

Optimal Production Policies for Multi-Stage Systems with Setup Costs and Uncertain Capacities

Juhwen Hwang • Medini R. Singh

Institute of Business Management, National Dong Hwa University, Hualien, Taiwan, 974

The Amos Tuck School of Business Administration, Dartmouth College, Hanover, New Hampshire 03755-1798

The increased complexity of modern manufacturing has led to uncertainties in production processes. Factors such as unplanned machine maintenance, tool unavailability, and complex process adjustments make it difficult to maintain a predictable level of output. To be effective, an appropriate production model must incorporate these uncertainties into the representation of the production process. This paper considers a one-time production of an application-specific product which must follow a fixed routing through the manufacturing system. The flow of items can be modeled as a multi-stage serial production line. The productive capacity is uncertain at each stage and the decision to produce at any stage incurs a significant setup cost. Semifinished products have little value and inability to satisfy the demand incurs a penalty for each unit of unmet demand. We show that the optimal production policy for this system can be characterized by two critical numbers, which can be computed apriori based on the cost parameters and distributional information for all downstream stages. Sensitivity of the critical numbers is also explored.

(Multi-stage Systems; Random Yield; Dynamic Programming Application; Production Decision; Setup Cost; Capacity Uncertainty)

1. Introduction

The increased sophistication of modern manufacturing processes in many high-tech industries has led to an increase in internal uncertainties in these manufacturing systems. The use of state-of-the-art technology, intricate equipment, complex tooling requirements, involved process control, reliance on specialized operator skills and greater adherence to performance specifications are all factors that make it difficult to maintain a predictable level of output. Often, the newer, more profitable products face even greater uncertainty because the organization has not yet accumulated enough learning experience with the equipment and processes to reduce the variance. Nevertheless, production needs to be carried out in an

economic manner, despite the inherent variabilities in the system.

We present a model for planning production in a multi-stage serial production system where the aggregate productive capacity at each stage is uncertain. That is, the output quantity at any stage is the minimum of the input quantity and the realized productive capacity, which is random. Production is carried out to satisfy an uncertain one-time demand for the final product. Each unit of unsatisfied demand incurs a penalty; so does the disposal of any unused material. A decision to produce at any stage incurs a setup cost plus a processing cost for each unit produced. Due to limited time until shipment, there is only one opportunity to produce at any stage, i.e., a production shortfall cannot be compensated

by another production run. Production control must decide, for each stage, how much to produce *after* the output from the immediately preceding stage becomes available. One must take into account capacity uncertainties at all downstream stages, together with the demand uncertainties and costs, to arrive at an economic production decision.

The motivation for this model comes from the production planning problem faced by many semiconductor foundries. These foundries manufacture application specific integrated circuit (ASIC) chips used in thousands of different devices from computers and mobile phones to microwave ovens and door chimes. The rising ubiquity of chips in consumer goods, automotive parts and telecommunication devices has assured a booming demand for foundries such as Taiwan Semiconductor Manufacturing and Singapore's Chartered Semiconductor Manufacturing. Unfortunately, it has also led to a greater variety of one-time orders and capacity problems. Because of multiple production stages and long lead-time in wafer fabrication, foundries often get only one chance to satisfy a time-sensitive order. The next order is likely to have a new design or may possibly be for a different product. The single-period multi-stage model presented here tries to address the production control problems in such environments.

Our analysis shows that the optimal production policy for each stage can be characterized simply by two critical numbers. If the available input exceeds the *lower* critical number, one tries to produce as much as possible, but no more than the *upper* critical number. If, on the other hand, available input is less than the lower critical number, one chooses not to produce. The inter-relationship among critical numbers and their sensitivity to various cost parameters as well as capacities and demand distributions are also explored.

There are two fundamentally different models in the literature to represent internal uncertainties. *Yield* models focus on output loss due to process imperfections while *capacity* models deal with production loss due to resource unavailability. In random yield models, one identifies the defective units *after* processing the entire input quantity and incurring the production cost. In uncertain capacity models, one may not be able to process the entire input material due to resource constraints; un-

used input incurs no production cost. For most yield models, including the binomial and proportional yield models, increasing the input batch size increases the expected output. This is not true for the uncertain capacity model where, no matter how large the input batch size, the expected output cannot exceed the average productive capacity.

Yield models have been studied rather extensively in recent years. Interested readers are referred to Yano and Lee (1995) for an excellent survey. We discuss here only those works which are most closely related to the present paper. Lee and Yano (1988) analyze a single-period, single-product, serial production system, similar to the one considered here, but without set-up cost. The demand is known and the yield at each stage is a random multiple of the input batch size. The optimal production policy for each stage is characterized by a single critical number representing the target input quantity. The policy stipulates that one should input the target quantity, if enough is available; otherwise one should input whatever is available. An identical result for the random capacity case can be obtained from our model by setting the setup costs at all stages to zero. Yano (1986) attempts an extension of the Lee and Yano (1988) model by incorporating a setup cost at each stage of production. For a single-stage problem, she demonstrates that a two-critical-number policy, similar to that proposed in this paper, is optimal. She also demonstrates the optimality of a similar policy for a two-stage system, but only under certain restrictive conditions. The form of the optimal policy for general serial production systems with random yield and setup costs remains an open research problem worthy of investigation. This paper addresses the same problem for the case where production stages are subjected to, not yield, but capacity uncertainties. Single stage random yield models with setup cost have been a subject of great interest in another context—when a rigid demand has to be satisfied by (possibly) multiple production runs, each incurring a setup cost (see, among others, Klein 1966, White 1967, Grosfeld-Nir and Gerchak 1994). Since our interest here is primarily in the multi-stage systems, interested readers are referred to Yano and Lee (1995) for a discussion of these single-stage models. The analysis of multi-stage multi-period random yield problems is quite involved even

without setup cost and is unlikely to yield a structurally simple policy. In fact, the dynamic problem is quite complex even for a single-stage problem with no setup cost, as discussed in Gerchak et al. (1988) and Henig and Gerchak (1990). As a result, recent research efforts have been focussed on the study of such systems under suboptimal but tractable operating policies (Tang 1990; Akella, Rajagopalan, and Singh 1992).

Clark and Scarf (1960) consider a multi-period problem where a serial system tries to meet stochastic demand at the lowest installation. A linear ordering and shipping cost is incurred at all stages, except at the highest installation, where a setup cost is also incurred. The optimal ordering policy is characterized by a single critical number at all stages except at the highest installation, where it is characterized by two critical numbers. For stochastic production/inventory systems with setup cost, this is the only work to our knowledge where a structured policy is shown to be optimal in a multi-stage multi-period scenario. It should be emphasized that in the Clark and Scarf model adding a setup cost at any other stage except at the highest installation destroys the structured nature of the optimal policy. Incorporating setup cost at multiple stages has been a long-standing challenge for researchers in inventory theory and their success has been limited only to deterministic systems (Zangwill 1966).

The aggregate capacity model used in this paper captures parsimoniously the cumulative impact of varying availabilities of numerous productive resources. Hopp et al. (1993) have used this representation to model the total amount of regular-time capacity available in any period. They demonstrate that the distribution of aggregate capacity plays a central role in setting production quotas for a pull manufacturing system. Ciarallo et al. (1994) consider finite and infinite-horizon models for a single-stage production system with uncertain capacity and uncertain demand. They show that a single critical number, which represents the order-up-to point, is sufficient to characterize the optimal policy in a multi-period setting. This is in contrast to yield models where, under similar circumstances, the optimal policy is known to have no structurally simple form (Henig and Gerchak 1990); the best one can do to characterize the optimal policy is to state when *not* to order. Finally,

when yield and capacity uncertainties are simultaneously present, the optimal policy retains the complexity of yield model, as shown recently by Wang and Gerchak (1996).

The mathematical instrument used to model the uncertainty in aggregate capacity here is identical to those in Ciarallo et al. In contrast to their single-stage, multi-period model, we analyze a multi-stage, single-period model with setup cost at each stage of production. In Ciarallo et al., order-up-to policies are derived from the analysis of a cost function which is quasi-convex. In spite of this nonconvexity, the cost-to-go is shown to be convex. The presence of setup cost in our model destroys the quasi-convex unimodal structure of the cost function so effectively utilized by Ciarallo et al. in their analysis. We derive the two-critical-number policy from the analysis of a cost function which is neither unimodal nor quasi-convex. In fact, the nature of this cost function changes from concave-increasing, to convex-decreasing, convex-increasing, simply increasing, and finally to concave-increasing.

The contribution of this paper is threefold. First, the optimality of a simple two-critical-number policy is established for an important class of manufacturing problems. Considering the unusual behavior of the cost function involved, our proof of optimality is somewhat novel. Second, the sensitivity of the critical numbers to cost parameters is explored, which reveal many interesting structural properties of the system. Finally, the impact of uncertainties on the production line is studied by changing the demand distribution and the capacity distribution stochastically. The interaction between internal uncertainties and demand uncertainty is also revealed by the recursive equation used for the computation of critical numbers.

This paper is organized as follows. Section 2 presents a mathematical description of the problem and its formulation as a dynamic optimization model. Section 3 analyzes a single-stage problem and shows that a two-critical-number policy is optimal. This result is extended in §4 to a multi-stage problem. The sensitivity of the critical numbers to cost parameters, as well as to capacity and demand uncertainties, is explored in §§5 and 6, respectively. A numerical example is presented in §7. The paper concludes with some final remarks in §8.

2. Problem Description and Formulation

Consider an N -stage serial production system as shown in Figure 1. Let the stages be numbered such that the final stage of production is denoted as stage 1 while the stage of production to be performed first is denoted as stage N . The production is aimed towards satisfying a single uncertain requirement or demand for a product at the final stage. Let Z be the demand random variable with c.d.f. $Q(z)$ and p.d.f. $q(z)$.

