Echelon Reorder Points, Installation Reorder Points, and the Value of Centralized Demand Information

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We consider a serial inventory system with N stages. The material flows from an outside supplier to stage N, then to stage $N-1$, etc., and finally to stage 1 where random customer demand arises. Each stage replenishes a stage-specific inventory position according to a stage-specific reorder point/order quantity policy. Two variations of this policy are considered. One is based on echelon stock, and the other installation stock. The former requires centralized demand information, while the latter does not. The relative cost difference between the two policies is called the value of centralized demand information. For fixed order quantities, we develop efficient algorithms for computing both the optimal echelon reorder points and the optimal installation reorder points. These algorithms enable us to conduct an extensive computational study to assess the value of centralized demand information and to understand how this value depends on several key system parameters, i.e., the number of stages, leadtimes, batch sizes, demand variability, and the desired level of customer service.
(*Supply Chain Management*; *Stochastic Demand*; *Batch Ordering*; *Centralized Information*; *Electronic Data Interchange*)

1. Introduction

We consider a serial inventory system with N stages. The material flows from an outside supplier to stage N, then to stage $N-1$, etc., and finally to stage 1, where random customer demand arises. Inventories are transferred from one stage to the next in batches. Each stage replenishes a stage-specific inventory position according to a stage-specific reorder point/order quantity policy: whenever the inventory position falls to or under the reorder point, the stage orders an integer multiple of a base quantity to bring the inventory position to above the reorder point. Throughout this paper, we assume that the (base) order quantities are fixed and the only decision variables are the reorder points.

We consider two variations of the reorder point/order quantity policy with different informational requirements. One is based on echelon stock: whenever the echelon stock at a stage falls to or below an *echelon reorder point*, an order is placed. A stage's echelon stock is the inventory position of the subsystem consisting of the stage itself as well as all the downstream stages. The other variation is based on installation stock: whenever the installation (or local) inventory position at a stage falls to or below an *installation reorder point*, an order is placed. Note that in order to continuously monitor its echelon stock, a stage must have access to the demand information at stage 1 on a real-time basis. Thus the echelon-stock policy requires centralized demand information. On the other hand, the installation-stock policy only requires local inventory information.

A key result of this paper is that the optimal echelon reorder points can be determined sequentially: first for stage 1, then for stage 2, and so on. This is based on an observation that the steady-state echelon inventory position at each stage can be replicated by the steady-state inventory position of a standard single-location reorder

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point/order quantity model with a *random reorder point*. These random reorder points at different stages satisfy a simple recursive equation that is also found in the Clark-Scarf model with base-stock policies. Our result, in a nutshell, is essentially that after a proper transformation, the batch-transfer model can be treated as a base-stock model for the purpose of determining the optimal echelon reorder points.

The determination of the optimal installation reorder points is more difficult. We establish easy-to-compute bounds on the optimal installation reorder points. An optimal solution can thus be found by a search. This, however, is computationally infeasible for some systems, especially the ones with many stages. In response, we develop a much more efficient heuristic algorithm. It uses an empirical observation based on our numerical experience and thus does not guarantee an optimal solution. However, numerical tests show that the heuristic solution is excellent.

The above results enable us to address a fundamental issue in supply chain management: the value of centralized demand information. As mentioned earlier, echelon-stock policies require centralized demand information, while installation stock policies only require local information. The relative cost difference between the two is a measure of the value of centralized demand information. In an extensive numerical study, we found that this value ranges from 0% to 9% with an average of 1.75%. The numerical examples also provide insights into how this value depends on several parameters of the system, i.e., the number of stages, leadtimes, batch sizes, demand variability, and the desired level of customer service.

There is an extensive literature on serial inventory systems. When there are no economies of scale in placing orders at all the stages except the most upstream stage, Clark and Scarf (1960) show that echelon base-stock policies are optimal in a periodic-review, finite-horizon setting. This result is later generalized to periodic-review, infinite-horizon models by Federgruen and Zipkin (1984). Recently, Chen and Zheng (1994b) offer an exceedingly simple proof of the optimality of the echelon base-stock policies that also applies to continuous-time models. In sharp contrast, when there are economies of scale at all stages, the optimal policies suddenly become extremely difficult to characterize (Clark and Scarf 1962). In fact, the optimal policies are still unknown to this date. (Interestingly, Chen 1996 characterized a heuristic policy for a two-stage, serial model that is guaranteed, a priori, to be within 6% of optimality.) As a result, research attention has been focused on reasonable and easily implementable heuristic policies such as the reorder point/order quantity policies. For this class of policies, De Bodt and Graves (1985) and Chen and Zheng (1994a) provide, respectively, an approximate and an exact cost-evaluation procedure. An algorithm is later developed by Chen and Zheng (1998) for determining near-optimal base order quantities. For a comprehensive review on reorder point/order quantity policies in multiechelon, stochastic inventory systems, see Axsäter (1993).

The value of information is a central issue in inventory management. It has long been recognized that demand information and inventory are substitutes for one another. For example, advanced warnings from customers of their orders reduce inventory (Hariharan and Zipkin 1995). So does the additional demand information acquired through various marketing means (Milgrom and Roberts 1988). The tradeoff between make-to-order and make-to-stock systems also reflects the value of demand information (see Federgruen and Katalan (1994) and Nguyen (1995) and the references therein). These examples focus on the value of better communication between the customer and the supplier. Similarly, it also creates value to improve the communication between supply-chain members. The proliferation of advanced information technologies (e.g., EDI) in supply chains suggests that more and more companies have come to realize the importance of better communication. Academic researchers have also started to pay attention (see, e.g., Lee et al. 1997). This paper focuses on a specific benefit of better communication between supply chain members: the benefit of the timely placement of orders due to the availability of echelon-stock information. (We thus ignore the other benefits such as shorter order processing times which may result from better communication.) As shown by Axsäter and Rosling (1993), echelon-stock policies are superior to installation-stock policies in serial systems. This theoretical result indicates that it is beneficial to have the echelon-stock information. We furnish this qualitative observation with extensive quantitative evidence.

The rest of the paper is organized as follows. Section 2 defines the model and notation. Section 3 develops the sequential algorithm for determining the optimal echelon reorder points. Section 4 aims to find the optimal installation reorder points. Section 5 reports the computational study. Section 6 concludes the paper.

2. Preliminaries

Consider a serial inventory system with N stages. Stage 1 orders from stage 2, 2 from 3, . . . , and stage N orders from an outside supplier with unlimited stock. The outside supplier is also called stage $N + 1$. Each stage represents a stocking point in a production system, a distribution system, or a hybrid one. The production/transportation leadtimes from one stage to the next are constant. Customers arrive at stage 1 according to a Poisson process. The demand sizes of the customers are independent, identically distributed random variables; and they are independent of the arrival process. Thus, the demand process is compound Poisson. When stage 1 runs out of stock, demand is backlogged. The system incurs linear holding costs at every stage, and linear backorder costs at stage 1. The objective is to minimize the long-run average total cost in the system.

The system has the following parameters:

λ = customer arrival rate,
D = demand size of a customer, a discrete random variable, $\mu = E[D]$,
L_i = leadtime from stage $i + 1$ to stage i, a nonnegative constant,
H_i = installation holding cost rate at stage i,
h_i = echelon holding cost rate at stage $i = H_i - H_{i+1} > 0$ with $H_{N+1} = 0$, and
p = backorder cost rate (at stage 1).

