K(\phi) \neq 0$.

A replenishment strategy π specifies all replenishment epochs and quantities for *all* products over an infinite horizon. For a given strategy, let $C_\pi(t)$ be the total cost in $[0, t)$; its long-run average cost C^*_π is given by

$$C^*_\pi = \limsup_{t \to \infty} \frac{1}{t} C_\pi(t).$$

We do not address the question whether an optimal strategy exists, i.e., whether $C^* \stackrel{\text{def}}{=} \inf_\pi C^*_\pi$ may be achieved or not. The existence question is, to our knowledge, open, even for the simplest of all joint cost structures.

Very few structural properties may be assumed without loss of optimality. The single exception, known to date, follows.

Zero-Inventory Ordering. Each strategy is dominated by one under which no product's inventory is ever replenished unless its inventory equals zero.

Optimality of zero-inventory ordering policies is well known for the single item EOQ and dynamic lot sizing models; see Wagner and Whitin (1958). We refer to Veinott (1969), and Zheng (Theorem 2.1) where this property is proven in a more general context. The following example shows that it may be necessary to replenish some of the items with *nonconstant*, i.e., *nonstationary* replenishment intervals.

Example 1. Consider a three-product problem with $d_1 = d_2 = d_3 = 2$ and $h_1 = \epsilon$, $h_2 = 1$ and $h_3 = 4/9$. Assume that the following nonseparable cost structure applies:

$$K(\{1\}) = \epsilon,\ K(\{2\}) = 1,\ K(\{3\}) = 1;$$

$$K(\{1, 2\}) = K(\{1, 3\}) = 1;$$

$$K(\{2, 3\}) = K(\{1, 2, 3\}) = 2 - \epsilon.$$

It is easy to verify that $K(\cdot)$ is monotone and submodular. We show that a stationary policy cannot be optimal, i.e., for any given policy with *constant* reorder intervals, there exists a *nonstationary* policy whose average cost is lower. Note first that a setup for product 1 may be arranged whenever a setup for product 2 and/or product 3 takes place, at *no* additional cost. Therefore, assuming to the contrary that a stationary policy *is* optimal, note that any such policy must be "nested" in the sense that its constant replenishment intervals of products 2 and 3 are multiples of product 1's interval. For, if a stationary policy were not nested, the holding cost of product 1 can be reduced strictly by ordering this product at *all* replenishment epochs of the others, without increasing *any* other cost component. The addition of repenishment epochs for product 1 improves the total cost, but transforms the stationary policy into a *nonstationary one.*

The argument is completed by showing that the best *nested* stationary policy is dominated by the following *nonstationary* cyclic rule R. Policy R applies zero-inventory ordering, replenishes items 2 and 3 at constant intervals $\tau_2 = 1$ and $\tau_3 = 3/2$, respectively, and replenishes product 1 whenever one of the other products is replenished. (Note that τ_2 and τ_3 equal the optimal EOQ-intervals for products 2 and 3, respectively. Indeed, the cost of product 1 is insignificant compared to that of the others and an insignificant amount is saved by a joint setup of products 2 and 3. This suggests it is efficient to replenish products 2 and 3 according to their own separate EOQ rules. It is the clearly optimal to order product 1 at zero additional cost, at least whenever one of the other items is replenished, i.e., at $t = 0, 1, 3/2, 2, 3, 4, 9/2, 5, \ldots$.) For $\epsilon = 0.1$, we compute in Appendix A the average cost of R as well as a *lower bound* for the cost of the best nested stationary policy and conclude that the former is lower than the latter.

Since C^* may not be achieved or even (infinitely closely) approached by policies of a simple structure, we first restrict ourselves to the class of power-of-two policies. Power-of-two policies are simple and easy to implement, and we show below that the best such policy is guaranteed to come within a few percentage points of a lower bound for C^*.

A power-of-two policy applies zero-inventory ordering and prescribes for each product $i \in N$ a replenishment interval T_i, such that replenishments occur at times $0, T_i, 2T_i, 3T_i, \ldots$. Moreover,

$$T_i = 2^{m_i} T_L \quad \text{for all } i \in N,$$

where m_i is a (possibly negative) integer and T_L is a base planning period. (T_L is sometimes predetermined but may be varied continuously in other settings.) A power-of-two policy is thus completely characterized by the replenishment interval vector $T = (T_1, T_2, \ldots, T_n)$. The problem of determining an optimal policy in this class may be formulated as:

$$\min K[T] + \sum_{i \in N} H_i T_i$$

subject to

$$T_i - 2^{m_i} T_L; \quad m_i \text{ integer and } i \in N,$$

where $H_i = 1/2\ h_i d_i$ ($i \in N$), and $K[T]$ denotes the system-wide average setup cost per unit of time.

For policies with constant but *arbitrary* replenishment intervals, no simple expression for $K[T]$ appears to prevail. A relatively simple expression may, however, be derived for power-of-two policies.

For a given power-of-two policy T let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a permutation of the indices $\{1, 2, \ldots, n\}$, such that

$$T_{\alpha_1}^{-1} \geq T_{\alpha_2}^{-1} \geq \ldots \geq T_{\alpha_n}^{-1},$$

i.e., the items' reorder frequencies are nonincreasing in this permutation. Observe that under T wherever item α_i is replenished, all the items $\{\alpha_1, \ldots, \alpha_{i-1}\}$ are replenished as well. Only one of the following n sets of items is therefore replenished at any replenishment epoch:

$$\{\alpha_1\}, \{\alpha_1, \alpha_2\}, \ldots, \{\alpha_1, \ldots, \alpha_i\}, \ldots, \{\alpha_1, \ldots, \alpha_n\}.$$

The order frequency of the set $\{\alpha_1, \ldots, \alpha_i\}$ is

$T_{\alpha_i}^{-1} - T_{\alpha_{i+1}}^{-1}$. (Set $T_{\alpha_{n+1}}^{-1} = 0$.)

Therefore

$$K[T] = \sum_{i=1}^{n} K(\{\alpha_1, \ldots, \alpha_i\})(T_{\alpha_i}^{-1} - T_{\alpha_{i+1}}^{-1}).$$

By rearranging the terms in this summation, we obtain:

$$K[T] = \sum_{i=1}^{n} [K(\{\alpha_1, \ldots, \alpha_i\}) - K(\{\alpha_1, \ldots, \alpha_{i-1}\})] T_{\alpha_i}^{-1}. \quad (1)$$

We now show that $K[T]$ may be viewed as the optimal objective function value of a special linear program.

Lemma 1. *Let*

$$\mathbf{K} \stackrel{\text{def}}{=} \left\{ \mathbf{k} \in R^N : \sum_{i \in S} k_i \leq K(S),\ S \subseteq N,\ \mathbf{k} \geq 0 \right\}.$$

K is referred to as the setup cost polyhedron.

a. *For any vector* $\mathbf{T} \in R^N$ ($\mathbf{T} \geq 0$), *let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be a permutation of the indices such that*

$$T_{\alpha_1}^{-1} \geq T_{\alpha_2}^{-1} \geq \ldots \geq T_{\alpha_n}^{-1}.$$

Then

$$\max_{\mathbf{k} \in \mathbf{K}} \sum_{i \in N} k_i / T_i = \sum_{i \in N} k_i^* / T_i,$$

where $\mathbf{k}^* \in \mathbf{K}$ *is defined by*

$$k_{\alpha_i}^* = K(\{\alpha_1, \ldots, \alpha_i\}) - K(\{\alpha_1, \ldots, \alpha_{i-1}\}),\ i = 1, \ldots, n.$$

b. *For any power-of-two vector* $\mathbf{T}$,

$$K[\mathbf{T}] = \max_{\mathbf{k} \in \mathbf{K}} \sum_{i \in N} k_i / T_i.$$

Thus, for any power-of-two policy $\mathbf{T}$ *there exists an allocation* $\mathbf{k} \in \mathbf{K}$ *of the joint cost structure, such that the long-run average setup costs under the separable allocated cost structure equal the actual average setup costs (under the joint cost structure).*

This lemma is due to a result of Edmonds (1970) with respect to a special class of polyhedral sets, the so-called "polymatroids."

Definition 1. Let E be a finite index set. A set function $f: 2^E \to R$ is a *rank function* on E if:

1. f is normalized: $f(\phi) = 0$;
2. f is nondecreasing: $f(T) \leq f(S)$. $T \subset S \subseteq E$;
3. f is submodular: $f(S \cup \{j\}) - f(S) \geq f(T \cup \{j\}) - f(T)$, $S \subset T \subseteq E$, $j \notin S$.

The following is a convenient alternative definition of submodularity:

$$f(S\cup T)+f(S\cap T)\leq f(S)+f(T), \quad S, T\subseteq E, \tag{20}$$

(see e.g., Nemhauser, Wolsey and Fisher 1978).

For any finite index set E and $\mathbf{x} \in R^E$, let $x(S) \stackrel{\text{def}}{=} \sum_{i\in S} x_i$, $S \subseteq E$.

Definition 2. A polyhedron $\mathbf{F} \subset R^E$ is called a polymatroid if there is a rank function f on ground set E such that

$$\mathbf{F} = F(E,f) \stackrel{\text{def}}{=} \{\mathbf{x}\in R^E: x(S) \leq f(S), S\subseteq E, \mathbf{x} \geq 0\}.$$

Note that the setup cost polyhedron $\mathbf{K}$ is a polymatroid, in view of the monotonicity and submodularity of the set function $K(.)$.

The proof of Lemma 1 is given in Appendix B, where we exhibit several properties of polymatroids as needed in the remainder of this paper. We conclude that an optimal power-of-two policy is obtained by solving

Problem JP

$$\min_{\mathbf{T}>0} \max_{\mathbf{k}\in\mathbf{K}} \sum_{i\in N} (k_i/T_i + H_i T_i)$$

subject to $T_i = 2^{m_i} T_L$ (m_i integer) $i \in N$.

For notational convenience and without loss of mathematical rigor, we use "max" and "min" rather than "sup" and "inf" regardless of whether the supremum or infimum is achieved.