At any stage, the productive capacity may be uncertain due to a number of factors such as machine failures, tool unavailability, time spent in process, parameter readjustments, etc. Let the random variable Y_n represent the available productive capacity at stage n with c.d.f. $F_n(y_n)$ and p.d.f. $f_n(y_n)$. We assume that all the random variables are mutually independent and their c.d.f.'s are continuous and twice differentiable.

The problem is to determine a *planned* production quantity, u_n , at each stage of production such that the expected total cost is minimized. The *actual* production output from a stage may be less than the planned production quantity. This will occur whenever the capacity realization, y_n , falls below the production target, u_n . Otherwise, the complete production target is accomplished successfully. In general, the production output of stage n , x_n , is given by $\min\{u_n, y_n\}$.

We assume that the initial inventory of all semi-finished items is zero; the analysis can be extended to accommodate positive initial inventories. The planned production quantity at stage n is obviously constrained by the actual production output from stage $n + 1$, which, in turn, depends upon the planned production quantity at stage $n + 1$, etc. That

is, $u_n \leq x_{n+1} = \min\{u_{n+1}, y_{n+1}\}$. We solve this problem dynamically by delaying the specification of the planned input quantity at a stage until the production output from the stage before becomes known. In other words, u_N, u_{N-1}, \dots, u_1 must be determined sequentially after receiving the output from the stage before.

The following costs are considered in this model. The decision to produce at stage n incurs an "out-of-pocket" setup cost K_n independent of the production quantity. In addition, each unit of item actually produced at stage n incurs a cost w_n . Each unit of unsatisfied demand incurs a penalty π . There is a cost h_n for disposing of a unit of item processed at stage n but not used by stage $n - 1$. The disposal cost for leftover raw material is represented by h_{N+1} . Since this model is especially applicable to one-time production of unique products, it is assumed that semi-finished items have little salvage value. Disposal of raw material and finished product, on the other hand, may bring a net cash inflow; this is modeled by allowing h_1 and h_{N+1} to be negative. All other parameters and costs are assumed to be nonnegative.

The following two cost conditions are necessary to ensure that it is profitable to produce and that production is motivated only by the desire to satisfy the demand.

CONDITION 1. $h_2 + \pi > w_1$.

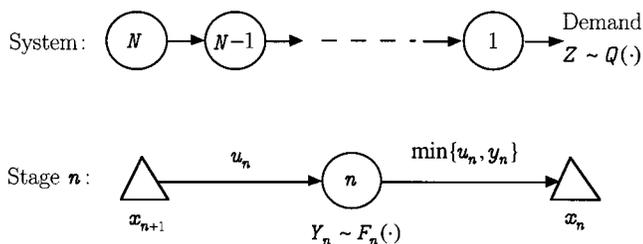
If Condition 1 did not hold, one would simply dispose of the input material at cost h_2 and incur a penalty π for not meeting the demand, rather than process an item at a higher cost w_1 . Clearly, this makes it unprofitable to produce anything at the final stage.

CONDITION 2. $w_n + h_n > h_{n+1}$ for $n = 1, \dots, N$.

This states that it is less expensive to dispose of an item at one stage than to process it and then dispose of it at the next stage.

Let $C_n(x_{n+1})$ be the expected cost of operating an n -stage system with available input x_{n+1} , assuming that the best input decision is used at stage n through stage 1. Then, $C_N(x_{N+1})$ represents the minimum expected cost to operate the whole system, where the available raw material is x_{N+1} at stage N . A dynamic program-

Figure 1 A Serial Production System



ming formulation for the problem can now be given as

$$C_0(x_1) \equiv E_Z\{h_1 \max\{0, x_1 - Z\} + \pi \max\{0, Z - x_1\}\}, \quad (1)$$

and

$$\begin{aligned} C_n(x_{n+1}) &= \min_{0 \leq u_n \leq x_{n+1}} E_{Y_n}\{h_{n+1}(x_{n+1} - \min\{u_n, Y_n\}) + K_n\delta(u_n) \\ &\quad + w_n \min\{u_n, Y_n\} + C_{n-1}(\min\{u_n, Y_n\})\}, \\ &\quad n = 1, \dots, N \end{aligned}$$

where

$$\delta(u_n) = \begin{cases} 1 & \text{if } u_n > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $E_Z\{\cdot\}$ signifies the expectation over random variable Z .

Let u_n^* be the optimal value of u_n . In our analysis we will often utilize the following representation for $C_n(x_{n+1})$:

$$\begin{aligned} C_n(x_{n+1}) &= \min_{0 \leq u_n \leq x_{n+1}} \{h_{n+1}x_{n+1} + K_n\delta(u_n) + \gamma_n(u_n)\}, \\ &\quad n = 1, \dots, N, \end{aligned} \quad (2)$$

where $\gamma_n(u_n)$ is a function only of u_n and is given by

$$\begin{aligned} \gamma_n(u_n) &= \int_0^{u_n} [(w_n - h_{n+1})y_n + C_{n-1}(y_n)]dF_n(y_n) + \bar{F}_n(u_n) \\ &\quad \times [(w_n - h_{n+1})u_n + C_{n-1}(u_n)], \end{aligned} \quad (3)$$

where

$$\bar{F}_n(u_n) = 1 - F_n(u_n).$$

The function $\gamma_n(u_n)$ plays a central role in defining the structure of the optimal policy for this problem. The following observation follows directly from (2).

LEMMA 1. *If $\gamma_n(0) < \gamma_n(u_n) + K_n$ for all u_n , then it is not worthwhile to produce at stage n . Thus, $u_n^* = 0$.*

The following lemma points to another intuitive result. If no input material is available at stage n , then the minimum expected cost for stage n through stage 1 is

simply the expected penalty for not meeting the demand.

LEMMA 2. $C_n(0) = \gamma_n(0) = \pi E[Z]$ for all n .

PROOF. From (2), $C_n(0) = \gamma_n(0)$ and from (3), $\gamma_n(0) = C_{n-1}(0)$. The result follows by induction, since $C_0(0) = \pi E[Z]$ from (1). Q.E.D.

These observations will be utilized during the analysis in coming sections.

3. The Single-Stage Problem

In this section we analyze the single-stage problem for the model introduced in §2. This analysis will provide important insights in understanding the multi-stage problem. We begin by rewriting (2) as

$$C_1(x_2) = \min_{0 \leq u_1 \leq x_2} \{h_2x_2 + K_1\delta(u_1) + \gamma_1(u_1)\}. \quad (4)$$

We first investigate the nature of $\gamma_1(\cdot)$ since it plays a central role in the minimization in (4). By using (1) and (3), we obtain the first two derivatives of $\gamma_1(u_1)$ as follows:

$$\gamma_1'(u_1) = \frac{d\gamma_1}{du_1} = \bar{F}_1(u_1)g_1(u_1), \quad (5)$$

$$\gamma_1''(u_1) = \frac{d^2\gamma_1}{du_1^2} = \bar{F}_1(u_1)(h_1 + \pi)q(u_1) - f_1(u_1)g_1(u_1), \quad (6)$$

where

$$g_1(u_1) = (h_1 + \pi)Q(u_1) + w_1 - h_2 - \pi. \quad (7)$$

$g_1(u_1)$ is nondecreasing since Conditions 1 and 2 together imply that $(h_1 + \pi) > 0$. Define S_1 such that $g_1(S_1) = 0$, i.e.,

$$S_1 = Q^{-1}\left(\frac{h_2 + \pi - w_1}{h_1 + \pi}\right). \quad (8)$$

S_1 is nonnegative and finite because $(h_2 + \pi - w_1) > 0$ and $(h_2 + \pi - w_1) < (h_1 + \pi)$ from Conditions 1 and 2. Then, $g_1(u_1)$ is negative in $(0, S_1)$ and positive in (S_1, ∞) . The nature of $\gamma_1(\cdot)$ can now be characterized using (5) and (6):

(1) For $u_1 \in (0, S_1)$, $\gamma_1'(u_1) \leq 0$ and $\gamma_1''(u_1) \geq 0$, hence γ_1 is decreasing and convex.

(2) For $u_1 \in (S_1, \infty)$, $\gamma_1'(u_1) \geq 0$ and hence γ_1 is

increasing. Nothing further can be concluded about the nature of γ_1 since γ_1' may alternate its sign in different sub-intervals.

From these observations, it is clear that $\gamma_1(u_1)$ attains its global minimum at $u_1 = S_1$. The behavior of $\gamma_1(u_1)$ is shown graphically in Figure 2.

Whenever $S_1 = 0$, the problem is trivial since it is optimal not to produce. Note that from (8), $S_1 > 0$ iff

$$h_2 + \pi - w_1 > (h_1 + \pi)Q(0),$$

or

$$w_1 + h_1 - h_2 < (h_1 + \pi) \Pr\{Z > 0\}.$$

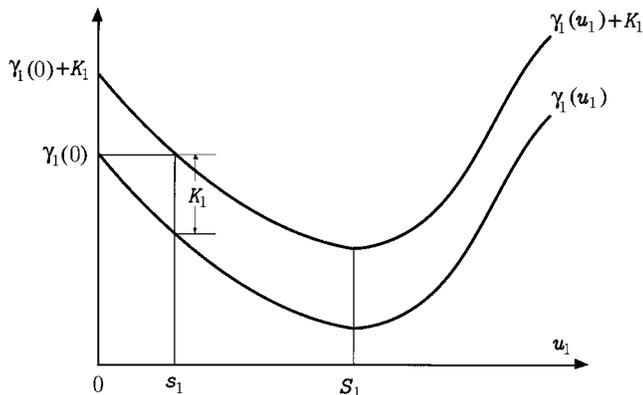
That is, if the effective cost of processing a unit, ($w_1 + h_1 - h_2$), exceeds the expected marginal saving derived from making a unit available, one is better off not processing anything.