For any time epoch t, define

$I_i(t)$ = echelon inventory at stage i
= on-hand inventory at stage i plus inventories on hand at, and in transit to, stages 1, . . . , $i - 1$,
$B(t)$ = backorder level at stage 1,
$IL_i(t)$ = echelon inventory level at stage $i = I_i(t) - B(t)$,
$IP_i(t)$ = echelon inventory position at stage i
= $IL_i(t)$ plus orders in transit to stage i,
$ES_i(t)$ = echelon stock at stage i
= $IP_i(t)$ plus outstanding orders of stage i that are backlogged at stage $i + 1$, and
$IS_i(t)$ = installation stock at stage i
= outstanding orders at stage i (in transit to stage i or backlogged at stage $i + 1$) plus on-hand inventory at stage i minus backlogged orders from stage $i - 1$
= $ES_i(t) - ES_{i-1}(t)$, $i \geq 2$; and $IS_1(t) = ES_1(t)$.

Therefore, the echelon stock at a stage changes when a customer demand occurs at stage 1. To continuously monitor its echelon-stock level, a stage must have access to the demand information at stage 1 on a real-time basis. On the other hand, the installation stock at a stage is local information.

The replenishment policy is of the following type. Each stage replenishes a stage-specific inventory position according to a stage-specific (R, nQ) policy: when the inventory position falls to or below a *reorder point* R, the stage orders a *minimum* integer multiple of Q (base quantity) from its upstream stage to increase the inventory position to above R. In case the upstream stage does not have sufficient on-hand inventory to satisfy this order, a partial shipment is sent with the remainder backlogged at the upstream stage. Throughout this paper, we assume that the base quantities are fixed and the reorder points are the only decision variables.

Let Q_i be the base quantity at stage i, $i = 1, \ldots, N$. We assume

$$Q_{i+1} = n_i Q_i, \quad i = 1, \ldots, N - 1,$$

where n_i is a positive integer. This integer-ratio constraint simplifies analysis significantly. It also simplifies material handling (e.g., packaging and bulk breaking) by restricting the shipments to each stage to be multiples of a fixed quantity which may represent a truckload or the size of a standard container. Moreover, the cost increase due to the constraint is likely to be insignificant because inventory costs tend to be insensitive to the choice of order quantities (Zheng (1992) and Zheng and Chen (1992)).

We consider two variations of the above (R, nQ) policy. One is based on *echelon stock*: each stage replenishes its echelon stock with an *echelon reorder point*. Let R_i be the echelon reorder point at stage i, $i = 1, \ldots, N$. Therefore, under an echelon-stock (R, nQ) policy, stage i orders a multiple of Q_i from stage $i + 1$ every time its echelon stock falls to or below R_i. The other variation is based on *installation stock*: each stage controls its installation stock with an *installation reorder point*. Let r_i be the installation reorder point at stage i, $i = 1, \ldots, N$. Therefore, under an installation-stock (R, nQ) policy, stage i orders a multiple of Q_i from stage $i + 1$ every time its installation stock falls to or below r_i. Note that echelon-stock (R, nQ) policies require centralized demand information, while installation-stock (R, nQ) policies only require local 'demand' information, i.e., orders from the immediate downstream stage. When every customer demands exactly one unit, i.e. the demand process is simple Poisson, each order by stage i is exactly of size Q_i, $i = 1, \ldots, N$. In this case, the (R, nQ) policy reduces to the (R, Q) policy.

We assume that the system starts with a plausible initial state. That is, the initial on-hand inventory at stage i is an integer multiple of Q_{i-1}, $i = 2, \ldots, N$. This is reasonable because each order placed by stage $i - 1$ is an integer multiple of Q_{i-1} and thus there is no incentive for stage i to keep a fraction of Q_{i-1} on hand. On the other hand, every shipment to stage i is an integer multiple of Q_i which is itself an integer multiple of Q_{i-1} due to the integer-ratio constraint, and every shipment leaving stage i (to stage $i - 1$) is an integer multiple of Q_{i-1}. This, together with the initial on-hand inventory at stage i, implies that the installation stock at stage i is always an integer multiple of Q_{i-1}. This observation is true whether an echelon-stock (R, nQ) policy or an installation-stock (R, nQ) policy is in place. Consequently, without any loss of generality, we restrict r_i to be an integer multiple of Q_{i-1}, $i \geq 2$. Of course, r_1 can be any integer. No such restrictions are placed on R_i, $i = 1, \ldots, N$.

We conclude this section with an important result from Axsäter and Rosling (1993). Two policies are said to be *identical* if they lead to the same ordering decisions (when and how much) at every stage. Axsäter and Rosling show that the installation-stock (R, nQ) policy, $(r_i, Q_i)_{i=1}^N$, is identical to the echelon-stock (R, nQ) policy, $(R_i, Q_i)_{i=1}^N$, if and only if

$$R_1 = r_1 \text{ and } R_i = R_{i-1} + Q_{i-1} + r_i,\ i = 2, \ldots, N. \quad (1)$$

Therefore, for any installation-stock (R, nQ) policy, there exists an identical echelon-stock (R, nQ) policy; but not vice versa since r_i must be an integer multiple of Q_{i-1} for $i \geq 2$. In short, installation-stock (R, nQ) policies are special cases of echelon-stock (R, nQ) policies. This result is quite intuitive. Suppose the demand process is simple Poisson, and consider any installation-stock (R, Q) policy. Here orders are 'nested' in the sense that every order epoch at stage i coincides with an order epoch at stages $i - 1, i - 2, \ldots, 1$. The installation stock at stage j after each order is $r_j + Q_j$ for all j. Consequently, just before stage i places an order, its echelon stock, which is the sum of the installation stocks at stages 1 to i, is $\sum_{j=1}^{i-1} (r_j + Q_j) + r_i$. Let this echelon stock level be R_i, $i = 1, \ldots, N$. It is easy to verify that the resulting echelon reorder points satisfy (1).

REMARK. As mentioned above, the paper assumes that the base quantities have already been determined and the only decision variables are the reorder points. The base quantities may be determined by exogenous factors such as the size of a standard container, or they may be a solution to a lot-sizing model. There are many lot-sizing models. Without capacity constraints, see Muckstadt and Roundy (1993) for a serial model without backlogging and Chen (1998) for one with backlogging. With capacity constraints, see Bahl et al. (1987) for a comprehensive review of lot-sizing models. Note that these models all assume deterministic demands. Chen and Zheng (1998) provide a *stochastic*, serial model for determining the lot sizes.

3. Echelon Reorder Points

In this section, we develop an efficient algorithm for determining the optimal echelon reorder points. The algorithm is based on an observation that in steady state, the echelon inventory position at each stage is essentially the inventory position of a standard single-stage (R, nQ) model with a *random* reorder point. The random reorder points at different stages satisfy a simple recursive equation that is also found in the Clark-Scarf model with base-stock policies. As a result, the optimal echelon reorder points can be determined sequentially by solving a sequence of standard single-stage problems.

We begin by reviewing some results from Chen and Zheng (1994a). At time t, the systemwide holding and backorder costs accrue at rate

$$\sum_{i=1}^{N} h_i I_i(t) + pB(t).$$

Since $I_i(t) = IL_i(t) + B(t)$ by definition, the above cost rate can be written as

$$\sum_{i=1}^{N} h_i IL_i(t) + (p + H_1)B(t).$$

It is convenient to introduce a time shift when assessing the holding and backorder costs. Let $l_i = \sum_{j=i}^{N} L_j$ for $i = 1, \ldots, N$ and $l_{N+1} \equiv 0$. Thus l_i is the total leadtime from the outside supplier to stage i. At time t, we charge the following cost rate:

$$\sum_{i=1}^{N} h_i IL_i(t + l_i) + (p + H_1)B(t + l_1).$$

Clearly, this shift in time does not affect the long-run average costs. Now suppose t is a time epoch in steady state. Let IP_i be $IP_i(t + l_{i+1})$ and IL_i be $IL_i(t + l_i)$, $i = 1, \ldots, N$. Let $B = B(t + l_1)$. Note that $B = (IL_1)^-$ by definition, where $(x)^- = \max\{0, -x\}$. Therefore, the long-run average system-wide cost is

$$C(\mathbf{R}) \stackrel{\text{def}}{=} E\left[\sum_{i=1}^{N} h_i IL_i + (p + H_1)B\right]$$

where $\mathbf{R} = (R_1, \ldots, R_N)$. The echelon reorder points that minimize the above expression are called optimal; they are denoted by $\mathbf{R}^* = (R_1^*, \ldots, R_N^*)$.

