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A CLASS OF EUCLIDEAN ROUTING PROBLEMS WITH GENERAL ROUTE COST FUNCTIONS*†

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In most vehicle routing problems, a given set of customers is to be partitioned into a collection of regions each of which is assigned to a single vehicle starting at a depot and returning there after visiting all of the region's customers exactly once in a route. In this paper we consider problem settings where the cost of a route may depend on its length ϑ as well as m, the number of points on the route, according to some general function $f(\vartheta, m)$, assumed to be nondecreasing and concave in ϑ.

We describe a class of $O(N \log N)$ or $O(N^2)$ heuristics and show under mild probabilistic assumptions that the solutions generated are asymptotically optimal. We also show that lower and upper bounds on the system-wide costs may be computed (with even simpler procedures) and that these bounds are asymptotically tight under the same assumptions.

1. Introduction. In most vehicle routing problems, a given set of geographically dispersed customers is to be partitioned into a collection of *regions* each of which is assigned to a single *vehicle* starting at a *depot* and returning there after visiting all of the region's customers exactly once in an efficient route. The costs are usually assumed to be proportional with the total distance driven by the vehicles, i.e., the total length of all routes (possibly in addition to fixed charges per route).

In this paper we consider problem settings where the cost of a route may depend on its length ϑ as well as on m, the number of customers included in the route, according to some general function $f(\vartheta, m)$ merely assumed to be nondecreasing and concave in ϑ. Our generalization is motivated by several optimization problems in the area of *integrated inventory control* and *vehicle routing* which may be reduced to special cases of the above described class of routing problems. In §4 we give an example of such an integrated model, and we refer the reader to several additional publications.

We consider the case where the number of vehicles, i.e., the number of regions, is fixed, as well as settings where the fleet size may be varied. It is assumed that the *capacity* of each vehicle is expressed in terms of an upper bound on the number of customers it may be assigned to; different vehicles may have different capacities. (As in previous papers on capacitated routing problems, we briefly discuss extensions where each customer is characterized by an integer weight and the vehicle capacity is expressed as a bound on the total weight of *all* customers in its region.) We assume that customers correspond to points in the plane and that the distance between any pair of customers is given by the Euclidean distance.

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We describe a class of relatively simple heuristics, with low complexity bounds and show under very mild probabilistic assumptions that the generated solutions are *asymptotically optimal*, as the number of customers increases to infinity. We also show that lower and upper bounds on the system-wide costs may be computed with even simpler procedures and that these bounds are *asymptotically tight* under the same assumptions.

To our knowledge and as mentioned above, the voluminous literature on the classical vehicle routing problem (VRP) (see Golden *et al.* 1977, Magnanti 1981 and Golden and Magnanti 1988 for some surveys) confines itself to the case where the (variable) cost of a route is strictly proportional with its length. (Some authors consider, on the other hand, additional operating constraints, e.g., with respect to the total distance travelled by each vehicle and permissible time-windows for each customer within which his delivery has to take place.) The proposed solution method falls in one of the following two categories:

1. *constructive and/or interchange heuristics unrelated to any specific mathematical programming formulation*: an initial feasible solution is constructed; this solution is sometimes used as the starting point of a local improvement procedure. With the exception of the recent paper by Haimovich and Rinnooy Kan (1985), discussed below, these methods fail to provide the user with an *ex ante* or even an *ex post* bound for the optimality gap of the generated solutions.

2. *algorithms based on (mixed) integer programming formulations*: many of these procedures have the distinct advantage of generating upon termination, bounds for the optimal solution value, in addition to a specific solution. They are, however, often difficult to implement since requiring the availability of (large-scale) linear programming codes to be used as subroutines, rather sophisticated matrix generators, etc. Bounds on computational requirements are usually unknown. The latter tend, however, to be large compared with, e.g., the first category of heuristics. No error bounds can be guaranteed even when the number of customers is large and, in general, no *ex ante* estimates for the minimal system-wide costs are available.

Haimovich and Rinnooy Kan (1985) showed that these difficulties and limitations can be overcome for certain stylized versions of the vehicle routing problem with Euclidean distances. Noticing that previous heuristics made no, or at best a limited use of the geometrical setting of the problem, they derived several simple and natural classes of heuristics, as well as easily computable bounds for the optimal solution value, both with strong asymptotic properties. Counter-balancing these advantages is the fact that the approach in Haimovich and Rinnooy Kan is restricted to settings with Euclidean distances and cannot easily be adopted to address operational routing constraints beyond the above elementary capacity constraint (e.g., time windows for individual deliveries). Many of the algorithms in categories 1 and 2 above are easily adopted to address a variety of such constraints. We refer to Magnanti (1981) and Golden and Magnanti (1988) for a detailed comparison between the various approaches in categories 1 and 2. Our procedures employ an approach which is based on that of Haimovich and Rinnooy Kan, and our results reduce to theirs for the special case where the cost of a route is strictly proportional with its length, i.e., $f(\vartheta, m) = \vartheta$, and where all vehicles are identical. We also use several of the results on general partitioning problems in Anily and Federgruen (1986).

In §2 we introduce some notation and describe the procedures which result in lower bounds on the optimal cost value. In §3 we describe a class of heuristics based on regional partitioning schemes as well as procedures for the computation of upper bounds on the optimal cost value. This section also contains our asymptotic analyses. §4 describes an application of our class of models to an infinite horizon integrated inventory control and vehicle routing problem.

2. Preliminaries; lower bounds. Let $X = \{x_1, \dots, x_N\}$ denote a set of N customers in the Euclidean plane, with r_i the distance between customer x_i and the depot x_0. The elements of X are numbered in ascending order of their radial distances, i.e., $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_N$. We choose the depot as the origin of the plane. Let L denote the number of vehicles available at the depot. L is sometimes treated as a decision variable and sometimes as a given parameter. The lth vehicle has capacity M_l^* $(l = 1, \dots, L)$ i.e., it may be assigned to at most M_l^* customers. The vehicles are numbered in ascending order of their capacities, i.e., $M_1^* \leqslant \cdots \leqslant M_L^* \stackrel{\text{def}}{=} M^*$. (Alternatively, the vehicle capacity may be determined by a maximum total load that may be carried on a route where customer x_i is to receive a delivery of w_i units (w_i integer). As pointed out in Haimovich and Rinnooy Kan 1985, this type of capacity constraint may easily be handled, provided it is allowed to satisfy a customer demand by more than one vehicle: we merely need to treat a customer with delivery size w as w customers with delivery size one.)

The cost of a route of length ϑ which visits m customers is given by $f(\vartheta, m)$ where $f\colon \mathscr{R}^2 \to \mathscr{R}$ is a general function. We assume without loss of generality that $f(0,0) = 0$. In addition we require that f be monotone in ϑ, i.e., $f \in F_0 = \{\varphi\colon \varphi$ is nondecreasing in $\vartheta\}$.

