AN ANTI-PASTA RESULT FOR MARKOVIAN SYSTEMS

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PASTA (Poisson Arrivals See Time Averages) is a term coined by R. Wolff in his well known 1982 paper. In keeping with Wolff's terminology, we use the term anti-PASTA to refer to the following converse of PASTA. Given that arrivals do indeed see time averages, when must the arrival process necessarily be Poisson? We show that anti-PASTA is satisfied in a pure-jump Markov process, provided that the arrival process corresponds to a subset of the Markov process jumps.

PASTA (Poisson Arrivals See Time Averages) is one of the most fundamental and widely used results in queueing theory (see Wolff 1982). However, the question remains as to what other types of arrival processes see time averages and under what circumstances?

This is an important issue from a practical as well as a theoretical viewpoint. The assumption that the fraction of arrivals (or departures) who find a continuous-time stochastic process in a given state equals the fraction of time that the process is in that state is very powerful for the analysis of many problems and especially for queueing systems. Therefore, it is important to understand when this assumption can or cannot be made.

Though not much has been published relating to this question, it is known that there are cases in which non-Poisson arrivals see time averages. In particular, Burke (1976) showed that in the M/M/1 queue with feedback, the composite stream of exogenous Poisson arrivals and feedback customers is not Poisson. Yet Disney, König and Schmidt (1984) showed that the work found by an arrival is identical to the virtual waiting time invariant distribution.

In this note, we address the following question: Under what circumstances does the coincidence of the limiting time and customer distributions in a queueing system imply that the arrival process is Poisson? This question was explored previously by König, Miyazawa, and Schmidt (1983). They considered several single station queueing systems with three types of service-time distribution: bounded, exponential, and Erlang. For these cases, they proved that if the limiting distribution of the number of customers in the system at an arbitrary epoch in time equals the limiting distribution at departure epochs for some set of service intensities with an accumulation point that allows for the existence of a stationary distribution, then the arrival process is Poisson.

Here, we consider an arbitrary continuous-time, pure jump Markov process and prove that if arrivals see time averages then the arrival stream must be Poisson (anti-PASTA). Since the state may be multidimensional, this result holds, for example, for arrivals at a node of a queueing network. An independent statement and proof of this result appeared in Walrand (1988, Theorem 2.10.6, part c). Related results have been reported subsequently in Melamed and Whitt (1989), Bremaud (1989) and Stidham and El-Taha (1989). The proof here differs from the one in Walrand in that it uses a time reversal argument and is based on an anti-PASTA result for departures given in Melamed (1982). Our proof is similar to one in Serfozo (1989) used in a different context.

1. THE THEOREM AND ITS PROOF

Let $X = \{X_t\}_{t=-\infty}^{\infty}$ be a Markov process with arbitrary measurable state-space (E, ϵ), right-continuous step function sample paths and regular transition structure.

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Let $\Theta \subset E^2 - \{(x, x): x \in E\}$ be a traffic set, that is, a set of transitions of X, and let K be the counting process associated with it. Consider a fixed interval $[t_1, t_2]$ and define $K_t^{[t_1,t)}$ and $K_t^{(t,t_2]}$ as the number of X transitions in Θ over the interval $[t_1, t)$ and $(t, t_2]$, respectively, for $t \in [t_1, t_2]$. Let $\{T_n\}_{n=-\infty}^{\infty}$ be the epochs at which X makes a transition in Θ, let $Y = \{X_{T_n}\}_{n=-\infty}^{\infty}$ be the X process embedded at these epochs and let $Y^- = \{X_{T_n^-}\}_{n=-\infty}^{\infty}$ be the process embedded just prior to jumps in Θ.

Thus, if X represents a queueing network and Θ corresponds to departures, then X_{T_n} is the network state seen by the nth departure. By right continuity of sample paths, this state is *left behind* by the departure, that is, the departing customer is excluded from the state. If Θ corresponds to arrivals, then $X_{T_n^-}$ is the network state seen by the nth arrival, i.e., the state just prior to the arrival where the arriving customer is excluded.

Let ξ be the equilibrium distribution of X and let ψ be the equilibrium distribution of Y. We will use the following result.

Theorem. (Melamed 1982, anti-PASTA for departures). *The equality $\psi = \xi$ holds if and only if X and K are pointwise independent, that is, for every interval $[t_1, t_2]$, X_t and $K_t^{(t_1,t]}$ are independent for each $t \in [t_1, t_2]$. Furthermore, $\psi = \xi$ implies that K is a Poisson process.*

Note that this result implies that if a departure stream sees the equilibrium state distribution, then the departure stream must be Poisson. We prove below that the same is true for arrival streams.

Let ψ^- be the equilibrium distribution of Y^-. Then ψ^- is the distribution of the state of the system seen by arrivals. Let $X^- = \{X_t^-\}_{t=-\infty}^{\infty}$ be the left-continuous version of X.

Theorem. (anti-PASTA for arrivals). *$\psi^- = \xi$ if and only if for every interval $[t_1, t_2]$, X_t^- and $K_t^{[t,t_2)}$ are independent for each $t \in [t_1, t_2]$. Furthermore, $\psi^- = \xi$ implies that K is a Poisson process.*

Remark. The independence of X_t^- and $K_t^{[t,t_2)}$ is identical to the lack of anticipation assumption (LAA) used by Wolff to prove PASTA. More specifically, LAA states that future jumps of the embedded process (that is, arrivals) are independent of the past of the observed process. Since our observed process is Markovian, the independence of future jumps and the present state ensures this condition.