2. A CHARACTERIZATION THEOREM FOR THE MODEL'S CONTINUOUS RELAXATION: A TWO-STAGE ALGORITHM

Consider the continuous relaxation of **JP** obtained by relaxing the power-of-two integrality constraints:

Problem RJP

$$c^* \stackrel{\text{def}}{=} \min_{\mathbf{T}>0} \max_{\mathbf{k}\in\mathbf{K}} \sum_{i\in N} (k_i/T_i + H_i T_i).$$

A vector $\mathbf{T}^*$ is an optimal solution of **RJP** if it achieves the minimum. Consider the dual optimization problem:

Problem JD

$$\underline{c}^* = \max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}>0} \sum_{i\in N} (k_i/T_i + H_i T_i).$$

A vector $\mathbf{k}^* \in \mathbf{K}$ is an optimal solution of **JD** if it achieves the maximum. Clearly $\underline{c}^* \leq c^*$. (See Lemma 36.1 in Rockafellar 1970 or Lemma C4 below.) We first give in Theorem 1 necessary and sufficient conditions for a pair $(\mathbf{T}^*, \mathbf{k}^*)$ to be optimal solutions for **RJP** and **JD**, respectively. Next we show in Theorem 2 that such optimal solutions indeed exist, that they can be computed efficiently and that **RJP** and **JD** have the same objective function value, i.e., $\underline{c}^* = c^*$, with

$$\begin{aligned}\max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}>0} \sum_{i\in N} (k_i/T_i + H_i T_i) &= \min_{\mathbf{T}>0} \sum_{i\in N} (k_i^*/T_i + H_i T_i)\\ &= \sum_{i\in N} (k_i^*/T_i^* + H_i T_i^*) = \max_{\mathbf{k}\in\mathbf{K}} \sum_{i\in N} (k_i/T_i^* + H_i T_i^*)\\ &= \min_{\mathbf{T}>0} \max_{\mathbf{k}\in\mathbf{K}} \sum_{i\in N} (k_i/T_i + H_i T_i).\end{aligned} \tag{3}$$

An alternative proof of (3) is given by Corollary 2.10 in Zheng. This proof is an adaptation of a classical minimax theorem (see Rockafellar) using the fact that the objective function is convex in $\mathbf{T}$ and concave in $\mathbf{k}$.

We first need some additional notation. Let $(N_1, N_2, \ldots, N_M)$ be a partition of N; let $K_1(S) \stackrel{\text{def}}{=} K(S)$, $S \subseteq N_1$; for $l = 2, \ldots, M$, define the set functions $K_l(.)$ by

$$K_l(S) \stackrel{\text{def}}{=} K\left(\bigcup_{j=1}^{l-1} N_j \cup S\right) - K\left(\bigcup_{j=1}^{l-1} N_j\right), \quad S \subseteq N_l.$$

We use $\mathbf{k}^l$ to denote the vector $\{k_i, i \in N_l\}$ and let $k(S) \stackrel{\text{def}}{=} \sum_{i\in S} k_i$ $S \subset N$; let

$$\mathbf{K}_l \stackrel{\text{def}}{=} \{\mathbf{k}^l \in R^{N_l}: k(S) \leq K_l(S), S \subseteq N_l, \mathbf{k}^l \geq 0\}.$$

It follows from Lemma B2a in Appendix B that $\mathbf{K}_l$ is a polymatroid ($l = 1, \ldots, M$).

The following characterization theorem generalizes that of Jackson, Maxwell and Muckstadt (1988) for first-order interaction cost structures, i.e., the classical joint replenishment problem. Let $H(S) \stackrel{\text{def}}{=} \sum_{i\in S} H_i$, $S \subseteq N$.

Theorem 1. (Characterization Theorem) *If the components of* $\mathbf{T}^* = (T_1^*, \ldots, T_n^*)$ *take on* M *distinct values* $T(1) < T(2) < \ldots < T(M)$, *and* $(N_1, N_2, \ldots, N_M)$ *is the partition of* N *with* $N_l = \{i \in N: T_i^* = T(l)\}$, *then* $\mathbf{T}^*$ *is optimal for* **RJP**, $\mathbf{k}^*$ *is optimal for* **JD** *and* $c^* = \underline{c}^*$ *if and only if the following hold for* $l = 1, \ldots, M$:

i. $T(l) = \sqrt{K_l(N_l)/H(N_l)}$,
ii. $k_i^* = H_i T^2(l)$,
iii. $K_l(S)/H(S) \geq T^2(l)$ *for all* $S \subseteq N_l$.

Proof. Let

$$f(\mathbf{T}, \mathbf{k}) \stackrel{\text{def}}{=} \sum_{i \in N} (k_i/T_i + H_i T_i).$$

For the sufficiency part, assume that i, ii and iii hold. For any $S \subseteq N_l$, by ii and iii, $k^*(S) = H(S)K_l(N_l)/H(N_l) \leqslant K_l(S)$, i.e., $\mathbf{k}^{*l} \in \mathbf{K}_l$ ($l = 1, \ldots, M$). It follows from Lemma B2b that $\mathbf{k}^* \in \mathbf{K}$. By Lemma C4, it suffices to show that $\max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, \mathbf{k}) = \min_{\mathbf{T}>0} f(\mathbf{T}, \mathbf{k}^*)$. By the definition of $\mathbf{T}^*$, we have

$$\max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, \mathbf{k}) = \max_{\mathbf{k} \in \mathbf{K}} \sum_{l=1}^{M} (k_i/T(l) + H_i T(l)).$$

Since $T(1)^{-1} > T(2)^{-1} > \ldots > T(M)^{-1}$ and $k^*(N_l) = K_l(N_l)$ ($l = 1, \ldots, M$), it follows from Lemma B3c that the above maximum can be reached by $\mathbf{k}^*$. Thus,

$$\max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, \mathbf{k}) = \sum_{l=1}^{M} (K_l(N_l)/T(l) + H(N_l)T(l))$$

$$= 2 \sum_{l=1}^{M} (K_l(N_l)H(G_l))^{1/2},$$

where the second equality follows from i. Hence,

$$\min_{\mathbf{T}>0} f(\mathbf{T}, \mathbf{k}^*) = \sum_{i=1}^{n} 2(k_i^* H_i)^{1/2}$$

$$= 2 \sum_{l=1}^{M} (K_l(N_l)H(N_l))^{1/2} = \max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, k).$$

To show the necessity part, let $\mathbf{T}^*$, $\mathbf{k}^*$, be an optimal solution of **RJP** and **JD**, respectively, and assume that $c^* = \underline{c}^*$. In view of Lemma C4, $(\mathbf{T}^*, \mathbf{k}^*)$ is a saddle point of $f(\mathbf{T}, \mathbf{k})$, i.e.,

$$\min_{\mathbf{T}>0} f(\mathbf{T}, \mathbf{k}^*) = f(\mathbf{T}^*, \mathbf{k}^*) = \max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, \mathbf{k}).$$

Since $T(1)^{-1} > T(2)^{-1} > \ldots > T(M)^{-1}$ and $\mathbf{k}^*$ is an optimal solution of $\max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, \mathbf{k})$, it follows from Lemma B3*c* that $k^*(N_l) = K_l(N_l)$, $= l\, 1, \ldots, M$. Thus,

$$\min_{\mathbf{T}>0} f(\mathbf{T}, \mathbf{k}^*) = \min_{\mathbf{T}>0} \sum_{l=1}^{M} \sum_{i \in N_l} (k_i^*/T_i + H_i T_i)$$

$$\leqslant \sum_{l=1}^{M} 2(K_l(N_l)H(N_l))^{1/2}$$

$$\leqslant \sum_{l=1}^{M} (K_l(N_l)/T(l) + H(N_l)T(l))$$

$$= \max_{\mathbf{k} \in \mathbf{K}} f(\mathbf{T}^*, \mathbf{k}),$$

where the first inequality follows by choosing $\mathbf{T}$ such that

$$T_i = (K(N_l)/H(N_l))^{1/2},\ i \in N_l\ (l = 1, \ldots, M).$$

We conclude that both inequalities hold as equalities. The fact that the second one does implies i, and the fact that the first one does implies for all $l = 1, \ldots, M$ that $\mathbf{k}^{*l}$ achieves the maximum in

$$\max\left\{\sum_{i \in N_l} 2(k_i H_i)^{1/2}: \sum_{i \in N_l} k_i = K_l(N_l)\right\};$$

hence

$$k_i^* = H_i K_l(N_l)/H(N_l) = H_i T(l), \quad i \in N_l.$$

This proves ii. Also, for any $S \subseteq N_l$,

$$H(S)K_l(N_l)/H(N_l)$$

$$= k^*(S) = k^*(N_1 \cup \ldots \cup N_{l-1} \cup S)$$

$$- k^*(N_1 \cup \ldots \cup N_{l-1})$$

$$\leqslant K(N_1 \cup \ldots \cup N_{l-1} \cup S)$$

$$- K(N_1 \cup \ldots \cup N_{l-1}) = K_l(S)$$

$$(l = 1, 2, \ldots, M),$$

thus proving iii.

Note that for any given $\mathbf{k} \in \mathbf{K}$, the inner minimization in **JD** may be carried out explicitly, i.e.,

Problem JD

$$\underline{c}^* = \max_{\mathbf{k} \in \mathbf{K}} \sum_{i \in N} 2(k_i H_i)^{1/2}.$$

Thus, **JD** is equivalent to the maximization of a separable concave function over a polymatroidal feasible region. Groenevelt (1985) developed two algorithms for this class of problems: the Decomposition Algorithm (DA) and the Bottom Up Algorithm (BUA). See, also, Federgruen and Groenevelt (1991) who show that a simple greedy procedure succeeds in terminating with an optimal solution on any given discrete grid; and Federgruen and Groenevelt (1988) for efficient implementations when the polymatroidal feasible region is generated by an underlying network. We now describe an implementation of the decomposition algorithm. This algorithm bears similarity to the Divide and Conquer Algorithm in Maxwell and Muckstadt (1985) for models with a separable cost structure but physical interdependencies between the items.

DAJD: Decomposition Algorithm for JD

Step 0. $M := 1$, $l := 1$, $N_1 = N$.
Repeat Step 1 until $l = M + 1$:

Step 1. Determine a set $S \subseteq N_l$ achieving $\min\{K_l(S) + u^l(N_l \backslash S): S \subseteq N_l\}$, where $u^l = \{H_i K(N_l)/H(N_l): i \in N_l\}$. If $S = N_l$, $k_i^* := u_i^l$ $(i \in N_l)$ and $l := l + 1$. Otherwise, replace N_l in the list $(N_1, \ldots, N_M)$ by S and $N_l \backslash S$ and renumber; $M := M + 1$.

