OPTIMAL VARIANCE STRUCTURES AND PERFORMANCE IMPROVEMENT OF SYNCHRONOUS ASSEMBLY LINES

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Contemporary management theories such as Just-in-Time and Total Quality Management emphasize variance reduction as a critical step in improving system performance. But little is said about how such efforts should be directed. Suppose a manager has only limited resources for variance reduction efforts. How should she allocate them among a set of competing activities? Which activity should receive the highest priority? We explore such questions in the context of a synchronous assembly line where processing times are variable, incomplete jobs are reworked at the end of the line, and the objective is to minimize the total expected work overload. Our results indicate that the station with the highest variance may not always be the best choice for variance reduction. Identifying the set of stations that should receive variance reduction in an optimal solution is not trivial. Moreover, the variances at these stations may not be reduced by the same amount or to the same level. We establish that the remaining variances among stations that receive variance reduction must conform to one of two preferred structures: equal variance or spike-shaped. Dominance results are presented to identify the set of stations and the amount of reduction in an optimal solution.

Variability in manufacturing systems leads to many operational inefficiencies such as higher rework costs, poor quality, lower throughput, higher work-in-process, longer cycle times, and reduced labor productivity. Identifying the causes of variability, assessing their impact, and eliminating them is central to process improvement efforts. Contemporary theories of operations management such as Just-in-Time and Total Quality Management point to variance reduction as a critical step in improving system performance. For example, Hopp and Spearman (1996) emphasize that "variability reduction is a key means for improving a manufacturing system." Similarly, Sarkar and Zangwill (1991) observe that "Just-in-Time (JIT) manufacturing procedures emphasize (among other things) reduction in variance." Central to Total Quality Management is the maxim that "variability is root of all evil" (Schonberger 1986) and that one should strive to reduce it continually.

A number of researchers have analyzed the benefits of reducing variance. For example, Sarkar and Zangwill (1991) demonstrate that reducing processing time variance can expand effective capacity and reduce inventory in a multi-item production facility. Gerchak and Parlar (1991) consider a continuous review inventory model that quantifies the benefits of investing to reduce the variability in lead-time. A recent book by Hopp and Spearman (1996) provides important insights into the benefits of variability reduction through a number of illustrative queueing models. Unfortunately, variance reduction efforts require resources that are scarce. When several competing alternatives for variance reduction are available, the allocation of these resources becomes critical and raises several interesting questions. For example, how should a manager allocate the precious resource among a set of competing stages or activities such that maximal improvement in system performance is achieved? Should one always start with the worst culprit—the station with the most variance? Does a variance reduction strategy that attempts to equalize the remaining variance among stations perform better than other strategies? We explore these questions in the context of a synchronous assembly line.

Consider a serial line where jobs move synchronously from one station to the next every T time units. Such lines are prevalent in automotive and appliance assembly where jobs move simultaneously on an automated conveyor. The processing time at each station is assumed to be normally distributed with mean μ, but stations differ in terms of variability. A number of factors contribute to processing time variability such as unforeseen disruptions, machine and tool failures, differences in processing requirements due to material and machine variations, operator availability, variation in worker skill and experience across different tasks, natural human variation in a repetitive manual work environment, and inconsistency in operator speed due to fatigue. Should the processing of a job at a station require longer than the cycle time T, the residual work is considered to be an *overload*. All overload work is completed at the end of the line at a rework station. Such a rework policy is often used in industry to avoid line slowdowns which result in reduced throughput. The objective is to minimize the total expected work overload (EWO) for a

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job. The EWO is a surrogate for rework cost and serves as a means to improve cost and quality. As Yano and Rachamadugu (1991) note, "minimizing the work overload contributes to reducing the total labor cost and improving product quality."

Processing time variance can be reduced by investing in preventive maintenance, SPC, worker training, better equipment and tools, etc. Suppose the manager for this line has a "bag of money" to allocate to variance reduction efforts among the stations. Assume that a dollar spent at a station reduces its processing time variance by a fixed amount that is independent of the choice of the station. That is, variance reduction is proportional to the amount of money spent and any two stations receiving the same amount of money will realize identical reduction in their variances. It is possible that learning effects bring increasing returns to investment in variance reduction. It is also possible that limits of available technology and current engineering know-how result in decreasing returns to investment in variance reduction. Since it is difficult to know in advance which of these two factors will play a dominant role, we assume a linear return on investment in variance reduction for all stations.

With a limited budget, the manager can accomplish only a finite amount of variance reduction, say ΔV. The *variance reduction problem* then is to apportion the total reduction ΔV among the set of stations, $\mathcal{S}$, on the assembly line, so that EWO is minimized. No matter how ΔV is apportioned, the final remaining variance on the line is always the same, say V, the sum of initial variances *less* the reduction. Consider two such apportionments, i.e., two feasible solutions to the variance reduction problem. These two otherwise identical lines, both with the same amount of total variability V, may perform quite differently depending upon how variability is distributed across various stations. This raises several interesting questions: Are there preferred *variance structures* that yield better line performance for a given level of variability? How do these structures change as the level of overall variability increases or decreases? Answers to these questions are important for a targeted variance reduction effort. They can also provide important insights into better design and planning of assembly lines.

We address these questions by analyzing the following *variance allocation problem*: How should the total variance V be allocated among the set of stations $\mathcal{S}$ such that the EWO is minimized? A similar question has been explored by Lau (1992) who investigates via a simulation study the impact of different variance structures on the throughput of an *asynchronous* line while keeping the total variance constant. He finds bowl-shaped and symmetric configurations of processing time variances desirable, which can be explained by the bowl-phenomenon (Hillier and Boling, 1966) and the well-known reversibility property (Muth 1979). However, Lau also makes the surprising observation that the optimal solution in some cases is a spike-shaped configuration where all (or nearly all) the variability is concentrated at only one station and all other stations have zero (or very close to zero) variability. Unfortunately, this work does not give an indication under what circumstances a spike-shaped configuration will be better than a bowl-shaped or symmetric configuration.

Our results indicate that there are two desirable variance structures for processing times on synchronous assembly lines: (i) the uniform configuration, where variability is evenly distributed among all the stations; and (ii) the spike-shaped configuration, where most of the variability is concentrated at only one station and all other stations have relatively little variability. These configurations are similar to those obtained by Lau (1992) for asynchronous lines. Our work not only establishes the desirability of these variance structures analytically, but it also establishes the circumstances under which each of these configurations is best. Specifically, we show that if the total amount of variability exceeds a critical level, then a spike-shaped configuration is optimal; otherwise, the uniform configuration is optimal. As the total variability increases beyond the critical level, the spike-shape becomes more pronounced; that is, the allocation at all stations except one approaches zero. Any further increase in the total variability beyond a point makes the line with uniform configuration the worst possible performer.

Given these results on variance allocation, we can summarize the lessons for variance reduction for our synchronous assembly line as follows: Unless the remaining variance after reduction is to be brought below the critical level, it is not optimal to equalize the variability at each station. Moreover, the station with the highest variance may receive no reduction at all in an optimal solution. The variance structure for stations that do receive variance reduction follow the configuration described above—their remaining variances are all identical except for possibly one station, which may be left with a larger variance than others.

The remainder of the paper is organized as follows. The next section contains a formal statement of the variance reduction problem and its relationship to the variance allocation problem. The key results for optimal variance structures are developed in § 2, first for a two-station line and then for a general n station line. We revisit the variance reduction problem in § 3, where results from § 2 are used to develop dominance results. These results provide important insight into the set of stations that receive variance reduction. An example is also presented to illustrate the structural properties. Section 4 demonstrates the validity of results when the normality assumption regarding processing time distribution is relaxed. It is shown that the validity of the results hinges on an important structural property of the expected work overload, not the normality assumption for the processing time distributions. We end the paper with some final thoughts on how the insights from this model can be applied to more realistic systems.

1. PROBLEM FORMULATION AND ANALYSIS

Consider a synchronous assembly line with a set of stations $\mathcal{S}$. This line is designed for a throughput of $1/T$, i.e., jobs spend T time units at each station. The processing time at each station is random with mean μ. The mean processing time at each station is kept the same so that the line is balanced on average. Workforce rules often preclude assigning more work to some workers than others. We assume that the processing times are normally distributed. Studies in the literature (Wilhelm 1987) support this assumption. Additionally, the normal distribution leads to analytically tractable results. In § 4 we show that similar results hold for a number of other distributions.

Consider a station that has processing time variance v. The EWO for this station is given by

$$f(v) \triangleq \int_T^\infty \frac{(t-T)}{\sqrt{2v\pi}} e^{-(t-\mu)^2/2v}\, dt$$
$$= \sqrt{v}\left[Z\left(\frac{T-\mu}{\sqrt{v}}\right) - \left(\frac{T-\mu}{\sqrt{v}}\right)\left(1 - P\left(\frac{T-\mu}{\sqrt{v}}\right)\right)\right],$$

where $Z(x)$ and $P(x)$ are the standard normal probability density and cumulative distribution functions evaluated at x, respectively. The function $f(v)$ is continuous and differentiable with $\lim_{v\to 0} f(v) = 0$.

Suppose that the processing time variance at station i is currently v_i^0. The larger the variance at a station, the greater the expected work overload. It is assumed that the cost of rework is proportional to the work overload (incomplete work). The total expected work overload can thus be used as a surrogate for the expected rework cost at the end of the line. To reduce the costly rework, the manager is interested in reducing the variability in processing time at some or all of the stations. Since any variance reduction effort requires resources, e.g., time, money, engineering expertise, etc., that are scarce, she would like to use them most efficiently. We assume that the amount of variance reduction achieved is directly proportional to the amount of resource invested. If the same amount of resource is invested at two different stations, it is likely to result in identical variance reduction (provided there is enough variance to be reduced). The variance reduction problem can then be posed as

$$(\mathcal{P}) \text{ Minimize } \sum_{i\in\mathcal{S}} f(v_i)$$

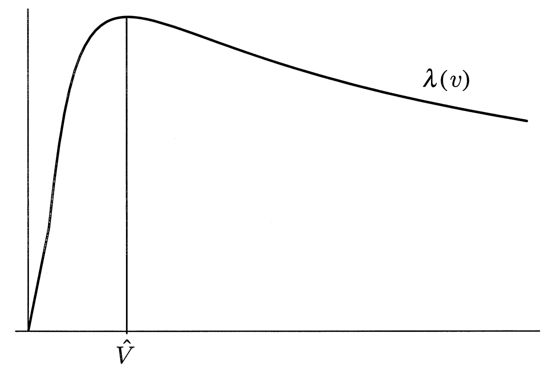
$$\text{subject to } \sum_{i\in\mathcal{S}} (v_i^0 - v_i) = \Delta V, \quad (1)$$

$$v_i \leqslant v_i^0, \quad i \in \mathcal{S}, \quad (2)$$

$$v_i \geqslant 0, \quad i \in \mathcal{S}, \quad (3)$$

where ΔV is the total amount of variance reduction possible based on the available resources. The objective function can be interpreted either as the total expected work overload accumulated by a job or as the total expected work overload across all stations for a cycle. It is assumed that the current total variance, $\sum_{i\in\mathcal{S}} v_i^0$, is larger than ΔV; otherwise, one simply reduces the variance to zero at all the stations and the problem is trivial.

Figure 1. $\lambda(v) = f'(v)$. The rate of change of the EWO at a station with mean processing time μ and processing time variance v.



The slope of expected work overload function,

$$\lambda(v) \triangleq f'(v) = \frac{Z\left(\frac{T-\mu}{\sqrt{v}}\right)}{2\sqrt{v}}, \quad (4)$$

plays a critical role in determining the optimal solution to problem ($\mathcal{P}$). Note from (4) that $\lambda(v) \geqslant 0$, and $\lambda(v) \to 0$ as $v \to 0$ and as $v \to \infty$. That is, $\lambda(v)$ is positive everywhere except at 0 and its value approaches zero as v goes to infinity. Let $\hat{V} \triangleq (T-\mu)^2$, then

$$\lambda'(v) = f''(v) = \frac{Z\left(\frac{T-\mu}{\sqrt{v}}\right)}{4v^{5/2}}(\hat{V} - v) \quad (5)$$

crosses 0 only at $v = \hat{V}$. Moreover, $\lambda'(v) > 0$ for $v < \hat{V}$ and $\lambda'(v) < 0$ for $v > \hat{V}$. Hence $\lambda(v)$ is a unimodal function that is strictly increasing on $(0, \hat{V})$, strictly decreasing on $(\hat{V}, \infty)$, and achieves its (global) maximum at $\hat{V}$. A graph of $\lambda(v)$ is shown in Figure 1.

Note that the set of constraints (1)–(3) is convex. If the objective function in problem ($\mathcal{P}$) was concave, an extreme point solution would be optimal (Zangwill 1960) and problem ($\mathcal{P}$) could be solved very efficiently. Unfortunately, as Equations (4) and (5) indicate, f is increasing on $(0, \infty)$, convex on $(0, \hat{V})$ and concave on $(\hat{V}, \infty)$. As a result, the objective function in problem ($\mathcal{P}$) is a complex multidimensional convex-concave function and finding its minimum over the constraint set (1)–(3) is a nontrivial task.