Now consider the minimization in (4), in particular the term $(K_1\delta(u_1) + \gamma_1(u_1))$. The nature of $(K_1 + \gamma_1(u_1))$ is identical to that of $\gamma_1(u_1)$ and it attains its global minimum at S_1 with value $K_1 + \gamma_1(S_1)$. If $\gamma_1(0) < K_1 + \gamma_1(S_1)$, it is not worthwhile to produce, as per Lemma 1. An excessively high setup cost makes the problem trivial, since one always chooses not to produce. If, on the other hand, $\gamma_1(0) > K_1 + \gamma_1(S_1)$, since γ_1 is decreasing in $(0, S_1)$, there exists a unique $s_1 \in (0, S_1)$ such that

$$\gamma_1(0) = K_1 + \gamma_1(s_1). \quad (9)$$

It follows (see Figure 2) from the definition of s_1 and the decreasing nature of $\gamma_1(u_1)$ over $(0, S_1)$ that

Figure 2 The Form of $\gamma_1(u_1)$



$$\gamma_1(0) \leq K_1 + \gamma_1(u_1) \quad \text{for } u_1 \leq s_1, \quad (10)$$

and

$$\gamma_1(0) > K_1 + \gamma_1(u_1) \quad \text{for } s_1 < u_1 \leq S_1. \quad (11)$$

Based upon the available input material x_2 , the optimal policy can now be characterized in terms of the two critical numbers s_1 and S_1 . For $x_2 \leq s_1$, it is not worthwhile to produce because the setup cost, K_1 , will offset the expected savings, $(\gamma_1(0) - \gamma_1(u_1))$, derived from production. This follows from (10), since $u_1 \leq x_2 \leq s_1$. For $s_1 < x_2 \leq S_1$, the advantage gained by producing, $(\gamma_1(0) - \gamma_1(u_1))$, can offset the setup cost, K_1 , provided one plans to produce more than s_1 . This follows from (11). In fact, since γ_1 is decreasing in (s_1, S_1) , it pays to set the production target as high as possible; the optimal policy simply inputs all the available material. Finally, for $x_2 > S_1$, one sets the production target at S_1 , the point where $(K_1 + \gamma_1(u_1))$ attains its global minimum.

The optimal policy for stage 1 and the nature of $\gamma_1(u_1)$ are now summarized in the following theorem.

THEOREM 1. For the model stated in §2, if it is worthwhile to produce, then

(1) the optimal policy for stage 1 is

$$u_1^*(x_2) = \begin{cases} 0 & \text{if } x_2 \in (0, s_1), \\ x_2 & \text{if } x_2 \in (s_1, S_1), \\ S_1 & \text{if } x_2 \in (S_1, \infty), \end{cases} \quad (12)$$

where critical numbers S_1 and s_1 are solutions to Eqs. (8) and (9), respectively, and satisfy the relationship, $0 \leq s_1 \leq S_1 < \infty$;

(2)

$$\gamma_1(u_1) \text{ is } \begin{cases} \text{decreasing and convex} & \text{in range } (0, S_1), \\ \text{increasing} & \text{in range } (S_1, \infty). \end{cases}$$

The upper critical number, S_1 is the production target, i.e., the largest quantity one would ever process at stage 1. The lower critical number, s_1 , represents the smallest input quantity one is willing to process at stage 1. Note that S_1 , given by (8), is the same newsboy solution one would expect if the problem were formulated without any capacity constraint. This observation is consistent with that in Ciarallo et al. regarding the order-up-to

point in a single-stage single-period model with capacity uncertainty. The lower critical number, s_1 , however, does depend on the capacity distribution. In effect, given an input quantity, the decision as to whether one should produce is intrinsically linked to capacity uncertainty. However, the production target is independent of capacity. Once a decision is made to produce, one simply hopes that enough capacity will be available.

The cost-to-go, $C_1(x_2)$, is obtained by substituting u_1^* from (12) into (4):

$$C_1(x_2) = \begin{cases} h_2x_2 + \gamma_1(0) & \text{if } x_2 \in (0, s_1), \\ h_2x_2 + K_1 + \gamma_1(x_2) & \text{if } x_2 \in (s_1, S_1), \\ h_2x_2 + K_1 + \gamma_1(S_1) & \text{if } x_2 \in (S_1, \infty). \end{cases}$$

One can easily show that, unlike $C_0(\cdot)$, $C_1(\cdot)$ is not convex. As a result, the mathematical structure of the multi-stage problem presented in the next section is quite different than that for a single-stage problem.

4. The Multi-Stage Problem

The analysis of the single-stage problem, presented in the last section, was considerably simplified because the terminal cost-to-go, $C_0(x_1)$, is convex. This is no longer true for the multi-stage problem that we analyze in this section. The key results of this paper are contained in

THEOREM 2. *For the model stated in §2, if it is worthwhile to produce, then*

(1) *the optimal policy for stage n is*

$$u_n^*(x_{n+1}) = \begin{cases} 0 & \text{if } x_{n+1} \in (0, s_n), \\ x_{n+1} & \text{if } x_{n+1} \in (s_n, S_n), \\ S_n & \text{if } x_{n+1} \in (S_n, \infty), \end{cases}$$

where critical numbers s_n and S_n are solutions to equations

$$\gamma_n(s_n) + K_n = \gamma_n(0), \quad (13)$$

and

$$\gamma_n'(S_n) = 0,$$

respectively, and satisfy the relationship: $0 \leq s_{n-1} \leq s_n \leq S_n \leq S_{n-1} \leq \infty$;

(2)

$$\gamma_n(\cdot) \text{ is } \begin{cases} \text{increasing and concave} & \text{in range } (0, s_{n-1}), \\ \text{decreasing and convex} & \text{in range } (s_{n-1}, S_n), \\ \text{increasing} & \text{in range } (S_n, S_{n-1}), \\ \text{increasing and concave} & \text{in range } (S_{n-1}, \infty). \end{cases}$$

PROOF. The proof will be by induction on n . Define $s_0 \equiv 0$ and $S_0 \equiv \infty$. Then, for stage 1 all the properties are true from §3. To prove that these properties hold for the general case, suppose Theorem 2 is true for stage $n - 1$. Then, the following properties must hold:

$$u_{n-1}^*(x_n) = \begin{cases} 0 & \text{if } x_n \in (0, s_{n-1}), \\ x_n & \text{if } x_n \in (s_{n-1}, S_{n-1}), \\ S_{n-1} & \text{if } x_n \in (S_{n-1}, \infty), \end{cases} \quad (14)$$

where critical numbers s_{n-1} and S_{n-1} satisfy

$$\gamma_{n-1}(s_{n-1}) + K_{n-1} = \gamma_{n-1}(0), \quad (15)$$

$$\gamma_{n-1}'(S_{n-1}) = 0, \quad (16)$$

$$s_{n-2} \leq s_{n-1} \leq S_{n-1} \leq S_{n-2}, \quad (17)$$

and

$$\gamma_{n-1}(\cdot) \text{ is } \begin{cases} \text{increasing and concave} & \text{in range } (0, s_{n-2}), \\ \text{decreasing and convex} & \text{in range } (s_{n-2}, S_{n-1}), \\ \text{increasing} & \text{in range } (S_{n-1}, S_{n-2}), \\ \text{increasing and concave} & \text{in range } (S_{n-2}, \infty). \end{cases} \quad (18)$$

Figure 3(a) illustrates the nature of $\gamma_{n-1}(u_{n-1})$ and the relationship among critical numbers as indicated in the induction hypotheses (15)–(18).

Since the nature of $\gamma_n(\cdot)$ plays a fundamental role in determining the form of the optimal policy, we first explore its behavior. To this end, we first differentiate $\gamma_n(u_n)$ in (3) to get

$$\gamma_n'(u_n) = \bar{F}_n(u_n)[w_n - h_{n+1} + C'_{n-1}(u_n)]. \quad (19)$$

The term $C'_{n-1}(\cdot)$ can be obtained by first substituting u_{n-1}^* from (14) into (2) to get

$$C_{n-1}(x_n) = \begin{cases} h_n x_n + \gamma_{n-1}(0) & \text{if } x_n \in (0, s_{n-1}), \\ h_n x_n + K_{n-1} + \gamma_{n-1}(x_n) & \text{if } x_n \in (s_{n-1}, S_{n-1}), \\ h_n x_n + K_{n-1} + \gamma_{n-1}(S_{n-1}) & \text{if } x_n \in (S_{n-1}, \infty), \end{cases} \quad (20)$$

which can be differentiated and then substituted into (19) to yield

$$\gamma'_n(u_n) = \begin{cases} \bar{F}_n(u_n)g_n(u_n) & \text{if } u_n \in (s_{n-1}, S_{n-1}), \\ \bar{F}_n(u_n)[w_n + h_n - h_{n+1}] & \text{otherwise,} \end{cases} \quad (21)$$

where

$$g_n(u_n) = w_n + h_n - h_{n+1} + \gamma'_{n-1}(u_n). \quad (22)$$

Differentiating $\gamma'_n(u_n)$, one readily obtains

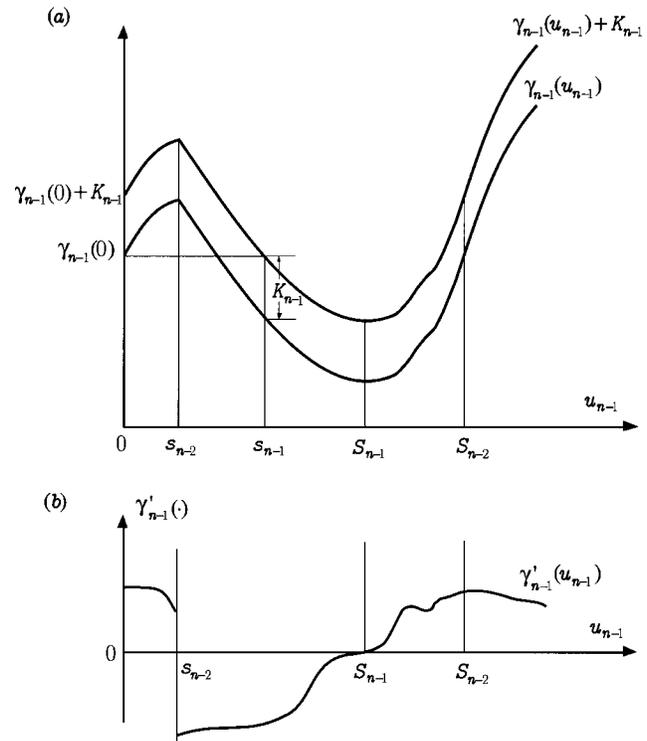
$$\gamma''_n(u_n) = \begin{cases} \bar{F}_n(u_n)\gamma''_{n-1}(u_n) - f_n(u_n)g_n(u_n) & \text{if } u_n \in (s_{n-1}, S_{n-1}), \\ -f_n(u_n)[w_n + h_n - h_{n+1}] & \text{otherwise.} \end{cases} \quad (23)$$