The **DAJD** algorithm consists of at most $(2N - 1)$ executions of Step 1 because at the end of Step 1 either l or M are increased by one and $M \leqslant N$. The set minimization problem in Step 1 is of the type $\min\{f(S) + u(E \backslash S): S \subseteq E\}$ with $f(\cdot)$ a rank function on a ground set $E \subseteq N$. (The set functions K_l are rank functions (see Lemma B2a). This represents one of the basic optimization problems regarding rank functions and polymatroids and is equivalent (see Edmonds 1970, and Frank and Tardos 1989) to the "*maximum element problem*," i.e., to determining $\max\{x(E): \mathbf{x} \in \mathbf{F}, \mathbf{x} \leqslant u\}$ with $\mathbf{F}$ the polymatroid $\{\mathbf{x} \geqslant 0: x(S) \leqslant f(S), S \subseteq E\}$. General polynomial algorithms based on the ellipsoid method are available for the minimal set or maximal element problem with general rank functions $f(\cdot)$; see Grötschel, Lovász and Schrijver (1981). A pseudopolynomial procedure is due to Cunningham (1985); see also Bixby, Cunningham and Topkis (1985). Significantly more efficient algorithms are available for a variety of special structures; see Federgruen and Zheng.

It is easy to verify from Groenevelt that **DAJD** is a correct implementation of his general decomposition algorithm; thus, in view of his Theorem 2, **DAJD** generates an optimal solution $\mathbf{k}^*$ for **JD**. Here we give a self-contained proof showing that in addition **DAJD** generates an optimal solution for **RJP** as well and that $\underline{c}^* = c^*$. We first need the following lemma.

Lemma 2. *Fix* $\gamma > 0$. *Let* $E \subseteq N$ *and* $\kappa(\cdot)$ *be a rank function on E. Let* $E_1 \subset E$ *be a set achieving* $\min\{\kappa(S) + \gamma H(E \backslash S), S \subset E\}$ *and* $E_2 \stackrel{\text{def}}{=} E \backslash E_1$. *Let*

$$\kappa_1(S) \stackrel{\text{def}}{=} \kappa(S), \; S \subset E_1,$$

$$\kappa_2(S) \stackrel{\text{def}}{=} \kappa(S \cup E_1) - \kappa(E_1), \quad S \subseteq E_2.$$

Then

a. $\kappa_1(E_1)/H(E_1) \leqslant \gamma \leqslant \kappa_2(E_2)/H(E_2)$,

b. *For any* $\underline{E}_1 \subset E_1$,
$[\kappa_1(E_1) - \kappa_1(\underline{E}_1)]/H(E_1 \backslash \underline{E}_1) \leqslant \gamma$,
For any $\underline{E}_2 \subset E_2$, $\kappa_2(\underline{E}_2)/H(\underline{E}_2) \geqslant \gamma$.

Proof. Since $E_1 \subset E$ is a set achieving $\min\{\kappa(S) + \gamma H(E \backslash S), S \subseteq E\}$,

$$\kappa(E_1) + \gamma H(E_2) \leqslant \begin{cases} \gamma H(E) \\ \kappa(E) \\ \kappa(\underline{E}_1) + \gamma H(E \backslash \underline{E}_1) \\ \kappa(E_1 \cup \underline{E}_2) + \gamma H(E_2 \backslash \underline{E}_2). \end{cases}$$

The inequalities of part a as well as part b follow by rearranging the terms of these inequalities.

Theorem 2. *Let* $\{N_1, \ldots, N_M\}$ *and* $\mathbf{k}^*$ *denote the partition and setup cost vector generated by* **DAJD**. *For* $l = 1, \ldots, M$ *define*

$$T(l) = [K_l(N_l)/H(N_l)]^{1/2}$$

and $T_i^* = T(l)$ *for all* $i \in N_l$. *The vector* $\mathbf{T}^*$ *is optimal for* **RJP**, $\mathbf{k}^*$ *is optimal for* **JD** *and* $c^* = \underline{c}^*$.

Proof. In view of Theorem 1, it suffices to show that

$$K_l(S)/H(S) \geqslant T^2(l) \quad \text{for all } S \subseteq N_l \quad (l = 1, \ldots, M); \tag{4}$$

$$K_1(N_1)/H(N_1) \leqslant K_2(N_2)/H(N_2) \leqslant \ldots \leqslant K_M(N_M)/H(N_M). \tag{5}$$

Fix $l = 1, \ldots, M$. Since $S^* = N_l$ achieves $\min\{K_l(S) + \gamma H(N_l \backslash S): S \subseteq N_l\}$ with $\gamma = K(N_l)/H(N_l)$ (see Step 1), (4) follows from algebra.

To prove (5), consider a pair of sets (N_l, N_{l+1}) $(l = 1, \ldots, M - 1)$. Note that N_l and N_{l+1} belong to the same set in the initial partition (the singleton $\{N\}$). Consider now the *last* iteration at which a partition $\{\tilde{N}_1, \ldots, \tilde{N}_{\tilde{M}}\}$ is generated in which one of the product sets (say $\tilde{N}_s$) contains N_l as well as N_{l+1}. In the iteration thereafter $\tilde{N}_s$ is partitioned into two sets $\underline{N}_s$ and $\bar{N}_s$ with $N_l \subseteq \underline{N}_s$, $N_{l+1} \subseteq \bar{N}_s$, and $\underline{N}_s$ achieves the minimum in $\min\{\tilde{K}_s(S) + u^s(\tilde{N}_s \backslash S): S \subseteq \tilde{N}_s\}$ with $u^s = \{H_i \tilde{K}_s(\tilde{N}_s)/H(\tilde{N}_s): i \in \tilde{N}_s\}$ and

$$\tilde{K}_s(S) = K\left(\bigcup_{i=1}^{s-1} \tilde{N}_i \cup S\right) - K\left(\bigcup_{i=1}^{s-1} \tilde{N}_i\right).$$

Apply Lemma 2 with $\gamma = [\tilde{K}_s(\tilde{N}_s)/H(\tilde{N}_s)]$, $E = \tilde{N}_s$, $\kappa(\cdot) = \tilde{K}_s(\cdot)$, $\underline{E}_1 = \underline{N}_s \backslash N_l$ and $\underline{E}_2 = N_{l+1}$, to conclude that

$$K_l(N_l)/H(N_l)=\left[K\left(\bigcup_{j=1}^{l}N_j\right)-K\left(\bigcup_{j=1}^{l-1}\Big/ N_j\right)\right]\Big/ H(N_l)$$

$$=\left[K\left(\left(\bigcup_{r=1}^{s-1}\tilde{N}_r\right)\cup \underline{N}_s\right)\right.$$

$$\left.-K\left(\left(\bigcup_{r=1}^{s-1}\tilde{N}_r\right)\cup \underline{N}_s\backslash N_l\right)\right]\Big/ H(N_l)$$

$$=[\tilde{K}_s(\underline{N}_s)-\tilde{K}_s(\underline{N}_s\backslash N_l)]/H(N_l)\leq\gamma$$

$$\leq[\tilde{K}_s(\underline{N}_s\cup N_{l+1})-\tilde{K}_s(\underline{N}_s)]/H(N_{l+1})$$

$$=\left[K\left(\left(\bigcup_{r=1}^{s-1}\tilde{N}_r\right)\cup \underline{N}_s\cup N_{l+1}\right)\right.$$

$$\left.-K\left(\left(\bigcup_{r=1}^{s-1}\tilde{N}_r\right)\cup \underline{N}_s\right)\right]\Big/ H(N_{l+1})$$

$$=K_{l+1}(N_{l+1})/H(N_{l+1})$$

We also observe the following.

Remark. There exists an allocation vector $\mathbf{k}^* \in \mathbf{K}$ of the joint cost structure such that the separable cost model with (separable) setup cost vector $\mathbf{k}^*$ is equivalent to **RJP** in the sense that the same vector of replenishment intervals T is optimal in both models and the average cost under this policy is identical in both models.

2.1. The Second Stage: Rounding Procedures

We now discuss rounding procedures which transform an optimal solution $\mathbf{T}$ for the continuous relaxation **RJP** into a power-of-two vector $\mathbf{T}^*$. We distinguish between the following two cases:

a. T_L, the base planning period, is given. In most practical planning problems, replenishment intervals are to be chosen as multiples of some convenient time period, say an hour, day or a week, i.e., in practice T_L is indeed usually *prespecified.*
b. T_L is variable.

RPFB: Rounding Procedure for Fixed Base Planning Period

For all $l = 1, \ldots, M$, find that unique integer m_l such that

$$2^{m_l-1/2}T_L\leq T(l)<2^{m_l+1/2}T_L,$$

and set the common reorder interval for N_l as $T^*(l) = 2^{m_l}T_L$.

Now we turn to the case where T_L is variable. Note that its choice may be restricted to the interval $[1, 2)$, without loss of generality. The rounding procedure embeds **RPFB** in an efficient one-dimensional minimization over T_L in $[1, 2)$. First, to normalize the intervals $\{T(l)\colon l = 1, \ldots, M\}$, compute, for each $l = 1, \ldots, M$, the (unique) integer m_l and the value $b(l)$ such that

$$1\leq b(l)\overset{\text{def}}{=}\frac{T(l)}{2^{m_l-1/2}}<2. \tag{6}$$

For a given $l = 1, \ldots, M$, note that if T_L is chosen such that $1 \leq T_L < b(l)$, $1 \leq (b(l)/T_L) < 2$, and hence

$$2^{m_l-1/2}T_L\leq T(l)<2^{m_l+1/2}T_L,$$

so that **RPFB** would set $T(l) = 2^{m_L}T_L$. Similarly, if T_L is chosen such that

$$b(l)\leq T_L<2,\ 1/2<\frac{b(l)}{T_L}\leq 1$$

and hence

$$2^{m_l-3/2}T_L<T(l)\leq 2^{m_l-1/2}T_L,$$

so that **RPFB** would set $T^*(l) = 2^{m_l-1}T_L$.