The distributions of IP_i and IL_i for $i = 1, \ldots, N$ can be determined recursively. Let D_i be the total demand in the interval $(t + l_{i+1}, t + l_i]$, $i = 1, \ldots, N$, which is also referred to as the leadtime demand at stage i. Since the demand process is compound Poisson, D_i and IP_i are independent. Moreover, $D_1, D_2, \ldots, D_N$ are independent since the demand process has independent increments. The following relationship is well known:

$$IL_i = IP_i - D_i, \quad i = 1, \ldots, N. \tag{2}$$

From Chen and Zheng (1994a), we have

$$IP_i = O_i[IL_{i+1}], \quad i = 1, \ldots, N - 1 \tag{3}$$

where

$$O_i[x] = \begin{cases} x & x \le R_i + Q_i, \\ x - nQ_i & \text{otherwise}, \end{cases}$$

where n is the largest integer so that $x - nQ_i > R_i$. (The reason for (3) is that $IL_{i+1} - IP_i$ is the on-hand inventory at stage $i + 1$ at time $t + l_{i+1}$, which is always a nonnegative integer multiple of Q_i. For details, see Chen and Zheng (1994a).) Equations (2) and (3) provide a top-down recursive procedure for determining the distributions of IP_i and IL_i, $i = 1, \ldots, N$. It is easy to see that the distribution of IP_N is uniform over $\{R_N + 1, \ldots, R_N + Q_N\}$ by simply imagining the whole system as a single stage. (Actually, this result requires some mild regularity conditions on the demand-size distribution, see, e.g., Hadley and Whitin (1961). A sufficient condition is $\Pr(D = 1) > 0$. Let us make this assumption.) From (2) with $i = N$, we have the distribution of IL_N. From (3) with $i = N - 1$, we have the distribution of IP_{N-1}, so on and so forth.

Now we are ready to establish new results. We first introduce some random variables. Define

$$\Pr(U_i = u) = \frac{1}{Q_i}, \quad u = 1, \ldots, Q_i, i = 1, \ldots, N$$

and

$$\Pr(Z_i = z) = \frac{1}{n_i}, \quad z = 0, \ldots, n_i - 1, i = 1, \ldots, N - 1.$$

(Recall that $Q_{i+1} = n_i Q_i$ and that n_i is a positive integer.) We assume that these uniform random variables are independent, and they are all independent of the demand process.

We write $X_1 \stackrel{d}{=} X_2$ if the two random variables X_1 and X_2 have the same distribution. Clearly, $X_1 + X_2 \stackrel{d}{=} X_1 + X_3$ if X_1 and X_2 are independent, X_1 and X_3 are independent, and $X_2 \stackrel{d}{=} X_3$. This fact will be used later.

As mentioned above, IP_N is uniform over $\{R_N + 1, \ldots, R_N + Q_N\}$. Thus, $IP_N \stackrel{d}{=} R_N + U_N$. We next show that the echelon inventory positions at the other stages can also be decomposed in a similar fashion. (It is clear that $ES_i \stackrel{d}{=} R_i + U_i$ for each stage i. But for $i \neq N$, the decomposition of IP_i, which is always less than or equal to ES_i, involves a random component in addition to U_i.) We begin with some intermediate results.

LEMMA 1. $U_{i+1} \stackrel{d}{=} Z_i Q_i + U_i$, $i = 1, \ldots, N - 1$.

PROOF. Follows from $Q_{i+1} = n_i Q_i$. □

LEMMA 2. *For* $i = 1, \ldots, N$, $O_i[X + U_i] \stackrel{d}{=} \min\{R_i, X\} + U_i$, *where X is any random variable independent of* U_i.

PROOF. Let $X = x$. Suppose $x \ge R_i$. It follows from the definition of $O_i[\cdot]$ that the distribution of $O_i[x + U_i]$

is uniform over $\{R_i + 1, \ldots, R_i + Q_i\}$. Thus, $O_i[x + U_i] \stackrel{d}{=} R_i + U_i = \min\{R_i, x\} + U_i$. Now suppose $x < R_i$. Again from the definition of $O_i[\cdot]$, $O_i[x + U_i] = x + U_i = \min\{R_i, x\} + U_i$. This completes the proof. □

We now introduce another sequence of random variables. Let $V_N \equiv R_N$. Define recursively

$$V_i = \min\{R_i, V_{i+1} + Z_iQ_i - D_{i+1}\},\quad i = 1, \ldots, N - 1. \tag{4}$$

Since the U's are independent of the demand process as well as the Z's, the V's are independent of the U's. From the above recursive definition, it is also clear that V_i is independent of D_i for $i = 1, \ldots, N$ and that V_{i+1} is independent of Z_i for $i = 1, \ldots, N - 1$.

THEOREM 1. $IP_i \stackrel{d}{=} V_i + U_i$, $i = 1, \ldots, N$.

PROOF. The theorem is clearly true for $i = N$. Suppose it holds for $i + 1$, i.e.,

$$IP_{i+1} \stackrel{d}{=} V_{i+1} + U_{i+1}.$$

Recall that V_{i+1} and U_{i+1} are independent. From Lemma 1, $U_{i+1} \stackrel{d}{=} Z_iQ_i + U_i$. Also recall that V_{i+1} is independent of U_i and Z_i, which implies that V_{i+1} is independent of $Z_iQ_i + U_i$. Therefore

$$IP_{i+1} \stackrel{d}{=} V_{i+1} + Z_iQ_i + U_i.$$

Since V_{i+1} is independent of D_{i+1} and since Z_i and U_i are independent of the demand process, the right side of the above equation is independent of D_{i+1}. This, together with (2) and the fact that IP_{i+1} is independent of D_{i+1}, leads to

$$IL_{i+1} = IP_{i+1} - D_{i+1} \stackrel{d}{=} V_{i+1} + Z_iQ_i + U_i - D_{i+1}.$$

From (3),

$$\begin{aligned} IP_i &= O_i[IL_{i+1}] \\ &\stackrel{d}{=} O_i[V_{i+1} + Z_iQ_i - D_{i+1} + U_i] \\ &\stackrel{d}{=} V_i + U_i \end{aligned}$$

where the last equality follows from Lemma 2 and the fact that $V_{i+1} + Z_iQ_i - D_{i+1}$ is independent of U_i. This completes the induction. □

Theorem 1 indicates that IP_i is precisely the steady-state inventory position of the standard single-stage (R, nQ) model with $R = V_i$ and $Q = Q_i$. A key feature here is that the reorder point of the single-stage model is *random*, and it is jointly determined by the control parameters (echelon reorder points and base quantities) at stages $i, i + 1, \ldots, N$ and the leadtime demands at stages $i + 1, \ldots, N$. We will refer to V_i as the *effective reorder point* at stage i. Consequently, the N-stage model can be *decomposed* into N single-stage (R, nQ) models with reorder point V_i and base quantity Q_i, $i = 1, \ldots, N$. The linkage between these single-stage models is captured by (4).