We first investigate the behavior of $\gamma_n(u_n)$ for $u_n \notin (s_{n-1}, S_{n-1})$. From Condition 2, the term $(w_n + h_n - h_{n+1})$ is always positive. Equations (21) and (23) then imply that $\gamma'_n > 0$ and $\gamma''_n < 0$, and hence γ_n is increasing and concave outside the interval (s_{n-1}, S_{n-1}) .

To explore the behavior of γ_n in (s_{n-1}, S_{n-1}) , we need first to understand the behavior of γ'_{n-1} since it controls the behavior of g_n . From the properties of γ_{n-1} indicated in (18), it can be inferred that (i) for $u_n \in (s_{n-2}, S_{n-1})$, $\gamma'_{n-1}(u_n) < 0$, $\gamma''_{n-1}(u_n) > 0$, and (ii) for $u_n > S_{n-1}$, $\gamma'_{n-1}(u_n) > 0$. These imply that $\gamma'_{n-1}(\cdot)$ is negative and increasing in (s_{n-2}, S_{n-1}) , crosses zero from below at S_{n-1} and remains positive for $u_n > S_{n-1}$, as illustrated in Figure 3(b).

Now consider the behavior of $g_n(u_n)$ over the interval of interest, (s_{n-1}, S_{n-1}) . From (22), $g_n(u_n) = \text{positive constant} + \gamma'_{n-1}(u_n)$. But γ'_{n-1} is increasing in (s_{n-2}, S_{n-1}) . From (17), (s_{n-1}, S_{n-1}) is a subinterval of (s_{n-2}, S_{n-1}) . Hence, g_n is increasing over (s_{n-1}, S_{n-1}) . Let $\gamma'(s^+)$ be

Figure 3 (a) The Form of $\gamma_{n-1}(u_{n-1})$ and (b) Its Derivative



the right-hand derivative of γ at s . Also, let $g(s^+)$ be the value of g as one approaches s from right. The zero-crossing property of $g_n(u_n)$ is of fundamental interest. But before we explore that, we propose the following lemma which rules out the circumstances under which the problem is trivial. The proof can be found in the Appendix.

LEMMA 3. *If $(w_n + h_n - h_{n+1}) > -\gamma'_{n-1}(s_{n-1}^+)$, then it is not worthwhile to produce at stage n . That is, $u_n^* = 0$.*

An economic interpretation for Lemma 3 can be given as follows. Recall that for the single-stage problem, an equivalent condition is $(w_1 + h_1 - h_2) > (\pi + h_1) \Pr\{Z > 0\}$ or $g_1(0) > 0$, which implies that $S_1 = 0$. Condition 2 and Lemma 3 together specify lower and upper bounds on quantity $(w_n + h_n - h_{n+1})$ which can be explained as follows. If $(w_n + h_n - h_{n+1}) < 0$, one has an unnecessary incentive to process the input at stage n , just for the sake of disposing the output at the next stage. On the other hand, as $(w_n + h_n - h_{n+1})$ increases,

it provides an increasing disincentive for production, and beyond a point, it may well become totally uneconomical to carry out production at stage n . In general, if the effective cost of processing a unit, $(w_n + h_n - h_{n+1})$, exceeds the maximum marginal benefit derived from making the unit available at the next stage, $-\gamma'_{n-1}(s_{n-1}^+)$, one simply chooses not to produce.

Consider $g_n(u_n)$ now. It is increasing over (s_{n-1}, S_{n-1}) . From Lemma 3 and (22), $g_n(s_{n-1}^+) < 0$. Also, $g_n(S_{n-1}) > 0$ by substituting (16) in (22) and using Condition 2. Hence, there exists an S_n , $s_{n-1} \leq S_n \leq S_{n-1}$, such that

$$g_n(S_n) = 0. \quad (24)$$

Clearly, $g_n(u_n) < 0$ for $u_n \in (s_{n-1}, S_n)$ and $g_n(u_n) > 0$ for $u_n \in (S_n, S_{n-1})$. Also, from (18), $\gamma''_{n-1}(u_n) > 0$ for $u_n \in (s_{n-2}, S_{n-1}) \supseteq (s_{n-1}, S_{n-1})$. It follows from (21) and (23) that (i) $\gamma'_n(S_n) = 0$, (ii) for $u_n \in (s_{n-1}, S_n)$, $\gamma'_n(u_n) < 0$ and $\gamma''_n(u_n) > 0$, and (iii) for $u_n \in (S_n, S_{n-1})$, $\gamma'_n(u_n) > 0$, but $\gamma''_n(u_n)$ may be positive or negative. Together, these properties imply that $\gamma_n(\cdot)$ (i) has a local minimum at S_n , (ii) is decreasing and concave over (s_{n-1}, S_n) , and (iii) is increasing over (S_n, S_{n-1}) . Recall that γ_n is increasing and concave for $u_n \leq s_{n-1}$ and $u_n \geq S_{n-1}$. This completes the characterization of $\gamma_n(\cdot)$.

We now characterize the form of the optimal policy. Note that $\gamma_n(u_n)$ for $u_n \in (0, s_{n-1})$ achieves its minimum at $\gamma_n(0)$ while $\gamma_n(u_n)$, $u_n \in (s_{n-1}, \infty)$, achieves its minimum at $\gamma_n(S_n)$. If $\gamma_n(0) \leq \gamma_n(S_n)$, it is optimal not to produce at stage n . If, on the other hand, $\gamma_n(S_n) \leq \gamma_n(0)$, i.e., S_n is the global minimum of $\gamma_n(\cdot)$, then it may be worthwhile to produce. The answer depends on the setup cost, K_n . If $K_n \geq \gamma_n(0) - \gamma_n(S_n)$, i.e., the setup is more expensive than the maximum benefit achievable from production, then one chooses simply not to produce, irrespective of the available input material x_{n+1} . If, on the other hand, $K_n < \gamma_n(0) - \gamma_n(S_n)$, there exists a unique $s_n \in (s_{n-1}, S_n)$ such that

$$\gamma_n(0) = K_n + \gamma_n(s_n). \quad (25)$$

Note that a point s_n satisfying (25) cannot lie in $(0, s_{n-1})$ since γ_n is increasing over this interval and $\gamma_n(s_n) < \gamma_n(0)$. It follows from the definition of s_n and the decreasing nature of $\gamma_n(u_n)$ over (s_{n-1}, S_n) that

$$\gamma_n(0) \leq K_n + \gamma_n(u_n) \quad \text{for all } u_n \leq s_n, \quad (26)$$

and

$$\gamma_n(0) > K_n + \gamma_n(u_n) \quad \text{for all } s_n < u_n \leq S_n. \quad (27)$$

Based on the available input material (the reasoning is similar to that in the single-stage problem), the optimal policy for the multiple-stage model can be summarized by (13). Q.E.D.

The lower critical number, s_n , represents the smallest input quantity one is willing to process at stage n . In this sense, s_n is a measure of the *effective* setup cost at stage n . The upper critical number, S_n , is the maximum desired output from stage n . The interrelationship among critical numbers is of significance for their efficient computation and for understanding how the system operates. An immediate corollary of Theorem 2 is

COROLLARY 1. *For the model stated in §2, the optimal operating policy is characterized by a sequence of critical numbers, $\{s_n, S_n\}$, such that*

$$0 \leq s_1 \leq s_2 \leq \dots \leq s_N \leq S_N \leq \dots \leq S_2 \leq S_1 < \infty.$$

This relationship has many intuitive implications for the operation of the serial system under study:

- (1) The lower critical number is zero at a stage *only* if it is zero at *all* downstream stages.
- (2) The lower critical number increases as one moves upstream.
- (3) The upper critical number decreases as one moves upstream. As a result, for $n = N - 1, \dots, 2, 1$, one has a binary choice: (i) input everything if $x_{n+1} > s_n$, or (ii) input nothing if $x_{n+1} < s_n$. In effect, the system has a single-critical-number policy for all stages except stage N , which must follow a two-critical-number policy.
- (4) The upper critical number at *any* stage is greater than the lower critical number for *all* stages. That is, the largest lower critical number is smaller than the smallest upper critical number.
- (5) The sequence of intervals $\{(s_n, S_n)\}$ is imbedded, i.e.,

$$(s_N, S_N) \subseteq (s_{N-1}, S_{N-1}) \subseteq \dots \subseteq (s_2, S_2) \subseteq (s_1, S_1) \subseteq (0, \infty). \quad (28)$$

5. Sensitivity Analysis

This section explores how the critical numbers, s_n and S_n , change as the cost parameters are varied. Whenever possible, we have tried to characterize the sensitivity quantitatively, but in many instances we could only give a qualitative characterization.