An efficient minimization over T_L in $[1, 2)$ thus starts by ranking the values $\{b(l)\colon l = 1, \ldots, M\}$ in nondecreasing order. For each $m = 1, \ldots, M$ let $l(m)$ be the corresponding set index, i.e., $b(l(m)) = b_m$. Let $\{b_1, \ldots, b_M\}$ be an enumeration of these values, arranged in nondecreasing order and set $b_0 = 1$ and $b_{M+1} = 2$. If $b_m \leq T_L < b_{m+1}$ $(0 \leq m < M)$, we have $T^*(l) = 2^{m_l-1}T_L$ for all l with $b(l) \leq b_m$ and $T^*(l) = 2^{m_l}T_L$ for all other $l = 1, \ldots, M$. Thus, let $2^{(m)}$ be the N vector defined by

$$2_i^{(m)}=\begin{cases}2^{m_l}, & \text{if } i\in N_l \text{ and } b(l)>b_m\\ 2^{m_{l}-1}, & \text{if } i\in N_l \text{ and } b(l)\leq b_m.\end{cases}$$

Hence $C_{T_L}(\mathbf{T}^*) = K'_m/T_L + H'_m\,T_L$, where

$$K'_m = K[2^{(m)}] \text{ (see (1))} \tag{7}$$

$$H'_m=\sum_{l:b(l)\leq b_m}H(N_l)2^{m_l-1}+\sum_{l:b(l)>b_m}H(N_l)2^{m_l}(m=0,\ldots,M). \tag{8}$$

Now $C_{T_L}(\mathbf{T}^*)$ is continuous in T_L and although convex on each of the intervals $[b_m, b_{m+1})$, $m = 0, \ldots, M$, it is not in general convex over the entire interval $[1, 2)$; see Appendix D.

It is therefore necessary to determine the minimum of $C_{T_L}(\mathbf{T}^*)$ on all $M + 1$ intervals $[b_m, b_{m+1})$ $(m = 0, \ldots, M)$.

RPVB: Rounding Procedure for Variable Base Planning Period

Step 1. For $l = 1, \ldots, M$ compute $b(l)$ and m_l as in (6); rank $\{b(l)\colon l = 1, \ldots, M\}$ in nondecreasing order.

Step 2. Initialize $m := 0$; $K := K_0'$ and $H := H_0'$ (see (7) and (8)).

$T_L = 1$; $x = +\infty$.

Step 3. If $-K/b_m^2 + H \geq 0$, then

$\{x := \min[x, K/b_m + Hb_m]$; $T_L := b_m$; go to Step 4$\}$;

If $-K/b_{m+1}^2 + H \leq 0$, then

$\{x := \min[x, K/b_{m+1}]$; $T_L := b_{m+1}$; go to Step 4$\}$;

$x := \min\{x, 2\sqrt{KH}\}$; $T_L := \sqrt{K/H}$.

Step 4. $m := m + 1$; $l := l(m)$; $K := K + 2^{-m_l}$ $[K(N_1 \cup \ldots \cup N_l) - K(N_1 \cup \ldots \cup N_{l-1})]$;

$H := H - H(N_l)\, 2^{m_l - 1}$; if $m \leq M$ go to Step 3, otherwise for $l = 1, \ldots, L$, set $T^*(l) = 2^{m_l} T_L$.

The first rounding procedure **RPFB** requires $O(n)$ operations. In the second rounding procedure, the total amount of work performed in Steps 3 and 4 is linear in n as well and is thus dominated by the computational effort involved in ranking the numbers $\{b(l)\colon l = 1, \ldots, M\}$ in Step 1, which is $O(n \log n)$.

The following theorem and corollary show that the two-stage algorithms result in optimal power-of-two policies. For the proof we refer to Zheng where this result is demonstrated in a more general setting.

These optimality results are similar to those obtained by Maxwell and Muckstadt (1985) and Roundy (1986) for the models considered ibid.

Theorem 3. *Assume that the base planning period T_L is fixed. The two-stage procedure* **DAJD-RPFB** *results in an optimal power-of-two policy.*

Corollary 1. *Assume that the base planning period T_L is variable. The two-stage procedure* **DAJD-RPVB** *results in an optimal power-of-two policy.*

We also observe the following corollary.

Corollary 2. *There exists an allocation vector* $\mathbf{k}^* \in \mathbf{K}$ *of the joint cost structure, such that the separable cost model with (separable) setup cost vector* $\mathbf{k}^*$ *is equivalent to the original model in the sense that the two models share the same optimal power-of-two policy and the long-run average cost under this policy is identical in both models.*

3. A LOWER BOUND THEOREM AND WORST CASE ANALYSIS

The following theorem demonstrates the benefit of solving the continuous relaxation **RJP** as an intermediate step in the solution procedure. This theorem bears some similarity to the lower bound theorems of Roundy (1985, 1986).

Theorem 4. (The Lower Bound Theorem) *Let c^* be the minimal objective function value in* **RJP**; *c^* is a lower bound for the average cost of any feasible schedule over any finite horizon.*

Proof. Let C be the sum of all costs incurred by a given feasible schedule over the time horizon $[0, \tau)$ $(\tau > 0)$; we will show that $C \geq c^*\tau$. For any $S \subseteq N$, let $J(S)$ be the number of times in $[0, \tau)$ that an order is placed specifically for S. Let J_i be the number of times in $[0, \tau)$ that product i is included in an order. Obviously $J_i = \sum_{S: i \in S} J(S)$. Then for any $\mathbf{k} \in \mathbf{K}$, the total setup cost

$$\sum_S K(S)J(S) \geq \sum_S \left(\sum_{i \in S} k_i\right) J(S)$$

$$= \sum_{i \in N} \left[k_i \sum_{S: i \in S} J(S)\right] = \sum_{i \in N} k_i J_i.$$

For any $i \in N$ and $t > 0$, let I_i^t be the inventory of product i, measured in units of $\tfrac{1}{2}d_i$, at time t and under the given replenishment schedule. The total holding cost over $[0, \tau)$ is given by $\sum_{i \in N} H_i \int_0^\tau I_i^t\, dt$. Since I_i^t can be increased only when product i is included in an order and decreases continuously at unit rate in-between orders, we have

$$C \geq \sum_{i \in N} \left(k_i J_i + H_i \int_0^\tau I_i^t\, dt\right)$$

$$\geq \tau \sum_{i \in N} 2(k_i H_i)^{1/2}.$$

The second inequality follows from a well known result for single product lot sizing problems (see e.g., Carr and Howe 1962). We conclude that

$$C \geq \tau \max_{\mathbf{k} \in \mathbf{K}} 2 \sum_{i \in N} (k_i H_i)^{1/2} = \tau c^*.$$

The following theorem shows that the average cost of the power-of-two policy generated by the two-stage procedure **DAJD-RPFB** is no more than 6% larger than c^*, the lower bound for the minimum system-wide average costs.

Theorem 5. *Let T_L be fixed. The two-stage procedure* **DAJD-RPFB** *generates an optimal power-of-two policy* $\mathbf{T}^*$ *with a worst case performance ratio* 1.061, *i.e.,* $C(\mathbf{T}^*)/c^* \leqslant 1.061$.

Proof. Let $\mathbf{T}$ be the optimal solution of **RJP** obtained by the **DAJD** algorithm and let $(\alpha_1, \ldots, \alpha_n)$ be a permutation of the product indices such that

$$T_{\alpha_1}^{-1} \geqslant T_{\alpha_2}^{-1} \geqslant \ldots \geqslant T_{\alpha_n}^{-1}.$$

Note that the components of $\mathbf{T}^*$ may be ranked in the same way. Assume that $\mathbf{T}$ takes on M distinct values $T(1) < \ldots < T(M)$ and let $N_l = \{i: T_i = T(l)\}$, $l = 1, \ldots, M$. Thus by Lemma 1,

$$\begin{aligned} c^* &= \sum_{i=1}^{n} H_i T_i + \sum_{i=1}^{n} [K(\{\alpha_1, \ldots, \alpha_i\}) \\ &\quad - K(\{\alpha_1 \ldots, \alpha_{i-1}\})] T_{\alpha_i}^{-1} \\ &= \sum_{l=1}^{M} [H(N_l)T(l) + K_l(N_l)/T(l)] \end{aligned}$$

$$\begin{aligned} C(T^*) &= \sum_{i=1}^{n} H_i T_i^* + \sum_{i=1}^{n} [K(\{\alpha_1, \ldots, \alpha_i\}) \\ &\quad - K(\{\alpha_1, \ldots, \alpha_{i-1}\})] T_{\alpha_i}^{*-1} \\ &= \sum_{l=1}^{M} [H(N_l)T^*(l) + K_l(N_l)/T^*(l)]. \end{aligned}$$

Thus, $C(\mathbf{T})$ and $C(\mathbf{T}^*)$ represent the sum of M independent EOQ cost functions evaluated at the minimizing intervals $T(l)$, $(l = 1, 2, \ldots, M)$ and intervals $T^*(l)$ with $1/\sqrt{2} \leqslant T^*(l)/T(l) \leqslant \sqrt{2}$, respectively. It is thus easy to verify and well known since Brown (1978) that each of the terms in $C(\mathbf{T}^*)$ is at most 6% larger than the corresponding term in $C(\mathbf{T})$.

A slight modification of the proof of the 98% theorem in Roundy (1986) exhibits that the worst case bound may be decreased to 2% by employing the second rounding procedure **RPVB**.

Theorem 6. *Let T_L be variable. The two-stage procedure* **DAJD-RPVB** *generates an optimal power-of-two policy with a worst case performance ratio* 1.021, *i.e.,* $C(\mathbf{T}^*)/c^* \leqslant 1.021$.

4. A DIRECT ALGORITHM FOR THE INTEGER PROBLEM JP

Upon inspection of the two-stage algorithm in Section 2, we find that much of the computational effort there is wasted if one is only interested in determining an efficient power-of-two policy. Consider, for example, a system with 100 items, separable setup costs, and for all $i = 1, \ldots, 100$, $H_i = 1$ and setup costs $K_i = (1 + \epsilon)^{2i}$, where $\epsilon = 10^{-100}$. Assume that the base planning T_L is fixed at $T_L = 1$. The first step of the algorithm requires more than 100 iterations to divide $N = \{1, 2, \ldots, 100\}$ into 100 singletons $\{1\}$, $\{2\}, \ldots, \{100\}$ with only slightly different replenishment intervals $T_i = (1 + \epsilon)^i$. In the second step, one finds that since for all $i = 1, \ldots, 100$, $1/\sqrt{2}T_L \leqslant T_i < \sqrt{2}\, T_L$, all T_i's are rounded to $T_i^* = 1$, i.e., the 100 singletons are regrouped into a single cluster.