For a given set of points $Y \subset X$, let $TSP(Y)$ represent the length of the optimal traveling salesman tour through the points in the set Y. Let $\chi = \{X_1, \dots, X_L\}$ denote a partition of the set of customers X into L nonempty subsets, or *routes* i.e., $\bigcup_{l=1}^{L} X_l = X$ and $X_l \cap X_k = \varnothing$ for $1 \leqslant k < l \leqslant L$. Also, let $m_l = |X_l|$ and $X_l^0 = X_l \cup \{x_0\}$. A partition $\chi = \{X_1, \dots, X_L\}$ is feasible if and only if the number of points in X_l does not exceed the capacity bound M_l^*, $l = 1, \dots, L$. Our objective is to find a feasible partition of minimum cost where the cost of a partition $\chi = \{X_1, \dots, X_L\}$ is given by

$$U^*(\chi) = \sum_{l=1}^{L} f\big(TSP(X_l^0), m_l\big). \tag{1}$$

Let $V^*(X)$ denote the minimal cost value:

$$V^*(X) = \min\{U^*(\chi) \colon \chi = \{X_1, \dots, X_L\} \text{ partitions } X \text{ and } m_l \leqslant M_l^*,\ l = 1, \dots, L\}. \tag{2}$$

This partitioning problem is NP-complete, even in the simplest case where $f(\vartheta, m) = \vartheta$. Exact determination of $V^*(X)$ is therefore in general intractable for all but the smallest size problems.

Instead, we concentrate on *heuristic* solution methods. For a given heuristic H applied to the set X, let $V^H(X)$ denote the cost of the generated solution and define the *relative error*

$$e^H(X) = \frac{V^H(X) - V^*(X)}{V^*(X)}.$$

If $X_{(N)}$ denotes the first N points of a randomly generated sequence $\{x_1, x_2, \dots\}$, we call H *asymptotically optimal* if $\lim_{N\to\infty} e^H(X_{(N)}) = 0$ almost surely.

Within a given partition $\chi = \{X_1, \dots, X_L\}$ we number the routes in nondecreasing order of their cardinalities, i.e. $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_L$ and assign the lth route to the lth vehicle (with capacity M_l^*; $l = 1, \dots, L$). In a given partition, let $l(i)$ denote the index of the route to which the ith customer is assigned; we refer to the index function $l(\cdot)$ as the "route index function."

A partition is called *consecutive* if it consists of consecutive sets, i.e., sets in which the indices of the customers are consecutive integers. (For example, $\chi = \{X_1, X_2\} = \{\{4,5\}; \{1,2,3\}\}$ is a consecutive partition of $X = \{x_1, \dots, x_5\}$.) A partition is called *monotone* if the group index function is nondecreasing. (Note that a monotone partition is consecutive; the partition χ above fails to be monotone but $\chi^* = \{X_1^*, X_2^*\} = \{\{1,2\}; \{3,4,5\}\}$ is.) A partitioning problem of the set X with capacities $\{M_1^*, \dots, M_L^*\}$ is determined by a cost function U which assigns a cost $U(\chi)$ to each partition χ; the partitioning problem consists of determining (P): $\min\{U(\chi)\colon \chi$ is feasible$\}$.

A partitioning problem is said to be *extremal* if the following two properties are satisfied:

1. an optimal monotone partition exists;
2. let $\chi = \{X_1, \dots, X_L\}$ be a monotone partition of X. The cost of a partition χ' obtained by transferring the highest indexed element of some set X_l $(1 \leqslant l < L)$ to X_{l+1} is less than or equal to the cost of χ.

Lower bounds. We derive two lower bounds for $V^*(X)$ which may be efficiently computed for general functions $f(\cdot,\cdot)$ satisfying general structural properties. Thus, for any partition χ of X define lower bounds for $U^*(\chi)$:

$$\underline{U}^1(\chi) = \sum_{l=1}^{L} f\Big(2 \sum_{j \in X_l} r_j/m_l, m_l\Big), \tag{3}$$

$$\underline{U}^2(\chi) = \sum_{l=1}^{L} f\Big(2 \max_{j \in X_l} r_j, m_l\Big). \tag{4}$$

Clearly, $\underline{U}^1(\chi) \leqslant \underline{U}^2(\chi) \leqslant U(\chi)$ since $2\sum_{j \in X_l} r_j/m_l \leqslant 2\max_{j \in X_l} r_j \leqslant TSP(X_l^0)$ $(l = 1, \dots, L)$ and since $f \in F_0$ is nondecreasing in ϑ.

Next define

$$(\underline{P}^1)\colon \underline{V}^1(X) = \min\big\{\underline{U}^1(\chi)\colon \chi \text{ partitions } X \text{ and } m_l \leqq M_l^*,\, l = 1, \dots, L\big\}, \tag{5}$$

$$(\underline{P}^2)\colon \underline{V}^2(X) = \min\big\{\underline{U}^2(\chi)\colon \chi \text{ partitions } X \text{ and } m_l \leqq M_l^*,\, l = 1, \dots, L\big\}. \tag{6}$$

One easily concludes:

LEMMA 1. $\underline{V}^1(X) \leqslant \underline{V}^2(X) \leqslant V^*(X)$.

The next lemma shows that $\underline{V}^2(X)$ is often only slightly superior a lower bound than $\underline{V}^1(x)$. The conditions of the lemma are, for example, satisfied in the model in §4 as well as the models discussed in Anily and Federgruen (1988) and Anily (1986).

LEMMA 2. *Let $f \in F_0$ be nondecreasing in m. Assume $\underline{P}^1$ has a monotone optimal partition $\chi = \{X_1, \dots, X_L\}$. Then*

$$\underline{V}^2(X) \leqslant \underline{V}^1(X) + f(2r_N, m_L) - f\left(\frac{2}{m_1} \sum_{i=1}^{m_1} r_i, m_1\right).$$

PROOF.

$$\underline{V}^2(X) \leqslant \underline{U}^2(\chi) = \sum_{l=1}^{L-1} f\left(2 \max_{i \in X_l} r_i, m_l\right) + f(2r_N, m_L)$$

$$\leqslant \sum_{l=1}^{L-1} f\left(\frac{2}{m_{l+1}} \sum_{i \in X_{l+1}} r_i, m_{l+1}\right) + f(2r_N, m_L)$$

$$= \underline{V}^1(X) - f\left(\frac{2}{m_1} \sum_{i \in X_1} r_i, m_1\right) + f(2r_N, m_L).$$

The last inequality follows from $m_l \leqslant m_{l+1}$ for all $l = 1, \ldots, L-1$ and f nondecreasing in both arguments. ∎

The partitioning problems $\underline{P}^1$ and $\underline{P}^2$ are known to be NP-complete for general functions $f(\cdot,\cdot)$. On the other hand, if f is concave in ϑ, an optimal *consecutive* partition exists, both for $\underline{P}^1$ and $\underline{P}^2$ which can be determined by computing a shortest path in an acyclic network; see Chakravarty *et al.* (1982) and Anily and Federgruen (1986). (For $\underline{P}^2$ an optimal consecutive partition exists for any $f \in F_0$.) The complexity of this shortest path algorithm depends on whether L is variable or fixed and whether the capacities M_l^* $(l = 1, \ldots, L)$ are identical or not (see Table 1), but even in the worst case the complexity is only quadratic in N. In addition, significant simplifications in the determination of an optimal partition may be obtained by exploiting important qualitative properties of an optimal partition which arise when f has additional properties beyond monotonicity and concavity in ϑ. See Anily and Federgruen (1986) for a detailed discussion.

We thus assume, throughout the remainder of this paper, that

Main cost assumption. $f \in F_1 = \{\varphi \in F_0: \varphi \text{ is concave in } \vartheta\}$.