5.1. Sensitivity Analysis for the Upper Critical Number, S_n

The upper critical number, S_n , was defined in §4 as the zero of the recursive function $\gamma'_n(\cdot)$ in (s_{n-1}, S_{n-1}) . The following lemma states that the upper critical number for any stage can be computed directly from model parameters in a non-recursive fashion. The proof can be found in the Appendix.

LEMMA 4. *The upper critical number, S_n , satisfies*

$$\sum_{k=1}^n \prod_{j=k}^n \bar{F}_j(S_n)(w_k + h_k - h_{k+1}) - \prod_{j=1}^n \bar{F}_j(S_n) \bar{Q}(S_n)(\pi + h_1) = 0, \quad \forall n. \quad (29)$$

Note that S_n is independent of the capacity distribution for stage n , but it does depend on the capacity distributions for all the downstream stages. Also, the absence of the setup cost parameters, K_i s, in (29) implies that the upper critical numbers do not depend on setup cost. The following theorem characterizes the sensitivity of upper critical numbers to changes in cost parameters (proof given in the Appendix).

THEOREM 3. *The sensitivity of the upper critical number, S_n , with respect to*

(1) *disposal cost, h_i , is*

$$\frac{\partial S_n}{\partial h_i} = \begin{cases} -Q(S_n) \prod_{j=1}^{n-1} \bar{F}_j(S_n) / \gamma''_{n-1}(S_n) & \text{if } i = 1, \\ -F_{i-1}(S_n) \prod_{j=i}^{n-1} \bar{F}_j(S_n) / \gamma''_{n-1}(S_n) & \text{if } 1 < i < n, \\ -F_{n-1}(S_n) / \gamma''_{n-1}(S_n) & \text{if } i = n, \\ 1 / \gamma''_{n-1}(S_n) & \text{if } i = n + 1, \\ 0 & \text{otherwise;} \end{cases}$$

(2) *unit production cost, w_i , is*

$$\frac{\partial S_n}{\partial w_i} = \begin{cases} - \prod_{j=i}^{n-1} \bar{F}_j(S_n) / \gamma''_{n-1}(S_n) & \text{if } i < n, \\ -1 / \gamma''_{n-1}(S_n) & \text{if } i = n, \\ 0 & \text{otherwise;} \end{cases}$$

(3) *penalty for not satisfying the demand, π , is*

$$\frac{\partial S_n}{\partial \pi} = \bar{Q}(S_n) \prod_{j=1}^{n-1} \bar{F}_j(S_n) / \gamma''_{n-1}(S_n);$$

(4) *setup cost, K_i , is*

$$\frac{\partial S_n}{\partial K_i} = 0.$$

Note that $\gamma''_{n-1}(S_n) > 0$ since $\gamma_{n-1}(u_n)$ is convex in (s_{n-1}, S_{n-1}) and $S_n \in (s_{n-1}, S_{n-1})$ by Theorem 2. We make the following observations based on Theorem 3:

(1) The upper critical number at a stage (i) *decreases* with an increase in disposal cost at a downstream inventory location, (ii) *increases* with an increase in disposal cost at the inventory location immediately upstream, and (iii) *remains unaffected* by changes in disposal cost at all other upstream inventory locations.

The sensitivity of the upper critical number at stage n , with respect to changes in disposal cost at a downstream inventory location i , $\partial S_n / \partial h_i$, is directly proportional to the probability that the entire input S_n gets through *all* the production stages following stage n , up to and including stage i , and is stopped at location i (due to insufficient capacity at stage $i - 1$, or insufficient demand, if $i = 1$). That is,

$$\frac{\partial S_n}{\partial h_i} \propto \Pr\{Y_{n-1} > S_n; Y_{n-2} > S_n; \dots;$$

$$Y_i > S_n; Y_{i-1} < S_n\} \quad \text{for } 1 < i \leq n,$$

and

$$\frac{\partial S_n}{\partial h_1} \propto \Pr\{Y_{n-1} > S_n; Y_{n-2} > S_n; \dots; Y_1 > S_n; Z < S_n\}.$$

Clearly, the farther the inventory location i from stage n , the smaller the likelihood that the entire input S_n reaches location i , and hence the weaker the dependence of S_n on h_i .

(2) The upper critical number at a stage *decreases* with an increase in production cost at that stage or at any

other downstream stages, but it remains unaffected by the changes in production costs at all upstream stages. Moreover, $\partial S_n / \partial w_i$ is proportional to the probability that input S_n is successfully processed at all stages following stage n , up to and including stage i . Based on observation 1,

$$\frac{\partial S_n}{\partial h_i} = \begin{cases} \Pr\{Y_{i-1} < S_n\} \partial S_n / \partial w_i & \text{if } 1 < i \leq n, \\ \Pr\{Z < S_n\} \partial S_n / \partial w_1 & \text{if } i = 1, \end{cases}$$

also, for $i < j \leq n$,

$$\frac{\partial S_n}{\partial w_i} = \Pr\{Y_i > S_n; Y_{i+1} > S_n; \dots; Y_{j-1} > S_n\} \frac{\partial S_n}{\partial w_j} < \frac{\partial S_n}{\partial w_j},$$

i.e., the upper critical number at a stage is far more sensitive to the changes in production cost at a nearby downstream stage than those at a stage farther downstream.

(3) The upper critical number at a stage *increases* as the penalty for not satisfying the demand increases. Moreover, the sensitivity $\partial S_n / \partial \pi$ is proportional to the probability that input S_n is successfully processed at all downstream stages, and still falls short of satisfying the demand. Based on observation 1, we have

$$\frac{\partial S_n}{\partial \pi} = -\Pr\{Z > S_n\} \frac{\partial S_n}{\partial w_1},$$

i.e., the upper critical number is less sensitive to changes in stockout penalty than to changes in production cost; the negative sign signifies that their effects are in opposite direction.

(4) The upper critical number is equally sensitive to changes in disposal cost at the immediately preceding inventory location and to the production cost at the stage under consideration. In fact the changes in either of these parameters affects the upper critical number the most, as can be seen from,

$$\frac{\partial S_n}{\partial w_n} = -\frac{\partial S_n}{\partial h_{n+1}} = -\frac{1}{\gamma''_{n-1}(S_n)} \\ = \text{Constant of proportionality.}$$

5.2. Sensitivity Analysis for Lower Critical Number, s_n
The following lemma will be helpful for exploring sensitivity results for the lower critical numbers (proof

given in the Appendix). It expresses the relationship between the lower critical numbers for a stage, n , and its immediately following stage using parameters of the stage n and function γ'_n .

LEMMA 5. For each stage n ,

$$\int_0^{s_{n-1}} \bar{F}_n(u_n) [w_n + h_n - h_{n+1}] du_n \\ + \int_{s_{n-1}}^{s_n} \gamma'_n(u_n) du_n = -K_n. \quad (30)$$

The first term of (30) is the area under γ'_n between 0 and s_{n-1} . This area is always positive due to Condition 2 (see Figure 3). Similarly, the second term of (30) is the area under γ'_n between s_{n-1} and s_n . This area is always negative since γ_n is decreasing in (s_{n-1}, S_n) by Theorem 2. The difference between the two areas always equals $-K_n$, a fact that will prove indispensable while exploring sensitivity results for lower critical numbers. We summarize below these sensitivity results; the proofs are somewhat involved and are relegated to the Appendix.

THEOREM 4. The lower critical number at a stage is zero if and only if the production at that stage incurs no setup cost and the same is true for all downstream stages. That is, $s_n = 0$ if and only if $K_i = 0$, for all $i \leq n$.

Whenever the lower critical number, s_n , is zero, the optimal policy at stage n is of *produce-up-to* form given by a single critical number, S_n ,

$$u_n^*(x_{n+1}) = \begin{cases} x_{n+1} & \text{if } x_{n+1} \leq S_n, \\ S_n & \text{if } x_{n+1} > S_n. \end{cases}$$

According to Theorem 4, a nonzero setup cost at a stage makes the lower critical number positive not only for that stage, but for all upstream stages. If the optimal policy for a stage is of *produce-up-to* form, the optimal policy for all downstream stages must also be of *produce-up-to* form.

The following theorem shows that any increase in the setup cost at a stage will also increase the effective setup costs at all the earlier stages.

THEOREM 5. An increase in setup cost at a stage leads to an increase in the lower critical number at that stage as well as at all preceding stages.

The lower critical number at a stage (i) increases with an increase in setup cost at that stage or at any other succeeding stages, and (ii) remains unaffected by setup cost changes at preceding stages.

THEOREM 6. *An increase in stockout penalty leads to a decrease in lower critical numbers for all stages.*

As the shortage penalty increases, one is more likely to carry out production (due to a decrease in lower critical number) and one tends to produce in a larger quantity (due to an increase in upper critical number) at all stages, to avoid the higher shortage penalty. A reduction in unit production cost has an analogous affect on all upstream stages as indicated in the theorem below.

THEOREM 7. *A decrease in the unit production cost at a stage leads to a decrease in the lower critical number at that stage as well at all preceding stages.*

The lower critical number at a stage (i) decreases with a decrease in unit production cost at that stage or at any other succeeding stages, and (ii) remains unaffected by the changes in unit production costs at preceding stages.

THEOREM 8. *A decrease in disposal cost at an inventory location leads to an increase in the lower critical number for the immediately succeeding stage.*

Any decrease in disposal cost at the inventory location before a stage increases the lower critical number but decreases the upper critical number at the immediately succeeding stage. However, the decrease in the disposal cost will increase the upper critical numbers at all the upstream stages.

6. The Impact of Uncertainties

We now turn our attention to examine the effect of demand and capacity uncertainties on the optimal policy. We explore this by examining changes in critical numbers as the demand or capacity distribution is increased stochastically. A random number with distribution $\hat{Q}(\cdot)$ is said to be stochastically larger than another random number with distribution $Q(\cdot)$ if $Q(z) \geq \hat{Q}(z), \forall z$ (Ross 1983). Our results are summarized in the following two theorems; their proofs can be found in the Appendix.