Any feasible solution to problem ($\mathcal{P}$) can be viewed in terms of stations that receive variance reduction and those which do not. Let $\mathcal{R}$ be the set of stations that receive variance reduction, i.e., $\mathcal{R} = \{i : v_i < v_i^0\}$. For the remaining $\mathcal{S}\backslash\mathcal{R}$ stations, the constraints in (2) are active. Corresponding to a choice of $\mathcal{R}$, there may possibly be many feasible solutions to problem ($\mathcal{P}$). The best solution, the one that apportions the variance reduction ΔV among the stations in $\mathcal{R}$ such that the total expected work overload is minimized, can be found from,

$$(\mathcal{P}_{\mathcal{R}}) \text{ Minimize } \sum_{i \in \mathcal{R}} f(v_i)$$

$$\text{subject to } \sum_{i \in \mathcal{R}} (v_i^0 - v_i) = \Delta V, \quad (6)$$

$$v_i \leq v_i^0, \quad i \in \mathcal{R}, \quad (7)$$

$$v_i \geq 0, \quad i \in \mathcal{R}. \quad (8)$$

Problem ($\mathcal{P}_{\mathcal{R}}$) is obtained from problem ($\mathcal{P}$) by using the fact that $v_i = v_i^0$ for $i \in \mathcal{S} \backslash \mathcal{R}$. This implies that (i) $\Sigma_{i \in \mathcal{S} \backslash \mathcal{R}} f(v_i^0)$ is constant, (ii) $\Sigma_{i \in \mathcal{S} \backslash \mathcal{R}} (v_i^0 - v_i) = 0$, and (iii) $v_i \geq 0$ for $i \in \mathcal{S} \backslash \mathcal{R}$. Given a choice of $\mathcal{R}$, problem ($\mathcal{P}_{\mathcal{R}}$) yields the best allocation for ΔV. If, on the other hand, problem ($\mathcal{P}_{\mathcal{R}}$) is found to be infeasible, then no feasible solution exists for problem ($\mathcal{P}$) such that the variance reduction is limited to the set of stations $\mathcal{R}$.

Unfortunately, solving problem ($\mathcal{P}_{\mathcal{R}}$) is not any easier than solving problem ($\mathcal{P}$), except for the reduced dimensionality, since $\mathcal{R} \subseteq \mathcal{S}$. Consider now a relaxation of problem ($\mathcal{P}_{\mathcal{R}}$) obtained by disregarding constraints (7), that is

$$(\mathcal{P}1_{\mathcal{R}}) \text{ Minimize } \sum_{i \in \mathcal{R}} f(v_i)$$

$$\text{subject to } \sum_{i \in \mathcal{R}} v_i = V(\mathcal{R}),$$

$$v_i \geq 0, \quad i \in \mathcal{R},$$

where $V(\mathcal{R}) = \Sigma_{i \in \mathcal{R}} v_i^0 - \Delta V$. Problem ($\mathcal{P}1_{\mathcal{R}}$) is a *variance allocation problem* that distributes the total variance $V(\mathcal{R})$ among the set of stations $\mathcal{R}$ such that the total expected work overload is minimized.

Our interest in the variance allocation problem ($\mathcal{P}1_{\mathcal{R}}$) is due to its close relationship to the variance reduction problem ($\mathcal{P}$) and its subproblem ($\mathcal{P}_{\mathcal{R}}$). Given a choice of $\mathcal{R}$, if the optimal solution to problem ($\mathcal{P}1_{\mathcal{R}}$) satisfies constraints (7), then it solves problem ($\mathcal{P}_{\mathcal{R}}$) optimally and we obtain the best solution to problem ($\mathcal{P}$) with the qualification that variance reduction is limited to the set of stations $\mathcal{R}$. By considering all possibilities of $\mathcal{R}$ and solving the resulting problem ($\mathcal{P}1_{\mathcal{R}}$), of which there are only a finite number, one can obtain the optimal solution to problem ($\mathcal{P}$).

For this approach to succeed, we need to address the following two problems: (a) how to solve problem ($\mathcal{P}1_{\mathcal{R}}$) efficiently, and (b) what happens if the solution to problem ($\mathcal{P}1_{\mathcal{R}}$) does not satisfy a constraint in (7). We address both these problems by analyzing structural properties of the variance allocation problem ($\mathcal{P}1_{\mathcal{R}}$). These properties, which we call *optimal variance structures*, are developed in § 2. They provide interesting insights into the nature of the variance allocation problem, ($\mathcal{P}1_{\mathcal{R}}$), and its relationship to problem ($\mathcal{P}$). By exploiting the optimal variance structures, we show in § 2 that the solution to problem ($\mathcal{P}1_{\mathcal{R}}$) is either trivial or can be obtained from a single variable optimization problem. This resolves problem (a). Problem (b) is resolved by Theorem 11 in § 3. According to this theorem, unless the optimal solution to problem ($\mathcal{P}1_{\mathcal{R}}$) is feasible to problem ($\mathcal{P}$), $\mathcal{R}$ can not be the set of stations that receives variance reduction in an optimal solution to problem ($\mathcal{P}$). The only exception is when $V(\mathcal{R})$ belongs to a narrow interval defined in Theorem 11 and problem ($\mathcal{P}1_{\mathcal{R}}$) has two local minima, one of which solves problem ($\mathcal{P}$).

2. OPTIMAL VARIANCE STRUCTURES FOR THE ALLOCATION PROBLEM

In this section we examine problem ($\mathcal{P}1_{\mathcal{R}}$) and characterize the structure of its optimal solution. Let n be the cardinal of the set $\mathcal{R}$. Then problem ($\mathcal{P}1_{\mathcal{R}}$) can be written as

$$(\mathcal{P}1) \text{ Minimize } \sum_{i=1}^{n} f(v_i)$$

$$\text{subject to } \sum_{i=1}^{n} v_i = V,$$

$$v_i \geq 0, \quad i = 1, 2, \ldots, n,$$

where we have suppressed the dependence of $V(\mathcal{R})$ on $\mathcal{R}$ for notational convenience, i.e., $V = V(\mathcal{R})$. Note that problem ($\mathcal{P}1$) is indifferent to the composition of the set $\mathcal{R}$ except for its cardinal and sum of initial variances. Two sets of stations, $\mathcal{R}_1$ and $\mathcal{R}_2$, with the same cardinal but different composition and different initial variances, will lead to the same solution if $\Sigma_{i \in \mathcal{R}_1} v_i^0 = \Sigma_{i \in \mathcal{R}_2} v_i^0$.

If $f(v)$ was convex, then problem ($\mathcal{P}1$) would be a special case of the resource allocation problem addressed by Zipkin (1980), Luss and Gupta (1975), and others, who minimize a convex nonlinear-additive objective function subject to a single linear constraint using a very efficient ranking algorithm. In fact in that case, problem ($\mathcal{P}1$) could be solved trivially since each nonlinear function is the same. Unfortunately, $f(v)$ is not convex as demonstrated in the last section. In this sense, problem ($\mathcal{P}1$) is very different than the classical resource allocation problems. Before we explore the solution to problem ($\mathcal{P}1$) for n stations, we investigate the optimal solution to a two station problem. The solution to the two station problem will provide crucial insights that will allow us to solve the n station problem.

2.1. Analysis of a Two-Station Line

For $n = 2$ problem ($\mathcal{P}1$) reduces to:

$$(\mathcal{P}1.1) \text{ Minimize } f(v_1) + f(v_2)$$

$$\text{subject to } v_1 + v_2 = V, \quad (9)$$

$$v_1, v_2 \geq 0.$$

While this problem is symmetric in v_1 and v_2, the following theorem states the optimal solution does not necessarily have a symmetric structure.

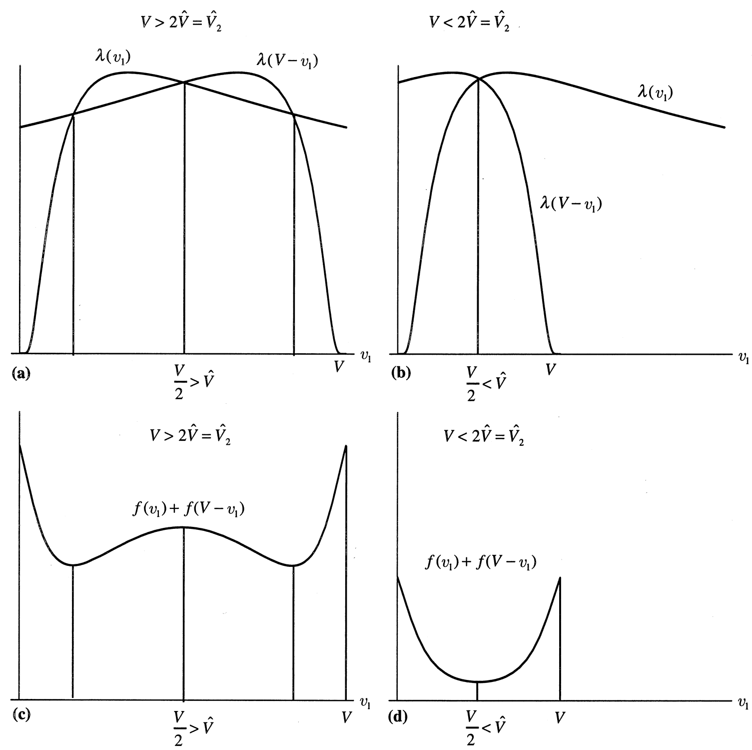
THEOREM 1. *For problem* ($\mathcal{P}1.1$) *there exists a critical variance level*

$$\hat{V}_2 \triangleq 2\hat{V} = 2(T - \mu)^2,$$

such that the equal variance solution is optimal if and only if the total variance $V \leq \hat{V}_2$. *Moreover, an optimal unequal variance solution satisfies the equation*

$$\lambda(v_1) = \lambda(v_2). \quad (10)$$

Figure 2. The effect of total variance on λ and $f(v) + f(V - v)$, the total expected work overload.



Notes: (a) and (b). $\lambda(v_1)$ and $\lambda(v_2) = \lambda(V - v_1)$ for $V > \hat{V}_2$ and $V < \hat{V}_2$, respectively.
(c) and (d). $f(v_1) + f(v_2) = f(v_1) + f(V - v_1)$ for $V > \hat{V}_2$ and $V < \hat{V}_2$, respectively.

PROOF. See Appendix A.1.

This result is not intuitive. Due to the symmetric structure of problem ($\mathcal{P}$1.1), a casual observer might have guessed that the equal variance solution would always be optimal. However, Theorem 1 states that this is only conditionally true. If the total variance is large enough, (i.e. greater than the critical variance, $\hat{V}_2$), then an unequal variance solution is optimal. This structure of the optimal solution is explained intuitively as follows. If an inherently large amount of variability exists, at least one station will have a significant amount of variance. If at least one station has significant variance, it is better to have more of the variance at the same station because the effect of marginal variability on a station that already has significant variance is much less detrimental than the effect of marginal variability on a station that has little or no variance.

A graphical interpretation of Theorem 1 can be given as follows. Note that Equations (9) and (10) define the first-order conditions for problem ($\mathcal{P}$1.1). Graphically, the intersection of $\lambda(v_1)$ with its reflection about the line $v_1 = V/2$ gives the first-order points as shown in Figures 2a and 2b. It is clear from these figures that one first order point will always be at $v_1 = V/2$. The other first order points always appear as symmetric pairs. Whether there is one first order point or several depends on whether the point of reflection $V/2$ is greater than or less than $\hat{V}$ as illustrated in Figures 2a and 2b, respectively. The corresponding objective functions $f(v_1) + f(v_2) = f(v_1) + f(V - v_1)$ are shown in Figures 2c and 2d, respectively. Note that in Figure 2d where $V < 2\hat{V} = \hat{V}_2$ that the global minimum has an equal variance structure, while in Figure 2c we have $V > 2\hat{V} = \hat{V}_2$ and the global minimum has an unequal variance structure.

Theorem 1 gives us a simple rule for determining the structure of the optimal solution to problem ($\mathcal{P}$1.1). For $V \leq \hat{V}_2$, the structure of the optimal solution is one where both stations have identical variance. For $V > \hat{V}_2$, the optimal solution has an unequal variance structure. If (v_1, v_2) is the optimal solution in this case then (v_1, v_2) satisfies Equations (9) and (10). Since the unequal variance solution (v_1, v_2) always appears as a symmetric pair, we will not distinguish (v_2, v_1) from its mirror image (v_1, v_2). The following theorem demonstrates that there is only one such pair satisfying the first-order conditions given by Equations (9) and (10).

THEOREM 2. *For any $V > \hat{V}_2$ there is a unique unequal variance allocation pair (v_1, v_2) satisfying the first-order conditions given by Equations* (9) *and* (10).

PROOF. See Appendix A.3.

Theorem 2 (in conjunction with Theorem 1) allows us to conclude that when the total variance V exceeds the critical variance $\hat{V}_2$, there is a unique unequal variance solution (ignoring the symmetric counterpart) that satisfies the first order conditions, and that unequal variance solution is also the global minimum. To summarize, we have shown for problem ($\mathcal{P}$1.1) that an unequal variance solution is optimal if and only if the total variance is large enough (greater than $\hat{V}_2$). Furthermore, if an unequal variance solution is optimal, it is the unique unequal variance solution (ignoring the symmetric counterpart) that satisfies the first order conditions. Using these results, we now address the general case of problem ($\mathcal{P}$1) with n stations.

2.2. Analysis of an *n* Station Line

In this section we extend the results of the previous section for instances of problem ($\mathcal{P}$1) when there are more than two stations. We first note that problem ($\mathcal{P}$1) has a symmetric structure. That is, if $(v_1, v_2, \cdots, v_n)$ is a feasible solution to problem ($\mathcal{P}$1) with objective function value $\sum_{i=1}^{n} f(v_i)$, then any permutation of $(v_1, v_2, \cdots, v_n)$ is also a feasible solution to problem ($\mathcal{P}$1) with the same objective function value. Therefore, the optimal solution will not be unique unless $v_i = V/n$ for all i. We assume (without loss of generality) that $v_1 \leq v_2 \leq \cdots \leq v_n$. The first-order conditions for problem ($\mathcal{P}$1) are given by

$$\lambda(v_i) - \lambda = 0, \quad i = 1, \ldots, n, \tag{11}$$

$$\sum_{i=1}^{n} v_i - V = 0. \tag{12}$$

Notice that Equation (11) is equivalent to $\lambda(v_i) = \lambda$. Thus, if $(v_1^*, v_2^*, \cdots, v_n^*, \lambda^*)$ satisfies the first-order conditions for this problem then given a "potential" λ^*, we need only find the v_i that solve $\lambda(v_i) = \lambda^*$ and then find combinations of the v_i that satisfy Equation (12).

While this approach may sound easy, it is not because the equation $\lambda(v) = \lambda^*$ is transcendental. Instead, by using some of the properties of $\lambda(v)$, we will characterize the structure of the optimal solution to problem ($\mathcal{P}$1). We now state our first result which characterizes a local optimum.