Proof. The proof will be accomplished by a time reversal argument. Let $\tilde{X} = \{\tilde{X}_t\}_{t=-\infty}^{\infty}$ be the time reversed version of X^- (about 0 to simplify notation), i.e., $\tilde{X}_t \equiv X_{-t}^-$, $-\infty < t < \infty$. So for an arbitrary interval $[t_1, t_2]$, the evolution of $\tilde{X}$ from $-t_2$ to $-t_1$ corresponds to the evolution of X^- from t_2 to t_1 (backward in time).

Similarly, for any $t \in [t_1, t_2]$, we define $\tilde{K}_{-t}^{(-t_2,-t]} \equiv K_t^{[t,t_2)}$ where $\tilde{K}$ has an associated traffic set $\tilde{\Theta}$ obtained from Θ by reversing all state pairs.

$\tilde{X}$ is right continuous (since X^- is left continuous) and has equilibrium distribution $\tilde{\xi} = \xi^-$ where ξ^- is the equilibrium distribution of X^- (see Kelly 1979). Since X and X^- are stochastically equivalent, $\xi = \xi^-$. Furthermore, the invariant distribution $\tilde{\psi}$ of the time reversed version $\tilde{Y}$ of Y^- equals ψ^-.

Assume that $\psi^- = \xi$. Then $\tilde{\psi} = \tilde{\xi}$, and thus, it follows from the previous theorem that $\tilde{X}_{-t}$ and $\tilde{K}_{-t}^{(-t_2,-t]}$ are independent for each $t \in [t_1, t_2]$ or equivalently that X_t^- is independent of $K_t^{[t,t_2)}$ for each $t \in [t_1, t_2)$. It also follows from the previous theorem that K is a Poisson process.

Conversely, the independence of X_t^- and $K_t^{[t,t_2)}$ implies that $\tilde{X}_{-t}$ and $\tilde{K}_{-t}^{(-t_2,-t]}$ are independent for each $t \in [t_1, t_2]$, and from the previous theorem we get $\tilde{\psi} = \tilde{\xi}$ or equivalently, $\psi^- = \xi$.

2. DISCUSSION

In applying the result of this paper, it is important to keep in mind that the observed process must be Markovian as well as the meaning of the phrase *a customer sees.* We illustrate these points by several examples.

First consider the single GI/GI/1 queue. For the state space to be Markovian, it must include the age (or excess) of the interarrival and service times in addition to the queue length. In this case, the theorem implies that if arrivals see the equilibrium distribution of this three-dimensional state space, the arrival stream must be Poisson. Note that the corresponding anti-PASTA result for departures (Melamed 1982) cannot be used in this case because departures always see (that is, immediately after their departure) the residual service time as identically zero and thus cannot see time averages.

As mentioned in the Introduction, there exist queuing networks where non-Poisson flows still see time averages. Specifically, consider a Jackson network with node set N and assume that $n_1, n_2 \in N$ are distinct nodes participating in a cycle (that is, customers can reach n_2 from n_1 and vice versa with positive probability). It is known that the equilibrium customer stream on arc (n_1, n_2) is not Poisson (Melamed 1979, Walrand and Varaiya 1981). Nevertheless, the customer stream on arc (n_1, n_2) sees time averages when observing the subnetwork $N - \{n_1\}$ just prior to arrival at n_2, and when observing the subnetwork $N - \{n_2\}$ just after arrival at n_2 (Melamed 1982). Note, however, that neither $N - \{n_1\}$ nor $N - \{n_2\}$ are Markovian, and so, no contradiction ensues.

At this point, it becomes important to clarify what is meant by the phrase *a customer sees*. Fundamentally, it means excluding the observing customer from the observed state. Mathematically, this notion can be captured by formulating a simple removal operator on the state space so as to exclude the customer in motion (see, for example, Melamed 1982, Section 6c). In special cases, the mathematical setup can be simplified further. For queuing systems observed just prior to external arrivals, it is enough to consider the left-continuous version of the state process to obtain the removal effect. For the case of departures, one analogously considers the right-continuous version. Thus, the observing customer is about to arrive or has just departed, respectively, so that in either case the customer in motion is excluded from the observed state.

When, however, the customer stream is neither external arrivals nor departures but a flow on an arc (n_1, n_2) as in the beginning of this section, then things get a bit more complicated. Taking left-continuous or right-continuous versions of the state process does not exclude the customer in motion; in the former, it is counted in node n_1 and in the latter in node n_2. The exclusion effect is attained rather drastically by removing the customer's node of residence, i.e., by removing n_1 from N in the left-continuous version or n_2 from N in the right-continuous one. The difference in results for external and internal flows is that the external flows (arrivals and departures) observe a Markovian network N, while the internal flows observe the non-Markovian subnetworks $N - \{n_1\}$ and $N - \{n_2\}$, respectively.

Finally, we point out that the approach of taking left- and right-continuous versions does not work for a direct feedback stream in a single node. Here, node removal is vacuous as it eliminates the system under consideration, so one must use a customer removal operation. However, in using this removal operation, the state space is defined as max {the number of customers in the system -1, 0}. Thus, the observed process in continuous time is not Markovian, and so, again no contradiction ensues.

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