THEOREM 9. *As the demand increases stochastically, the upper critical number increases but the lower critical number decreases at every stage of the system.*

An increase in demand tends to increase the expected shortage penalty. To avoid this extra penalty, one is less reluctant to start production and is willing to produce more.

THEOREM 10. *A stochastic increase in capacity at any stage leads to an increase in the upper critical numbers for all upstream stages, but the upper critical numbers for all downstream stages, including that for the current stage, remain unchanged.*

As capacity decreases stochastically at a stage, there is larger probability that capacity will take on smaller values. Since it is more likely that only a smaller input quantity can be processed at this stage, the maximum desired output from upstream stages are curtailed.

The capacity changes at a stage, however, do not affect the maximum desired output from that stage. To explain this, suppose that the maximum desired output for a stage was determined assuming an infinite capacity at that stage. In case of capacity uncertainty, one simply hopes to produce this desired amount. Reducing the maximum desired output in anticipation of a low capacity realization guarantees lower output for all capacity realizations. This can be no better than letting the realized capacity limit the output.

7. A Numerical Example

The purpose of this example is (i) to demonstrate that the two critical numbers can be computed efficiently, (ii) to validate that the sequence of critical numbers, $\{s_n, S_n\}$, is imbedded, and (iii) to illustrate the nature of $\gamma_n(\cdot)$ numerically.

Consider a three-stage serial production system with the following costs,

$$\begin{aligned} w_3 &= 30 & w_2 &= 10 & w_1 &= 15 \\ h_4 &= 10 & h_3 &= 20 & h_2 &= 25 & h_1 &= 50 & \pi &= 200. \\ K_3 &= 25000 & K_2 &= 0 & K_1 &= 45000. \end{aligned}$$

Demand as well as capacities are assumed to follow a lognormal density function

$$\phi(a|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma a} \exp\left[\frac{-(\ln a - \mu)^2}{2\sigma^2}\right] \text{ for all } a > 0,$$

where μ and σ are parameters of the lognormal distribution. That is, $q(z) = \phi(z|\mu_d, \sigma_d)$ and $f_n(y_n) = \phi(y_n|\mu_n, \sigma_n)$, $n = 1, 2, 3$. The parameters of these distributions are

$$\begin{aligned} \mu_3 = 8.5 \quad \mu_2 = 8.3 \quad \mu_1 = 8.5 \quad \mu_d = 7.3 \\ \sigma_3 = 0.2 \quad \sigma_2 = 0.5 \quad \sigma_1 = 0.3 \quad \sigma_d = 0.5. \end{aligned}$$

The upper critical numbers, S_n s, can be computed using (29). Note that this computation need not be carried out in a recursive fashion, i.e., the upper critical numbers can be obtained independent of each other. In fact, since the implementation of the optimal policy requires only S_3 (see Observation 3 for Corollary 1), there is no need to compute any other upper critical numbers. For the purpose of comparison, we report below all the S_n s

$$S_1 = 2434, \quad S_2 = 2177, \quad S_3 = 1708.$$

The lower critical numbers are calculated recursively using (13) to obtain

$$s_1 = 214, \quad s_2 = 231, \quad s_3 = 453.$$

Since $S_3 = 1708$, the search for s_1 was limited between zero and 1708. Similarly, after finding $s_1 = 214$, the search for s_2 was limited between 214 and 1708. Note also that s_2 is greater than zero despite the fact that K_2 is zero. This is because the effective setup cost at stage 2 is greater than zero due to a positive setup cost at stage 1. Observe also that the critical numbers satisfy the monotonicity property indicated in Corollary 1.

8. Concluding Remarks

A major goal of this investigation was to explore how capacity uncertainties affect production decisions in a multi-stage system with setup costs. Despite the unwieldy nature of the cost functions involved, we were able to establish the optimality of a simple two-critical-number policy. The critical numbers for successive stages were shown to be monotonic. This fact was exploited for efficient computation of the critical numbers. The monotonicity of the critical numbers also lead us to the interesting conclusion that production for all but the

very first stage can effectively be controlled using only the lower critical numbers. In our effort to understand how production decisions are affected by cost parameters, we explored the sensitivity of the critical numbers to these parameters. The impact of demand and capacity uncertainties were also studied by letting their distributions change stochastically. These results provided us with further insights on the interrelationship among various stages of the system.

Appendix

PROOF OF LEMMA 3. Suppose $(w_n + h_n - h_{n+1}) > -\gamma'_{n-1}(s_{n-1}^+)$ then, from (22), $g_n(s_{n-1}^+) > 0$. Since $g_n(u_n)$ is increasing over interval (s_{n-1}, S_{n-1}) , it remains positive over this entire interval. As a result, from (21), $\gamma'_n(u_n)$ is always positive. That is, $\gamma_n(0) < \gamma_n(u_n)$ for all u_n , and it is not worthwhile to produce at stage n . Q.E.D.

PROOF OF LEMMA 4. Consider $\gamma'_n(u_n)$ for $u_n \in (s_{n-1}, S_{n-1})$. Substituting (22) into (21),

$$\begin{aligned} \gamma'_n(u_n) = \bar{F}_n(u_n)(w_n + h_n - h_{n+1}) + \bar{F}_n(u_n)\gamma'_{n-1}(u_n), \\ \text{for } u_n \in (s_{n-1}, S_{n-1}). \end{aligned} \quad (\text{A.1})$$

Similarly, for $u_n \in (s_{n-2}, S_{n-2})$, $\gamma'_{n-1}(u_n)$ is given by

$$\gamma'_{n-1}(u_n) = \bar{F}_{n-1}(u_n)(w_{n-1} + h_{n-1} - h_n) + \bar{F}_{n-1}(u_n)\gamma'_{n-2}(u_n),$$

which can be substituted in (A.1) to yield

$$\begin{aligned} \gamma'_n(u_n) = \bar{F}_n(u_n)(w_n + h_n - h_{n+1}) + \bar{F}_n(u_n)\bar{F}_{n-1}(u_n)(w_{n-1} + h_{n-1} - h_n) \\ + \bar{F}_n(u_n)\bar{F}_{n-1}(u_n)\gamma'_{n-2}(u_n), \text{ for } u_n \in (s_{n-1}, S_{n-1}). \end{aligned}$$

Note that this substitution is valid because $(s_{n-1}, S_{n-1}) \subseteq (s_{n-2}, S_{n-2})$ as indicated in (28). By recursive substitution, we can obtain $\gamma'_n(u_n)$ in terms of the model parameters alone. The result is

$$\begin{aligned} \gamma'_n(u_n) = \sum_{k=1}^n \prod_{j=k}^n \bar{F}_j(u_n)(w_k + h_k - h_{k+1}) \\ - \prod_{j=1}^n \bar{F}_j(u_n)\bar{Q}(u_n)(\pi + h_1), \quad u_n \in (s_{n-1}, S_{n-1}) \end{aligned} \quad (\text{A.2})$$

where we have used the definition of $g_i(u_i)$ given in (7). Since the upper critical number, S_n , is the solution to equation $\gamma'_n(u_n) = 0$, $u_n \in (s_{n-1}, S_{n-1})$, the result follows. Q.E.D.

PROOF OF THEOREM 3. Using the implicit-function theorem, the sensitivity of S_n with respect to any parameter p can be expressed by

$$\frac{\partial S_n}{\partial p} = -\frac{\partial \gamma'_n(S_n)/\partial p}{\partial \gamma'_n(S_n)/\partial S_n}.$$

From (23), for $u_n \in (s_{n-1}, S_{n-1})$,

$$\frac{\partial \gamma'_n(S_n)}{\partial S_n} = \gamma''_n(S_n) = \bar{F}_n(S_n)\gamma''_{n-1}(S_n) - f_n(S_n)g_n(S_n).$$

But $g_n(S_n) = 0$ from definition of S_n , which makes the last term of the above equation vanish. Hence

$$\frac{\partial S_n}{\partial p} = -\frac{\partial \gamma'_n(S_n)/\partial p}{\bar{F}_n(S_n)\gamma''_{n-1}(S_n)}. \quad (A.3)$$

The sensitivity of S_n to changes in cost parameters can now be obtained by taking partial derivative of (29) with respect to h_i , w_i , π and K_i respectively and then substituting the results into (A.3). Q.E.D.

PROOF OF LEMMA 5. By the definition of s_n ,

$$-K_n = \gamma_n(s_n) - \gamma_n(0) = \int_0^{s_n} \gamma'_n(u_n) du_n.$$

From (21), $\gamma'_n(u_n)$ can be decomposed into two subintervals $(0, s_{n-1})$ and (s_{n-1}, s_n) instead of $(0, s_n)$. Hence, (25) can be rewritten as (30). Q.E.D.

PROOF OF THEOREM 4. From (9), $s_1 = 0$ if and only if $K_1 = 0$ since $s_1 \in (0, S_1)$ and γ_1 is decreasing in $(0, S_1)$ by Theorem 2. Assume $K_1 = 0$. From Theorem 2, γ_2 is decreasing in $(0, S_2)$. Then, from (25), $s_2 = 0$ if and only if $K_2 = 0$ since $s_2 \in (0, S_2)$ by definition. If $K_1 \neq 0$, then $s_1 > 0$. By Theorem 2, $s_2 > s_1 > 0$. Therefore K_1 must be zero to have $s_2 = 0$. To proceed by induction, assume $s_{n-1} = \dots = s_1 = 0$ if and only if $K_{n-1} = \dots = K_1 = 0$. By Theorem 2, γ_n is decreasing in $(0, S_n)$. By the definition of the lower critical number, $s_n = 0$ if and only if $K_n = s_{n-1} = 0$ since $s_{n-1} \in (0, S_n)$ by definition. Suppose $K_i \neq 0$ for some $i < n$. Then, $s_{n-1} > 0$ by assumption. By Theorem 2, $s_n > s_{n-1} > 0$. Therefore, K_i for all $i < n$ must be zero to have $s_n = 0$. Q.E.D.