LEMMA 1. *Any unequal variance solution that is a local optimum for problem* ($\mathcal{P}$1) *is such that the variance at each station must take one of the two values,* $v_l < \hat{V}$ *or* $v_h > \hat{V}$.

PROOF. If $(v_1^*, v_2^*, \cdots, v_n^*, \lambda^*)$ satisfies the first-order conditions, then $\lambda(v_i) = \lambda^*$ for all i. It was shown in § 1 that $\lambda(v)$ is strictly increasing on $(0, \hat{V})$ and strictly decreasing on $(\hat{V}, \infty)$. Thus given a value of λ^*, it is clear that $\lambda(v) = \lambda^*$ has at most two solutions, which we call v_l and v_h. Since the maximum of $\lambda(v)$ occurs at $\hat{V}$, it must be true that $v_l < \hat{V}$ and $v_h > \hat{V}$. □

According to Lemma 1, the optimal solution to problem ($\mathcal{P}$1) is such that k of the n stations will have variance v_l, and the remaining $n - k$ stations will have variance v_h. Since

$$k v_l + (n - k) v_h = V,$$

the optimization problem ($\mathcal{P}$1) can be reformulated as

$$(\mathcal{P}1') \text{ Minimize } kf(v_l) + (n - k) f\left(\frac{V - k v_l}{n - k}\right)$$

$$\text{subject to } 0 \leq v_l \leq \frac{V}{n},$$
$$1 \leq k \leq n - 1,$$
$$k \text{ integer.}$$

Problem ($\mathcal{P}1'$) is a nonlinear-integer program with two decision variables: v_l, the low variance; and k, the number of stations with the low variance. Each of the remaining $n - k$ stations has high variance $v_h = (V - kv_l)/(n - k)$. In formulating problem ($\mathcal{P}1'$) we have excluded the possibility of $k = 0$, i.e., no stations with low variance and n stations with high variance, which corresponds to the equal variance solution. Instead we account for the equal variance solution by noting that for any value of k if $v_l = V/n$ then $v_h = V/n$ and we get the equal variance solution.

Problem ($\mathcal{P}1'$) is significantly simpler than problem ($\mathcal{P}$1) since for a given value of k, the optimal value of v_l can be determined using a univariate search. Let $v_l^*[k]$ be the value of v_l that minimizes the objective function in problem ($\mathcal{P}1'$) for a given k. The optimal solution to problem ($\mathcal{P}1'$) can then be found by comparing the objective function value for all $(k, v_l^*[k])$ pairs. Unfortunately, for a large number of stations, this procedure becomes tedious since it must be repeated $n - 1$ times. The following result allows us to perform the search only once and still obtain the optimal solution.

THEOREM 3. *In an optimal unequal variance solution to problem* ($\mathcal{P}$1), *all stations except one have low variance.*

PROOF. We prove the result by contradiction. Suppose that an unequal variance solution is optimal, k stations have low variance, and $k < n - 1$. Form a two-station subproblem by considering stations $k + 1$ and $k + 2$ in isolation. We let $V_{(k)} \triangleq v_k + v_{k+1}$. Clearly $V_{(k+1)} = v_{k+1} + v_{k+2} = 2v_h > 2\hat{V} = \hat{V}_2$. Thus by Theorem 1, an unequal variance solution is better for the two-station subproblem of stations $k + 1$ and $k + 2$ considered in isolation. But if the solution to this two-station subproblem can be improved, this implies that the original solution to the n station problem is not optimal. This fact is a result due to the separability of the objective function from problem ($\mathcal{P}$1). Thus we have a contradiction, and therefore it must be true that that $k = n - 1$. □

The consequence of Theorem 3 is that given an instance of problem ($\mathcal{P}$1), we can determine the structure of the optimal solution. Either an equal variance solution is optimal, or an unequal variance solution is optimal with $n - 1$ stations having low variance. In either case, to find the optimal values of the v_i, we only need to apply a univariate search to problem ($\mathcal{P}1''$),

$$(\mathcal{P}1'') \underset{0 \leq v_l \leq V/n}{\text{Minimize}} \ \eta(v_l) \triangleq (n - 1) f(v_l) + f(V - (n - 1) v_l).$$

That is, if v_l^* is the optimal solution to problem ($\mathcal{P}1''$), then $v_i^* = v_l^*$ for $i = 1, \ldots, n - 1$ and $v_n^* = V - (n - 1)v_l^*$.

An important question still remains to be answered. When does the optimal solution to an n station problem have an unequal variance structure? Recall that for a two-station problem, an unequal variance solution is optimal if $V > \hat{V}_2$. Before giving an analogous result for the n station problem, we first define

$$v_a = \frac{V^2 + 2(n-1)V\hat{V} - V\sqrt{V^2 - 4(n-1)\hat{V}^2}}{2(n-1)(V + n\hat{V})}, \quad (13)$$

and

$$v_b = \frac{V^2 + 2(n-1)V\hat{V} + V\sqrt{V^2 - 4(n-1)\hat{V}^2}}{2(n-1)(V + n\hat{V})}. \quad (14)$$

THEOREM 4. *Let $\widehat{V}_n \triangleq 2\sqrt{n-1}\ \hat{V}$ and $\hat{V}_n \triangleq n\hat{V}$. The following holds for the optimization problem ($\mathcal{P}1''$):*

(1) *If $V \geqslant \hat{V}_n$, then:*
   (a) *the equal-variance solution, $v_l = V/n$, is a local maximum of $\eta(v_l)$;*
   (b) *in the interval $0 \leqslant v_l < V/n$, there exists only one unequal-variance solution that satisfies the first-order conditions and $\eta(v_l)$ achieves its global minimum at this point;*
   (c) *$v_1^* < v_a < V/n$.*

(2) *If $\widehat{V}_n < V < \hat{V}_n$, then*
   (a) *the equal-variance solution, $v_l = V/n$, is a local minimum of $\eta(v_l)$;*
   (b) *in the interval $0 \leqslant v_l < V/n$, there exists either (i) no unequal-variance solution that satisfies the first-order conditions, or (ii) two unequal-variance solutions that satisfy the first-order conditions—a local minimum in the interval $(0, v_a)$ and a local maximum in the interval (v_a, v_b);*
   (c) *either $v_l^* = V/n$, or $v_l^* < v_a < V/n$.*

(3) *If $V \leqslant \widehat{V}_n$ then the equal variance solution, $v_l = V/n$, is the only solution that satisfies the first-order conditions, and this solution is the global minimum of $\eta(v_l)$.*

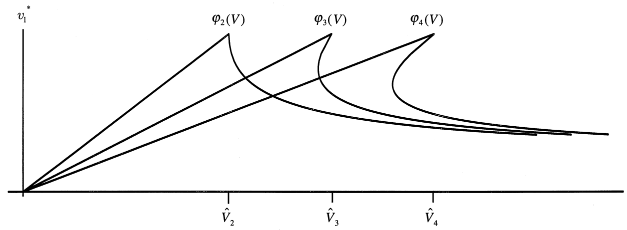
PROOF. See Appendix A.4.

Theorem 3 characterizes the *structure* of an optimal unequal variance solution but says nothing about when such a solution may be optimal. Theorem 4, on the other hand, gives conditions when an unequal variance solution is ($V \geqslant \widehat{V}_n$) or may be ($\widehat{V}_n < V < \hat{V}_n$) optimal. If $V \leqslant \widehat{V}_n$, we are able to conclude that the equal variance solution is optimal. Unlike Theorem 1 for the two-station problem, we are not always able to determine what type of solution (equal-variance or unequal-variance) will be optimal. To further understand the implications of Theorem 4 and the differences between the solutions to the two-station problem and the n station problem, consider the function $\varphi_2(V)$ which gives the optimal lower variance to a two station problem with variance V. That is,

$$v_l^* = \varphi_2(V)$$

implies that v_l^* must satisfy $\lambda(v_l^*) = \lambda(V - v_l^*)$. The function $\varphi_2(V)$ has a unique value for each V as shown in Figure 3. Consistent with Theorem 1, $\varphi_2(V)$ is linear with

Figure 3. $\varphi_n(V)$. Candidate optimal low variances to problem ($\mathcal{P}1$) as a function of the total variability V.



slope 1/2 (denoting the equal-variance solution) for $V < \hat{V}_2$ and decreases for $V > \hat{V}_2$.

Now consider all first-order points that are candidates for the optimal solution to an n station problem as characterized by Theorem 3. Let $\varphi_n(V)$ be a relation that gives all v_l^* that satisfy the first-order conditions of problem ($\mathcal{P}1''$), that is $\lambda(v_l^*) = \lambda(V - (n-1)v_l^*)$. So,

$$\begin{aligned}\varphi_n(V) = v_l^* &\Leftrightarrow \lambda(v_l^*) = \lambda(V - (n-1)v_l^*)\\ &\Leftrightarrow \lambda(v_l^*) = \lambda((V + v_l^*) - nv_l^*)\\ &\Leftrightarrow v_l^* \quad = \varphi_{n+1}(V + v_l^*).\end{aligned}$$

Equating v_l^* with $\varphi_n(V)$ we obtain

$$\varphi_n(V) = \varphi_{n+1}(V + \varphi_n(V)).$$

By induction it is easily shown that

$$\varphi_{k+2}(V + k\varphi_2(V)) = \varphi_2(V). \quad (15)$$

In words, this identity says that if v_l^* is the optimal solution to a two-station problem with total variance V, then it satisfies the first-order conditions to a $k+2$ station problem with total variance $(V + kv_l^*)$.

Graphs of φ_3 and φ_4 obtained from Equation (15) are also shown in Figure 3. An interesting phenomenon occurs for $n > 2$. The graph of φ_n "bends back" at $\hat{V}_n$. This behavior is predicted by Theorem 4. When V is slightly less than $\hat{V}_n$ we see that there are three points that satisfy the first order conditions, as predicted in part 2b of Theorem 4. In other words, for a V in this range, we do have three potential v_l^* satisfying Equation (15), each of which is pairwise optimal for all two-station subproblems. The equal variance solution in this range may not be optimal as noted in Theorem 4. To illustrate this point, consider a three-station problem with cycle time $T = 12$, mean processing time $\mu = 10$ for which the critical variance is $\hat{V}_3 = n(T - \mu)^2 = 12$. Suppose the total variance is $V = 11.57$. There are three points that satisfy the first order conditions—the equal-variance solution, (11.57/3, 11.57/3, 11.57/3) and two unequal-variance solutions, (2.863, 2.863, 5.844) and (3.443, 3.443, 4.684). The equal-variance solution is a local minimum, (3.443, 3.443, 4.684) is a local maximum, and (2.863, 2.863, 5.844) is the global minimum. For the specified T and μ, the optimal solution to a two-station problem has equal variance if $V < \hat{V}_2 = 8$ and has unequal-variance otherwise. Notice that all pairwise combinations formed from these solutions are optimal.

To summarize, if $V > \hat{\hat{V}}_n$, then an unequal-variance solution is optimal, and this solution is such that $n - 1$ stations have low variance, v_l, and the remaining station has high variance level $v_h = V - (n - 1)v_l$. For $V < \overset{\frown}{V}_n$, the equal-variance solution is optimal. When $\overset{\frown}{V}_n < V < \hat{\hat{V}}_n$, we make no claim whether an equal or unequal-variance solution is optimal. However, if an unequal-variance solution is optimal, it must be true that $n - 1$ stations are allocated a low variance level. In any case, the optimal solution can be found by solving the univariate optimization problem ($\mathcal{P}1''$).

2.3. The Worst Solution to Problem ($\mathcal{P}1$)

Recall that the equal variance solution may be a local maximum to problems ($\mathcal{P}1.1$) and ($\mathcal{P}1$). The following theorem indicates that it can potentially be the worst choice to problem ($\mathcal{P}1.1$).

THEOREM 5. *For problem* ($\mathcal{P}1.1$), *there exists an upper critical variance level* $\hat{\hat{V}}_2 > \hat{V}_2$ *such that the equal variance solution is the global maximum if and only if* $V > \hat{\hat{V}}_2$. *The critical variance level* $\hat{\hat{V}}_2$ *is given by*

$$\hat{\hat{V}}_2 = \left(\frac{T - \mu}{\hat{\hat{x}}}\right)^2 \approx 3.83(T - \mu)^2,$$

where $\hat{\hat{x}}$ *is the unique positive solution to*

$$x + Z(x) + xP(x) = \sqrt{2}Z(x\sqrt{2}) + 2xP(x\sqrt{2}). \tag{16}$$

PROOF. See Appendix A.5.

Consider $f(v) + f(V - v)$ as a function of v for a given $V > \hat{V}_2$. It was shown in the proof of Theorem 1 that the equal variance solution is a local maximum for this function. Whether or not it is a global maximum depends upon whether the total variance exceeds the upper critical variance level, $\hat{\hat{V}}_2$, according to Theorem 5. This point is illustrated in Figures 4a and 4b, which show $f(v) + f(V - v)$ for $\hat{V}_2 < V < \hat{\hat{V}}_2$ and $V > \hat{\hat{V}}_2$, respectively.

The following result, analogous to Theorem 5, indicates as to when the equal variance solution is the worst solution to problem ($\mathcal{P}1$).