We pause here to make the following two remarks.

REMARK 1. There is a very different decomposition method in the multiechelon, stochastic inventory literature. It is based on the observation that the *actual* leadtime for each order placed by a stage consists of two parts: the production/transportation leadtime plus the delay (or retard) experienced by the order at the upstream stage. This delay is usually random and its characteristics depend on the control parameters at the upstream stages. Therefore, one way to decompose a multistage system into single-stage ones is to try to characterize the *actual leadtimes* at all stages. In contrast, our decomposition method hinges on the characterization of the *effective reorder points*.

REMARK 2. Equation (4) is a generalization of a similar equation found in the Clark-Scarf model with base-stock policies. This is hardly surprising since echelon base-stock policies are special cases of echelon-stock (R, nQ) policies: an (R, nQ) policy with $Q = 1$ reduces to a base-stock policy with order-up-to level $S = R + 1$. Now suppose $Q_i = 1$, i.e., stage i follows a base-stock policy with order-up-to level $S_i = R_i + 1$, for $i = 1, \ldots, N$. In this case, Theorem 1 becomes $IP_i \stackrel{d}{=} V_i + 1$. Since $n_i = 1$ which implies $Z_i \equiv 0$, (4) becomes

$$V_i = \min\{R_i, V_{i+1} - D_{i+1}\},\quad i = 1, \ldots, N - 1$$

with $V_N \equiv R_N$, which is equivalent to

$$IP_i \stackrel{d}{=} \min\{S_i, IP_{i+1} - D_{i+1}\},\quad i = 1, \ldots, N - 1$$

with $IP_N \equiv S_N$. This is precisely the recursive equation for the Clark-Scarf model (see Chen and Zheng 1994b).

With the above characterization of the steady-state echelon inventory positions and the recursive equation (4) linking the effective reorder points, we are now ready to develop an algorithm for identifying the optimal echelon reorder points.

Let us call

$$\sum_{j=1}^{i} h_j IL_j + (p + H_1)B$$

the *echelon cost* at stage i; it includes all the holding and backorder costs incurred in the subsystem consisting of stages $1, \ldots, i$. From the recursive equations (2) and (3), we know that IP_i uniquely determines the distributions of $IL_i, IL_{i-1}, \ldots, IL_1$, and $B = (IL_1)^-$. Given $IP_i = y$, the (conditional) expected echelon cost at stage i is

$$G_i(y) \stackrel{\text{def}}{=} E\left[\sum_{j=1}^{i} h_j IL_j + (p + H_1)B \,\middle|\, IP_i = y\right], \quad i = 1, \ldots, N.$$

Since $IP_i \stackrel{\text{d}}{=} V_i + U_i$ (Theorem 1) and only the effective reorder points V_i depend on the decision variables (i.e., echelon reorder points), it is natural to express the expected echelon cost at a stage as a function of its effective reorder point. Define

$$\tilde{G}_i(y) = EG_i(y + U_i), \quad i = 1, \ldots, N$$

which represents the (conditional) expected echelon cost at stage i given the stage's effective reorder point is y. Thus, the expected echelon cost at stage i is $E\tilde{G}_i(V_i)$. In particular,

$$C(\mathbf{R}) = E\tilde{G}_N(V_N) = \tilde{G}_N(R_N) \tag{5}$$

since $V_N \equiv R_N$.

The cost function $C(\mathbf{R})$ exhibits some useful properties. Note that

$$\begin{aligned} C(\mathbf{R}) &= \sum_{j=2}^{N} h_j E[IL_j] + EG_1(IP_1) \\ &= \sum_{j=2}^{N} h_j E[IL_j] + E\tilde{G}_1(V_1) \\ &= \sum_{j=2}^{N} h_j E[IL_j] + E\tilde{G}_1(\min\{R_1, V_2 + Z_1Q_1 - D_2\}). \end{aligned} \tag{6}$$

From (4), $V_2, \ldots, V_N$ are independent of R_1. As a result, the first term in (6), $\Sigma_{j=2}^{N} h_j E[IL_j]$, is independent of R_1 (see also (2) and Theorem 1). Therefore, the optimal R_1 minimizes

$$E\tilde{G}_1(\min\{R_1, V_2 + Z_1Q_1 - D_2\}). \tag{7}$$

Since $IL_1 = IP_1 - D_1$ and IP_1 is independent of D_1,

$$G_1(y) = E[h_1(y - D_1) + (p + H_1)(y - D_1)^-]$$

which is convex. Therefore, $\tilde{G}_1(\cdot)$ is also convex. Let $\bar{Y}_1$ be a finite minimum point of $\tilde{G}_1(\cdot)$. Notice that $\tilde{G}_1(\min\{R_1, W\})$ is minimized at $R_1 = \bar{Y}_1$ for any value of W. Consequently, (7) is minimized at $R_1 = \bar{Y}_1$. In other words, the optimal reorder point at stage 1 is independent of the reorder points at the upstream stages. Similarly, when stage 1 uses its optimal reorder point, the optimal reorder point at stage 2 is again independent of the reorder points at stages 3 to N, so on and so forth. This fact is formally established in Lemma 3 and Theorem 2 below.

Let $\mu_i = E[D_i]$, $i = 1, \ldots, N$. From the definition of $G_i(\cdot)$, $i = 1, \ldots, N$, we have from (2) and (3)

$$\begin{aligned} G_{i+1}(y) &= h_{i+1}(y - \mu_{i+1}) + EG_i(IP_i \,|\, IP_{i+1} = y) \\ &= h_{i+1}(y - \mu_{i+1}) + EG_i(O_i[y - D_{i+1}]), \\ &\qquad i = 1, \ldots, N - 1. \end{aligned}$$

Thus, for $i = 1, \ldots, N - 1$,

$$\begin{aligned} \tilde{G}_{i+1}(y) &= h_{i+1}E(y + U_{i+1} - \mu_{i+1}) \\ &\quad + EG_i(O_i[y + U_{i+1} - D_{i+1}]) \\ &\stackrel{\text{Lemma 1}}{=} h_{i+1}E(y + U_{i+1} - \mu_{i+1}) \\ &\quad + EG_i(O_i[y + Z_iQ_i + U_i - D_{i+1}]) \\ &\stackrel{\text{Lemma 2}}{=} h_{i+1}E(y + U_{i+1} - \mu_{i+1}) \\ &\quad + E\tilde{G}_i(\min\{R_i, y + Z_iQ_i - D_{i+1}\}). \end{aligned} \tag{8}$$

The following sequence of functions are parallel to $\tilde{G}_i(\cdot)$, $i = 1, \ldots, N$. Let $\bar{G}_1(y) = \tilde{G}_1(y)$ for all y. Suppose $\bar{G}_i(\cdot)$ is defined, $i = 1, \ldots, N$. Let $\bar{Y}_i$ be a finite minimum point of $\bar{G}_i(\cdot)$. Define recursively

$$\begin{aligned} \bar{G}_{i+1}(y) &= h_{i+1}E(y + U_{i+1} - \mu_{i+1}) \\ &\quad + E\bar{G}_i(\min\{\bar{Y}_i, y + Z_iQ_i - D_{i+1}\}), \\ &\quad i = 1, \ldots, N - 1. \end{aligned} \tag{9}$$

LEMMA 3. (a) $\bar{G}_i(\cdot)$ *is convex*, $i = 1, \ldots, N$;

(b) $[\bar{G}_i(y+1) - \bar{G}_i(y)] \to h_i$ (*resp.*, $-(p + H_{i+1})$) *as* $y \to +\infty$ (*resp.*, $-\infty$), $i = 1, \ldots, N$;

(c) $\tilde{G}_i(y) \geq \bar{G}_i(y)$, $\forall y$, $i = 1, \ldots, N$;

(d) *If* $R_j = \bar{Y}_j$ *for* $j = 1, \ldots, i-1$ *then* $\tilde{G}_i(y) = \bar{G}_i(y)$, $\forall y$, $i = 1, \ldots, N$.