While this example may appear extreme, note that in an optimal power-of-two vector *very few distinct* components are used. For example, if the number of distinct replenishment intervals were larger than 10, then some products would be replenished at least $2^{10} = 1{,}024$ times less frequently than others, a situation that could hardly be economical in any real-world application. It is, therefore, natural to search for a decomposition algorithm in which the number of iterations (partitions) is limited to the small number of distinct components in an optimal power-of-two vector.

Indeed, in this section we develop a direct algorithm which generates an optimal solution for **JP** *directly* rather than via a rounding procedure applied to the solution of **RJP**, its continuous relaxation. This algorithm is based on the following characterization theorem giving necessary and sufficient conditions for a vector T^* to be optimal in the integer program **JP**. These conditions bear considerable similarity to the ones for the characterization theorem in Section 2.

Theorem 7. *Assume that the vector* $\mathbf{T}^* = (T_1^*, T_2^*, \ldots, T_n^*)$ *takes on M distinct power-of-two values* $T(1) < T(2) < \ldots < T(M)$, *and* $(N_1, N_2, \ldots, N_M)$ *is a partition of N with* $N_l = \{i \in N: T_i^* = T(l)\}$, *then T^* is an optimal solution for* **JP** *if and only if the following two conditions hold for* $l = 1, 2, \ldots, M$. *Define the set functions* $K_l(\cdot)$ *by*

$$K_l(S) = K\left(\bigcup_{j=1}^{l-1} N_j \cup S\right) - K\left(\bigcup_{j=1}^{l-1} N_j\right), \quad S \subseteq N_l,\ l = 1, \ldots, M.$$

Then

i. $1/\sqrt{2}\,(K_l(N_l)/H(N_l))^{1/2} \leqslant T(l) \leqslant \sqrt{2}\,(K_l(N_l)/H(N_l))^{1/2}$;

ii. $1/\sqrt{2}\,T(l) \leqslant (K_l(S)/H(S))^{1/2}$, $((K_l(N_l) - K_l(S))/H(N_l \backslash S))^{1/2} \leqslant \sqrt{2}\,T(l)$, $S \subseteq N_l$.

Proof. We first prove the necessity part. Let $\mathbf{T}^*$ be an optimal solution of **JP**. Assume to the contrary that for some j,

$$(K_j(N_j)/H(N_j))^{1/2} < 1/\sqrt{2}\ T(j).$$

Let $\mathbf{T}'$ be defined by $T_i' = 1/2T(j)$, $i \in N_j$; $T_i' = T_i^*$, $i \notin N_j$. Let

$$C_l(t) \overset{\text{def}}{=} K_l(N_l)/t + H(N_l)t,\ l = 1, \ldots, M.$$

It follows from (1) and $T(1) < T(2) < \ldots < T(M)$ that

$$C(\mathbf{T}^*) = \sum_{l=1}^{M} C_l(T(l)),$$

$$C(\mathbf{T}') = \sum_{l \neq j} C_l(T(l)) + C_j(1/2\ T(j)).$$

It is easy to verify that $C_j(1/2T(j)) < C_j(T(j))$ since $T(j) > \sqrt{2}\ (K_j(N_j)/H(N_j))^{1/2}$ (see Figure 1). We thus have $C(\mathbf{T}') < C(\mathbf{T}^*)$, which contradicts the optimality of $\mathbf{T}^*$. The proof of the second inequality in i is analogous.

To prove ii, assume to the contrary that for some j and $S \subset N_j$, $(K_j(S)/H(S))^{1/2} < 1/\sqrt{2}\ T(j)$. Define $\mathbf{T}'$ by $T_i' = 1/2T_i^*(i \in S)$ and $T_i' = T_i^*$ otherwise. Then

$$\begin{aligned} C_j(T(j)) &= K_j(N_j)/T(j) + H(N_j)T(j) \\ &= K_j(S)/T(j) + H(S)T(j) + (K_j(N_j) \\ &\quad - K_j(S))/T(j) + H(N_j \backslash S)T(j) \\ &> K_j(S)/(1/2T(j)) + H(S)(\tfrac{1}{2}T(j)) \\ &\quad + (K_j(N_j) - K_j(S))/T(j) + H(N_j \backslash S)T(j) \end{aligned}$$

where the inequality follows from $T(j) > \sqrt{2(K_j(S)/H(S))}$ as above. Thus

$$\begin{aligned} C(\mathbf{T}^*) &= \sum_{l \neq j} C_l(T(l)) + C_j(T(j)) \\ &> \sum_{l \neq j} C_l(T(l)) + K_j(S)/(\tfrac{1}{2}T(j)) \\ &\quad + H(S)(\tfrac{1}{2}T(j)) \\ &\quad + (K_j(N_j) - K_j(S))/T(j) + H(N_j \backslash S)/T(j) \\ &= C(\mathbf{T}') \end{aligned}$$

where the last equality follows from (1). This contradicts the optimality of $\mathbf{T}^*$.

Now we prove the sufficiency part. Assume that $\mathbf{T}^*$ satisfies i and ii. Let the optimal value of **JP** be C^+, $D = \{T > 0;\ T_i = 2^{m_i}T_L,\ (m_i \text{ integer})\}$. Then

$$\begin{aligned} C^+ &= \min_{\mathbf{T} \in \mathbf{D}} \max_{\mathbf{k} \in \mathbf{K}} \sum_{i \in N} (k_i/T_i + H_i T_i) \\ &\geqslant \min_{\mathbf{T} \in \mathbf{D}} \max_{\{\mathbf{k} \in \mathbf{K}, \mathbf{k}^l \in \mathbf{K}_l, l=1,\ldots,M\}} \sum_{i \in N} (k_i/T_i + H_i T_i) \\ &= \min_{\mathbf{T} \in \mathbf{D}} \sum_{l=1}^{M} \max_{\{\mathbf{k}^l \in \mathbf{K}_l, l=1,\ldots,M\}} \sum_{i \in N_l} (k_i/T_i + H_i T_i) \end{aligned}$$

where the last equality follows from Lemma B1. Therefore

$$C^+ \geqslant \sum_{l=1}^{M} C_l^+$$

where for subproblem $\mathbf{JP}_l$

$$C_l^+ = \min_{\mathbf{T} > 0} \max_{\mathbf{k} \in \mathbf{K}_l} \sum_{i \in N_l} (k_i/T_i + H_i T_i)$$

subject to $T_i = 2^{m_i}T_L(m_i\text{: integer})$, $i \in N_l$.

We first show that $\{T_i = T(l),\ i \in N_l\}$ is an optimal solution for subprogram $\mathbf{JP}_l$, $l = 1, \ldots, M$. Then

$$C_l^+ = K_l(N_l)/T(l) + H(N_l)T(l)\ (l = 1, \ldots, M)$$

and

$$C^+ \geqslant \sum_{l=1}^{M} C_l^+ = \sum_{l=1}^{M} \{K_l(N_l)/T(l) + H(N_l)T(l)\} = C(\mathbf{T}^*)$$

in view of (1), thus proving the optimality of $\mathbf{T}^*$.

Fix $l = 1, \ldots, M$ and an optimal solution $\mathbf{T} \in R^{N_l}$ for $\mathbf{JP}_l$. Assume that $\mathbf{T}$ takes on distinct values $T(l_1) < T(l_2) < \ldots < T(l_a)$ with a corresponding partition $(N_{l_1}, N_{l_2}, \ldots, N_{l_a})$. By the above-proved necessity part of this theorem:

$$\sqrt{K_l(N_{l_1})/H(N_{l_1})} \leqslant \sqrt{2}T(l_1).$$

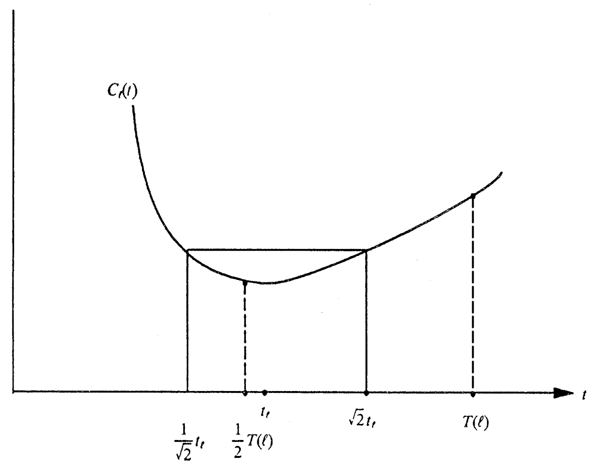

Figure 1. $t_l \equiv (K(G_l)/H(G_l))^{1/2}$.

On the other hand, by ii, letting $S = N_{l_1} \subseteq N_l$,

$$1/\sqrt{2}\,T(l) \leq \sqrt{K_l(N_{l_1})/H(N_{l_1})}.$$

Therefore, $1/\sqrt{2}\,T(l) \leq \sqrt{2}T(l_1)$ or $1/2T(l) \leq T(l_1)$. The equality holds if and only if

$$\sqrt{K_l(N_{l_1})/H(N_{l_1})} = 1/\sqrt{2}\,T(l) = \sqrt{2}\,T(l_1).$$

In this case, if we replace $T(l_1)$ by $T(l)$, the objective value in $\mathbf{JP}_l$ will not change. Therefore, we assume without loss of generality that $T(l_1) \geq T(l)$. By a similar argument, $T(l_a) \leq T(l)$, i.e., $T_i = T(l)$ for all $i \in N_l$.

Combining Theorem 7 and Lemma 2, it is easy to verify that the following modified decomposition algorithm solves **JP** for any fixed base planning period.

DAJP: Decomposition Algorithm for JP

Let $\tau_j \overset{\text{def}}{=} 2^{j-1/2}T_L$, j integer. Let l be the unique integer such that $\tau_l^2 \leq K(N)/H(N) < \tau_{l+1}^2$. For any partition $\{N_a, N_{a+1}, \ldots, N_b\}$ of $N(a < b)$, let

$$K_i(S) \overset{\text{def}}{=} K\left(\bigcup_{j=a}^{i-1} N_j \cup S\right) - K\left(\bigcup_{j=a}^{i-1} N_j\right), S \subseteq N_i, i = a, \ldots b;$$

let $\mathbf{K}_i$ be the polymatroid:

$$\mathbf{K}_i = \{\mathbf{k}^i \in R^{N_i}: k(S) \leq K(S), \quad S \subseteq N_i, \mathbf{k}^i \geq 0\}.$$

Step 0. Determine a set $S \subseteq N$ achieving min $\{K(S) + \tau_l^2\, H(N\backslash S)\}$; $a := l - 1$, $b := l$; $N_a := S$, $N_b := N\backslash S$.