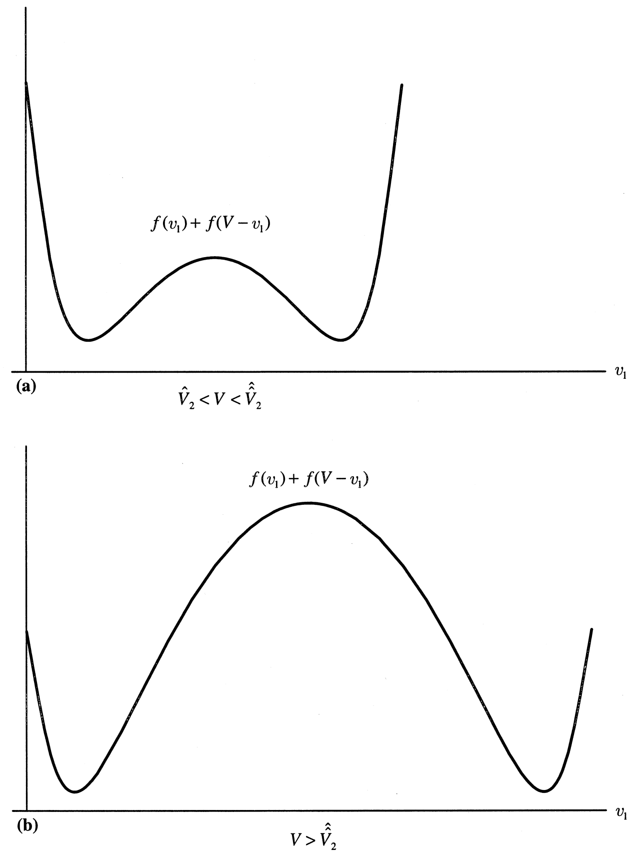
THEOREM 6. *For problem* ($\mathcal{P}1$) *there exists an upper critical variance level* $\hat{\hat{V}}_n > \hat{V}_n$ *such that if* $V > \hat{\hat{V}}_n$, *then the equal-variance solution is the global maximum. The upper critical variance level is given by*

$$\hat{\hat{V}}_n = (n - 1)\hat{\hat{V}}_2.$$

PROOF. See Appendix A.7.

The following theorem demonstrates that when the variance is sufficiently large, one should prefer a configuration that places most of the variance at as few stations as possible.

Figure 4. $f(v_1, v_2) = f(v_1) + f(V - v_1)$ for $\hat{V}_2 < V < \hat{\hat{V}}_2$ and $V > \hat{\hat{V}}_2$. These graphs show the quality of the equal variance solution to problem ($\mathcal{P}1.1$) as a function of the total variability, V. (N. B., vertical axes not to the same scale.)



THEOREM 7. *For the problem* ($\mathcal{P}1$), *if the total variance* V *exceeds the upper critical variance* $\hat{\hat{V}}_n$, *then the optimal solution where* k *stations have low variance and* $n - k$ *stations have high variance is worse than the optimal solution where* $k + 1$ *stations have low variance and* $n - k - 1$ *stations have high variance*.

PROOF. See Appendix A.8.

3. THE VARIANCE REDUCTION PROBLEM

We now address the variance reduction problem ($\mathcal{P}$). Our solution approach to problem ($\mathcal{P}$) involves repeated solution of the variance allocation problem ($\mathcal{P}1_{\mathcal{R}}$)—a task that can now be performed very efficiently. We first present the properties of the optimal solution for the variance reduction problem. Our interest is in identifying (i) *where* to reduce the variance, and (ii) by *how much*. Theorems 8–10 address the first question by specifying where *not to reduce* the variance in an optimal solution to problem ($\mathcal{P}$). Theorem 11 addresses the second question by characterizing the structure of the optimal variance reduction.

THEOREM 8. *Consider two sets of stations, $\mathcal{R}_1$ and $\mathcal{R}_2$, such that $\mathcal{R}_1 \supset \mathcal{R}_2$. If the optimal solution to the variance allocation problem ($\mathcal{P}1_{\mathcal{R}_1}$) is feasible to the variance reduction problem ($\mathcal{P}_{\mathcal{R}_1}$) then $\mathcal{R}_2$ cannot be the set of stations where variance reduction occurs in an optimal solution to the variance reduction problem ($\mathcal{P}$).*

PROOF. Let $v^*(1)$ be the optimal solution to problem ($\mathcal{P}1_{\mathcal{R}_1}$). Suppose $v^*(1)$ is feasible to problem ($\mathcal{P}_{\mathcal{R}_1}$), as per the assumption of the theorem. Then $v^*(1)$ also solves problem ($\mathcal{P}_{\mathcal{R}_1}$) optimally. Now consider problem ($\mathcal{P}_{\mathcal{R}_2}$), which may or may not have a feasible solution. If problem ($\mathcal{P}_{\mathcal{R}_2}$) is infeasible then no solution exists for problem ($\mathcal{P}$) such that variance reduction occurs only among the set of stations $\mathcal{R}_2$, and the theorem is true. If problem ($\mathcal{P}_{\mathcal{R}_2}$) is feasible, then let $v^*(2)$ be the optimal solution to problem ($\mathcal{P}_{\mathcal{R}_2}$).

Let $\bar{v}^*(k)$ be the optimal solution to problem ($\mathcal{P}$) when variance reduction is restricted to the set of stations $\mathcal{R}_k$. Such a solution can be constructed from the optimal solution to problem ($\mathcal{P}_{\mathcal{R}_k}$) as follows:

$$\bar{v}_i^*(k) = \begin{cases} v_i^*(k) & \text{if } i \in \mathcal{R}_k, \\ v_i^0 & \text{if } i \in \mathcal{S}\backslash\mathcal{R}_k. \end{cases}$$

We need to show that $\sum_{i\in\mathcal{S}} f(\bar{v}_i^*(2)) > \sum_{i\in\mathcal{S}} f(\bar{v}_i^*(1))$. Note that

$$\begin{aligned}\sum_{i\in\mathcal{S}} f(\bar{v}_i^*(2)) - \sum_{i\in\mathcal{S}} f(\bar{v}_i^*(1)) &= \sum_{i\in\mathcal{R}_2} f(v_i^*(2)) + \sum_{i\in\mathcal{S}\backslash\mathcal{R}_2} f(v_i^0) \\ &\quad - \sum_{i\in\mathcal{R}_1} f(v_i^*(1)) \\ &\quad - \sum_{i\in\mathcal{S}\backslash\mathcal{R}_1} f(v_i^0) \\ &= \sum_{i\in\mathcal{R}_2} (f(v_i^*(2)) - f(v_i^*(1))) \\ &\quad + \sum_{i\in\mathcal{R}_1\backslash\mathcal{R}_2} (f(v_i^0) \\ &\quad - f(v_i^*(1))) \\ &= \sum_{i\in\mathcal{R}_1} f(v_i'(1)) \\ &\quad - \sum_{i\in\mathcal{R}_1} f(v_i^*(1)),\end{aligned}$$

where $v'(1)$ is a solution to problem ($\mathcal{P}_{\mathcal{R}_1}$) defined as

$$v_i'(1) = \begin{cases} v_i^*(2) & \text{if } i \in \mathcal{R}_2, \\ v_i^0 & \text{if } i \in \mathcal{R}_1\backslash\mathcal{R}_2. \end{cases}$$

Note that $v'(1)$ is feasible to problem ($\mathcal{P}_{\mathcal{R}_1}$). But $v^*(1)$ is optimal to problem ($\mathcal{P}_{\mathcal{R}_1}$) and hence has a lower objective function value than any other feasible solution to problem ($\mathcal{P}_{\mathcal{R}_1}$). As a result, $\sum_{i\in\mathcal{R}_1} f(v_i'(1)) > \sum_{i\in\mathcal{R}_1} f(v_i^*(1))$, which implies that $\sum_{i\in\mathcal{S}} f(\bar{v}_i^*(2)) > \sum_{i\in\mathcal{S}} f(\bar{v}_i^*(1))$. □

Let $\mathcal{R}^*$ be the set of stations that receive variance reduction in the optimal solution to the variance reduction problem ($\mathcal{P}$). Let $\mathcal{R}$ be any set of stations. If the optimal solution to the variance allocation problem ($\mathcal{P}1_{\mathcal{R}}$) is feasible to problem ($\mathcal{P}_{\mathcal{R}}$) then, according to Theorem 8, $\mathcal{R}^* \not\subset \mathcal{R}$.

THEOREM 9. *Let $\{\bar{v}_i, i \in \mathcal{S}\}$ be a feasible solution to the variance reduction problem ($\mathcal{P}$). Let $\{v_i, i \in \mathcal{R}\}$ be an optimal solution to the variance allocation problem ($\mathcal{P}1_{\mathcal{R}}$). If $\sum_{i\in\mathcal{S}} f(\bar{v}_i) < \sum_{i\in\mathcal{R}} f(v_i) + \sum_{i\in\mathcal{S}\backslash\mathcal{R}} f(v_i^0)$ then $\mathcal{R}^* \underline{\not\subset} \mathcal{R}$.*

PROOF. Define

$$\tilde{v}_i = \begin{cases} v_i & \text{if } i \in \mathcal{R}, \\ v_i^0 & \text{if } i \in \mathcal{S}\backslash\mathcal{R}. \end{cases}$$

Then, according to the theorem, $\sum_{i\in\mathcal{S}} f(\bar{v}_i) < \sum_{i\in\mathcal{S}} f(\tilde{v}_i)$. If $\mathcal{R}^* = \mathcal{R}$ then $\tilde{v}$ is optimal to problem ($\mathcal{P}$) and $\sum_{i\in\mathcal{S}} f(\tilde{v}_i) \leqslant \sum_{i\in\mathcal{S}} f(\bar{v}_i)$, a contradiction. To show that $\mathcal{R}^* \not\subset \mathcal{R}$, let $\grave{\mathcal{R}} \subset \mathcal{R}$. Consider the following optimization problem:

$$(\mathcal{P}(\mathcal{R}))\ \text{Minimize} \sum_{i\in\mathcal{S}} f(v_i)$$

$$\text{subject to} \sum_{i\in\mathcal{S}} (v_i^0 - v_i) = \Delta V,$$

$$\begin{aligned} v_i &\leqslant v_i^0, \quad i \in \mathcal{R}, \\ v_i &= v_i^0, \quad i \in \mathcal{S}\backslash\mathcal{R}, \\ v_i &\geqslant 0, \quad i \in \mathcal{S}. \end{aligned} \tag{17}$$

Problem ($\mathcal{P}(\mathcal{R})$) is a restatement of problem ($\mathcal{P}$) under the assumption that only stations in $\mathcal{R}$ can potentially receive any variance reduction. Note that the feasible region of problem ($\mathcal{P}(\mathcal{R})$) is a subset of the feasible region of problem ($\mathcal{P}1_{\mathcal{R}}$). In the absence of constraints (17), problem ($\mathcal{P}(\mathcal{R})$) is solved optimally by $\tilde{v}$. Let $v^*(\mathcal{R})$ be the optimal solution to problem ($\mathcal{P}(\mathcal{R})$). Similarly, define problem ($\mathcal{P}(\grave{\mathcal{R}})$) as above and its optimal solution as $v^*(\grave{\mathcal{R}})$. Since $\grave{\mathcal{R}} \subset \mathcal{R}$, the feasible region of problem ($\mathcal{P}(\grave{\mathcal{R}})$) is a subset of feasible region of problem ($\mathcal{P}(\mathcal{R})$). Hence,

$$\sum_{i\in\mathcal{S}} f(\bar{v}_i) < \sum_{i\in\mathcal{S}} f(\tilde{v}_i) \leqslant \sum_{i\in\mathcal{S}} f(v_i^*(\mathcal{R})) \leqslant \sum_{i\in\mathcal{S}} f(v_i^*(\grave{\mathcal{R}})).$$

The first inequality follows from the statement of the theorem. The other inequalities follow because the feasible region of ($\mathcal{P}(\grave{\mathcal{R}})$) is a subset of the feasible region of ($\mathcal{P}(\mathcal{R})$), which is a subset of the feasible region of ($\mathcal{P}1_{\mathcal{R}}$). From the above inequality we conclude that $\mathcal{R}^* \neq \grave{\mathcal{R}}$. □

According to Theorem 9, if the objective function value for the variance reduction problem ($\mathcal{P}$) corresponding to the optimal solution for the variance allocation problem ($\mathcal{P}1_{\mathcal{R}}$) is worse than that for any feasible solution to problem ($\mathcal{P}$), then $\mathcal{R}$ or any of its subsets cannot be the set of stations where variance reduction occurs in an optimal solution to problem ($\mathcal{P}$). Note that both Theorems 8 and 9 rely on the optimal solution to problem ($\mathcal{P}1_{\mathcal{R}}$) to rule out the subsets of $\mathcal{R}$ as candidates for $\mathcal{R}^*$. While Theorem 8 is based solely on the feasibility of this solution to problem ($\mathcal{P}$), Theorem 9 relies only on the objective function value.

THEOREM 10. *If* $\sum_{i\in\mathcal{R}} v_i^0 < \Delta V$ *then* $\mathcal{R}^* \not\subseteq \mathcal{R}$.

PROOF. If $\mathcal{R}$ is the set of stations where variance reduction occurs in an optimal solution to problem ($\mathcal{P}$), then it must satisfy the feasibility conditions (1) and (3):

$$\Delta V = \sum_{i\in\mathcal{S}} (v_i^0 - v_i) = \sum_{i\in\mathcal{R}} (v_i^0 - v_i) \leqslant \sum_{i\in\mathcal{R}} v_i^0,$$

which contradicts the assumption in the theorem. Since $\mathcal{R}$ is not feasible to problem ($\mathcal{P}$), it can not be optimal. Now consider, $\tilde{\mathcal{R}} \subset \mathcal{R}$. Clearly, $\sum_{i\in\tilde{\mathcal{R}}} v_i^0 < \sum_{i\in\mathcal{R}} v_i^0 < \Delta V$. But, as shown above, $\tilde{\mathcal{R}}$ is feasible to problem ($\mathcal{P}$) only if $\sum_{i\in\tilde{\mathcal{R}}} v_i^0 \geqslant \Delta V$. Hence, there does not exist a feasible solution to problem ($\mathcal{P}$) such that variance reduction is limited to the subset of stations $\mathcal{R}$. □

We now consider the structure of the optimal variance reduction. If the optimal solution to the variance allocation problem ($\mathcal{P}1_{\mathcal{R}^*}$) is feasible to the variance reduction problem ($\mathcal{P}$), then the final variances among stations in $\mathcal{R}^*$ inherit the optimal variance structure for the allocation problem. What if problem ($\mathcal{P}1_{\mathcal{R}^*}$) is not feasible to problem ($\mathcal{P}$)? Is it possible? If so, what is the structure of final variances among stations in $\mathcal{R}^*$? The following theorem addresses these questions.