PROOF. See Appendix. □

THEOREM 2. $\mathbf{R}^* = (\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_N)$ *and* $C(\mathbf{R}^*) = \bar{G}_N(\bar{Y}_N)$.

PROOF. Take any $\mathbf{R}$. From Lemma 3(c) and the definition of $\bar{Y}_N$, $\tilde{G}_N(R_N) \geq \bar{G}_N(R_N) \geq \bar{G}_N(\bar{Y}_N)$. Thus, from (5), $C(\mathbf{R}) \geq \bar{G}_N(\bar{Y}_N)$. In other words, $\bar{G}_N(\bar{Y}_N)$ is a lower bound on the long-run average cost of any feasible echelon-stock (R, nQ) policy. But from Lemma 3(d), this lower bound can be achieved if $\mathbf{R} = (\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_N)$. This completes the proof. □

Now we have an algorithm for determining the optimal echelon reorder points. The algorithm solves N single-stage (R, nQ) models sequentially. It starts at stage 1. The single-stage model for stage 1 has the holding-backorder cost function $G_1(\cdot)$. The long-run average cost of the (R_1, nQ_1) policy in this model is $\bar{G}_1(R_1) = \tilde{G}_1(R_1)$. The minimum point of this function, $\bar{Y}_1$, is the optimal echelon reorder point at stage 1. Now suppose stage 1 uses its optimal echelon reorder point. The single-stage model for stage 2 has the holding-backorder cost function $G_2(\cdot)$. The long-run average cost of the (R_2, nQ_2) policy in this model is $\tilde{G}_2(R_2)$. Since $R_1 = \bar{Y}_1$ we have from Lemma 3(d) $\tilde{G}_2(R_2) = \bar{G}_2(R_2)$. The minimum point of $\bar{G}_2(\cdot)$, $\bar{Y}_2$, is the optimal echelon reorder point at stage 2. Continue in this fashion until the optimal echelon reorder point at stage N is found.

The above algorithm is reminiscent of the sequential procedure for determining the optimal base-stock levels in the Clark-Scarf model (see Rosling (1989) and Chen and Zheng (1994b)). In fact, if $Q_i = 1$ for $i = 1, \ldots, N$, then the above algorithm reduces to the algorithm in Chen and Zheng (1994b). Both algorithms determine the optimal values of the decision variables (reorder points or base-stock levels) sequentially, starting at the bottom stage and ending at the top stage. For each stage i, a standard single-stage model is solved. This single-stage model assumes that all the downstream stages behave optimally and that the immediate upstream stage (stage $i+1$) has infinite supply. The difference is that a single-stage newsboy model is solved for the Clark-Scarf model, whereas a single-stage (R, nQ) model is solved for the batch-transfer model.

4. Installation Reorder Points

As we will see, optimal installation reorder points are much more difficult to find than their echelon counterparts. In this section, we present bounds on the optimal installation reorder points. These bounds can then be used to search for an optimal solution. This search procedure, however, is computationally infeasible for some problem instances, especially those with many stages. In response, we develop a heuristic algorithm based on a numerical observation. Although the heuristic algorithm does not guarantee to find the optimal installation reorder points, numerical tests suggest that the heuristic solution is excellent.

In §2, we mentioned that installation-stock (R, nQ) policies are special cases of echelon-stock (R, nQ) policies and that the two policies are identical if and only if (1) is satisfied. Also recall that r_i, the installation reorder point at stage i, is an integer multiple of Q_{i-1}, the base quantity at stage $i-1$, for $i \geq 2$. Therefore, the problem of finding the optimal installation reorder points can be formulated as:

$$\mathcal{P} \quad \min C(\mathbf{R})$$
$$\text{s.t.} \quad R_{i+1} - R_i = m_i Q_i, \quad m_i \text{ integer},$$
$$i = 1, \ldots, N-1.$$

Let $\mathbf{R}^0 = (R_1^0, \ldots, R_N^0)$ be an optimal solution to $\mathcal{P}$. Let $\mathbf{r}^* = (r_1^*, \ldots, r_N^*)$ be the corresponding optimal installation reorder points, i.e., $r_1^* = R_1^0$ and $r_i^* = R_i^0 - R_{i-1}^0 - Q_{i-1}$ for $i \geq 2$.

Due to its constraints, problem $\mathcal{P}$ is difficult to solve. Below, we present bounds on $\mathbf{R}^0$, which can be used to search for an optimal solution. For the sake of brevity, we only present the results informally and refer interested readers to Chen (1995) for formal proofs.

We begin with bounds on R_1^0. Recall that $\bar{G}_1(\cdot)$ is convex and $\bar{Y}_1$ is its minimum point. Let y_i be the ith minimum point of $\bar{G}_1(\cdot)$, $i = 1, 2, \ldots$. Thus $y_1 = \bar{Y}_1$. Let $S = \{y_1, \ldots, y_{Q_1}\}$, which is a contiguous set (see, e.g., Federgruen and Zheng 1992). Let R_1^- be the smallest element of S. Let $R_1^+ = \bar{Y}_1 + Q_1 - 1$.

THEOREM 3. $R_1^- \le R_1^0 \le R_1^+$.

We proceed to bound R_N^0. Let C_f be the long-run average cost of a feasible installation-stock (R, nQ) policy. Suppose C_f is finite. Thus $C_f \ge C(\mathbf{R}^*) = \bar{G}_N(\bar{Y}_N)$ (Theorem 2). Define

$$R_N^- = \min\{y \mid \bar{G}_N(y) \le C_f\}$$

and

$$R_N^+ = \max\{y \mid \bar{G}_N(y) \le C_f\}.$$

Since $\lim_{y\to\pm\infty} \bar{G}_N(y) = +\infty$ by Lemma 3(b), both R_N^- and R_N^+ exist and are finite.

THEOREM 4. $R_N^- \le R_N^0 \le R_N^+$.

Similarly, one can bound the remaining reorder points. Take any $i = 2, \ldots, N-1$. Imagine that the N-stage, serial system is composed of two subsystems: an upper subsystem consisting of stages $i+1, \ldots, N$ and a lower subsystem consisting of the remaining stages. First, consider the installation holding costs incurred in the upper subsystem. The installation inventory at stage j ($\ge i+1$) has two parts: the inventory on hand at stage j and the inventory in transit from stage j to stage $j-1$. Ignore the first part. Note that it takes each unit of inventory L_{j-1} units of time to travel from stage j to stage $j-1$, and that the average demand rate is $\lambda\mu$ units per unit of time, which is also the throughput rate from stage j to stage $j-1$. By Little's law, the average inventory in between stage j and stage $j-1$ is $\lambda\mu L_{j-1}$. Since the installation holding cost rate is H_j at stage j, the following is a lower bound on the long-run average holding cost incurred in the upper subsystem:

$$\sum_{j=i+1}^{N} H_j \lambda\mu L_{j-1}. \tag{10}$$

Now consider the lower subsystem. It is an i-stage serial system with an echelon holding cost rate H_i at stage i, an echelon holding cost rate h_j at stage $j < i$, and a backorder cost rate p. (For this truncated system, the echelon holding cost rate at stage $i+1$, or the "outside supplier," is zero.) Therefore, the holding and backorder costs in the i-stage system accrue at the following rate:

$$H_i IL_i + \sum_{j=1}^{i-1} h_j IL_j + (p + H_1)B$$

$$= H_{i+1} IL_i + \left\{ \sum_{j=1}^{i} h_j IL_j + (p + H_1)B \right\}.$$

Recall that the terms inside the curly brackets are the *echelon cost* at stage i, and its expected value is $E\tilde{G}_i(V_i)$. Since $IL_i \overset{d}{=} V_i + U_i - D_i$, the long-run average cost in the lower subsystem is

$$H_{i+1} E(V_i + U_i - D_i) + E\tilde{G}_i(V_i) \ge E\bar{\bar{G}}_i(V_i) \tag{11}$$

where

$$\bar{\bar{G}}_i(y) \overset{\text{def}}{=} H_{i+1}\left[y + \frac{Q_i + 1}{2} - \mu_i \right] + \bar{G}_i(y).$$

The inequality in (11) follows from Lemma 3(c).