Step 1. If $N_b = \phi$, then go to Step 2, otherwise, find a set $S \subseteq N_b$ achieving min $\{K_b(S) + \tau_b^2\, H(N_b\backslash S)\}$; $N_{b+1} := N_b\backslash S$, $N_b := S$; $b := b + 1$, go back to Step 1.

Step 2. If $N_a = \phi$, then go to Step 3, otherwise find a set $S \subseteq N_a$ achieving min $\{K_a(S) + \tau_a^2\, H(N_a\backslash S)\}$; $N_{a-1} := S$, $N_a := N_a\backslash S$; $a := a - 1$, go back to Step 2.

Step 3. For the final partition $(N_a, N_{a+1}, \ldots, N_b)$: let $T(m) = 2^m T_L$ be the common reorder interval of N_m, $m = a, \ldots, b$. **T** is an optimal solution of **JP**.

Theorem 8. *The* **DAJP** *algorithm generates an optimal power-of-two policy* **T**.

Proof. Let $(N_a, N_{a+1}, \ldots, N_b)$ be the final partition generated by the algorithm. We clearly have $T(a) < T(a + 1) < \ldots < T(b)$. In view of Theorem 7, it thus suffices to verify that conditions i and ii in the theorem hold.

Consider a set N_j with $a \leq j < l$. (The proof for the case $j \geq l$ is analogous.) Let $\tilde{N} = N_a \cup \ldots \cup N_j$; N_j was generated in an execution of Step 2 by determining min $\{K(S) + \tau_j^2\, H(\tilde{N}\backslash S)$: $S \subseteq \tilde{N}\}$. Apply Lemma 2 with $E = \tilde{N}$, $E_1 = \tilde{N}\backslash N_j$ and $E_2 = N_j$ to conclude that

$$\tau_j^2 = 1/2\ T^2(j) \leq [K(\tilde{N}) - K(\tilde{N}\backslash N_j)]/H(N_j)$$
$$= K_j(N_j)/H(N_j)$$

and

$$1/2\ T^2(j) \leq [K(\tilde{N}\backslash N_j \cup S) - K(\tilde{N}\backslash N_j)]/H(S)$$
$$= K_j(S)/H(S)$$

for all $S \subseteq N_j$, thus verifying the second inequality in i and the first one in ii.

Now consider the previous execution of Step 2 in which set N_{j+1} is created again by determining min $\{K(S) + \tau_{j+1}^2\, H(\tilde{N}\backslash S)$: $S \subseteq \tilde{N}\}$, now with $\tilde{N} = N_a \cup \ldots \cup N_{j+1}$. Apply Lemma 2 with $E = \tilde{N}$, $\underline{E}_1 = N_a U \ldots U N_{j-1} = \tilde{N}\backslash N_{j+1}\backslash N_j$ to conclude that

$$[K(N_a \cup \ldots \cup N_j) - K(N_a \cup \ldots \cup N_{j-1})]/H(N_j)$$
$$\leq \tau_{j+1}^2 = 2T^2(j),$$

thus verifying the first inequality in i. Apply Lemma 2 again with $\underline{E}_1 = N_a \cup \ldots \cup N_{j-1} \cup S$ $(S \subseteq N_j)$ to conclude that

$$[K_j(N_j) - K_j(S)]/H(N_j\backslash S)$$
$$= [K(N_a \cup \ldots \cup N_{j-1} \cup N_j)$$
$$- K(N_a \cup \ldots \cup N_{j-1} \cup S)]/H(N_j\backslash S)$$
$$\leq \tau_{j+1}^2 = 2\ T^2(j),$$

thus verifying the second inequality in ii.

We conclude this section with a brief discussion of the complexity of the two-stage algorithm as well as that of the direct algorithm **DAJP**.

Recall that the two-stage algorithm consists of the decomposition method **DAJD** followed by one of two rounding procedures **RPFB** or **RPVB**, depending upon whether the base planning period is fixed or variable, respectively. In Section 2 we verified that the complexity of these rounding procedures is $O(n)$ and $O(n \log n)$, respectively. This is, in most cases, dominated by the complexity of the first stage, the decomposition algorithm.

As pointed out in Section 2, the **DAJD** algorithm requires at most $2n - 1$ iterations. The complexity of the **DAJD** algorithm is thus $O(nQ)$ with Q as the time required to find a set S minimizing $f(S) + u(E\backslash S)$, where f is a rank function defined on a ground set $E \subseteq N$, and $u \in R^E$ is a constant vector. See Section 2 for a discussion of the magnitude of Q.

As far as the integrated algorithm **DAJP** is concerned, it is easy to verify that the number of basic iterations is given by $1 + L(= b - a + 2)$, the number of distinct power-of-two values in the optimal replenishment vector, which is small in all practical problems. If, e.g., L were larger than 10, some products are replenished every day (say), while others are replenished no more frequently than once in close to three years. The complexity of **DAJP** is clearly $O(LQ)$ and can be argued to be $O(Q)$.

5. A NUMERICAL EXAMPLE

We illustrate the algorithms presented in the previous sections by the example in Rosenblatt and Kaspi.

Example 2. A supermarket buys five different items from a single supplier. Items 1 and 2 must be transported under very cold conditions (a refrigerated truck). Item 3 must be transported under warm conditions (a regular truck). Items 4 and 5 can be transported in any truck, however, transporting item 4 under very cold conditions requires an additional $5 packaging cost. Transportation cost in a refrigerated truck is $60 and in a regular truck $50. In addition to transportation costs there are packaging costs for each item; see Table I. It is easy to verify that the setup costs are monotone and submodular.

The **DAJD** algorithm proceeds as follows.

Iteration 1. $l = 1$, $M = 1$, $N_1 = N$. $K_1(N_1) = 200$, $H(N_1) = 17{,}500$,

$$u^1 = \frac{200}{17{,}500}(1{,}000, 8{,}000, 2{,}500, 3{,}000, 3{,}000)$$

$$\approx (11.43, 91.43, 28.57, 34.29, 34.29).$$

The set $S^* = \{2, 4, 5\}$ achieves

$$\min\{K_1(S) + u^1(N_1 \backslash S), S \subseteq N_1\}.$$

We obtain the new partition $\{N_1, N_2\}$ with $N_1 = \{2, 4, 5\}$ and $N_2 = \{1, 3\}$.

Iteration 2. $l = 1$, $M = 2$, $N_1 = \{2, 4, 5\}$. $K_1(N_1) = 110$, $H(N_1) = 14{,}000$,

$$u^1 = (u_2^1, u_4^1, u_5^1) = \frac{110}{14{,}000}(8{,}000, 3{,}000, 3{,}000)$$

$$\approx (62.86, 23.57, 23.57).$$

The set $S^* = \{2, 4, 5\}$ achieves

$$\min\{K_1(S) + u^1(N_1 \backslash S), S \subseteq N_1\}.$$

The partition $\{N_1, N_2\}$ is maintained.

Table I
Data for Example 2

Item No.	Packaging Cost	d_i	h_i	H_i
1	30	1,000	2	1,000
2	20	8,000	2	8,000
3	15	5,000	1	2,500
4	20 (cold) 15 (warm)	3,000	2	3,000
5	10	6,000	1	3,000

Iteration 3. $l = 2$, $M = 2$, $N_2 = \{1, 3\}$.

$$K_2(N_2) = K(N) - K(N_1) = 200 - 110 = 90,$$

$$H(N_2) = 3{,}500,$$

$$u^2 = (u_1^2, u_3^2) = \frac{90}{3500}(1{,}000, 2{,}500) = (25.71, 64.29).$$

The set $S^* = \{3\}$ achieves $\min\{K_2(S) + u^2(N_2 \backslash S), S \subseteq N_2\}$. We obtain the new partition $\{N_1, N_2, N_3\}$ with $N_1 = \{2, 4, 5\}$, $N_2 = \{3\}$, $N_3 = \{1\}$.

We thus obtain the lower bound

$$c^* = \sum_{i=1}^{3} 2\sqrt{K_i(N_i)H(N_i)} = 2(\sqrt{110 \times 14000}$$

$$+ \sqrt{60 \times 2500} + \sqrt{30 \times 1000}) = 3602.94.$$

The vector $\mathbf{T} = (0.1732, 0.0886, 0.1549, 0.0886, 0.0886)$ is optimal for **RJP**. If the base planning period $T_L = 0.1$, we obtain, after rounding, $\mathbf{T}^* = (0.2, 0.1, 0.2, 0.1, 0.1)$ and $C(\mathbf{T}^*) = 110/0.1 + 90/0.2 + 14{,}000 \times 0.1 + 3{,}500 \times 0.2 = 3{,}650$. If T_L is variable, we obtain the vector $\mathbf{T}^{**} = (0.1718, 0.859, 0.1718, 0.859, 0.859)$ and $C(T^{**}) = 3{,}608.02$. We have $(C(\mathbf{T}^*) - c^*)/c^* \approx 1.31\%$ and $(C(\mathbf{T}^{**}) - c^*)/c^* \approx 0.15\%$.

For $T_L = 0.1$, the **DAJP** algorithm determines $\mathbf{T}^*$ by generating:

Iteration 1. $N_{-1} = S = \phi$, $N_0 = N$;
Iteration 2. $N_0 = S = \{2, 4, 5\}$, $N_1 = \{1, 3\}$;
Iteration 3. $N_1 = S = \{1, 3\}$, $N_2 = \phi$.

6. A COMPARISON BETWEEN POWER-OF-TWO AND FIXED PARTITION STRATEGIES

In this section, we make a brief comparison between power-of-two policies and fixed partition strategies, the second restriction approach reviewed in the Introduction of this paper. Fixed partition strategies may dominate power-of-two policies: A trivial example arises when the setup cost structure is independent, in which case, the fixed partition strategy, which replenishes each item by itself according to its specific EOQ rule, is optimal, and the average cost of an optimal power-of-two policy may exceed the optimal cost.

The following example shows, on the other hand, that the average cost of an optimal fixed partition strategy may exceed that of an optimal power-of-two policy by as much as 20%. The determination of the worst case gap between these two classes of strategies remains an open question.