THEOREM 11. *Suppose* $\mathcal{R}^*$ *is the set of stations that receive variance reduction in an optimal solution to the variance reduction problem* ($\mathcal{P}$). *Suppose* n *is the cardinal of* $\mathcal{R}^*$ *and* $V = V(\mathcal{R}^*) = \sum_{i\in\mathcal{R}^*} v_i^0 - \Delta V$. *Then the optimal variance reduction is such that either* (*a*) $v_i^* = V/n$ *for all* $i \in \mathcal{R}^*$, *or* (*b*) $v_i^* = v^* < \hat{V}$ *for all* $i \in \mathcal{R}^*$ *except one, say* j, *for which* $v_j^* > \hat{V}$ *and* $\lambda(v_i^*) = \lambda(v_j^*)$, $\forall i, j \in \mathcal{R}^*$. *Moreover, the optimal solution to the variance allocation problem* ($\mathcal{P}1''$) *always solves the variance reduction problem* ($\mathcal{P}$) *except when* $\check{V}_n < V < \hat{V}_n$ *and problem* ($\mathcal{P}1''$) *has two local minima*, *one of which solves problem* ($\mathcal{P}$).

PROOF. From the Kuhn-Tucker conditions, the optimal solution to problem ($\mathcal{P}$) must satisfy

$$\lambda(v_i^*) = \lambda^* \quad \text{if } i \in \mathcal{R}^*,$$

$$\lambda(v_i^*) < \lambda^* \quad \text{if } i \in \mathcal{S}\backslash\mathcal{R}^*.$$

But $\lambda(v)$ is strictly increasing on $(0, \hat{V})$ and strictly decreasing on $(\hat{V}, \infty)$. Thus given a value of λ^*, the solution to $\lambda(v_i^*) = \lambda^*$ is either (i) $v_i^* = v_l < \hat{V}$ or (ii) $v_i^* = v_h > \hat{V}$. Suppose that of the n stations in $\mathcal{R}^*$, k stations have high variance, v_h, and $(n - k)$ stations have low variance, v_l. We now prove, by contradiction, that $k \leqslant 1$. Suppose $k > 2$. Then there exist two stations i, j such that $v_i^* < v_i^0$, $v_j^* < v_j^0$ and $v_i^* = v_j^* = v_h > \hat{V}$. Let $v_i' = v_i^* - \epsilon$, $v_j' = v_j^* + \epsilon$ where $0 < \epsilon < \min(v_h - \hat{V}, v_j^0 - v_j^*)$. Consider an alternative solution to problem ($\mathcal{P}$) where v_i' and v_j' has been substituted for v_i^* and v_j^* respectively. This solution is feasible. Consider the function

$$L(\epsilon) = f(v_i^* - \epsilon) + f(v_j^* + \epsilon).$$

Note that

$$L'(\epsilon) = -\lambda(v_i^* - \epsilon) + \lambda(v_j^* + \epsilon)$$
$$= -\lambda(v_h - \epsilon) + \lambda(v_h + \epsilon) < 0,$$

since $\lambda(v)$ is strictly decreasing on $[\hat{V}, \infty]$. That is, the objective function value can be further improved. This contradicts the fact that v_i^* and v_j^* are part of the optimal solution. Hence in $\mathcal{R}^*$, no more than one station can have the high variance; all others must have low variance. Note that in the statement of the theorem, cases (a) and (b) correspond to $k = 0$ and $k = 1$, respectively.

Let the stations in $\mathcal{R}^*$ be labeled such that $v_1^0 \leqslant v_2^0 \leqslant \ldots \leqslant v_n^0$. Then, by virtue of the result just proven, problem ($\mathcal{P}$) can be expressed as

$$(\mathcal{P}''_{\mathcal{R}^*}) \ \underset{0\leqslant v_l \leqslant V/n}{\text{Minimize}} \ \eta(v_l) = (n-1)f(v_l) + f(V - (n-1)v_l)$$

$$\text{subject to } \frac{V - v_n^0}{n-1} \leqslant v_l \leqslant v_1^0. \quad (18)$$

Note that the feasibility conditions $v_i^* \leqslant v_i^0$, $\forall i \in \mathcal{R}^*$ are equivalent to $v_l \leqslant v_1^0$ and $v_h = (V - (n-1)v_l) \leqslant v_n^0$. Except for constraint (18), problem ($\mathcal{P}''_{\mathcal{R}^*}$) is identical to problem ($\mathcal{P}1''$). The first-order points of $\eta(v_l)$ are of interest since minimization must occur at one of these points. The possibility of the minimand occurring at a boundary is excluded since that would imply that one of the stations in $\mathcal{R}^*$ did not receive variance reduction—a contradiction to the definition of $\mathcal{R}^*$. If $V < \check{V}_n$ or if $V > \hat{V}_n$ then $\eta(v_l)$ has only one local minimum—in the interval $[0, V/n]$ as indicated in Theorem 4, parts (1b) and (3), respectively. In these cases, the optimal solution to problems ($\mathcal{P}''_{\mathcal{R}^*}$) and ($\mathcal{P}1''$) must occur at the same point, since there is no other first-order point. In contrast, if $\check{V}_n < V < \hat{V}_n$, then $\eta(v_l)$ has either one or two local minima in the interval $[0, V/n]$, as indicated in Theorem 4, part (2). In the case of two local minima, it is possible that the minimand of problem ($\mathcal{P}1''$) does not fall in the interval specified by constraint (18). In this case, the optimal solution for problems ($\mathcal{P}1''$) and ($\mathcal{P}''_{\mathcal{R}^*}$) are different, and they correspond to the two local minima of $\eta(v_l)$. □

That is, Theorem 11 states that the optimal solution to the variance reduction problem ($\mathcal{P}$) is such that among the stations that receive variance reduction, at most one can have its final variance larger than $\hat{V}$, all others must have the same final variance that is smaller than $\hat{V}$. Though the variance allocation problem ($\mathcal{P}1_{\mathcal{R}^*}$) may not solve problem ($\mathcal{P}$) if $V(\mathcal{R}^*)$ is in the interval $(\check{V}_n, \hat{V}_n)$, the final variances among stations in $\mathcal{R}^*$ always inherit the optimal variance structure for the allocation problem. The result below follows directly from Theorem 11.

COROLLARY 1. *Let* $\mathcal{R}$ *be a set of stations with cardinal* n *and* $V(\mathcal{R})$ *outside the interval* $(\check{V}_n, \hat{V}_n)$. *If the optimal solution to the variance allocation problem* ($\mathcal{P}1_{\mathcal{R}}$) *is not feasible to problem* ($\mathcal{P}_{\mathcal{R}}$), *then* $\mathcal{R}$ *cannot be the set of stations where variance reduction takes place in an optimal solution to the variance reduction problem* ($\mathcal{P}$).

Based on these results, we have developed a branch-and-bound algorithm for determining the optimal solution to the variance reduction problem ($\mathcal{P}$). For the sake of brevity, we present here only an outline of the algorithm. It illustrates how the results of this section can be utilized to identify the set of stations in the optimal solution. Let each node in the branch-and-bound tree correspond to a set of stations $\mathcal{R}$ where variance reduction may possibly occur. The algorithm selects a node with the highest cardinal so that if fathomed, the largest number of subsets are eliminated. For any node, the allocation problem is solved only if the initial variance is large enough to absorb the available variance reduction, otherwise the node is fathomed. If the optimal solution to ($\mathcal{P}1_{\mathcal{R}}$) yields a worse objective function value than that of the incumbent best solution, the node is fathomed. Otherwise, the feasibility of this solution to problem ($\mathcal{P}$) is checked. If feasible, this node represents a better solution than those found before; the incumbent is updated and the node is fathomed. Only when the solution to ($\mathcal{P}1_{\mathcal{R}}$) is infeasible to ($\mathcal{P}$), its objective function value better than the incumbent, and $V(\mathcal{R})$ in the critical range $(\tilde{V}_n, \hat{V}_n)$, that we explore alternative minimum for $\eta(v)$. If an alternative minimum exists and is found to be both feasible and better than the incumbent, we update the incumbent solution and fathom the node. When a node does not lead to a better solution and it cannot be fathomed, it is branched by creating descendants. Descendant nodes are created in a manner that rules out generation of duplicate nodes as well as those fathomed earlier. When a node is fathomed, all its subsets as well as those nodes with worse lower bound are removed from the set of unfathomed nodes. The algorithm stops when all branched nodes are fathomed.

EXAMPLE. We now present an example to illustrate the nature of the optimal solution. Consider a four-station assembly line with $T = 12$, $\mu = 10$, $v_1^0 = 5$, $v_2^0 = 10$, $v_3^0 = 16$, and $v_4^0 = 25$. Suppose the amount of variance reduction desired is $\Delta V = 19.5$. An optimal solution to this problem can be found as $v_1^* = 3$, $v_2^* = 3$, $v_3^* = 5.5$ and $v_4^* = 25$. That is, stations 1, 2, and 3 receive 2, 7, and 10.5 units of variance reduction, respectively. Station 4, which has the largest initial variance, receive no reduction. Note that the remaining variance at stations receiving variance reduction follows the optimal variance structure: All stations have their variances reduced to the same level (three units), except one, which has its variance reduced to a higher level (5.5 units). An alternative optimal solution can also be constructed from this solution as $v_1^* = 3$, $v_2^* = 5.5$, $v_3^* = 3$ and $v_4^* = 25$, resulting in the same expected work overload. In this solution, stations 2 and 3 receive 4.5 and 13 units of variance reduction, respectively, instead of 7 and 10.5 units of reduction received in the first solution. Whenever an alternative like this exists, one should choose a solution that reduces variance at an upstream station before reducing the variance at a downstream station. This will reduce the risk that an unfinished task upstream hinders the completion of a task downstream on a synchronous line. This is consistent with the observation in Hopp and Spearman (1996) that "variability early in a line is more disruptive than variability late in a line," when one considers congestion effects in an asynchronous line.

4. SENSITIVITY TO PROCESSING TIME DISTRIBUTION

So far we have assumed that the processing times are normally distributed. This assumption was motivated by analytical tractability as well as realism (a combination of uncertainties often add up to a symmetric unimodal processing time distribution with distinctive peak). At this point it is natural to ask the following: To what extent do the results of the last three sections depend on the normality assumption? Will the structural properties still hold if the processing times are *not* normally distributed? The use of the normal distribution in obtaining these results makes the results *potentially* suspect for two reasons. First, the normal distribution allows for negative processing times, especially at high levels of variability. Second, the unimodal, symmetric shape of the normal distribution may not always be appropriate to represent the processing time.

We address these questions and issues by exploring the nature of the expected work overload function $f(v)$. If $f(v)$ were convex, an equal variance solution would be optimal for the variance allocation problem ($\mathcal{P}1$). In this case, the variance reduction problem ($\mathcal{P}$) could be solved by reducing the variance incrementally at the station(s) with the highest variance. On the other hand, if $f(v)$ were concave, an extreme point solution would be optimal for both problems ($\mathcal{P}$) and ($\mathcal{P}1$). The optimal allocation for problem ($\mathcal{P}1$) will be such that a single station receives the entire variance. The variance reduction problem ($\mathcal{P}$) could be solved by reducing the variance incrementally at the station(s) with the lowest nonzero variance. Recall that when processing times are normally distributed, $f(v)$ is convex for $v < \hat{V}$ and concave otherwise. All structural results for allocation problem ($\mathcal{P}1$) are directly attributable to this convex-concave nature of $f(v)$. When the total variability is small, one is allocating variance in the convex region of $f(v)$ and an equal variance solution is optimal. Conversely, when the total variability is large, i.e., $V > \hat{V}_n$, one station gets most of the variance due to the concavity of $f(v)$ for large v. However, all other stations do not get zero variance because $f(v)$ is not concave for small v. The result is a hybrid of an extreme point solution and an equal-variance solution. An optimal solution to the variance reduction problem ($\mathcal{P}$) preserves this structure; all stations that receive variance reduction get their variances reduced to the same level, with the possible exception of one station that may have its final variance larger than the others.

It is clear that the increasing, convex-concave nature of $f(v)$ is central to all arguments leading to the structural results in §§ 2 and 3. To demonstrate the generality of these results for other processing time distributions, it will be enough to show that the resulting expected work overload function has an increasing, convex-concave shape. Unfortunately, it is often difficult, if not impossible, to find a closed form expression for the expected work overload as a function of variance. We present below the analytical results for the uniform and shifted exponential distributions. We have verified numerically that similar results hold for lognormal, beta, and gamma distributions.

4.1. Uniform Distribution

Suppose the processing time at each station is uniformly distributed with mean μ. Consider a station with variance v. As variance is reduced, the support of the distribution, $[\mu - \sqrt{3v}, \mu + \sqrt{3v}]$, shrinks symmetrically around mean μ. If the half-spread of the distribution, $\sqrt{3v}$, falls below the slack in processing time, $(T - \mu)$, one always finishes work within the cycle time. For this reason, it is not worthwhile to reduce the variance at any station below the level $(T - \mu)^2/3$. The expected work overload is given by

$$f(v) = \int_T^{\mu+\sqrt{3v}} \frac{t - T}{2\sqrt{3v}}\, dt = \begin{cases} 0 & \text{if } 0 \le v < \dfrac{(T-\mu)^2}{3}, \\ \dfrac{1}{4}\left[\dfrac{(T-\mu)^2}{\sqrt{3v}} - 2(T-\mu) + \sqrt{3v}\right] & \text{otherwise.} \end{cases} \tag{19}$$

The following theorem indicates that the structural results derived for normal distribution hold for uniform distribution.

THEOREM 12. *For the expected work overload function* $f(v)$ *given by* (19) *the following holds*:

(*a*) $f(v)$ *is* 0 *on the interval* $[0, (T - \mu)^2/3]$, *convex-increasing on the interval* $[(T - \mu)^2/3, \hat{V}]$ *and concave-increasing on the interval* $[\hat{V}, \infty]$ *where* $\hat{V} \triangleq (T - \mu)^2$.

(*b*) *For the two-station problem*, ($\mathcal{P}$1.1), *there exists a critical variance level* $\hat{V}_2 \triangleq 2\hat{V}$, *such that the equal-variance solution is optimal if an donly if the total variance* $V \le \hat{V}_2$.