Thus let χ_1 and χ_2 be optimal partitions for $\underline{P}^1$ and $\underline{P}^2$ as determined by one of the algorithms in Anily and Federgruen (1986). Define the (possibly empty) sets

$$X_j^{(m)} = \{x_i: \chi_j \text{ assigns } x_i \text{ to a set of cardinality } m\}, \quad m = 1, \ldots, M^*,\ j = 1, 2. \tag{7}$$

For a general function $f \in F_1$, the convex hulls of the sets $\{X_j^{(m)}: m = 1, \ldots, M^*\}$ may overlap. However, if $f \in F_1$ has antitone differences, an optimal monotone partition exists both for $\underline{P}^1$ and $\underline{P}^2$, so that the convex hulls of the sets $\{X_j^{(m)}: m = 1, \ldots, M^*\}$ are contained in disjoint rings; see Figure 1. (A function $\varphi: \mathscr{R}^2 \to \mathscr{R}$ is said to have *antitone differences* if $\varphi(\vartheta + \Delta, m) - \varphi(\vartheta, m)$ is nonincreasing in m for $\Delta > 0$; see Anily and Federgruen 1986, Theorem 2 and Theorem 8.) In this case we refer to the set $X_j^{(m)}$ as a ring.

Anily and Federgruen (1986) establish sufficient conditions with respect to the function f under which the partitioning problems $\underline{P}^1$ and $\underline{P}^2$ are *extremal* in addition to possessing an optimal monotone partition. (One such condition is that $f \in F_1$ has antitone differences while being concave in m.) When the partitioning problem is extremal, an optimal partition may be found by a special algorithm (the Extremal Partitioning Algorithm) the complexity of which is only linear in N; see Table 1. Moreover, the optimal partition is independent of the specific choice of the function $f(\cdot,\cdot)$. For example, in the special case where the fleet size is variable and all vehicles

TABLE 1

The Complexity of Computing the Optimal Partition

Problem type	# of sets	Capacity constraints		# of elementary operations	# of evaluations of the cost function
optimal consecutive partition exists	given	yes	identical	$4NLM^*$	NM^*
			nonidentical	no efficient algorithm is known	
		no		$N^2L/2$	$N^2/2$
	variable	yes	identical	NM^*	NM^*
			nonidentical	no efficient algorithm is known	
		no		N^2	$N^2/2$
optimal monotone partition exists	given	yes	identical	$4NLM^*$	NM^*
			nonidentical	$4NLM^*$	NM^*
		no		$N^2L/2$	$N^2/2$
	variable	yes	identical	NM^*	NM^*
			nonidentical	$2N^2M^*$	NM^*
		no		N^2	$N^2/2$
problem is extremal	given	yes	identical	$4\min\{L, \lceil N/M^* \rceil\}$	0
			nonidentical	$4\min\{L, \lceil N/2 \rceil\}$	0
		no		1	0
	variable	yes	identical	$4\lceil N/M^* \rceil$	0
			nonidentical	$4\lceil N/2 \rceil$	0
		no		1	0

*The complexity counts given in Table 1 assume the points are numbered in ascending order of their attribute values.

identical ($M_l^* = M^*$ for all l), extremality implies that an optimal partition exists under which the first N *modulo* (M^*) points are assigned to a single set and all other points to consecutive sets of cardinality M^*; this is easily verified from the definition of extremality. Thus, in addition to the ring $X_j^{(M^*)}$ at most one other ring is nonempty, and if a second ring is needed, it contains at most $M^* - 1$ points assigned to a single set. The above mentioned conditions may, e.g., be used to establish extremality of all the $\underline{P}^1$ problems discussed in §4 and Anily (1986).

The regional partitioning heuristics discussed in the next section operate independently on each of the (nonempty) collections $X_j^{(m)}$, $m = 1, \ldots, M^*$. All points in a collection $X_j^{(m)}$ are assigned to a route with exactly m retailers. This feature is shared

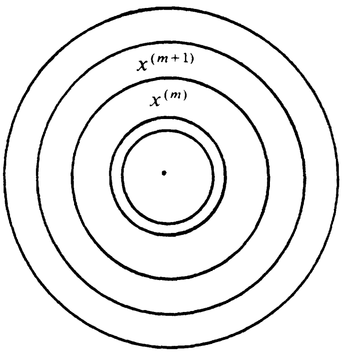

FIGURE 1. The Sets $X^{(m)}$.

with the heuristics proposed for the combined inventory allocation and routing problem in Federgruen, Rinnooy Kan and Zipkin (1985).

3. Heuristics and upper bounds; asymptotic behaviour. As mentioned in the Introduction, regional partitioning procedures exploit the geometrical setting of the problem. These heuristics partition the plane into geometrically compact regions each containing (no more than) a given number of points.

Haimovich and Rinnooy Kan (1985) consider three classes of regional partitioning schemes for the classical routing problem treated in their paper: the Rectangular Partitioning Procedure (RPP), the Polar Regional Partitioning Scheme (PRP), and the Circular Regional Partitioning Scheme (CRP). The (RPP) was introduced in Karp (1977) as the basis for a heuristic method of the TSP. Given N points in a rectangle and an integer $t \geqslant 2$, the (RPP) divides the original rectangle into 2^k subrectangles with $k = \lceil \log_2\{(N-1)/(t-1)\}\rceil$ each containing at most t points and such that each side of a subrectangle passes through one of the points in X or coincides with a side of the original rectangle; see Figure 2. The procedure alternates between vertical and horizontal cuts; see Karp (1977) for details.

In the (PRP) one considers a circle centered at the depot (the origin) which contains the set X. Karp's partitioning procedure is applied to this circle, with circular concentric arcs substituting for horizontal cuts and circle radii for vertical cuts; see Marchetti Spaccamela *et al.* (1985) for details. The (CRP) differs from the (PRP) in that the circle is first partitioned into a given number h ($h = O(N^{1/2})$) of equal sectors; each sector is then partitioned into several subregions by circular cuts, such that all of them contain exactly t points with the possible exception of the subregion closest to the depot. The collection of subregions containing less than t points (there are at most h such subregions) are then repartitioned by radial cuts into at most

(RPP)

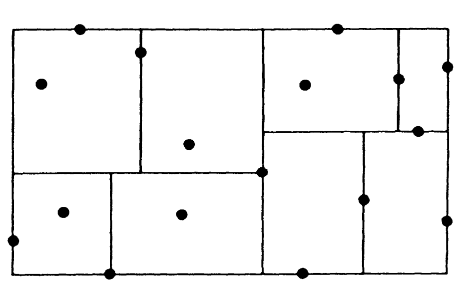

FIGURE 2. $N = 17$; $t = 3$; $2^k = 8$.

$(h - 1)$ subregions containing exactly t points and possibly one containing fewer than t points.

We now describe a heuristic for the model with *general* route-costs which is based on a slight modification of the (CRP). While it appears that the asymptotic performance analysis is most easily established for this specific partitioning scheme, it would be of interest to derive asymptotically optimal heuristics which are based on (slight modification of) the (RPP) and (PRP) as well. For the classical model in which the route cost is given by the length of the route $(f(\vartheta, m) = \vartheta)$ it is known from Haimovich and Rinnooy Kan (1985) that the three types of partitioning schemes result in solutions with quite similar asymptotic properties. We expect the same to be true when the route cost is given by a general function $f(\cdot, \cdot)$.

To design a heuristic and an associated upper bound on $V^*(X)$ we start by determining an optimal partition for either $\underline{P}^1$ or $\underline{P}^2$ by one of the procedures discussed in Anily and Federgruen (1986). Next, we apply the following modification of the (CRP) (which we will refer to as the Modified Circular Region Partitioning scheme (MCRP)) separately to each of the sets $X^{(m)}$, $m = 1, \ldots, M^*$ as defined in (7). In this section we assume that M^* is independent of N. Fix $m = 1, \ldots, M^*$.