Let C_f be again the long-run average cost of a feasible installation-stock (R, nQ) policy. Suppose it is finite. Define

$$R_i^- = \min\left\{ y \mid \bar{\bar{G}}_i(y) \le C_f - (\lambda\mu) \sum_{j=i+1}^{N} H_j L_{j-1} \right\}.$$

Now consider any feasible solution to $\mathcal{P}$ with $R_i < R_i^-$. From (10) and (11), the long-run average cost of this feasible solution is greater than or equal to

$$\sum_{j=i+1}^{N} H_j \lambda\mu L_{j-1} + E\bar{\bar{G}}_i(V_i).$$

Since $V_i \le R_i < R_i^-$ where the first inequality follows from (4), one can show that the above lower bound is greater than C_f. Therefore, the considered feasible solution cannot be optimal.

THEOREM 5. $R_i^0 \ge R_i^-$, $i = 2, \ldots, N-1$.

The feasible region of $\mathcal{P}$ can be further reduced by eliminating redundant solutions. Let $\mathbf{R} = (R_1, \ldots, R_N)$ be a feasible solution to $\mathcal{P}$. Suppose $R_i + Q_i > R_{i+1} + Q_{i+1}$ for some $i = 1, \ldots, N-1$. Note that $IL_{i+1}(t) \le IP_{i+1}(t) \le R_{i+1} + Q_{i+1}$ for any t. Therefore, $IL_{i+1}(t) < R_i + Q_i$. From (3), $IP_i(t) = IL_{i+1}(t)$, suggesting that the on-hand inventory at stage $i+1$, $IL_{i+1}(t) - IP_i(t)$, is always zero. As a result, the material flow from stage $i+1$ to stage i is completely controlled by stage $i+1$ and the upstream stages. Now consider an alternative policy with $\mathbf{R}' = (R_1, \ldots, R_{i-1}, R_i', R_{i+1}, \ldots, R_N)$ where $R_i' = R_{i+1}$

$+ Q_{i+1} - Q_i$. It is easy to verify that $\mathbf{R}'$ is also a feasible solution to $\mathcal{P}$. Note that under this new policy, stage $i + 1$ still does not hold any inventory. Thus, the material flow from stage $i + 1$ to stage i is still completely controlled by stages $i + 1, \ldots, N$ whose control parameters are the same as before. Since the control parameters at stages $1, \ldots, i - 1$ are also the same as before, the new policy does not change the material flow in the system. Therefore, replacing R_i with R_i' does not affect the system performance. It is in this sense that the original solution $\mathbf{R}$ is redundant. Consequently, we can add the following constraints to $\mathcal{P}$:

$$R_i + Q_i \le R_{i+1} + Q_{i+1}, \quad i = 1, 2, \ldots, N - 1. \quad (12)$$

With the bounds in Theorems 3, 4 and 5 and the additional constraints in (12), one can find an optimal solution to $\mathcal{P}$ by a search. Of course, we need an initial solution, whose long-run average cost, C_f, is needed in order to compute the bounds. Next, we provide a heuristic algorithm for finding a good initial solution. In fact, as later numerical examples show, this initial solution is already very close to being optimal.

The heuristic algorithm is based on the following empirical observation. In many numerical examples, we found that $\mathbf{R}^*$, the optimal echelon reorder points, contains useful information about $\mathbf{r}^*$, the optimal installation reorder points. Obviously, if $\mathbf{R}^*$ is a feasible solution to $\mathcal{P}$, then the problem is solved: $r_1^* = R_1^*$ and $r_i^* = R_i^* - R_{i-1}^* - Q_{i-1}$ for $i \ge 2$. What is interesting is that even when $\mathbf{R}^*$ is infeasible, one can still obtain r_i^*, $i \ge 2$, by rounding $R_i^* - R_{i-1}^* - Q_{i-1}$ to an integer multiple of Q_{i-1}. Formally, take any $i \ge 2$. Let r_i^- (resp., r_i^+) be the maximum (resp., minimum) integer multiple of Q_{i-1} that is less (resp., larger) than or equal to $R_i^* - R_{i-1}^* - Q_{i-1}$. Then, r_i^* is either r_i^- or r_i^+. Based on this empirical observation, we restrict the installation reorder points at stages $2, \ldots, N$ to the following set:

$$(r_2, \ldots, r_N) \in \mathcal{R}_{-1} \overset{\text{def}}{=} \{r_2^-, r_2^+\} \times \cdots \times \{r_N^-, r_N^+\}.$$

Note that if $R_i^* - R_{i-1}^* - Q_{i-1}$ is an integer multiple of Q_{i-1}, then $r_i^- = r_i^+$; otherwise, $r_i^- + Q_{i-1} = r_i^+$. Thus, the set $\{r_i^-, r_i^+\}$ contains at most two points, which implies that there are at most 2^{N-1} combinations of installation reorder points in $\mathcal{R}_{-1}$.

For any fixed installation reorder points at stages $2, \ldots, N$, it is rather easy to find the optimal reorder point at stage 1. Take any $(r_2, \ldots, r_N) \in \mathcal{R}_{-1}$. Let r_1 be the reorder point at stage 1. Let $R_1 = r_1$ and $R_i = R_{i-1} + Q_{i-1} + r_i$ for $i \ge 2$. Since $IL_j \overset{\text{d}}{=} V_j + U_j - D_j$ and $\tilde{G}_1(\cdot) = \bar{G}_1(\cdot)$, we have from (6)

$$C(r_1 | r_2, \ldots, r_N) \overset{\text{def}}{=} C(\mathbf{R}) = \sum_{j=2}^{N} h_j E[V_j + U_j - D_j] + E\bar{G}_1(V_1).$$

Now suppose we change r_1 to $r_1' = r_1 + \delta$, where δ is any integer. This causes every echelon reorder point to shift by δ: $R_i' = R_i + \delta$ for $i = 1, \ldots, N$. This shift leads to a new set of effective reorder points V_i', $i = 1, \ldots, N$. Since $V_i' = V_i + \delta$ for $i = 1, \ldots, N$ (see (4)),

$$\begin{aligned} &C(r_1 + \delta | r_2, \ldots, r_N) \\ &= \sum_{j=2}^{N} h_j E[V_j' + U_j - D_j] + E\bar{G}_1(V_1') \\ &= \sum_{j=2}^{N} h_j E[V_j + \delta + U_j - D_j] + E\bar{G}_1(V_1 + \delta) \end{aligned}$$

which is clearly convex in δ since $\bar{G}_1(\cdot)$ is convex.