(*c*) *In an optimal unequal-variance solution to problem* ($\mathcal{P}$1), *all stations except one have low variance*.

(*d*) *There exists a critical variance level* $\hat{V}_n \triangleq n\hat{V}$, *such that an unequal-variance solution is optimal to problem* ($\mathcal{P}$1) *if* $V > \hat{V}_n$.

PROOF. See Appendix A.9.

The dominance results derived in §3 remains true for the uniform distribution as well.

4.2. Shifted Exponential Distribution

Suppose the processing time at each station follows a shifted exponential distribution. The random variable Y is said to have a shifted exponential distribution if $Y = \alpha + X$, α is a constant, and X is exponentially distributed. If X has mean θ, then Y has mean $\mu = \alpha + \theta$ and variance $v = \theta^2$, which implies that

$$\alpha = \mu - \sqrt{v},$$
$$\theta = \frac{1}{\sqrt{v}}.$$

For the shifted exponential distribution, the expected work overload at a station with mean processing time μ and variance v is given by

$$f(v) = \int_T^{\infty} \frac{t - T}{\sqrt{v}}\, e^{-(t-\mu+\sqrt{v})/\sqrt{v}}\, dt = \frac{\sqrt{v}}{e}\, e^{-(T-\mu)/\sqrt{v}}. \tag{20}$$

THEOREM 13. *For the expected work overload function* $f(v)$ *given by* (20) *the following holds*:

(*a*) $f(v)$ *is convex-increasing on the interval* $[0, \hat{V}]$ *and concave-increasing on the interval* $[\hat{V}, \infty]$ *where*

$$\hat{V} \triangleq \left(\frac{\sqrt{5} - 1}{2}\right)^2 (T - \mu)^2.$$

(*b*) *For the two-station problem*, ($\mathcal{P}$1.1), *there exists a critical variance level* $\hat{V}_2 \triangleq 2\hat{V}$ *such that the unequal-variance solution is optimal if* $V > \hat{V}_2$.

(*c*) *In an optimal unequal-variance solution to problem* ($\mathcal{P}$1), *all stations except one have low variance*.

(*d*) *There exists a critical variance level* $\hat{V}_n \triangleq n\hat{V}$, *such that an unequal-variance solution is optimal to problem* ($\mathcal{P}$1) *if* $V > \hat{V}_n$.

PROOF. See Appendix A.10.

Note that the structure of the optimal allocation depends only upon *normalized slack processing time* per unit of standard deviation

$$s \triangleq \frac{T - \mu}{\sqrt{V/n}}, \tag{21}$$

and not on individual model parameters—T, μ, n, and V. The quantity s is dimensionless and can be viewed as a measure of processing flexibility. This is a fundamental quantity which governs the transition from an equal to an unequal variance structure. For the case of normally or uniformly distributed processing times, an unequal variance solution is optimal to the variance allocation problem if $s < 1$. For the shifted exponential distribution, the same is true if $s < (\sqrt{5} + 1)/2$. A similar quantity, $\alpha = 2(T - \mu)/v$, plays a fundamental role in determining the percentage of jobs requiring rework in Hsu's (1992) assembly line model.

5. CONCLUSIONS

This paper has analyzed, in the context of a synchronous assembly line, how to allocate a limited resource to a set of competing alternatives for variance reduction. It is shown that to focus variance reduction efforts among stations with the largest variability may not be optimal. While the choice of best stations that should receive variance reduction is complex, the structure of remaining variance among these stations can be obtained from a related variance allocation problem. Our analysis revealed that if the total remaining variance among stations receiving reduction is sufficiently large, the remaining variance is best left in a spike-shaped configuration; efforts to equalize them can be the worst solution.

Our model assumed that the total variance reduction can be apportioned to stations in any desired quantities. In reality, investment in variance reduction efforts may result in discrete amounts of reduction. However, even in circumstances when variance reductions are not infinitessimally divisible, our results provide targets at which reduction efforts should be aimed. Our model also assumed that stations respond identically to the variance reduction efforts. This assumption was made not only to simplify the analysis but also to reflect the fact that it is often difficult to know in advance how much variance reduction will be realized at a station as a result of an investment. In the absence of such information it may not be unreasonable to assume that a dollar spent on one station is as good as a dollar spent on another.

In situations where the remaining variance among stations receiving variance reduction is spike-shaped, the high variance station should ideally be located toward the end of the line. That is, given a choice, one should strive to reduce the variance at upstream stations first. This approach has several advantages. The resulting work overload at these high variance stations would have little disruptive effect on other stations. Moreover, the jobs requiring rework can be sent to the rework station with minimal delay. For variance reduction on asynchronous lines, Hopp and Spearman (1996) recommend a similar approach and note that "there tends to be greater leverage for variability reduction applied to the front end of a line than the back end."

The results of this paper can thus be used to guide performance improvement efforts in many realistic settings. They can also provide important insights into the design and planning of stochastic assembly lines. Finally, the mathematical results are applicable to a wide class of separable resource allocation problems with convex-concave structure.

APPENDIX A

A.1. Proof of Theorem 1

We recast problem ($\mathcal{P}$1.1) as a single variable optimization problem by substituting $V - v_1$ for v_2. (In what follows we drop the subscript on v_1 because it is the only decision variable.) The new optimization problem is given by

$$(\mathcal{P}1.1') \ \underset{0 \le v \le V}{\text{Minimize}}\, f(v) + f(V - v).$$

The function $f(v) + f(V - v)$ is the objective function from problem ($\mathcal{P}$1.1) with $V - v$ substituted for v_2. The first station receives v units of variance and the second station receives the remaining $V - v$ units of variance.

Differentiating $f(v) + f(V - v)$ with respect to v we obtain

$$f'(v) - f'(V - v) = \lambda(v) - \lambda(V - v).$$

Clearly, the first derivative vanishes at v satisfying

$$\lambda(v) = \lambda(V - v). \tag{22}$$

An optimal solution to problem ($\mathcal{P}1.1'$) must take one of the following two forms:

1. $v = V/2$, the equal-variance solution, or
2. $v \neq V/2$, an unequal-variance solution.

We first show that for $V > \hat{V}_2$, the equal-variance solution cannot be optimal. We will prove this by showing that for $V > \hat{V}_2$, the equal-variance solution is a local maximum and hence cannot be a global minimum. First note that $v = V/2$ satisfies Equation (22) and as a result must be either a local minimum, a local maximum, or a point of inflection. But

$$\begin{aligned} f''(V/2) + f''(V/2) &= 2f''(V/2) \\ &= \frac{\sqrt{2}Z\left(\frac{T-\mu}{\sqrt{V/2}}\right)}{V^{5/2}}\,(2(T-\mu)^2 - V) \\ &= \frac{\sqrt{2}Z\left(\frac{T-\mu}{\sqrt{V/2}}\right)}{V^{5/2}}\,(\hat{V}_2 - V) \\ &< 0 \quad \text{for } V > \hat{V}_2. \end{aligned}$$

Hence for $V > \hat{V}_2$ the equal-variance solution is a local maximum and therefore cannot be a global minimum. This proves that for $V > \hat{V}_2$, an unequal-variance solution is optimal.

It remains to be shown that if $(v^*, V - v^*)$ is an optimal unequal-variance solution, then $V > \hat{V}_2$. We first demonstrate that the extreme points $v = 0$ and $v = V$ cannot be optimal. Since

$$f'(0) - f'(V) = -\lambda(V) < 0,$$

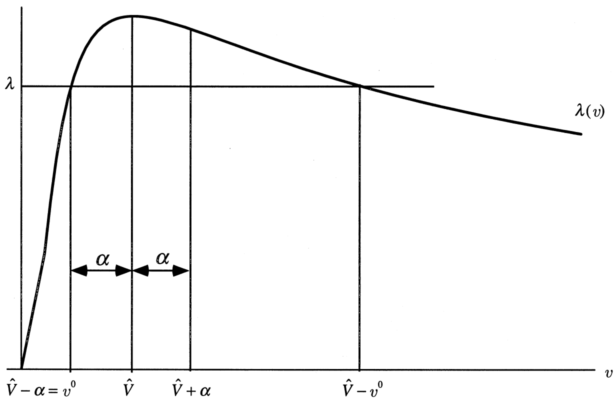
then $f(v) + f(V - v)$ is decreasing on the interval $[0, \epsilon]$ for some $\epsilon > 0$. Therefore $v = 0$ cannot be the global minimum. Likewise, it follows that

$$f'(V) - f'(0) = \lambda(V) > 0,$$

which implies that $v = V$ cannot be the global minimum, either. As a result we know that the optimal solution must be such that $0 < v^* < V/2$. We must show that if $(v^*, V - v^*)$ is an optimal unequal-variance solution, then $V > \hat{V}_2$. In fact we will show that a more general result holds—for any unequal-variance solution $(v^0, V - v^0)$ satisfying the first-order condition $\lambda(v^0) = \lambda(V - v^0)$, it must be true that $V > \hat{V}_2$.

Assume that $v^0 < V - v^0$ without loss of generality. For any solution $(v^0, V - v^0)$, which satisfies the first order conditions, we know that $\lambda(v^0) = \lambda(V - v^0) = \lambda$ as shown in Figure 5. Because $\lambda(v)$ attains its maximum at $v = \hat{V}$ (see § 1), it must be true that $v^0 < \hat{V} < V - v^0$. Let $\alpha = \hat{V} - v^0$. In A.2 we show that $\lambda(\hat{V} - \alpha) < \lambda(\hat{V} + \alpha)$ for $0 <$

Figure 5. $\lambda(v)$ for the Proof of Theorem 1.



$\alpha < \hat{V}$. Since $\lambda(v)$ is strictly decreasing for $v > \hat{V}$ (see § 1), then $V - v^0 > \hat{V} + \alpha$. But

$$V = v^0 + (V - v^0) > (\hat{V} - \alpha) + (\hat{V} + \alpha) = 2\hat{V} = \hat{V}_2,$$

and hence for any unequal-variance solution $(v^0, V - v^0)$, $v^0 > 0$ that satisfies the first-order conditions, it must but true that $V > \hat{V}_2$.

For the case when $V = \hat{V}_2$, it can be shown that the first, second, and third derivatives of $f(v) + f(V - v)$ vanish at the equal-variance solution. However, the fourth derivative of $f(v) + f(V - v)$ is positive and hence for $V = \hat{V}_2$, the equal-variance solution is the global minimum. □

A.2. Proof that $\lambda(\hat{V} - \alpha) < \lambda(\hat{V} + \alpha)$

If $0 \leqslant \alpha \leq \hat{V} = (T - \mu)^2$ then

$$\lambda(\hat{V} + \alpha) > \lambda(\hat{V} - \alpha)$$

$$\Leftrightarrow \frac{e^{-(T-\mu)^2/2(\hat{V}+\alpha)}}{2\sqrt{2\pi}\sqrt{\hat{V} + \alpha}} > \frac{e^{-(T-\mu)^2/2(\hat{V}-\alpha)}}{2\sqrt{2\pi}\sqrt{\hat{V} - \alpha}}$$

$$\Leftrightarrow e^{\alpha\hat{V}/(\hat{V}+\alpha)(\hat{V}-\alpha)} > \sqrt{\frac{\hat{V} + \alpha}{\hat{V} - \alpha}}$$

$$\Leftrightarrow e^{2\alpha\hat{V}/(\hat{V}+\alpha)(\hat{V}-\alpha)} > 1 + \frac{2\alpha}{\hat{V} - \alpha}.$$

But

$$e^{2\alpha\hat{V}/(\hat{V}+\alpha)(\hat{V}-\alpha)} > 1 + \frac{2\alpha\hat{V}}{(\hat{V} + \alpha)(\hat{V} - \alpha)} + \frac{2\alpha^2\hat{V}^2}{(\hat{V} + \alpha)^2(\hat{V} - \alpha)^2},$$

because $0 \leqslant \alpha \leqslant \hat{V}$.

Therefore $\lambda(\hat{V} + \alpha) > \lambda(\hat{V} - \alpha)$ if

$$1 + \frac{2\alpha\hat{V}}{(\hat{V} + \alpha)(\hat{V} - \alpha)} + \frac{2\alpha^2\hat{V}^2}{(\hat{V} + \alpha)^2(\hat{V} - \alpha)^2} > 1 + \frac{2\alpha}{\hat{V} - \alpha}$$

$$\Leftrightarrow \frac{2\alpha\hat{V}}{(\hat{V} + \alpha)^2(\hat{V} - \alpha)^2} + \frac{2\alpha^2\hat{V}^2}{(\hat{V} + \alpha)^2(\hat{V} - \alpha)^2} > \frac{2\alpha}{\hat{V} - \alpha}$$

$$\Leftrightarrow 2\alpha\hat{V}(\hat{V}^2 - \alpha^2) + 2\alpha^2\hat{V}^2 > 2\alpha(\hat{V}^2 - \alpha^2)(\hat{V} + \alpha)$$

$$\Leftrightarrow \hat{V}^3 - \alpha^2\hat{V} + \alpha\hat{V}^2 > \hat{V}^3 + \alpha\hat{V}^2 - \alpha^2\hat{V} - \alpha^3$$

$$\Leftrightarrow 0 > -\alpha^3.$$ □

A.3. Proof of Theorem 2

Let $\eta(v) = f(v) + f(V - v)$. Substituting the first-order condition, (10), in the second derivative

$$\frac{\partial^2\eta}{\partial v^2} = \frac{Z\left(\frac{T - \mu}{\sqrt{v}}\right)}{4v^{5/2}}(\hat{V} - v) + \frac{Z\left(\frac{T - \mu}{\sqrt{V - v}}\right)}{4(V - v)^{5/2}}(\hat{V} - (V - v))$$

yields

$$\frac{\partial^2\eta}{\partial v^2} = \frac{\lambda(v)((V + 2\hat{V})v^2 - (V^2 + 2V\hat{V})v + V^2\hat{V})}{2v^2(V - v)^2}.$$

Follow the proof for Theorem 4 below using $n = 2$. □

A.4. Proof of Theorem 4

For the optimization problem ($\mathscr{P}1''$), substituting the first-order condition

$$\frac{Z\left(\frac{T - \mu}{\sqrt{v}}\right)}{2\sqrt{v}} = \lambda(v) = \lambda(V - (n - 1)v) = \frac{Z\left(\frac{T - \mu}{\sqrt{V - (n - 1)v}}\right)}{2\sqrt{V - (n - 1)v}}$$

in the second derivative

$$\frac{\partial^2\eta}{\partial v^2} = (n - 1)\frac{Z\left(\frac{T - \mu}{\sqrt{v}}\right)}{4v^{5/2}}(\hat{V} - v) + (n - 1)^2\frac{Z\left(\frac{(T - \mu)}{\sqrt{V - (n - 1)v}}\right)}{4(V - (n - 1)v)^{5/2}} \cdot (\hat{V} - (V - (n - 1)v))$$

yields

$$\frac{\partial^2\eta}{\partial v^2} = \frac{(n - 1)\lambda(v)Q_n(v)}{2v^2(V - (n - 1)v)^2},$$

where

$$Q_n(v) = ((n - 1)(V + n\hat{V}))v^2 - (V^2 + 2(n - 1)V\hat{V})v + V^2\hat{V}.$$

Because $\eta(v)$ is continuous and differentiable, the sequence of first-order points for this function must alternate between local minimum and local maximum. Since $\eta'(0) < 0$, i.e., $\eta(v)$ is initially decreasing, the smallest first order point must be a local minimum. If the first-order points are arranged in an increasing order, then the second derivative evaluated at successive first-order points must alternate in sign, i.e., +, −, + etc. Since $n > 1$ and $\lambda(v) > 0$, $\forall v > 0$, the number of sign changes for the function

$Q_n(v)$, can be used as an indicator for the number (and nature) of first-order points for the function $\eta(v)$. In particular, because $Q_n(V/n) = (V/n)^2(n\hat{V} - V)$, V/n is a local minimum (local maximum) when $V < (>) \, n\hat{V}$.