Modified Circular Region Partitioning Scheme (MCRP)

Step 1. If $X^{(m)} \neq \varnothing$, $n_m := |X^{(m)}|$, $R_m := \max\{r_i \mid x_i \in X^{(m)}\}$ and $q_m := \lfloor n_m/(m\lfloor n_m^{1/2} \rfloor)\rfloor$. Otherwise, go to Step 4.

Step 2. Partition the circle with radius R_m into $\lfloor n_m^{1/2} \rfloor$ consecutive sectors containing mq_m points each and potentially one additional sector containing $n_m - \lfloor n_m^{1/2} \rfloor mq_m$ points. Let K_m denote the number of generated sectors. (Note $K_m = \lfloor n_m^{1/2} \rfloor$ or $K_m = \lfloor n_m^{1/2} \rfloor + 1$.) Let $S_k^{(m)}$ denote the kth generated sector, $k = 1, \ldots, K_m$.

Step 3. For each $k = 1, \ldots, K_m$ partition the sector $S_k^{(m)}$ by circular cuts such that each of the subregions contains m retailers and denote by $S_{k,l}^{(m)}$ the lth subregion in the kth sector, $l = 1, \ldots, |S_k^{(m)}|/m$.

Step 4. For each of the generated subregions, determine the optimal traveling salesman tour through the depot and the m points in the subregion.

We first make a number of preliminary observations regarding the (MCRP). In order to assess the computational complexity of the (MCRP), note that for a given value of $m = 1, \ldots, M^*$ Steps 2 and 3 can be implemented efficiently by first ranking the points in $X^{(m)}$ according to their angle coordinate and then in each sector separately according to their radial distances. The number of operations is thus bounded by $Cn_m \log n_m$ with an appropriate constant C. The total number of operations required for Steps 2 and 3 is thus bounded by

$$C \sum_{m=1}^{M^*} n_m \log n_m \leqslant C \max\left\{ \sum_{m=1}^{M^*} y_m \log y_m \,\middle|\, \sum_{m=1}^{M^*} y_m = N \right\} = CN \log N = O(N \log N)$$

and independent of M^*! (The first equality follows from the fact that the maximum of a convex optimization problem is achieved in an extreme point of the feasible region.) The total number of subregions generated is of course bounded by N, while the determination of the optimal traveling salesman tour for a given subregion is $O(1)$, as $N \to \infty$. The total amount of work required by Step 4 is thus linear in N.

We conclude that the asymptotic computational requirements of the (MCRP) are dominated by the amount of work required to determine the lower bounds $\underline{V}^1(X)$ or $\underline{V}^2(X)$; see Table 1.

When the sets $X^{(m)}$ are obtained by determining an optimal partition for $\underline{P}^i$ $(i = 1, 2)$ and the latter is extremal, additional observations may be made for the special case where all vehicles are identical, i.e., $M_l^* = M^*$ ($l = 1, \ldots, L$). When the fleet size L is fixed, at most *three* of the sets $X^{(m)}$ are nonempty, as is easily verified from Anily and Federgruen (1986). When L is variable, at most *two* of the sets $X^{(m)}$ are nonempty since, as observed above, the N *modulo* (M^*) points closest to the depot are assigned to $X^{(1)}$ while all the others are assigned to $X^{(M^*)}$.

Finally, one easily verifies from the proof of Theorem 3 below that the (MCRP) remains asymptotically optimal if in Step 4 heuristic rather than optimal traveling salesman tours are determined as long as a TSP heuristic is employed whose worst case relative error is bounded. This simplification was pointed out by Haimovich and Rinnooy Kan (1985) for their model (where $f(\vartheta, m) = \vartheta$). Note first that each of the generated subregions consists of m points. (Recall that n_m is a multiple of m.) The number of points assigned to a sector is $O(n_m^{1/2})$ as follows from the following lemma:

LEMMA 3. *The number of points in each sector* $S_k^{(m)}$ ($k = 1, \ldots, K_m$) *is bounded from above by* $m\lfloor n_m^{1/2} \rfloor + 3$.

PROOF. q_m satisfies the following inequalities:

$$q_m m \lfloor n_m^{1/2} \rfloor \leqslant n_m < (q_m + 1) m \lfloor n_m^{1/2} \rfloor. \tag{8}$$

Note that the number of points in each of the sectors $S_k^{(m)}$, $k = 1, \ldots, \lfloor n_m^{1/2} \rfloor$ satisfies the following inequalities:

$$\begin{aligned} |S_k^{(m)}| &= q_m m \leqslant n_m / \lfloor n_m^{1/2} \rfloor \leqslant \left(\lfloor n_m^{1/2} \rfloor + 1 \right)^2 / \lfloor n_m^{1/2} \rfloor \leqslant \lfloor n_m^{1/2} \rfloor + 2 + 1/\lfloor n_m^{1/2} \rfloor \\ &\leqslant m \lfloor n_m^{1/2} \rfloor + 3, \qquad k = 1, \ldots, \lfloor n_m^{1/2} \rfloor. \end{aligned}$$

If $K_m > \lfloor n_m^{1/2} \rfloor$ we get that

$$|S_{K_m}| = n_m - \lfloor n_m^{1/2} \rfloor q_m m < m \lfloor n_m^{1/2} \rfloor \quad \text{by inequality (8).}$$

Thus, $|S_k^{(m)}| \leqslant m\lfloor n_m^{1/2} \rfloor + 3$, $k = 1, \ldots, K_m$. ∎

Clearly, the cost of the generated solution is given by

$$V^H(X) = \sum_{m=1}^{M^*} \left\{ \sum_{k=1}^{K_m} \sum_{l=1}^{|S_k^{(m)}|/m} f\left(TSP\left(S_{k,l}^{0(m)} \right), m \right) \right\}. \tag{9}$$

When deriving an explicit upper bound for $V^H(X)$ and when analyzing the asymptotic properties of our heuristic, we need for each $m = 1, \ldots, M^*$ an upper bound for $\Sigma TSP(S_{k,l}^{(m)})$ the sum of the lengths of the traveling salesman tours through the points of each of the generated subregions in $X^{(m)}$. It follows first from Theorem 3 in Karp (1977) that

$$\sum_{k=1}^{K_m} \sum_{l=1}^{|S_k^{(m)}|/m} TSP\left(S_{k,l}^{(m)} \right) \leqslant TSP\left(X^{(m)} \right) + \Pi^{(m)MCRP} \qquad (m = 1, \ldots, M^*), \tag{10}$$

where $\Pi^{(m)MCRP}$ is the total perimeter of the generated subregions in $X^{(m)}$, $m = 1, \ldots, M^*$. Explicit $O(n_m^{1/2})$ bounds for both terms to the right of (10) may be obtained from the following two lemmas.