PROOF OF THEOREM 5. Suppose the setup cost at stage i increases from K_i to \hat{K}_i . Let the corresponding functions $\gamma_n(\cdot)$, $C_n(\cdot)$, etc., be represented by $\hat{\gamma}_n(\cdot)$, $\hat{C}_n(\cdot)$, etc. Similarly, let the new critical numbers be represented by \hat{s}_n and \hat{S}_n for all n . Recall that the upper critical numbers are not affected by changes in setup costs (Property 4 of Theorem 3), i.e., $S_n = \hat{S}_n$ for all n . Since the change at stage i does not affect decision for any of the downstream stage, hence $s_j = \hat{s}_j$ for all $j < i$. From (25), for stage i

$$\gamma_i(s_i) - \gamma_i(\hat{s}_i) = \gamma_i(0) - K_i - \hat{\gamma}_i(0) + \hat{K}_i > 0,$$

since $\gamma_i(0) = \hat{\gamma}_i(0)$ by Lemma 2 and $K_i < \hat{K}_i$. Observe from (2) and (3) that γ_i is not a function of K_i , i.e., $\gamma_i(u_i) = \hat{\gamma}_i(u_i)$ for all u_i . Then, $\gamma_i(\hat{s}_i) = \hat{\gamma}_i(\hat{s}_i) < \gamma_i(s_i)$. By Theorem 2, γ_i is decreasing in (\hat{s}_{i-1}, S_i) , and both s_i and \hat{s}_i belong to interval (s_{i-1}, S_i) . Hence, $\hat{s}_i > s_i$ since $\gamma_i(\hat{s}_i) < \gamma_i(s_i)$.

For stage $i + 1$, using (3), we can derive

$$\begin{aligned} & \hat{\gamma}_{i+1}(u_{i+1}) - \gamma_{i+1}(u_{i+1}) \\ &= \int_0^{u_{i+1}} [\hat{C}_i(y_{i+1}) - C_i(y_{i+1})] dF_{i+1}(y_{i+1}) \\ & \quad + \bar{F}_{i+1}(u_{i+1})[\hat{C}_i(u_{i+1}) - C_i(u_{i+1})], \quad \forall u_{i+1}. \end{aligned} \quad (A.4)$$

Consider (A.4) for $u_{i+1} \in (\hat{s}_i, S_i)$. Since $\hat{s}_i > s_i$ and $\hat{S}_i = S_i$, the first term

can be divided into three subintervals $(0, s_i)$, (s_i, \hat{s}_i) and (\hat{s}_i, u_{i+1}) . Then, substitute (20) into (A.4) to get

$$\begin{aligned} & \hat{\gamma}_{i+1}(u_{i+1}) - \gamma_{i+1}(u_{i+1}) \\ &= \int_0^{s_i} [\hat{\gamma}_i(0) - \gamma_i(0)] dF_{i+1}(y_{i+1}) \\ & \quad + \int_{s_i}^{\hat{s}_i} [\hat{\gamma}_i(0) - \gamma_i(y_{i+1}) - K_i] dF_{i+1}(y_{i+1}) \\ & \quad + \int_{\hat{s}_i}^{u_{i+1}} [\hat{K}_i - K_i] dF_{i+1}(y_{i+1}) + \bar{F}_{i+1}(u_{i+1})[\hat{K}_i - K_i]. \end{aligned} \quad (A.5)$$

Observing that $\gamma_i(0) = \hat{\gamma}_i(0)$ from Lemma 2, the first term becomes zero. The integrand of the second term can be rewritten as $\gamma_i(s_i) - \gamma_i(y_{i+1})$ by the definition of s_i . By Theorem 2, γ_i is decreasing in $(s_i, \hat{s}_i) \subseteq (s_{i-1}, S_i)$. Therefore, $\gamma_i(s_i) > \gamma_i(y_{i+1})$ for $y_{i+1} \in (s_i, \hat{s}_i)$. Hence, the second term of (A.5) is also positive. The last two terms of (A.5) are positive since $\hat{K}_i > K_i$. As a result, $\hat{\gamma}_{i+1}(u_{i+1}) > \gamma_{i+1}(u_{i+1})$ in (\hat{s}_i, S_{i+1}) . By Lemma 2 and from the definition of the lower critical number, we have

$$\gamma_{i+1}(s_{i+1}) = \gamma_{i+1}(0) - K_{i+1} = \hat{\gamma}_{i+1}(0) - K_{i+1} = \hat{\gamma}_{i+1}(\hat{s}_{i+1}).$$

Hence, $\gamma_{i+1}(s_{i+1}) = \hat{\gamma}_{i+1}(\hat{s}_{i+1}) > \gamma_{i+1}(\hat{s}_{i+1})$ since $\hat{s}_{i+1} \in (\hat{s}_i, S_{i+1})$ by Theorem 2. Notice that γ_{i+1} is decreasing in $(\hat{s}_i, S_{i+1}) \subseteq (s_i, S_{i+1})$ by Theorem 2. Therefore, $s_{i+1} < \hat{s}_{i+1}$.

To proceed by induction, suppose $s_{n-1} < \hat{s}_{n-1}$ and $\gamma_{n-1}(u_{n-1}) \leq \hat{\gamma}_{n-1}(u_{n-1})$ in (\hat{s}_{n-1}, S_n) for $n > i$. Using a similar argument as above, one can show that $\hat{\gamma}_n(u_n) > \gamma_n(u_n)$ in (\hat{s}_{n-1}, S_n) . Using Lemma 2 and Theorem 2, it follows that $\gamma_n(s_n) = \hat{\gamma}_n(\hat{s}_n) > \gamma_n(\hat{s}_n)$. But γ_n is decreasing in $(\hat{s}_{n-1}, S_n) \subseteq (s_{n-1}, S_n)$, hence $s_n < \hat{s}_n$. Finally, to complete the proof by induction, note from (28) that $\hat{\gamma}_n(u_n) > \gamma_n(u_n)$ in $(\hat{s}_n, S_{n+1}) \subseteq (\hat{s}_{n-1}, S_n)$. Q.E.D.

PROOF OF THEOREM 6. Suppose the penalty cost π increases to $\hat{\pi}$, and the associated $\gamma_n(\cdot)$, s_n and S_n become $\hat{\gamma}_n(\cdot)$, \hat{s}_n and \hat{S}_n , respectively. Consider the relationship between \hat{s}_n and \hat{s}_{n-1} for the case when the penalty is $\hat{\pi}$ from Lemma 5

$$\int_0^{\hat{s}_{n-1}} \bar{F}_n(u_n)[w_n + h_n - h_{n+1}] du_n + \int_{\hat{s}_{n-1}}^{\hat{s}_n} \hat{\gamma}'_n(u_n) du_n = -K_n. \quad (A.6)$$

Since $-K_n$ is a constant, the difference between the left-hand-sides of (30) and of (A.6) is zero. We will prove the theorem by induction. For each stage n , we will show that unless $s_n > \hat{s}_n$ is true, the difference between (30) and (A.6) will turn out to be negative. Hence, by contradiction, $s_n > \hat{s}_n$.

For stage 1, the first terms of both (30) and (A.6) vanish since $s_0 = \hat{s}_0 = 0$. Suppose $s_1 \leq \hat{s}_1$. Then, by subtracting (A.6) from (30), we have

$$\int_0^{s_1} [\hat{\gamma}'_1(u_1) - \gamma'_1(u_1)] du_1 + \int_{s_1}^{\hat{s}_1} \hat{\gamma}'_1(u_1) du_1 = 0. \quad (A.7)$$

Since $\hat{\pi} > \pi$, thereby $\gamma'_1(u_1) > \hat{\gamma}'_1(u_1)$ in $(0, s_1) \subset (0, \infty)$ from (A.2).

Then, the first term of (A.7) is negative. The second term is also negative since the integrand, $\hat{\gamma}'_i(u_i)$, is negative in $(s_i, \hat{s}_i) \subseteq (0, \hat{s}_i)$ by Theorem 1. Hence, the left-hand side of (A.7) becomes negative. This contradicts (A.7). Therefore, $s_1 > \hat{s}_1$.

Consider stage n . To proceed with induction, assume $s_{n-1} > \hat{s}_{n-1}$. In order to prove $s_n > \hat{s}_n$ by contradiction, suppose $s_n \leq \hat{s}_n$. Hence, $\hat{s}_{n-1} < s_{n-1} < s_n < \hat{s}_n$ since $s_{n-1} \leq s_n$ by Theorem 2. Subtract (A.6) from (30)

$$\begin{aligned} & - \int_{\hat{s}_{n-1}}^{s_{n-1}} \bar{F}_n(u_n)[w_n + h_n - h_{n+1}] du_n + \int_{\hat{s}_{n-1}}^{s_{n-1}} \hat{\gamma}'_n(u_n) du_n \\ & + \int_{s_{n-1}}^{s_n} [\hat{\gamma}'_n(u_n) - \gamma'_n(u_n)] du_n + \int_{s_n}^{\hat{s}_n} \hat{\gamma}'_n(u_n) du_n = 0. \end{aligned} \quad (\text{A.8})$$

The first term of (A.8) is negative since $w_n + h_n - h_{n+1} > 0$ by Condition 2. By Theorem 2, $\hat{\gamma}'_n$ is negative in $(\hat{s}_{n-1}, \hat{s}_n)$, which contains intervals (\hat{s}_{n-1}, s_{n-1}) and (s_{n-1}, \hat{s}_n) . Therefore, the second and fourth terms of (A.8) are both negative. From (A.2), consider γ'_n in (s_{n-1}, S_{n-1}) and $\hat{\gamma}'_n$ in $(\hat{s}_{n-1}, \hat{S}_{n-1})$. Since $\hat{\pi} > \pi$, hence $\hat{\gamma}'_n(u_n) < \gamma'_n(u_n)$ in $(s_{n-1}, S_{n-1}) = (s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})$ since $S_{n-1} < \hat{S}_{n-1}$ by Theorem 3. By Theorem 2, (s_{n-1}, S_{n-1}) contains interval (s_{n-1}, s_n) . Therefore, the third term of (A.8) is also negative. Hence, the left-hand side of (A.8) is negative. This contradicts the equality in (A.8). As a result, $\hat{s}_n < s_n$. Q.E.D.