Note that $Q_n(v)$ is quadratic in v with $Q_n(0) > 0$, $Q_n'(0) < 0$ and $Q_n''(v) > 0$. These imply that $Q_n(v)$ is a convex, parabolic function that is positive and decreasing at $v = 0$ and that can cross zero at most twice. That is, the function $\eta(v)$ can have at most three first-order points. The zero-crossing properties of $Q_n(v)$ are governed by its discriminant

$$(V^2 + 2(n-1)V\hat{V})^2 - 4(n-1)(V + n\hat{V})V^2\hat{V}$$
$$= V^2(V^2 - 4(n-1)\hat{V}^2).$$

For $V < 2\sqrt{n-1}\,\hat{V}$, there is no real root for $Q_n(v) = 0$ and $Q_n(v)$ is positive for all v. The equal-variance solution, $v = V/n$, must be the global minimum because there is no other first-order point. This is true for $V = 2\sqrt{n-1}\,\hat{V}$ also, because $Q_n(v) = 0$ has two repeated real roots and $Q_n(v)$ remains weakly positive for all v.

For $V > 2\sqrt{n-1}\,\hat{V}$, equation $Q_n(v) = 0$ has two real, positive roots given by v_a and v_b. We claim that for the multistage problem under consideration, whenever $Q_n(v) = 0$ has two distinct real roots, the equal-variance solution is strictly greater than the smaller of the two roots, i.e., $V/n > v_a$ for $n \geqslant 2$. To show this, first note that the parabolic curve $Q_n(v)$ achieves its minimum at

$$v_m = \frac{V^2 + 2(n-1)V\hat{V}}{2(n-1)(V + n\hat{V})}.$$

Clearly, $v_a < v_m < v_b$, i.e., the minimum lies between the two roots. Now suppose that $v_m > V/n$, then

$$\frac{V^2 + 2(n-1)V\hat{V}}{2(n-1)(V + n\hat{V})} > \frac{V}{n}$$
$$\Rightarrow nV^2 + 2n(n-1)V\hat{V} > 2(n-1)(V^2 + nV\hat{V})$$
$$\Rightarrow nV^2 > 2(n-1)V^2$$
$$\Rightarrow n < 2,$$

a contradiction. Hence it must be true that $v_m \leqslant V/n$ and, as a result, $v_a < V/n$. Now let

$$x \triangleq n\sqrt{V^2 - 4(n-1)\hat{V}^2}$$

and

$$y \triangleq (n-2)V.$$

Then $x < (\geqslant) \, y \Leftrightarrow v_b < (\geqslant) \, V/n$ and $-x < (\geqslant) \, y \Leftrightarrow v_a < (\geqslant) \, V/n$. This can be seen from

$$\pm n\sqrt{V^2 - 4(n-1)\hat{V}^2} < (\geqslant)(n-2)V$$
$$\Leftrightarrow V(nV + 2n(n-1)\hat{V} \pm n\sqrt{V^2 - 4(n-1)\hat{V}^2}) < (\geqslant) 2(n-1)V(V + n\hat{V})$$
$$\Leftrightarrow \frac{V^2 + 2(n-1)V\hat{V} \pm V\sqrt{V^2 - 4(n-1)\hat{V}^2}}{2(n-1)(V + n\hat{V})} < (\geqslant) \frac{V}{n}.$$

Moreover,

$$V < (\geqslant) \, \hat{V}_n$$
$$\Rightarrow V^2 < (\geqslant) \, n^2\hat{V}^2$$
$$\Rightarrow (n^2 - (n-2)^2)V^2 < (\geqslant) \, 4n^2(n-1)\hat{V}^2$$
$$\Rightarrow n^2(V^2 - 4(n-1)\hat{V}^2) < (\geqslant) \, (n-2)^2V^2$$
$$\Rightarrow x^2 < (\geqslant) \, y^2$$
$$\Rightarrow (x+y)(x-y) < (\geqslant) \, 0.$$

Now, $V > \hat{V}_n$ implies that either (a) $x > y$ and $-x < y$ or, (b) $x < y$ and $-x > y$. That is, either (a) $v_b > V/n$ and $v_a < V/n$ or, (b) $v_b < V/n$ and $v_a > V/n$. Condition (b) cannot hold since $v_a < v_b$ and $v_a < V/n$. Hence $V > \hat{V}_n \Rightarrow v_a < V/n < v_b$, i.e., the equal variance solution lies between the two roots. In the interval $(0, V/n)$, $Q_n(v)$ changes sign only once when it goes from + to − at v_a. Recall that the smallest first-order point for $\eta(v)$ must be a local minimum and that V/n is a local maximum for $V > n\hat{V}$. Thus, for $V > n\hat{V} = \hat{V}_n$, there are exactly two first-order points for $\eta(v)$, the minimizing variance v_l^*, which must be smaller than v_a and V/n, a local maximum.

Similarly, $V = \hat{V}_n$ implies that either $-x = y$ or $x = y$, i.e., either $v_a = V/n$ or $v_b = V/n$. But $v_a < V/n$, hence $V = \hat{V}_n \Rightarrow v_a < V/n = v_b$. In the interval $(0, V/n)$, $Q_n(v)$ changes sign only once when it goes from + to − at v_a and as $v \to V/n$, $Q_n(v)$ approaches zero from below. Hence $\eta(v)$ achieves its global minimum at a first-order point, v_l^*, that lies in the interval $(0, v_a)$ and $\eta(v)$ is increasing in the interval $(v_l^*, V/n)$.

Finally, $V < \hat{V}_n$ implies that either (a) $x > y$ and $-x > y$ or, (b) $x < y$ and $-x < y$. That is, either (a) $v_b > V/n$ and $v_a > V/n$ or, (b) $v_b < V/n$ and $v_a < V/n$. But condition (a) cannot hold since $v_a < V/n$. Hence $V < \hat{V}_n \Rightarrow v_b < V/n$ i.e., the equal variance solution is larger than both the roots. In the interval $(0, V/n)$, $Q_n(v)$ changes sign twice (+, −, +) at v_a and v_b, respectively. The number of first order points for $\eta(v)$ cannot exceed three. Recall that the smallest first-order point for $\eta(v)$ must be a local minimum, that V/n is also a local minimum, and that the successive first-order points must alternate between local maximum and local minimum. These observations rule out the possibility of $\eta(v)$ having just two first-order points in the interval $(0, V/n)$. Hence for $2\sqrt{n-1}\,\hat{V} < V \leqslant n\hat{V}$, one of the following must be true:

1. $\eta(v)$ has three first order points in the interval $(0, V/n)$. The first-order point that falls in the interval (v_a, v_b) is a local maximum; other two are local minimum, one of which minimizes $\eta(v)$.

2. $\eta(v)$ has only one first-order point in the interval $(0, V/n)$—the global minimum at V/n. □

A.5. Proof of Theorem 5

The only candidate points for the global maximum to problem ($\mathcal{P}1.1$) are first-order points and extreme points. In Theorem 2 it was shown that if $V > \hat{V}_2$ then there are three points that satisfy the first-order conditions. One of these points is the equal-variance solution, which is a local maximum. The other two points are a symmetric pair of unequal-variance solutions that are the global minima. Therefore, the only other candidate for the global maximum would be an extreme point. It is shown in A.6 that the equal variance solution is worse than an extreme point solution if and only if $V > \hat{\hat{V}}_2$. Thus, for $V > \hat{\hat{V}}_2$, the equal-variance solution must be the global maximum. □

A.6. Proof that the Equal Variance Allocation is Worse than an Extreme Point Allocation if and only if $V > \hat{\hat{V}}_2$

Define $f_{\text{EV}}(V) = 2f(V/2)$ and $f_{\text{EP}}(V) = f(V)$. The function $f_{\text{EV}}(V)$ is the EWO for the equal variance (EV) solution with total variance V while $f_{\text{EP}}(V)$ is the EWO for an extreme point (EP) solution with total variance V. If we let $x = (T - \mu)/\sqrt{V}$ then the equation $f_{\text{EP}}(V) = f_{\text{EV}}(V)$ reduces to Equation (16). Since $V > 0$, and there is a one-to-one relationship between positive x and positive V, we need only find positive roots to Equation (16). To show that $\hat{\hat{x}}$ is the unique positive root to Equation (16), we consider the equation $D(V) \triangleq f_{\text{EP}}(V) - f_{\text{EV}}(V)$. Now since

$$\lim_{V\to 0} D(V) = 0^+$$

and

$$\lim_{V\to\infty} D(V) = \lim_{V\to\infty}\left(\sqrt{\frac{V}{\pi}}\frac{(\sqrt{2}-2)}{2} - \frac{T-\mu}{2}\right) = -\infty < 0,$$

there is at least one root to $D(V) = 0$, and therefore at least one root to Equation (16). To show that there is a unique root to $D(V)$, we show that there exists a number β such that $D'(V) > 0$ if and only if $V < \beta$. This fact guarantees that there is only one root to $D(V) = 0$. Now, consider the first derivative of D,

$$D'(V) = \frac{Z\left(\dfrac{T-\mu}{\sqrt{V/2}}\right)}{\sqrt{2V}}\left(\frac{e^{(T-\mu)^2/2V}}{\sqrt{2}} - 1\right).$$

It is clear that $D'(V) > 0$ if and only if $e^{(T-\mu)^2/2V}/\sqrt{2} > 1$. But $e^{(T-\mu)^2/2V}/\sqrt{2} > 1$ if and only if $V < \beta \equiv (T-\mu)^2/\log 2$. Thus we have shown that there is exactly one root to $D(V) = 0$. This in turn implies that there is exactly one root, $\hat{\hat{x}}$, to Equation (16). So, $f_{\text{EV}}(V)$ will grow at a slower rate than $f_{\text{EP}}(V)$ until $V = \beta$ and then $f_{\text{EV}}(V)$ will grow at a faster rate than $f_{\text{EP}}(V)$, cross $f_{\text{EP}}(V)$ at $\hat{V}_2$ ($> \beta$), and $f_{\text{EV}}(V)$, will be greater than $f_{\text{EP}}(V)$ thereafter. Through numerical techniques, we have found that $\hat{\hat{x}} \approx 0.511$, which implies that $\hat{\hat{V}}_2 \approx 3.83(T-\mu)^2$. (Note that $\hat{\hat{V}}_2 > \hat{V}_2$.) □

A.7. Proof of Theorem 6

The proof is in two parts. We first show that an extreme point solution cannot be the global maximum. We then show that an interior point, unequal variance solution cannot be the global maximum. To show that an extreme point cannot be the global maximum when $V > \hat{\hat{V}}_n$, we first observe that an extreme point solution has $v_1 = 0$. (Recall that we have assumed without loss of generality that $v_i < v_{i+1}$, $i = 1, \ldots, n-1$.) Furthermore, it must be true that $v_n \geq V/(n-1) > \hat{\hat{V}}_2$. If we form a two-station subproblem by considering stations 1 and n in isolation, then the total variance for this subproblem is greater than $V/(n-1) > \hat{\hat{V}}_2$. By Theorem 5, the equal-variance solution to this two-station subproblem has a larger objective function value than an extreme point solution. Since (v_1, v_n) is not the global maximum to this two-station subproblem, then the original extreme point solution cannot be a global maximum to problem ($\mathcal{P}1$).