LEMMA 4 (See Theorem 2 in Haimovich and Rinnooy Kan 1985). *If a set of points X is contained in a connected planar region with area A and finite perimeter Π, then*

$$TSP(X) \leqslant (2|X|A)^{1/2} + 1.5\Pi.$$

LEMMA 5. *Let $\Pi^{(m)MCRP}$ be the total perimeter of the subregions generated by $(MCRP)$ in $X^{(m)}$, $m = 1, \ldots, M^*$. Then*

$$\Pi^{(m)MCRP} \leqslant R_m\big((4\pi + 2)\lfloor n_m^{1/2}\rfloor + 10\pi + 2\big), \qquad m = 1, \ldots, M^*.$$

PROOF. Fix $m = 1, \ldots, M^*$. Let C be the circle centered at the origin with radius R_m. Also, let

Π_1 = the total length of all circle cuts in $X^{(m)}$;

Π_2 = the perimeter of $C = 2\pi R_m$;

Π_3 = the total length of all radial cuts in $X^{(m)}$.

The total perimeter $\Pi^{(m)}$ of the subregions generated for $X^{(m)}$ is given by $\Pi^{(m)} = 2\Pi_1 + \Pi_2 + 2\Pi_3$ since each circle cut is adjacent to two subregions within a sector and each radial cut is adjacent to two sectors. The number of circle cuts performed in the kth sector is given by $|S_k^{(m)}|/m - 1$ which in view of Lemma 3 is bounded by $\lfloor n_m^{1/2}\rfloor + 2$. The total length of all circle cuts is thus bounded by $(\lfloor n_m^{1/2}\rfloor + 2)$ times the perimeter of C; hence, $\Pi_1 \leqslant 2\pi R_m(\lfloor n_m^{1/2}\rfloor + 2)$. Moreover, $\Pi_3 \leqslant K_m R_m \leqslant (\lfloor n_m^{1/2}\rfloor + 1)R_m$. The lemma now follows by simple algebra. ∎

The following theorem derives an explicit upper bound for $V^H(X)$. First, we need the following definitions:

Let $f_m^0 = \lim_{\vartheta \downarrow 0} f(\vartheta, m)$, $m = 1, \ldots, M^*$ (which exists since $f \in F_1$). Also, let

$$\underline{f}(\vartheta) = \min_{1 \leqslant m \leqslant M^*} \{f(\vartheta, m)\},$$

$$\alpha_m = \left(1 - \frac{1}{m}\right)\left((2\pi)^{1/2} + 6\pi + 3\right), \quad \beta_m = \frac{3}{2}\left(1 - \frac{1}{m}\right)(10\pi + 2) \quad \text{and}$$

$$\overline{V}^H(X) = \sum_{m=1}^{M^*} \sum_{k=1}^{K_m} \sum_{l=1}^{|S_{k,l}^{(m)}|/m} f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\right)$$

$$+ \sum_{m=1}^{M^*} \frac{n_m}{m}\left[f\left(\frac{m\alpha_m R_m}{\sqrt{n_m}} + \frac{\beta_m R_m}{n_m}, m\right) - f_m^0\right].$$

THEOREM 1 (upper and lower bounds). *For any set X*

$$\underline{V}^1(X) \leqslant \underline{V}^2(X) \leqslant V^*(X) \leqslant V^H(X) = \overline{V}^H(X).$$

PROOF. The first two inequalities follow from Lemma 1. $V^H(X)$ represents the cost of a feasible solution; therefore, $V^*(X) \leqslant V^H(X)$. It remains to be shown that $V^H(X) \leqslant \overline{V}^H(X)$. Observe from Theorem 1 in Haimovich and Rinnooy Kan (1985) that

$$V^H(X) = \sum_{m=1}^{M^*} \sum_{k=1}^{K_m} \sum_{l=1}^{|S_k^{(m)}|/m} f\left(TSP\left(S_{k,l}^{0(m)}\right), m\right) \tag{11}$$

$$\leqslant \sum_{m,k,l} f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i + \left(1 - \frac{1}{m}\right) TSP\left(S_{k,l}^{(m)}\right), m\right).$$

Since $f \in F_1$,

$$f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i + \left(1 - \frac{1}{m}\right) TSP\left(S_{k,l}^{(m)}\right), m\right) \tag{12}$$

$$\leqslant f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\right) + f\left(\left(1 - \frac{1}{m}\right) TSP\left(S_{k,l}^{(m)}\right), m\right) - f_m^0.$$

Let $\Omega(m) = (1 - 1/m)\sum_{k=1}^{K_m}\sum_{l=1}^{|S_k^{(m)}|/m} TSP(S_{k,l}^{(m)})$ for $m = 1,\ldots, M^*$. Substituting (12) into (11) we obtain

$$V^H(X) \leqslant \sum_{m,k,l} f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\right) + \sum_{m,k,l} \left[f\left(\left(1 - \frac{1}{m}\right) TSP\left(S_{k,l}^{(m)}\right), m\right) - f_m^0\right]$$

$$\leqslant \sum_{m,k,l} f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\right)$$

$$+ \sum_{m=1}^{M^*} \max\left\{ \sum_{k,l} \left[f(\xi_{k,l}, m) - f_m^0\right] : \sum_{k,l} \xi_{k,l} = \Omega(m)\right\}.$$

Since $f \in F_1$ the maxima within { } are achieved by equalizing all of the $\xi_{k,l}$, i.e., $\xi_{k,l} = m\Omega(m)/n_m$ for all k, l ($m = 1,\ldots, M^*$). Thus,

$$V^H(X) \leqslant \sum_{m,k,l} f\left(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\right) + \sum_{m=1}^{M^*} \frac{n_m}{m}\left[f\left(\frac{m\Omega(m)}{n_m}, m\right) - f_m^0\right].$$

In view of (10) Lemma 3, Lemma 4 and Lemma 5, we have:

(13)

$$\Omega(m) \leqslant \left(1 - \frac{1}{m}\right) TSP\left(X^{(m)}\right) + \frac{3}{2}\left(1 - \frac{1}{m}\right) \Pi^{(m)MCRP}$$

$$\leqslant \left(1 - \frac{1}{m}\right)\left(2\pi n_m R_m^2\right)^{1/2} + \frac{3}{2}\left(1 - \frac{1}{m}\right) R_m\left((4\pi + 2)\lfloor n_m^{1/2} \rfloor + 10\pi + 2\right)$$

$$\leqslant R_m\left(\alpha_m \sqrt{n_m} + \beta_m\right), \qquad m = 1,\ldots, M^*.$$

The inequality $V^H(X) \leqslant \bar{V}^H(X)$ thus follows from these bounds and $f \in F_0$. ■

The bound $\bar{V}^H(X)$ is used below to prove that the (MCRP) is asymptotically optimal. It should be noted that alternative and potentially more accurate upper bounds may be derived which can be computed after Step 3 of the (MCRP), i.e., ignoring the relatively time consuming Step 4 in which the traveling salesman tours need to be determined for all of the generated subregions. Such upper bounds on the system-wide costs (combined with the lower bound $\underline{V}^1(X)$ or $\underline{V}^2(X)$ may be sufficient for many planning problems.