Example 3. Consider the setup cost structure $K(S) = \max_{i \in S} K_i$, where $K_i = 3^i$, $i = 1, \ldots, n$; let $h_i = 3^{-i}$ and $d_i = 2$ ($i = 1, \ldots, n$). Verify that the vector **T** defined by $T_1 = (K_1/H_1)^{1/2} = 1$ and $T_i = ((K_i - K_{i-1})/H_i)^{1/2}$ ($i = 2, \ldots, n$) solves **RJP**. (Note that $T_1 < T_2 < \ldots < T_n$ and apply Theorem 1.) The optimal objective function value in **RJP**, c^* is thus given by

$$c^* = 2\left(\sum_{i=2}^{n} [(K_i - K_{i-1})H_i]^{1/2} + 1\right)$$
$$= 2\left(\sum_{i=2}^{n} [(3^i - 3^{i-1})3^{-i}]^{1/2} + 1\right)$$
$$= 2(\sqrt{2/3}(n-1) + 1).$$

It follows from Theorem 4 that the average cost of an optimal power-of-two policy $\mathbf{T}^*$ is given by

$$C(\mathbf{T}^*) \leq (1.021)2(\sqrt{2/3}(n-1)+1).$$

Below we verify that the strategy under which each item is replenished by itself, according to its own EOQ rule, is an optimal fixed partition strategy. The average cost C^F under this strategy is given by:

$$C^F = 2\sum_{i=1}^{n} (K_i H_i)^{1/2} = 2n$$

and note that

$$\lim_{n\to\infty} \frac{C^F}{C(\mathbf{T}^*)} = \frac{2}{(1.021)2\sqrt{2/3}} \doteq 1.2.$$

To verify that the partition $\{1, \ldots, n\}$ constitutes an optimal partition, it suffices to show for any $S \subseteq \{1, \ldots, n\}$ with $|S| \geq 2$ and $n^+ = \max\{i: i \in S\}$, the minimal cost $C(S\backslash\{n^+\}, \{n^+\})$ of the partition $\{S\backslash\{n^+\}, \{n^+\}\}$ is less than $C(\{S\})$, the minimal average cost of the partition $\{S\}$. Let n^- be the next to highest index in S, then

$$1/2C(S\backslash\{n^+\}, \{n^+\}) = 1 + \left(3^{n^-} \sum_{i \in S, i \leq n^-} H_i\right)^{1/2}$$
$$\leq 1 + (3^{n^+-1}(H(S) - 3^{-n^+}))^{1/2}$$
$$\leq (3^{n^+}H(S))^{1/2} = 1/2C(\{S\})$$

where the last inequality is equivalent to

$$1 + 2[3^{n^+-1}(H(S) - 3^{-n^+})]^{1/2} + 3^{n^+-1}(H(S) - 3^{-n^+}) \leq 3^{n^+}H(S)$$

which holds if and only if

$$2[3^{n^+-1}(H(S) - 3^{-n^+})]^{1/2} \leq 2[3^{n^+-1}(H(S) - 3^{-n^+})]$$

and the latter inequality is immediate from

$$3^{n^+-1}[H(S) - 3^{-n^+}] \geq 1.$$

APPENDIX A

Cost Evaluations for Example 1

The average cost C^R of policy R is clearly given by

$$C^R = \frac{1}{3}[K(\{1,2,3\}) + 2K(\{1,2\}) + K(\{1,3\})]$$
$$+ h_2\tau_2 + h_3\tau_3 + \frac{1}{3}h_1(5/2)$$
$$= (2 - \epsilon + 2 + 1)/3 + (1 + 2/3 + 5/6\epsilon)$$
$$= \frac{10}{3} + \epsilon/2.$$

In particular, for $\epsilon = 0.1$, $C^R = 3.3833$.

A stationary policy with constant reorder intervals T_1, T_2 and T_3 is called *nested* if $T_2 = mT_1$ and $T_3 = nT_1$ for integers $m, n \geq 1$. Under such a policy the system regenerates every lT_1 time units with l the least common multiple of m, n. Let $C(m, n, T_1)$ be the average cost of this policy and $\bar{C}(m, n) \overset{\text{def}}{=} \inf_{T_1>0} C(m, n, T_1)$

$$C(m, n, T_1) = \frac{1}{lT_1}[K(\{1,2,3\}) + \left(\frac{l}{m} - 1\right)K(\{1,2\})$$
$$+ \left(\frac{l}{n} - 1\right)K(\{1,3\})$$
$$+ \left(l - \frac{l}{m} - \frac{l}{n} + 1\right)K(\{1\}] + \sum_{i=1}^{3} h_i T_i$$
$$= \frac{1}{lT_1}\left\{(2-\epsilon) + \left(\frac{l}{m} - 1\right) + \left(\frac{l}{n} - 1\right)\right.$$
$$\left. + \left(l - \frac{l}{m} - \frac{l}{n} + 1\right)\epsilon\right\} + \left(m + \frac{4}{9}n + \epsilon\right)T_1$$
$$= \frac{1}{T_1}\left\{\left(\frac{1}{m} + \frac{1}{n}\right) + \left(1 - \frac{1}{m} - \frac{1}{n}\right)\epsilon\right\}$$
$$+ \left(m + \frac{4}{9}n + \epsilon\right)T_1.$$

Thus, when $\epsilon = 0.1$,

$$\bar{C}(m,n) = 2\left\{\left[\left(\frac{1}{m}+\frac{1}{n}\right)+\left(1-\frac{1}{m}-\frac{1}{n}\right)0.1\right]\cdot\left[m+\frac{4}{9}n+0.1\right]\right\}^{1/2}$$

$$= 2\left\{\left(0.1+\frac{0.9}{m}+\frac{0.9}{n}\right)\left(m+\frac{4}{9}n+0.1\right)\right\}^{1/2}$$

$$= 2\left\{\left(1.31+\left(\frac{0.4}{m}+\frac{0.4}{9}\right)n+0.9\left(m+0.1\right)\frac{1}{n}+0.1m+\frac{0.09}{m}\right\}^{1/2}.$$

It remains to be shown that $\inf_{m,n} \bar{C}(m, n) > C^R$. Note first that

$$\bar{C}(1,n) = 2\left\{1.31+0.4444n+\frac{0.99}{n}+0.19\right\}^{1/2}.$$

Since $\bar{C}(1, n)^2$ is convex in n and achieves its continuous minimum at $n^* = 1.49$, we have

$$\bar{C}(1,n) \geq \min\{\bar{C}(1,1); \bar{C}(1,2)\}$$
$$= \min\{3.426; 3.396\} > 3.396 > C^R.$$

For $m \geq 2$ we have

$$\bar{C}(m,n)$$
$$\geq 2\left\{1.31+\inf_{n\geq 1}\left[\left(\frac{0.4}{m}+\frac{0.4}{9}\right)n+0.9(m+0.1)\frac{1}{n}\right]+0.1m+\frac{0.09}{m}\right\}^{1/2}$$
$$= 2\left\{2\left[\left(\frac{0.4}{m}+\frac{0.4}{9}\right)(m+0.1)0.9\right]^{1/2}+0.1m+\frac{0.09}{m}+1.31\right\}^{1/2}.$$

The right-hand side is clearly increasing for $m \geq 2$. Thus, $\inf_{m\geq 2, n\geq 1} \bar{C}(m, n) = 3.414 > C^R$ as well.

APPENDIX B

Polymatroids: A Special Class of Polyhedral Sets

In this appendix, we exhibit several properties of polymatroids needed to prove Lemma 1 as well as several key results in this paper. We first recall some elementary properties of rank functions.

Lemma B1. *Let E, E_1 be finite sets with $E_1 \subset E$; let f, g be rank functions on E. Then*

a. *αf is a rank function on E for any $\alpha > 0$;*
b. *$f + g$ is a rank function on E;*
c. *the set function $\bar{f}$ defined on $E \backslash E_1$ by $\bar{f}(S) = f(E_1 \cup S) - f(E_1)$ $(S \subseteq (E \backslash E_1))$ is a rank function on $E \backslash E_1$.*

Proof. Parts a and b are immediate. To prove c, note that for any $T \subset S \subseteq (E \backslash E_1)$ and $j \in E \backslash E_1$ with $j \notin S$:

$$\bar{f}(S \cup \{j\}) - \bar{f}(S) = f(E_1 \cup S \cup \{j\}) - f(E_1 \cup S)$$
$$\leq f(E_1 \cup T \cup \{j\}) - f(E_1 \cup T)$$
$$= \bar{f}(T \cup \{j\}) - \bar{f}(T).$$

Lemma B2. *Let $(E_1, E_2, \ldots, E_m)$ be a partition of a finite index set E, and let f be a rank function on the ground set E. For each $l = 1, \ldots, m$ define the set function f_l: $2^{E_l} \rightarrow R$ by*

$$f_l(S) = f\left(\bigcup_{i=1}^{l-1} E_i \cup S\right) - f\left(\bigcup_{i=1}^{l-1} E_i\right), \quad S \subseteq E_l.$$

Then

a. *f_l is a rank function on $E_l (l = 1, 2, \ldots, m)$;*
b. *let $\mathbf{x} \in R^E$ and for each $l = 1, \ldots, m$, let $\mathbf{x}^l = (x_i: i = E_l)$, $l = 1, \ldots, m$; if $\mathbf{x}^l \in F(E_l, f_l)$ for all $l = 1, \ldots, m$, then $\mathbf{x} \in F(E, f)$.*

Proof. Part a is immediate from part c of Lemma B1. To prove part b, fix $S \subseteq E$ and let $S_l = S \cap E_l$, $l = 1, \ldots, m$. We show by induction that

$$\sum_{l=1}^{l'} x(S_l) \leq f\left(\bigcup_{l=1}^{l'} S_l\right), \quad l' = 1, \ldots, m.$$

It is clear that the inequality holds for $l' = 1$ since $\mathbf{x}^1 \in F(E_1, f)$. Assume that the inequality holds for l', $1 \leq l' < m$. Then

$$\sum_{l=1}^{l'+1} x(S_l) = \sum_{l=1}^{l'} x(S_l) + x(S_{l'+1})$$
$$\leq f\left(\bigcup_{l=1}^{l'} S_l\right) + f\left(\bigcup_{l=1}^{l'} E_l \cup S_{l'+1}\right) - f\left(\bigcup_{l=1}^{l'} E_l\right)$$
$$\leq f\left(\bigcup_{l=1}^{l'+1} S_l\right).$$

The last inequality follows from (2), and

$$\left(\bigcup_{l=1}^{l'} E_l\right) \cup \left(\bigcup_{l=1}^{l'+1} S_l\right) = \bigcup_{l=1}^{l'} E_l \cup S_{l'+1},$$

$$\left(\bigcup_{l=1}^{l'} E_l\right) \cap \left(\bigcup_{l=1}^{l'+1} S_l\right) = \bigcup_{l=1}^{l'} S_l.$$

Now, consider a linear program of the form

$$\max w(\mathbf{x}) = \sum_{i \in E} c_i x_i \quad (c_i \geq 0, i \in E = \{1, \ldots, n\}) \qquad (9)$$

subject to

$$\mathbf{x} \in \mathbf{F} = F(E, f).$$

Edmonds showed that an optimal solution may be obtained by the following greedy procedure, provided that $\mathbf{F}$ is a polymatroid.