We now show that an interior point, unequal variance solution cannot be the global maximum. Suppose that an interior point, unequal variance solution was the global maximum. Then by Lemma 1, it would have k stations with variance v_l and $n-k$ stations with variance v_h. Since it is an unequal-variance solution, $1 \leq k \leq n-1$. If $v_l + v_h > \hat{\hat{V}}_2$, then by Theorem 5, the equal-variance solution must be the worst solution to this two-station subproblem, which implies that this unequal-variance solution to the n station problem cannot be the global maximum. We now show that $v_l + v_h > \hat{\hat{V}}_2$. Let $v_l = V/n - \Delta$. Then $v_h = V/n + k\Delta/(n-k)$. So $v_l + v_h = 2V/n + (2k-n)\Delta/(n-k)$. First consider the case when $2k - n < 0$. Then

$$\begin{aligned}
v_l + v_h &= 2V/n + (2k-n)\Delta/(n-k)\\
&> \frac{2V}{n} + \frac{(2k-n)V}{(n-k)n}, \text{ since } \Delta < \frac{V}{n}\\
&= \frac{V}{n-k} > \frac{(n-1)}{n-k}\hat{\hat{V}}_2\\
&\geq \hat{\hat{V}}_2.
\end{aligned}$$

Now consider the case when $2k - n \geq 0$. Then

$$\begin{aligned}
v_l + v_h &= 2V/n + (2k-n)\Delta/(n-k)\\
&\geq \frac{2V}{n}\\
&> \frac{2(n-1)}{n}\hat{\hat{V}}_2\\
&\geq \hat{\hat{V}}_2 \text{ since } \frac{2(n-1)}{n} \geq 1 \quad \text{for } n \geq 2.
\end{aligned}$$

□

A.8. Proof of Theorem 7

Consider the optimal solution to the problem where k stations have low variance and $n - k$ stations have high variance. Let this solution be

$$\bar{v}^{(1)} = (\underbrace{v^*_{l(k)}, \cdots, v^*_{l(k)}}_{k}, \underbrace{v^*_{h(k)}, \cdots, v^*_{h(k)}}_{n-k}).$$

Now consider $\bar{v}^{(2)}$, which is a solution to the problem where $k + 1$ stations have a low variance of $v^*_{l(k)}$ and $n - k - 1$ stations have high variance $v'_{h(k)} = (V - (k + 1)v^*_{l(k)})/(n - k - 1)$. That is,

$$\bar{v}^{(2)} = (\underbrace{v^*_{l(k)}, \cdots, v^*_{l(k)}}_{k+1}, \underbrace{v'_{h(k)}, \cdots, v'_{h(k)}}_{n-k-1}).$$

These two solutions have the same variance assigned to each of the first k stations. Therefore, consider the $n - k$ station subproblem with total variance $V' = V - kv^*_{l(k)}$ formed by considering the last $n - k$ stations of either one of these solutions. The solution to this subproblem formed from $\bar{v}^{(1)}$ is

$$\bar{v}'^{(1)} = (\underbrace{v^*_{h(k)}, \cdots, v^*_{h(k)}}_{n-k}).$$

The solution to this subproblem formed from $\bar{v}^{(2)}$ is

$$\bar{v}'^{(2)} = (\underbrace{v^*_{l(k)}, v'_{h(k)}, \cdots, v'_{h(k)}}_{n-k-1}).$$

Note that $\bar{v}'^{(1)}$ is an equal-variance solution to this subproblem. If we can demonstrate that the total variance for this subproblem, V', is greater than the upper critical variance level for this subproblem, then by Theorem 6 it will be true that $\bar{v}'^{(2)}$ is a better solution to the subproblem than $\bar{v}'^{(1)}$. But,

$$\begin{aligned} V' &= V - kv^*_{l(k)} \\ &> \hat{\hat{V}}_n - kv^*_{l(k)} \\ &= (n-1)\hat{\hat{V}}_2 - kv^*_{l(k)} \\ &> (n-1)\hat{\hat{V}}_2 - k\hat{\hat{V}}_2 \\ &= (n-k-1)\hat{\hat{V}}_2 \\ &= \hat{\hat{V}}_{n-k}. \end{aligned}$$

Hence the total variance for the subproblem is greater than the upper critical variance for the subproblem. Therefore $\bar{v}'^{(1)}$ is a worse solution to the subproblem than $\bar{v}'^{(2)}$. This implies that $\bar{v}^{(1)}$ is a worse solution to the original problem than $\bar{v}^{(2)}$, i.e., $\sum_{i=1}^n f(\bar{v}_i^{(1)}) > \sum_{i=1}^n f(\bar{v}_i^{(2)})$. Recall that $\bar{v}^{(2)}$ is a solution which has $k + 1$ stations with low variance and $n - k - 1$ stations with high variance. If we let

$$\bar{v}^{(3)} = (\underbrace{v^*_{l(k+1)}, \cdots, v^*_{l(k+1)}}_{k+1}, \underbrace{v^*_{h(k+1)}, \cdots, v^*_{h(k+1)}}_{n-k-1})$$

be the optimal solution to the problem with total variance V and $k + 1$ stations with low variance and $n - k - 1$ stations with high variance, then it must be true that $\sum_{i=1}^n f(\bar{v}_i^{(2)}) > \sum_{i=1}^n f(\bar{v}_i^{(3)})$. Therefore $\sum_{i=1}^n f(\bar{v}_i^{(1)}) > \sum_{i=1}^n f(\bar{v}_i^{(2)}) > \sum_{i=1}^n f(\bar{v}_i^{(3)})$, i.e., the optimal solution with k stations having low variance is worse than the optimal solution with $k + 1$ stations having low variance. □

A.9. Proof of Theorem 12

(a) From Equation (19), the first two derivatives of the expected work overload function are obtained as

$$\lambda(v) = f'(v) = \begin{cases} 0 & \text{if } 0 \leqslant v < \dfrac{(T-\mu)^2}{3}, \\ \dfrac{\sqrt{3}}{8v^{3/2}}\left(v - \dfrac{(T-\mu)^2}{3}\right) & \text{if } v \geqslant \dfrac{(T-\mu)^2}{3}, \end{cases}$$

and

$$\lambda'(v) = f''(v) = \begin{cases} 0 & \text{if } 0 \leqslant v < \dfrac{(T-\mu)^2}{3}, \\ \dfrac{\sqrt{3}}{16v^{5/2}}((T-\mu)^2 - v) & \text{if } v \geqslant \dfrac{(T-\mu)^2}{3}. \end{cases}$$

Except for $v < (T - \mu)^2/3$, $\lambda(v)$ is positive everywhere and its value approaches zero as v goes to infinity. This implies that $f(v)$ is increasing in v except for $v < (T - \mu)^2/3$, when it is zero. Let $\hat{V} \triangleq (T - \mu)^2$. Then for $v \geqslant (T - \mu)^2/3$, $\lambda'(v)$ crosses 0 only at $v = \hat{V}$. Moreover, $\lambda'(v) > 0$ for $v < \hat{V}$ and $\lambda'(v) < 0$ for $v > \hat{V}$. Hence $\lambda(v)$ is a unimodal function that is strictly increasing on $((T - \mu)^2/3, \hat{V})$, strictly decreasing on $(\hat{V}, \infty)$, and achieves its (global) maximum at $\hat{V}$. The expected work overload function f is convex on $((T - \mu)^2/3, \hat{V})$ and concave on $(\hat{V}, \infty)$. □

(b) The proof is identical to that for Theorem 1. Only that part of the proof that is dependent upon distributional assumption will be presented here. To show that for $V > \hat{V}_2$, the equal-variance solution cannot be a global minimum, note that the second derivative of the objective function, $f''(v) + f''(V - v)$, evaluated at $v = V/2$ gives

$$\begin{aligned} 2f''(V/2) &= \frac{\sqrt{3}}{4\sqrt{2}V^{5/2}}(2(T-\mu)^2 - V) \\ &= \frac{\sqrt{3}}{4\sqrt{2}V^{5/2}}(\hat{V}_2 - V) \\ &< 0 \quad \text{for } V > \hat{V}_2. \end{aligned}$$

Hence for $V > \hat{V}_2$ the equal variance solution is a local maximum and cannot be a global minimum.

To show that $\lambda(\hat{V} - \alpha) < \lambda(\hat{V} + \alpha)$, recall that $\lambda(v) = 0$ on $[0, (T - \mu)^2/3]$ and $\hat{V} = (T - \mu)^2$. Thus, we require that $\alpha \leqslant \frac{2}{3}(T - \mu)^2$. Now,

$$\lambda(\hat{V}-\alpha) < \lambda(\hat{V}+\alpha)$$

$$\Leftrightarrow \frac{1}{8}\left(\frac{\sqrt{3}}{\sqrt{\hat{V}-\alpha}} - \frac{\hat{V}}{\sqrt{3}(\hat{V}-\alpha)^{3/2}}\right)$$

$$< \frac{1}{8}\left(\frac{\sqrt{3}}{\sqrt{\hat{V}+\alpha}} - \frac{\hat{V}}{\sqrt{3}(\hat{V}+\alpha)^{3/2}}\right)$$

$$\Leftrightarrow 3(\hat{V}-\alpha) - \hat{V} < 3\,\frac{(\hat{V}-\alpha)^{3/2}}{\sqrt{\hat{V}+\alpha}} - \hat{V}\left(\frac{\hat{V}-\alpha}{\hat{V}+\alpha}\right)^{3/2}$$

$$\Leftrightarrow 2\hat{V} - 3\alpha < \sqrt{\frac{\hat{V}-\alpha}{\hat{V}+\alpha}}\left(3(\hat{V}-\alpha) - \hat{V}\,\frac{\hat{V}-\alpha}{\hat{V}+\alpha}\right)$$

$$\Leftrightarrow (2\hat{V}-3\alpha)(\hat{V}+\alpha) < \sqrt{\frac{\hat{V}-\alpha}{\hat{V}+\alpha}}\,(\hat{V}-\alpha)(2\hat{V}+3\alpha)$$

$$\Leftrightarrow (2\hat{V}-3\alpha)^2(\hat{V}+\alpha)^2 < \frac{\hat{V}-\alpha}{\hat{V}+\alpha}\,(\hat{V}-\alpha)^2(2\hat{V}+3\alpha)^2$$

$$\Leftrightarrow (2\hat{V}-3\alpha)^2(\hat{V}+\alpha)^3 < (\hat{V}-\alpha)^3(2\hat{V}+3\alpha)^2$$

$$\Leftrightarrow 9\alpha^2 < 5\hat{V}^2$$

$$\Leftrightarrow \alpha < \sqrt{\frac{5}{9}}\,\hat{V},$$

which gives us our result since we know that $\alpha \le \frac{2}{3}\hat{V} < \sqrt{\frac{5}{9}}\,\hat{V}$. □

(c) Identical to the proof of Theorem 3.

(d) We prove the result by contradiction. Suppose $V > \hat{V}_n$ but the equal-variance solution is optimal. Consider any two stations in isolation. The total variance from these two stations is $2V/n > 2\hat{V} = \hat{V}_2$. So for this two-station problem considered in isolation, an unequal-variance solution must be optimal from results in part (b) above. Since objective function in problem ($\mathcal{P}1$) is separable, this means that the equal-variance solution cannot be optimal to the problem ($\mathcal{P}1$). □

A.10. Proof of Theorem 13

The proofs are similar to those for the normal and uniform distributions. The convex-concave nature of the function $f(v)$ is proved below. The rest of the proof is omitted. From (20), the first two derivatives of the expected work overload function is obtained as

$$\lambda(v) = f'(v) = \frac{e^{(T-\mu)/\sqrt{v}}}{2e\sqrt{v}}\left(1 + \frac{T-\mu}{\sqrt{v}}\right), \tag{23}$$

and

$$\lambda'(v) = f''(v) = \frac{e^{-(T-\mu)/\sqrt{v}}}{4e\,v^{3/2}}\left[\frac{(T-\mu)^2}{v} - \frac{T-\mu}{\sqrt{v}} - 1\right]. \tag{24}$$

Note from (23) that $\lambda(v) \geq 0$, and $\lambda(v) \to 0$ as $v \to 0$, or as $v \to \infty$. Hence $f(v)$ is increasing. The term in the braces of (24) is negative if

$$0 \leq \frac{T-\mu}{\sqrt{v}} < \frac{1+\sqrt{5}}{2}$$

and positive if

$$\frac{T-\mu}{\sqrt{v}} \geq \frac{1+\sqrt{5}}{2}.$$

That is, $\lambda'(v) < 0$ for $v > \hat{V}$ and $\lambda'(v) > 0$ for $v < \hat{V}$ where

$$\hat{V} = \left(\frac{\sqrt{5}-1}{2}\right)^2 (T-\mu)^2.$$

Hence $\lambda(v)$ is a unimodal function that is strictly increasing on $(0, \hat{V})$, strictly decreasing on $(\hat{V}, \infty)$, and achieves its (global) maximum at $\hat{V}$. Also, $f(v)$ is convex-increasing on the interval $[0, \hat{V}]$ and concave-increasing on the interval $[\hat{V}, \infty]$. □

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REFERENCES

Gerchak, Y., M. Parlar. 1991. Investing in reducing lead-time randomness in continuous-review inventory models. *Engrg. Costs and Production Econom.* **21** 191–197.

Hillier, F. S., R. W. Boling. 1966. The effect of some design factors on the efficiency of production lines with variable operations times. *J. Industrial Engrg.* **17** 651–658.

Hopp, W. J., M. L. Spearman. 1996. *Factory Physics: Foundations of Manufacturing Management*. Irwin, Chicago, IL.

Hsu, L. Y. 1992. The design of an assembly line with stochastic task times. M. S. Thesis. Sloan School of Management, Massachusetts Institute of Technology, Cambridge, MA.

Lau, H.-S. 1992. On balancing variances of station processing times in unpaced lines. *European J. Oper. Res.* **56** 345–356.

Luss, H., S. K. Gupta. 1975. Allocation of effort resources among competing activities. *Oper. Res.* **23** 360–366.

Muth, E. J. 1979. The reversibility property of production lines. *Management Sci.* **25** 152–158.

Sarkar, D., W. I. Zangwill. 1991. Variance effects in cyclic production systems. *Management Sci.* **37** 444–453.

Schonberger, R. J. 1986. *World Class Manufacturing: The Lessons of Simplicity Applied*. Free Press, New York.

Wilhelm, W. E. 1987. On the normality of operation times in small-lot assembly systems: a technical note. *Internat. J. Prod. Res.* **25** 145–149.

Yano, C. A., R. Rachamadugu. 1991. Sequencing to minimize work overload in assembly lines with product options. *Management Sci.* **37** 572–586.

Zangwill, W. I. 1968. Minimum concave cost flows in certain networks. *Management Sci.* **14** 429–450.

Zipkin, P. H. 1980. Simple ranking methods for allocation of one resource. *Management Sci.* **26** 34–43.