Since $f \in F_1$, $\partial^+ f(\vartheta, m)/\partial\vartheta$, the right side partial derivative of f with respect to ϑ exists for all $\vartheta > 0$ (see Rockafeller 1970) and is nonincreasing. Thus, let $\delta_m = \partial^+ f(r_1, m)/\partial\vartheta$, $m = 1, \dots, M^*$. Define

$$\bar{V}^{(2)}(X) = \sum_{m=1}^{M^*} \sum_{k=1}^{K_m} \sum_{l=1}^{|S_k^{(m)}|/m} f\Bigl(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\Bigr) + \sum_{m=1}^{M^*} R_m \delta_m \bigl(\alpha_m \sqrt{n_m} + \beta_m\bigr).$$

THEOREM 2. *$V^H(X) \leqslant \bar{V}^{(2)}(X)$ for any set $X = \{x_1, \dots, x_N\}$.*

PROOF. It follows from (11), $f \in F_1$ and the definition of δ_m that

$$\begin{aligned} V^H(X) &\leqslant \sum_{m,k,l} f\Bigl(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\Bigr) + \sum_{m=1}^{M^*} \delta_m \Bigl(1 - \frac{1}{m}\Bigr) \sum_{k,l} TSP\bigl(S_{k,l}^{(m)}\bigr) \\ &= \sum_{m,k,l} f\Bigl(2m^{-1} \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i, m\Bigr) + \sum_{m=1}^{M^*} \delta_m \Omega(m), \end{aligned}$$

where $\Omega(m)$ is as defined in the proof of Theorem 1. Use (13) to complete the proof. ∎

The following theorem shows that $(MCRP)$ is asymptotically optimal:

THEOREM 3. *Let $\{x_1, x_2, \dots\}$ be a sequence of random points whose radial distances are i.i.d. Let $\eta = Ef(2r)$.*

(a) $\lim_{N\to\infty} \inf(1/N)\underline{V}^2(X_{(N)}) \geqslant \lim_{N\to\infty} \inf(1/N)\underline{V}^1(X_{(N)}) \geqslant \eta/M^*$ a.s.

(b) If, in addition, $\eta > 0$ and the radial distances are uniformly bounded by ρ, any implementation of the $(MCRP)$ is asymptotically optimal and the lower bounds $\underline{V}^1, \underline{V}^2$ and upper bounds V^H and $\bar{V}^H$ are asymptotically accurate, i.e., $\lim_{N\to\infty}[\bar{V}^H(X_{(N)}) - \underline{V}^1(X_{(N)})]/\underline{V}^1(X_{(N)}) = 0$ a.s.

PROOF. (a) Note first that since $f \in F_1$, $f(\cdot, m)$ is continuous for all $m = 1, \dots, M^*$, except possibly for $\vartheta = 0$. Thus $f(\vartheta)$ is continuous for $\vartheta > 0$ so that f is integrable and η exists. For any given N and a given realization $X_{(N)}$, let $\chi = \{X_1, \dots, X_L\}$ be a partition achieving $\underline{V}^1(X_{(N)})$. Thus, since $f \in F_1$,

$$\begin{aligned} \underline{V}^1(X_{(N)}) &= \sum_{l=1}^{L} f\Bigl(2 \sum_{i:\, x_i \in X_l} r_i/|X_l|, |X_l|\Bigr) \geqslant \sum_{l=1}^{L} \frac{1}{|X_l|} \sum_{i:\, x_i \in X_l} f(2r_i, |X_l|) \\ &\geqslant \frac{1}{M^*} \sum_{l=1}^{L} \sum_{i:\, x_i \in X_l} f(2r_i) = \frac{1}{M^*} \sum_{i=1}^{N} f(2r_i) \quad a.s. \end{aligned}$$

Part (a) now follows from the law of large numbers.

(b) In view of part (a) and Theorem 1 it suffices to show that

$$\lim_{N\to\infty} \frac{1}{N}\bigl[\bar{V}^H(X_{(N)}) - \underline{V}^1(X_{(N)})\bigr] = 0 \quad \text{a.s.}$$

We write

$$\bar{V}^H(X_{(N)}) = W_1(X_{(N)}) + W_2(X_{(N)}), \quad \text{where}$$

$$W_1(X_{(N)}) = \sum_{m,k,l} f\left(2m^{-1} \sum_{i_{x_i} \in S_{k,l}^{(m)}} r_i, m\right), \quad \text{and}$$

$$W_2(X_{(N)}) = \sum_{m=1}^{M^*} \frac{n_m}{m}\left[f\left(\frac{m\alpha_m R_m}{\sqrt{n_m}} + \frac{\beta_m R_m}{n_m}, m\right) - f_m^0\right].$$

We first show that $\lim_{N\to\infty}(1/N)W_2(X_{(N)}) = 0$ almost surely. Fix a realization of the sequence $\{x_1, x_2, \dots\}$. Since $f \in F_0$ and the expressions within [ ] are nonnegative, assume to the contrary that for some $m = 1, \dots, M^*$,

$$\lim_{N\to\infty} \sup \frac{n_m(N)}{mN}\left[f\left(\frac{m\alpha_m R_m(N)}{(n_m(N))^{1/2}} + \frac{\beta_m R_m(N)}{n_m(N)}, m\right) - f_m^0\right] = \gamma > 0.$$

Let $\{N_k\}_{k=1}^{\infty}$ *be a sequence of integers such that*

1. $\lim_{k\to\infty} n_m(N_k)$ exists;
2. $$\lim_{k\to\infty} \frac{n_m(N_k)}{N_k m}\left[f\left(\frac{m\alpha_m R_m(N_k)}{(n_m(N_k))^{1/2}} + \frac{\beta_m R_m(N_k)}{n_m(N_k)}, m\right) - f_m^0\right] = \gamma.$$

If $\lim_{k\to\infty} n_m(N_k) = \infty$, we have, in view of $n_m/N \leqslant 1$, $R_m \leqslant \rho$, the definition of f_m^0 and $f \in F_1$,

$$0 < \gamma \leqslant \frac{1}{m} \lim_{k\to\infty}\left[f\left(\frac{m\alpha_m \rho}{(n_m(N_k))^{1/2}} + \frac{\beta_m \rho}{n_m(N_k)}, m\right) - f_m^0\right] = 0.$$

Thus $\lim_{k\to\infty} n_m(N_k) < \infty$. But then, since $f \in \mathrm{F}_0$, we have

$$0 < \gamma \leqslant \lim_{k\to\infty} \frac{n_m(N_k)}{mN_k}\left[f(m\alpha_m\rho + \beta_m\rho, m) - f_m^0\right] = 0,$$

which leads to a contradiction.

It remains to be shown that

$$\lim_{N\to\infty} \frac{1}{N}\left[W_1(X_{(N)}) - \underline{V}^1(X_{(N)})\right] = 0 \quad \text{a.s.} \tag{14}$$

Once again fix a realization of the process $\{x_1, x_2, \dots\}$ and an integer N. We need to distinguish between two cases.

Case 1. The $(MCRP)$ is implemented after determining *an optimal partition* χ *for* $\underline{P}^1$: let $\chi = \{X_1, \dots, X_L\}$ be the optimal partition of $X_{(N)}$ [which achieves the lower

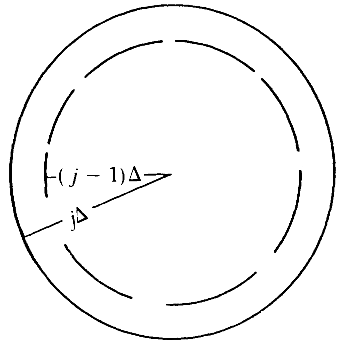

FIGURE 3. Ring C_j.

bound $\underline{V}^1(X_{(N)})$ and let $R_m = \max\{r_i : x_i \in X^{(m)}\}$ $(m = 1,\ldots, M^*)$. Let

$$W_1^{(m)} = \sum_{k=1}^{K_m} \sum_{l=1}^{|S_k^{(m)}|/m} f\left(2 \sum_{i:\, x_i \in S_{k,l}^{(m)}} r_i/m, m\right), \qquad m = 1,\ldots, M^*,$$

$$\underline{V}^{1(m)} = \sum_{l:\, m_l = m} f\left(2 \sum_{i:\, x_i \in X_l} r_i/m, m\right), \qquad m = 1,\ldots, M^*.$$

To prove (14) it suffices to show that

$$\lim_{N\to\infty} \frac{1}{N}\left[W_1^{(m)} - \underline{V}^{1(m)}\right] = 0 \quad \text{for all } m = 1,\ldots, M^*.$$

Thus, fix m $(1 \leqslant m \leqslant M^*)$. In the following we drop the superscripts (m) whenever possible.