PROOF OF THEOREM 7 AND THEOREM 8. Can be proved using similar mathematical arguments as for Theorem 6. Removed due to space limit.

PROOF OF THEOREM 9. Suppose that the demand distribution changes from $Q(z)$ to $\bar{Q}(z)$ such that $\bar{Q}(z) \leq \bar{Q}(z)$ for all z , and the associated $g_n(\cdot)$, $\gamma_n(\cdot)$, s_n and S_n become $\hat{g}_n(\cdot)$, $\hat{\gamma}_n(\cdot)$, \hat{s}_n and \hat{S}_n , respectively. We first show inductively that $S_n \leq \hat{S}_n$ for all stages. By substituting (21) into (A.2) for $u_n \in (s_{n-1}, S_{n-1})$, one obtains

$$\begin{aligned} g_n(u_n) &= (w_n + h_n - h_{n+1}) + \sum_{k=1}^{n-1} \prod_{j=k}^{n-1} \bar{F}_j(u_n)(w_k + h_k - h_{k+1}) \\ & - \prod_{j=1}^{n-1} \bar{F}_j(u_n) \bar{Q}(u_n)(\pi + h_1). \end{aligned} \quad (\text{A.9})$$

Since $\hat{s}_0 = s_0 \equiv 0$ and $\hat{S}_0 = S_0 \equiv \infty$, both g_1 and \hat{g}_1 are defined in $(0, \infty)$. Since $\bar{Q}(\cdot) \leq \bar{Q}(\cdot)$, $g_1(\cdot) \geq \hat{g}_1(\cdot)$ in $(0, \infty)$. Then, $\hat{g}_1(S_1) \leq g_1(S_1) = 0$ by the definition of S_1 . Hence, $S_1 \leq \hat{S}_1$ since \hat{g}_1 is increasing and $\hat{g}_1(\hat{S}_1) = 0$. By induction, assume $S_{n-1} \leq \hat{S}_{n-1}$. For stage n ,

- (1) If $S_n < \hat{s}_{n-1}$: By Theorem 2, $S_n \leq \hat{S}_n$ since $\hat{s}_{n-1} \leq \hat{S}_n$.
- (2) If $S_n > \hat{s}_{n-1}$: From (A.9), consider g_n in (s_{n-1}, S_{n-1}) and \hat{g}_n in $(\hat{s}_{n-1}, \hat{S}_{n-1})$. Since $\bar{Q}(\cdot) \leq \bar{Q}(\cdot)$, $g_n(\cdot) \geq \hat{g}_n(\cdot)$ in $(\max\{s_{n-1}, \hat{s}_{n-1}\}, S_{n-1}) = (s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})$. By Theorem 2, both S_n and \hat{S}_n belong to $(\max\{s_{n-1}, \hat{s}_{n-1}\}, S_{n-1})$. Hence, $\hat{g}_n(S_n) \leq g_n(S_n) = 0$ from (22). Then, $\hat{S}_n \geq S_n$ since \hat{g}_n is increasing in $(S_n, \hat{S}_{n-1}) \subseteq (\hat{s}_{n-1}, \hat{S}_{n-1})$.

Now, we need to prove that $s_n \geq \hat{s}_n$ for all stages. From (A.2), while the demand density shifts from $Q(z)$ to $\bar{Q}(z)$, we have $\hat{\gamma}'_n(u_n)$

$\leq \gamma'_n(u_n)$ in $\{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})\}$ since $\hat{g}_n \leq g_n$ in the same range. This result is analogous to the one that appeared in the proof of Theorem 6, where the penalty cost increases from π to $\hat{\pi}$. Following the proof of Theorem 6, one can show that $s_n \geq \hat{s}_n$. Q.E.D.

PROOF OF THEOREM 10. Assume the density of the capacity at stage i changes from $F_i(y_i)$ to $\bar{F}_i(y_i)$ such that $\bar{F}_i(y_i) \leq \bar{F}_i(y_i)$ for all y_i , and the associated $g_n(\cdot)$, s_n and S_n become $\hat{g}_n(\cdot)$, \hat{s}_n and \hat{S}_n , respectively. From (A.9), $g_i(u_i)$ is not a function of $F_i(\cdot)$. Hence, from (24) $S_i = \hat{S}_i$. From (21), if $u_i \in \{(s_{i-1}, S_i) \cap (\hat{s}_{i-1}, \hat{S}_i)\}$ then

$$\gamma'_i(u_i) - \hat{\gamma}'_i(u_i) = [\bar{F}_i(u_i) - \bar{F}_i(u_i)]g_i(u_i) \geq 0,$$

since $g_i(u_i) \leq 0$ by Theorem 2. From (22), if $u_{i+1} \in \{(s_{i-1}, S_i) \cap (\hat{s}_{i-1}, \hat{S}_i)\}$ then

$$g_{i+1}(u_{i+1}) - \hat{g}_{i+1}(u_{i+1}) = \gamma'_i(u_{i+1}) - \hat{\gamma}'_i(u_{i+1}) \geq 0.$$

By recursive substitution and (28), we obtain $\gamma_n(u_n) \geq \hat{\gamma}_n(u_n)$ and $g_n(u_n) \geq \hat{g}_n(u_n)$ in $\{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})\}$ for $n > i$.

By induction, assume $S_{n-1} \leq \hat{S}_{n-1}$ for $n - 1 > i$. Then, $g_n(u_n) \geq \hat{g}_n(u_n)$ in $\{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})\}$. For stage n , if $S_n < \hat{s}_{n-1}$, then $S_n \leq \hat{S}_n$ since $\hat{s}_{n-1} \leq \hat{S}_n$ by Theorem 2. If, on the other hand, $S_n > \hat{s}_{n-1}$, then $S_n \in \{(s_{n-1}, S_{n-1}) \cap (\hat{s}_{n-1}, \hat{S}_{n-1})\}$ by Theorem 2. Hence, $\hat{g}_n(S_n) \leq g_n(S_n) = 0$ from (22). Therefore, $\hat{S}_n \geq S_n$ since \hat{g}_n is increasing in $(S_n, \hat{S}_{n-1}) \subseteq (\hat{s}_{n-1}, \hat{S}_{n-1})$. Q.E.D.

References

- Akella, R., S. Rajagopalan, and M. R. Singh, "Part Dispatch in Random Yield Multi-Stage Flexible Test Systems for Printed Circuit Boards," *Oper. Res.*, 40 (1992), 776-789.
- Ciarallo, F. W., R. Akella, and T. E. Morton, "A Periodic Review, Production Planning Model with Uncertain Capacity and Uncertain Demand—Optimality of Extended Myopic Policies," *Management Sci.*, 40 (1994), 320-332.
- Clark, A. J. and H. Scarf, "Optimal Policies for a Multi-Echelon Inventory Problem," *Management Sci.*, 6 (1960), 475-490.
- Gerchak, Y., R. G. Vickson, and M. Parlar, "Periodic Review Production Models with Variable Yield and Uncertain Demand," *IIE Transactions*, 20 (1988), 144-150.
- Grosfeld-Nir, A. and Y. Gerchak, "Production to Order with Random Yields: Single-Stage Multiple Lot-Sizing," University of Waterloo, Waterloo, Ontario, 1994.
- Heing, M. and Y. Gerchak, "The Structure of Periodic Review Policies in the Presence of Random Yield," *Oper. Res.*, 38 (1990), 634-643.
- Hopp, W. J., M. L. Sperman, and I. Duenyas, "Economic Production Quotas for Pull Manufacturing Systems," *IIE Transactions*, 25 (1993), 71-78.
- Klein, M., "Markovian Decision Models for Reject Allowance Problems," *Management Sci.*, 12 (1966), 349-358.
- Lee, H. L. and C. A. Yano, "Production Control in Multistage Systems with Variable Yield Losses," *Oper. Res.*, 36 (1988), 269-278.
- Ross, S. M., *Stochastic Processes*, John Wiley & Sons, Inc., New York, 1983.

- Tang, C. S., "The Impact of Uncertainty of a Production Line," *Management Sci.*, 36 (1990), 1518-1531.
- Wang, Y. and Y. Gerchak, "Periodic Review Production Models with Variable Capacity, Random Yield, and Uncertain Demand," *Management Sci.*, 42 (1996), 130-137.
- White, L. S., "Bayes Markovian Decision Models for a Multiperiod Reject Allowance Problem," *Oper. Res.*, 15 (1967), 857-865.
- Yano, C. A., "Optimal Policies for a Serial System with Setup Costs and Variable Yields," T.R. 86-24, Dept. of Industrial and Operations Engineering, University of Michigan, Ann Arbor, Michigan, 1986.
- and H. L. Lee, "Lot-Sizing with Random Yields: A Review," *Oper. Res.*, 43 (1995), 311-334.
- Zangwill, W. I., "A Deterministic Multiproduct, Multifacility Production and Inventory Models," *Oper. Res.*, 14 (1966), 486-507.

Accepted by Hau L. Lee; received May 6, 1994. This paper has been with the authors 22 months for 2 revisions.