THEOREM 6. *$C(r_1 | r_2, \ldots, r_N)$ is convex in r_1 for any fixed installation reorder points at the upstream stages.*

Here is the heuristic algorithm. First, determine r_i^- and r_i^+, $i = 2, \ldots, N$, by rounding $R_i^* - R_{i-1}^* - Q_{i-1}$ down and up to an integer multiple of Q_{i-1}. Then, for each point in $\mathcal{R}_{-1}$, find the optimal corresponding r_1. The heuristic solution is the installation reorder points that have the lowest long-run average cost. The computational complexity of this algorithm is about 2^{N-1} times the effort of evaluating a single policy.

5. Numerical Examples

This section reports an extensive computational study designed to test the heuristic algorithm developed in §4 for finding installation reorder points and to assess the value of centralized demand information.

We assume that the demand-size distribution is geometric, i.e.,

$$\Pr(D = x) = (1 - \alpha)^{x-1}\alpha, \quad x = 1, 2, \cdots$$

where $0 < \alpha \le 1$. Thus $E[D] = 1/\alpha$ and $\text{Var}[D] = (1 - \alpha)/\alpha^2$. (Such a demand process is also referred to as a stuttering Poisson process in the literature. If $\alpha = 1$ then $\Pr(D = 1) = 1$ and the demand process becomes simple Poisson.) Let $D(1)$ be the total demand in one unit of time. It can be easily verified that $E[D(1)] = \lambda/\alpha$ and $\text{Var}[D(1)] = \lambda(2 - \alpha)/\alpha^2$ with a coefficient of variation

$$\text{cv} = \frac{\text{Std}[D(1)]}{E[D(1)]} = \sqrt{\frac{2 - \alpha}{\lambda}}.$$

The serial model is now completely specified by N, α, λ, p, h_i, L_i, and Q_i for $i = 1, \ldots, N$. To simplify the design of the computational study, we assume that 1) $h_i = 1/N$ for $i = 1, \ldots, N$ (thus $H_1 \equiv 1$), 2) $L_i = L$ for $i = 1, \ldots, N$, and 3) $(Q_1, \ldots, Q_N) = (mQ_1^0, \ldots, mQ_N^0)$ where m is a positive integer and $(Q_1^0, \ldots, Q_N^0)$ is fixed for each value of N. Table 1 summarizes the ranges of key parameters. The four values of cv are achieved with four different combinations of α and λ: $(\alpha, \lambda) = (1, 4)$, $(1, 1)$, $(0.4, 0.4)$, $(0.4, 0.1)$. Moreover, $(Q_1^0, \ldots, Q_N^0)$ are given in Table 2. There are a total of 1,536 examples.

For each example, we computed the optimal echelon reorder points by using the sequential algorithm developed in §3, and the *heuristic* installation reorder points by using the heuristic algorithm developed in §4. Due to the computational requirement of the search algorithm suggested in §4, we only computed the *optimal* installation reorder points for the examples with $N = 2$, 3, 4 for a total of 768 examples.

To test the heuristic algorithm, we compared the average cost of the heuristic solution with the average cost of the optimal installation reorder points for each of the 768 examples. Table 3 summarizes the results. The first column is intervals of the relative difference between the two costs, i.e., (heuristic cost—optimal cost)/optimal cost; the second column is the corresponding number of examples. The average relative difference is only 0.03%. This suggests that the heuristic solution is excellent.

To assess the value of centralized demand information (hereafter, value of information), we compared the average cost of the optimal echelon reorder points with the average cost of the optimal/heuristic installation reorder points for each of the 1,536 examples. (For the examples with $N = 2, 3, 4$, we used the average cost of the optimal installation reorder points. For the remaining examples, we used the heuristic solution.) We defined the relative difference between the two costs, i.e., (installation cost—echelon cost)/echelon cost, as a measure of the value of information. The numerical examples lead to the following observations.

(1) The value of information has a fairly wide range. The highest value is about 9%, which is achieved by an example with $N = 6$, $\text{cv} = \frac{1}{2}$, $L = 4$, $m = 4$, and $p = 20$. The mean is about 1.75%. Figure 1 depicts for each value of N, the histogram of the relative cost differences.

(2) It is of interest to see how the value of information depends on several key parameters of the model, i.e., N, L, m, cv, and p. Unfortunately, the individual examples do not provide discernible patterns. But if we

Table 1 Parameters

N	2, 3, 4, 6, 8, 10
cv	1/2, 1, 2, 4
p	5, 10, 15, 20
L	1, 2, 3, 4
m	1, 2, 3, 4

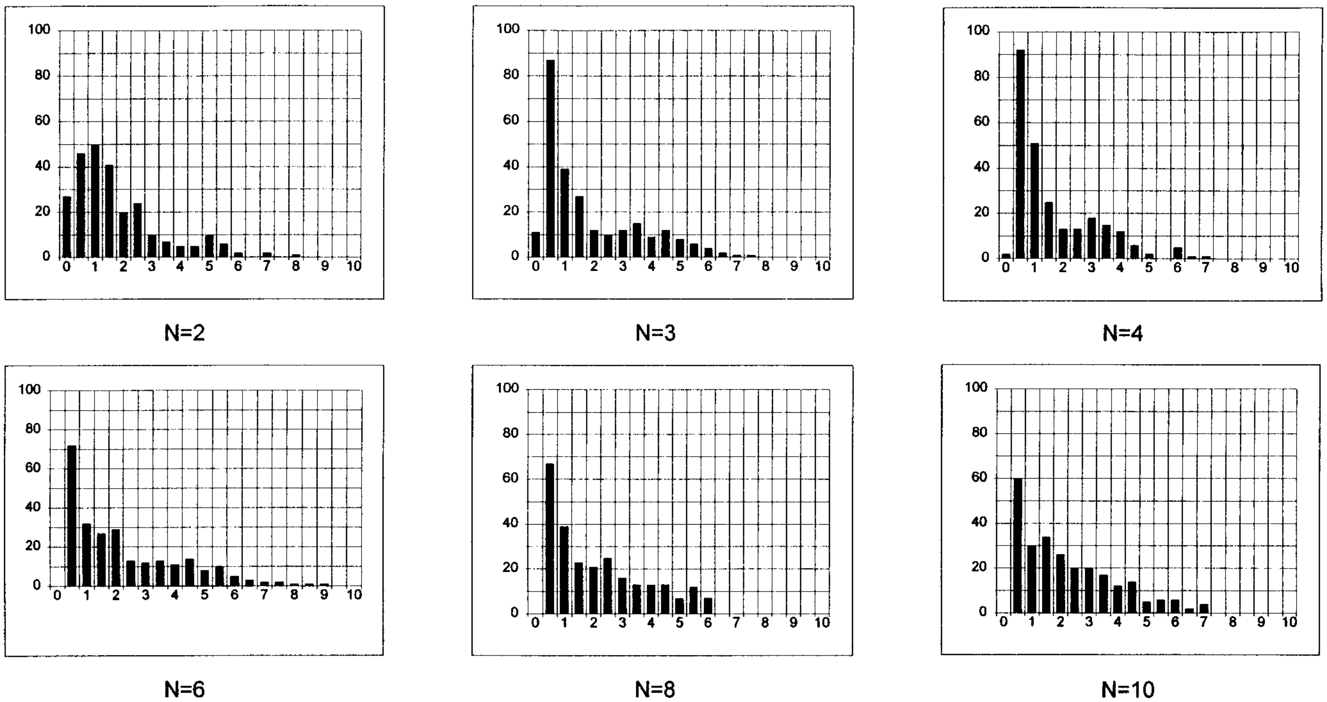
Table 2 Base Quantities

N	$(Q_1^0, \ldots, Q_N^0)$
2	(8, 32)
3	(8, 16, 32)
4	(8, 8, 16, 32)
6	(8, 8, 16, 16, 32, 32)
8	(8, 8, 8, 16, 16, 16, 32, 32)
10	(8, 8, 8, 8, 16, 16, 16, 32, 32, 32)

Table 3 Performance of the Heuristic Solution

Rel. Diff. (%)	No. of Examples
0	756
(0, 0.5]	1
(0.5, 1]	2
(1, 1.5]	0
(1.5, 2]	3
(2, 2.5]	1
(2.5, 3]	4
(3, 3.5]	0
(3.5, 4]	1

Figure 1 Value of Centralized Demand Information



X axis: value of information in percent.
Y axis: number of examples.

group the examples by the parameters, the aggregates seem to indicate some trends. Table 4 suggests that a) the value of information seems to exhibit an upward trend as N, or L, or m increases; b) increases in cv tend to decrease the value of information; and c) the value of information is highest for the extreme (high or low) values of p.