Consider the circle which is centered at the origin and has radius R_m. Partition this circle into $\lceil n_m^{1/4}\rceil$ so-called "major circle rings" as follows. Define $\Delta = R_m/\lceil n_m^{1/4}\rceil$. Let C_1 be the closed circle, centered at the origin with radius Δ. For $j = 2,\ldots,\lceil n_m^{1/4}\rceil$ let C_j denote the half-open circle ring bordered by the two circles centered at the origin, with radius $(j-1)\Delta$ and $j\Delta$ and with only the outer circle included in the ring; see Figure 3.

Also let $C_{j,k}$ be the intersection of C_j with the kth sector $(j = 1,\ldots,\lceil n_m^{1/4}\rceil$, $k = 1,\ldots, K_m)$.

The proof is established by the determination of an upper bound $U \geqslant W_1^{(m)}$ and a lower bound $L \leqslant \underline{V}^{1(m)}$ which, unlike $W_1^{(m)}$ and $\underline{V}^{1(m)}$, are easily compared with each other. Define $E_j = \{l$: the lowest indexed point in X_l belongs to $C_j\}$.

Note that the collection $\{E_j,\ j = 1,\ldots,\lceil n_m^{1/4}\rceil\}$ partitions $\{l: X_l \in X^{(m)}\}$. Also let $\nu_j = |\{i: x_i \in X^{(m)}$ and x_i belongs to $C_j\}|$ $(j = 1,\ldots,\lceil n_m^{1/4}\rceil)$. Since χ is a consecutive partition, all the points in C_j belong to some set X_l with $l \in E_j$ with the possible exception of the $(m-1)$ closest points in C_j; see Figure 4. Hence

$$(15) \qquad m|E_j| \geqslant \nu_j - m + 1, \qquad j = 1,\ldots,\lceil n_m^{1/4}\rceil.$$

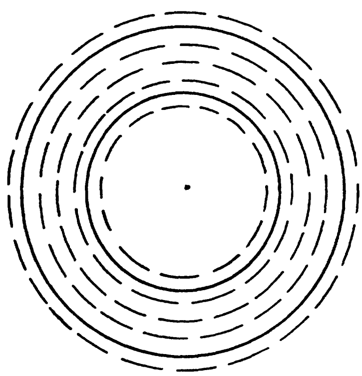

FIGURE 4. The ring between the bold-faced circles represents a circle ring C_j. The rings between the dotted circles represent the consecutive sets in χ of cardinality m for which the intersection with C_j is nonempty.

Moreover, since $f \in F_0$ and in view of (15)

$$(16)\qquad \underline{V}^{1(m)} = \sum_{j=1}^{\lceil n_m^{1/4}\rceil} \sum_{l \in E_j} f\left(2 \sum_{x_i \in X_l} r_i/m, m\right) \geqslant \sum_{j=1}^{\lceil n_m^{1/4}\rceil} |E_j| f(2(j-1)\Delta, m)$$

$$\geqslant \sum_{j=1}^{\lceil n_m^{1/4}\rceil} (\nu_j/m - 1) f(2(j-1)\Delta, m) \stackrel{\text{def}}{=} L.$$

Similarly, define for all $k = 1, \ldots, K_m$ and $j = 1, \ldots, \lceil n_m^{1/4}\rceil$: $D_{j,k} = \{l$: the highest indexed point in $S_{k,l}$ belongs to $C_{j,k}\}$ and $D_j = \bigcup_{k=1}^{K_m} D_{j,k}$. Note that the collection $\{D_{j,k}: j = 1, \ldots, \lceil n_m^{1/4}\rceil\}$ partitions the set of subregions generated in the kth sector $\{k = 1, \ldots, K_m\}$. Also let $\nu_{j,k} = |\{i: x_i \in X^{(m)}$ and x_i belongs to $C_{j,k}\}|$ $(j = 1, \ldots, \lceil n_m^{1/4}\rceil, k = 1, \ldots, K_m)$.

Since within a given sector the sets $\{S_{k,l}\}$ are all consecutive and since they all have cardinality m, only the points in $C_{j,k}$ and possibly the $(m-1)$ highest indexed points in $C_{j-1,k}$ may belong to a set $S_{k,l}$ with $l \in D_{j,k}$; see Figure 5. Thus,

$$\nu_{j,k} + (m-1) \geqslant m|D_{j,k}| \qquad \left(j = 1, \ldots, \lceil n_m^{1/4}\rceil, k = 1, \ldots, K_m\right),$$

and hence, in view of $K_m \leqslant \sqrt{n_m} + 1$,

$$(17)\qquad \nu_j + (m-1)\left(\sqrt{n_m} + 1\right) \geqslant \sum_{k=1}^{K_m} \left(\nu_{j,k} + (m-1)\right) \geqslant m|D_j|.$$

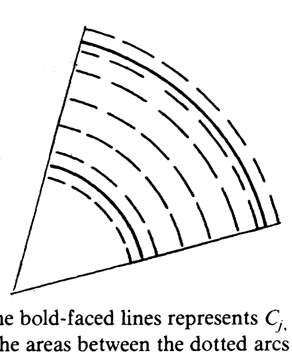

FIGURE 5. The area between the bold-faced lines represents $C_{j,k}$: the intersection of the circle ring C_j with the kth sector. The areas between the dotted arcs represent the subregions of cardinality m in the kth sector—$S_{k,l}^{(m)}$—for which the intersection with $C_{j,k}$ is nonempty.

Moreover, since $f \in F_0$ and in view of (17)

$$W_1^{(m)} = \sum_{j=1}^{\lceil n_m^{1/4} \rceil} \sum_{k=1}^{K_m} \sum_{l \in D_{j,k}} f\left(2 \sum_{x_i \in S_{k,l}} r_i/m, m\right) \leqslant \sum_{j=1}^{\lceil n_m^{1/4} \rceil} |D_j| f(2j\Delta, m) \tag{18}$$

$$\leqslant \sum_{j=1}^{\lceil n_m^{1/4} \rceil} m^{-1} v_j f(2j\Delta, m) + \left(\sqrt{n_m} + 1\right) \sum_{j=1}^{\lceil n_m^{1/4} \rceil} f(2j\Delta, m)$$

$$\leqslant \sum_{j=1}^{\lceil n_m^{1/4} \rceil} m^{-1} v_j f(2j\Delta, m)$$

$$+ \Delta^{-1}\left(\sqrt{n_m} + 1\right) \int_\Delta^{\rho+\Delta} f(2\vartheta, m)\, d\vartheta \overset{\text{def}}{=} U.$$