Greedy Procedure for (9)

Step 0. Let $(\alpha_1, \ldots, \alpha_n)$ be a permutation of the variable indices such that

$$c_{\alpha_1} \geq c_{\alpha_2} \geq \ldots \geq c_{\alpha_n}.$$

Step 1. Set $x_{\alpha_1} = f(\{\alpha_1\})$ and

$$x_{\alpha_i} = f(\{\alpha_1 \ldots, \alpha_i\}) - f(\{\alpha_1, \ldots, \alpha_{i-1}\}) \qquad (i = 2, \ldots, n). \quad (10)$$

In the greedy procedure the variables are thus ranked in nonincreasing order of their coefficients in the objective function. Treating the variables in this order, they are sequentially fixed at the highest achievable value. It is easy to verify (see Edmonds 1970), that these highest achievable values are given by the right-hand sides of (10).

Lemma B3. *Let* $\mathbf{F} = F(E, f)$ *be a polymatroid.*

a. *The greedy procedure* (10) *solves* (9).
b. *If the coefficients in the objective function of* (9) *are distinct, then the optimal solution is unique.*
c. *Assume that the coefficients in the objective function of* (9) *take on m distinct values* $c(1) > c(2) > \ldots > c(m)$ *and let* $(E_1, \ldots, E_m)$ *be a partition of* $E = \{1, 2, \ldots, n\}$, *such that* $c_i = c(l)$ $(i \in E_l)$, $l = 1, \ldots, m$. $\bar{\mathbf{x}}$ *is an optimal solution of* (9) *if and only if*

$$\sum_{i \in E_l} \bar{x}_i = f\left(\bigcup_{j=1}^{l} E_j\right) - f\left(\bigcup_{j=1}^{l-1} E_j\right) \quad l = 1, \ldots, m.$$

Proof. Part a is due to Edmonds (see, also, Welsh 1976 and Frank and Tardos 1989). We repeat its proof to facilitate those of parts b and c. Let

$$E_l = \{\alpha_l\}, f_l(\{\alpha_l\}) = f(\{\alpha_1, \ldots, \alpha_l\}) - f(\{\alpha_1, \ldots, \alpha_{l-1}\}) \qquad \text{for } l = 1, \ldots, n.$$

The solution $\mathbf{x}^*$ generated by the greedy procedure is feasible in (2), in view of Lemma B2b applied to this partition and the set of rank functions. Its objective function value is

$$w(\mathbf{x}^*) = \sum_{l=1}^{n} c_{\alpha_l}(f(\{\alpha_1, \ldots, \alpha_l\}) - f(\{\alpha_1, e\alpha_{l-1}\}))$$

$$= \sum_{l=1}^{n} (c_{\alpha_l} - c_{\alpha_{l+1}}) f(\{\alpha_1, \ldots, \alpha_l\}) \quad (\text{set } c_{\alpha_{n+1}} = 0),$$

To prove that $\mathbf{x}^*$ is optimal it is thus sufficient to identify a feasible solution $\mathbf{y}^*$ for the dual problem:

$$\text{minimize} \sum_{S \subseteq E} f(S) y_S$$

subject to

$$\sum_{S: i \in S} y_S \geq c_i \quad (i \in E = \{1, 2, \ldots, n\})$$

with an identical objective function value. Let $y^*_S = c_{\alpha_i} - c_{\alpha_{i+1}}$ if $S = \{\alpha_1, \ldots, \alpha_i\}$ $(i = 1, \ldots, n)$ and $y^*_S = 0$ otherwise; $\mathbf{y}^*$ is dual feasible and its objective value equals $w(\mathbf{x}^*)$.

b. When the c-coefficients are distinct, $y^*_{\{\alpha_1, \ldots, \alpha_i\}} > 0$, $i = 1, \ldots, n$. Consider an optimal solution $\tilde{\mathbf{x}}$ of (9). By the complementary slackness theorem we must have

$$\sum_{l=1}^{i} \tilde{x}_{\alpha_l} = f(\{\alpha_1, \ldots, \alpha_i\}) \quad (i = 1, \ldots, n),$$

and hence $\tilde{\mathbf{x}} = \mathbf{x}^*$.

c. To prove the sufficiency part, it is, in view of part a, sufficient to verify that $\bar{\mathbf{x}}$ has the same objective function value as $\mathbf{x}^*$, the solution generated by the greedy procedure.

$$\sum_{i=1}^{n} c_i x^*_i = \sum_{i=1}^{n} c_{\alpha_i}(f(\{\alpha_1, \ldots, \alpha_i\}) - f(\{\alpha_1, \ldots, \alpha_{i-1}\}))$$

$$= \sum_{l=1}^{m} c(l)\left(f\left(\bigcup_{j=1}^{l} E_j\right) - f\left(\bigcup_{j=1}^{l-1} E_j\right)\right)$$

$$= \sum_{l=1}^{m} c(l) \sum_{i \in E_l} \bar{x}_i = \sum_{i=1} c_i \bar{x}_i.$$

To prove the necessity part, note that the vector $(\sum_{i \in E_1} x^*_i, \ldots, \sum_{i \in E_m} x^*_i)$ needs to be optimal in the aggregated linear program:

$$\text{maximize} \sum_{l=1}^{m} c(l)x_l$$

subject to

$$\sum_{l\in R} x_l \leq f\Big(\bigcup_{l\in R} E_l\Big), \quad R \subseteq \{1, \ldots, m\}$$

$\mathbf{x} \geq 0$, and apply part b.

Since the greedy procedure achieves the optimum in the linear program (9) for any polymatroid **F**, and since **K** is a polymatroid (see above), this establishes Lemma 1.

APPENDIX C

The Saddle Point Lemma

In this appendix we exhibit three equivalent saddle point conditions.

Lemma C4. *Let θ be an arbitrary subset of some Euclidean space and* $f: \Theta \times \mathbf{K} \rightarrow R$, $\mathbf{T}^* \in \Theta$, $\mathbf{k}^* \in \mathbf{K}$. *Then the following statements are equivalent*:

i. $\mathbf{T}^*$, $\mathbf{k}^*$ *is a saddle point of f over* $\Theta \times \mathbf{K}$, *i.e.*,

$$f(\mathbf{T}, \mathbf{k}^*) \geq f(\mathbf{T}^*, \mathbf{k}^*) \geq f(\mathbf{T}^*, \mathbf{k}) \quad \text{for all } \mathbf{T} \in \Theta,\ \mathbf{k} \in \mathbf{K}.$$

ii. $\min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}^*) \geq \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}^*, \mathbf{k})$

iii. $\min_{\mathbf{T}\in\Theta} \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}, \mathbf{k}) = \max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k})$,

$\mathbf{T}^*$ and $\mathbf{k}^*$ *are optimal solutions for* $\min_{\mathbf{T}\in\Theta} \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}, \mathbf{k})$ *and* $\max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k})$, *respectively, that is*

$$\max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}^*, \mathbf{k}) = \min_{\mathbf{T}\in\Theta} \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}, \mathbf{k})$$

and

$$\min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}^*) = \max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}).$$

Proof

i $\Rightarrow$ ii: immediate

$$\text{ii} \Rightarrow \text{iii}: \max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}) \geq \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}^*) \geq$$

$$\geq \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}^*, \mathbf{k}) \geq$$

$$\geq \min_{\mathbf{T}\in\Theta} \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}, \mathbf{k}) \geq \max_{\mathbf{T}\in\Theta} \min_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}, \mathbf{k}).$$

The second inequality is ii, all other inequalities hold generally for any f, Θ and **K**. Since the two extremes of the inequalities are the same, all these inequalities must hold as equalities, hence iii.

$$\text{iii} \Rightarrow \text{i}: \quad f(\mathbf{T}, \mathbf{k})^* \geq \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}^*)$$

$$= \max_{\mathbf{k}\in\mathbf{K}} \min_{\mathbf{T}\in\Theta} f(\mathbf{T}, \mathbf{k}) = \min_{\mathbf{T}\in\Theta} \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}, \mathbf{k})$$

$$= \max_{\mathbf{k}\in\mathbf{K}} f(\mathbf{T}^*, \mathbf{k}) \geq f(\mathbf{T}^*, \mathbf{k})\ \mathbf{T} \in \Theta, \mathbf{k} \in \mathbf{K}.$$

APPENDIX D

Properties of $C(\mathbf{T}^*)$ as a Function of the Base Planning Period T_L

Lemma D5. $C_{T_L}(\mathbf{T}^*)$ *is continuous in* T_L.

Proof. It suffices to show continuity in the finite set of points $\{b_r: r = 1, \ldots, M\}$. Fix $r = 1, \ldots M$. Note from (7) and (8) that

$$C_{b_r}(\mathbf{T}^*) - \lim_{T_L \uparrow b_r} C_{T_L}(\mathbf{T}^*)$$

$$= (K'_r - K'_{r-1})/b_r + (H'_r - H'_{r-1})b_r$$

$$= [K(N_1 \cup \ldots \cup N_r) - K(N_1 \cup \ldots \cup N_{r-1})]$$

$$\cdot 2^{-m_r}/b(r) - H(N_r)2^{m_r - 1}b(r)$$

$$= K(N_r)/T(r)\sqrt{2} - H(N_r)T(r)/\sqrt{2} = 0.$$

In general, $C(\mathbf{T}^*)$ fails to be convex. (Note that

$$\frac{d^+C_{b_m}(\mathbf{T}^*)}{dT_L} - \frac{d^-C_{b_m}(\mathbf{T}^*)}{dT_L}$$

$$= (K'_{m-1} - K'_m)/b_m^2 + H'_m - H'_{m-1} \leq 0).$$

ACKNOWLEDGMENT

The research of the first author was partially supported by National Science Foundation grant ECS-8604409.

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