Discussion. Under installation-stock (R, nQ) policies, the orders placed by a stage become the 'demand' at the upstream stage. Since every stage orders in batches, the demand process at an upstream stage is different from the actual customer demand at stage 1. (Lee et al. (1997) call this the order-batching effect.) As the batch sizes increase, this distortion increases. It is thus not surprising that as m increases, the value of information increases. As N or L increases, the length of the serial channel increases. The numerical examples thus suggest that centralized demand information is more valuable in longer channels. This is intuitive. The relationship between the value of information and

Table 4 Value of Information and System Parameters

N	Average (%)	L	Average (%)	m	Average (%)	cv	Average (%)	p	Average (%)
2	1.50	1	1.50	1	1.50	1/2	3.09	5	1.85
3	1.61	2	1.68	2	1.74	1	1.90	10	1.65
4	1.42	3	1.92	3	1.88	2	0.97	15	1.70
6	2.03	4	1.91	4	1.87	4	1.05	20	1.80
8	1.93								
10	2.01								

demand variability is surprising at first glance. Intuitively speaking, as demand becomes more unpredictable, it should be more beneficial to share the demand information at stage 1 with the upstream stages. The numerical examples suggest the reverse. One possible explanation is that as demand variability increases, the total cost increases so much that the (relative) value of information actually decreases. Finally, we can interpret p as a measure of the desired level of customer service: a larger value of p signals a higher level of service. As p increases, it becomes more important to replenish inventories in a timely fashion and thus the value of information should increase. Then, why is the value of information so high when $p = 5$? The reason might be, again, that a small value of p leads to a lower total cost, resulting in an increase in the (relative) value of information.

Note. We implicitly assumed above that both the echelon-stock policy and the installation-stock policy use the same base quantities. This assumption is certainly valid if the base quantities are not decision variables, i.e. they are determined by exogenous factors. Otherwise, the two policies may call for different base quantities. In this case, the above assessment of the value of information is approximate. But it is reasonable to believe that the approximation is good since inventory costs tend to be insensitive to the choice of base quantities (Zheng and Chen 1992). Finally, it is interesting to note that the above findings are similar to those found by Krajewski et al. (1987) in a large scale simulation study of complex manufacturing environments.

6. Closing Remarks

This paper studies the reorder point/order quantity policy, or the (R, nQ) policy, in serial inventory systems. A key finding is that a serial system with N stages can be decomposed into N single-stage systems, each with a random reorder point. This observation leads to efficient algorithms for determining both the optimal echelon reorder points and the optimal installation reorder points. Although the paper confines itself to continuous-time models with compound Poisson demand processes, it is straightforward to extend all the results to *discrete-time* models with independent and identically distributed demands.

The paper has provided the first extensive numerical evidence on the value of centralized demand information, which is defined to be the relative cost difference between echelon-stock and installation-stock policies. (Echelon-stock policies use centralized demand information, while installation-stock policies use only local information.) In a pool of 1,536 examples, it is found that the value of information has a fairly wide range with the highest value of 9% and a mean of 1.75%. The numerical examples also suggest that the value of information tends to increase as a result of increases in the number of stages, the leadtimes, or the batch sizes. Interestingly, higher demand variability decreases the value of information, and extreme levels of customer service (either high or low) tend to increase the value.[1]

[1] The author would like to thank Paul Zipkin, an Associate Editor and two referees for their helpful comments on a previous version of this paper.

Appendix

Proof of Lemma 3. (a) Recall that $G_1(y) = E[h_1(y - D_1) + (p + H_1)(y - D_1)^-]$. Thus $G_1(\cdot)$ is convex, implying that $\tilde{G}_1(\cdot)$ is convex. Since $\bar{G}_1(y) = \tilde{G}_1(y)$ by definition, Lemma 3(a) holds for $i = 1$. Now suppose $\bar{G}_i(\cdot)$ is convex. Thus $\bar{G}_i(\min\{\bar{Y}_i, y\})$ is convex in y. This, together with (9), implies that $\bar{G}_{i+1}(\cdot)$ is convex. This completes the induction.

(b) Let $(x)^+ = \max\{0, x\}$. Since $x + (x)^- = (x)^+$, $G_1(y) = E[h_1(y - D_1)^+ + (p + H_2)(y - D_1)^-]$. It is then clear that $[G_1(y + 1) - G_1(y)] \to h_1$ (resp., $-(p + H_2)$) as $y \to +\infty$ (resp., $-\infty$). From the definition of $\bar{G}_1(\cdot)$, Lemma 3(b) holds for $i = 1$. Now suppose it holds for i. To show that the lemma holds for $i + 1$, note that the induction assumption implies that $E\bar{G}_i(\min\{\bar{Y}_i, y + Z_iQ_i - D_{i+1}\})$ as a function of y becomes flat as $y \to +\infty$ and becomes linear with slope $-(p + H_{i+1})$ as $y \to -\infty$. Thus, from (9)

$$\lim_{y\to+\infty} [\bar{G}_{i+1}(y + 1) - \bar{G}_{i+1}(y)] = h_{i+1} + 0 = h_{i+1}$$

and

$$\lim_{y\to-\infty} [\bar{G}_{i+1}(y + 1) - \bar{G}_{i+1}(y)] = h_{i+1} - (p + H_{i+1}) = -(p + H_{i+2}).$$

This completes the induction.

(c) By definition, Lemma 3(c) holds (with equality) for $i = 1$. Now suppose it holds for i. Thus, for any W,

$$\tilde{G}_i(\min\{R_i, W\}) \geq \bar{G}_i(\min\{R_i, W\}) \geq \bar{G}_i(\min\{\bar{Y}_i, W\}). \qquad (13)$$

From (8),

$$\tilde{G}_{i+1}(y) \overset{(13)}{\geq} h_{i+1}E(y + U_{i+1} - \mu_{i+1}) + E\bar{G}_i(\min\{\bar{Y}_i, y + Z_iQ_i - D_{i+1}\}) = \bar{G}_{i+1}(y). \qquad (14)$$

Thus Lemma 3(c) holds for $i + 1$. This completes the induction.

(d) Lemma 3(d) clearly holds for $i = 1$. Now suppose it holds for i. Suppose $R_j = \bar{Y}_j$ for $j = 1, \ldots, i$. By the induction assumption, $\tilde{G}_i(y) = \bar{G}_i(y)$ for all y. This, together with $R_i = \bar{Y}_i$, implies that the two inequalities in (13) now hold as equalities. As a result, the inequality in (14) becomes an equality. This completes the induction. □

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