We thus obtain since $f \in F_1$,

$$W_1^{(m)} - \underline{V}^{1(m)} \leqslant m^{-1} \sum_{j=1}^{\lceil n_m^{1/4} \rceil} v_j [f(2(j-1)\Delta + 2\Delta, m) - f(2(j-1)\Delta, m)]$$

$$+ \sum_{j=1}^{\lceil n_m^{1/4} \rceil} f(2(j-1)\Delta, m) + \Delta^{-1}\left(\sqrt{n_m} + 1\right) \int_\Delta^{\rho+\Delta} f(2\vartheta, m)\, d\vartheta$$

$$\leqslant m^{-1} n_m \left[f\left(\frac{2\rho}{\lceil n_m^{1/4} \rceil}, m \right) - f_m^0 \right] + \lceil n_m^{1/4} \rceil f(2\rho, m)$$

$$+ R_m^{-1} \lceil n_m^{1/4} \rceil \left(\sqrt{n_m} + 1\right) \int_\Delta^{\rho+\Delta} f(2\vartheta, m)\, d\vartheta.$$

Thus

$$\limsup_{N \to \infty} \frac{1}{N} \left[W_1^{(m)} - \underline{V}^{1(m)} \right] \leqslant m^{-1} \limsup_{N \to \infty} \left\{ \frac{n_m(N)}{N} \left[f\left(2\rho/\lceil n_m^{1/4} \rceil, m\right) - f_m^0 \right] \right\} = 0$$

where the last equality is verified in complete analogy to the above proof for $\lim_{N \to \infty} (1/N) W_2(X_{(N)}) = 0$. Note, as a corollary that

$$\lim_{N \to \infty} \frac{\underline{V}^2(X_{(N)})}{\underline{V}^1(X_{(N)})} = 1 \quad \text{a.s.} \tag{19}$$

Case 2. *The* (MCRP) *has been implemented after determining an optimal partition* χ *of* $X_{(N)}$ *for* $\underline{P}^2$: A similar proof shows that $\lim_{N \to \infty} (1/N)[W_1(X_{(N)}) - \underline{U}^1(\chi)] = 0$, where $\underline{U}^1$ is defined in (3). Hence

$$\lim_{N \to \infty} \frac{V^H(X_{(N)})}{\underline{U}^1(\chi)} = 1.$$

Since $\underline{V}^2(X_{(N)}) \geqslant \underline{U}^1(\chi) \geqslant \underline{V}^1(X_{(N)})$ for all N, we have in view of (19) that

$$\lim_{N \to \infty} \frac{\underline{U}^1(\chi)}{\underline{V}^1(X_{(N)})} = 1$$

and hence in view of Theorem 3(a),

$$\lim_{N\to\infty} \frac{V^H(X_{(N)})}{\underline{V}^1(X_{(N)})} = \lim_{N\to\infty} \frac{V^H(X_{(N)})}{\underline{U}^1(\chi)} \cdot \lim_{N\to\infty} \frac{\underline{U}^1(\chi)}{\underline{V}^1(X_{(N)})} = 1 \quad \text{a.s.}$$

The following observations are in order. The assumption that the sequence $\{x_1, x_2, \dots\}$ has radial distances which are independent and identically distributed is needed only to assure that the lower bound $\underline{V}^1$ grows linearly in N almost surely. Recall that $\underline{V}^1$ is the sum of at least N/M^* cost terms associated with the sets in a partition χ which minimizes $\underline{P}^1$. If $f_m^0 > 0$ ($m = 1, \dots, M^*$), we have $\underline{V}^1 \geqslant N/M^* = \min\{f_m^0 \colon m = 1, \dots, M^*\}$ under *any* stochastic process $\{x_1, x_2, \dots\}$. If $f_m^0 = 0$ for some $m = 1, \dots, M^*$, an assumption is needed to preclude "heavy" concentration of points near to the origin. Observe, in addition, that the i.i.d. assumption regarding the process $\{r_1, r_2, \dots\}$ is only needed to conclude that

$$\lim_{N\to\infty} \inf \frac{1}{N} \sum_{i=1}^{N} \underline{f}(r_i) > 0 \quad \text{almost surely.}$$

This condition is, e.g., satisfied for general stationary sequences; see Loève (1977) with $\eta > 0$. (The sequence $\{r_1, r_2, \dots\}$ is stationary if all finite-dimensional distributions $(r_{t+1}, \dots, r_{t+k})$ are independent of t.)

Similarly, the condition that the radial distances are uniformly bounded is unnecessarily strong. One merely needs that $R_{\max} = r_N$ does not grow "too fast" as $N \to \infty$ (almost surely). For example, when the radial distances are i.i.d. with cdf $R(\cdot)$, rather simple conditions with respect to $R(\cdot)$ can be invoked; see, e.g., David (1970, §9.3). Even more specifically, when the radial distances are normally distributed, $R_{\max} = O((\log N)^{1/2})$ a.s.; see David (1970, (9.3.9)).

4. An example. Consider a one-warehouse-multiple retailer system in which at each retailer x_i customer demands for a given product occur at a constant deterministic rate μ_i, with $\mu_i = k_i\mu$ for integers $k_i \geqslant 1$ and a given base rate $\mu > 0$ ($i = 1, \dots, N$). All stock enters the system through the depot from where it is distributed, in efficient routes, to (some of) the retailers via a fleet of vehicles, each with a given load capacity of b units.

Inventories are kept at the retailers but not at the depot. (A different, somewhat more complex, application of our class of routing models arises in systems with central inventories; see Anily 1986, Chapter 5.) Inventory carrying costs are incurred at a rate h^+ per unit, per unit of time. Transportation costs include a fixed cost c per route driven and variable costs proportional with the total (Euclidean) distance driven. (The cost per mile is normalized as one.) We wish to determine replenishment strategies enabling all retailers to meet their demands while minimizing long-run average transportation and inventory carrying costs.

Define a *demand point* as a point in the plane facing a demand rate of μ. Each retailer x_i ($i = 1, \dots, N$) with demand rate $k_i\mu$ may be replaced by k_i independent *demand points* (all located at the same geographic point). We restrict ourselves to the class of strategies which *partition* the demand points into a collection of L regions such that each time one of the demand points in a given region receives a delivery, this delivery is made by one of the vehicles visiting all other demand points in the region as well. (See Anily and Federgruen 1988 for a discussion of this restriction.) In view of limited vehicle fleet sizes and other considerations pointed out in Anily and Federgruen (1988), we specify that a vehicle may be dispatched to a given region at most f^* times per unit of time.

The resulting problem may be reduced to a special case of a routing problem with general route cost function $f(\vartheta, m)$ where

$$f(\vartheta, m) = \begin{cases} h^{+}\mu m/(2f^{*}) + f^{*}(\vartheta + c) & \text{if } \vartheta + c \leqslant \mu m h^{+}/(2f^{*2}), \\ [2h^{+}\mu m(\vartheta + c)]^{1/2} & \text{if } \mu m h^{+}/(2f^{*2}) \leqslant \vartheta + c \leqslant b^{2}h^{+}/(2\mu m), \\ h^{+}b/2 + \dfrac{\mu m}{b}(\vartheta + c), & \text{otherwise.} \end{cases}$$

See Lemma 1 and formula (3) in Anily and Federgruen (1988). Clearly, $f \in F_1$; the above stated algorithms and bounds thus apply and possess the above determined asymptotic optimality and accuracy properties. Moreover, it may be verified that the partitioning problem $\underline{P}^1$ is extremal; this results in additional simplifications as discussed above and in Anily and Federgruen (1988). The latter publication also reports on a numerical study of the performance of the heuristics and bounds for problems of *moderate* size.

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