A Note on the Convexity of Service-Level Measures of the (r, q) System

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This note gives a simple proof that in a (r, q) system the average outstanding backorders and the average stockouts per unit time are jointly convex in the two control variables q and r.
(*Convexity; Convex; Inventory; Backorders; Stockouts*)

A well-known method of inventory control is the reorder-point/order-quantity system, or (r, q) system, where an order of constant size q is placed whenever the inventory position (that is, stock on hand plus on order minus backorders) drops to a fixed reorder point r. Zipkin (1986) proves that, when stockouts are backordered, the average outstanding backorders, denoted by $B(q, r)$, is a convex function of the two control variables q and r. He also shows (under Poisson demand and an additional mild assumption) that the average stockouts per unit time, denoted by $A(q, r)$, is also convex in q and r. His proof is based on explicitly expressing $B(q, r)$ and $A(q, r)$ in terms of integrations involving probability distribution function of demand in a lead time, taking second partial derivatives and then checking for the nonnegative definiteness of the Hessian matrix. In this note, we present a much simpler proof that has a strong intuitive appeal.

Let $\mathbf{D}$ and $\mathbf{I}$ be two random variables denoting demand in a lead time and the inventory position, respectively. Then the average outstanding backorders $B(q, r)$ can by expressed as

$$B(q, r) = E_{\mathbf{D},\mathbf{I}}[\max(\mathbf{D} - \mathbf{I}, 0)], \qquad (1)$$

where E is the expectation operator. As has been done in Zipkin, here it is assumed that $\mathbf{I}$ is a random variable uniformly distributed on the interval $(r, r + q]$, and that $\mathbf{D}$ and $\mathbf{I}$ are independent of each other. These conditions are met when cumulative demand is described by a nondecreasing stochastic process with stationary increments and *continuous* sample paths (Zipkin 1986, Serfozo and Stidham 1978). We can substitute $\mathbf{I}$ by $r + q\mathbf{U}$ with $\mathbf{U}$ being uniform on (0, 1] and rewrite $B(q, r)$ as

$$B(q, r) = E_{\mathbf{D},\mathbf{U}}[b(q, r, \mathbf{D}, \mathbf{U})], \qquad (2)$$

where $b(q, r, \mathbf{D}, \mathbf{U}) = \max(\mathbf{D} - r - q\mathbf{U}, 0)$. To prove the convexity of $B(q, r)$, suffice it to show that $b(q, r, \mathbf{D}, \mathbf{U})$ is convex in (q, r) for any fixed values of $(\mathbf{D}, \mathbf{U})$. From the facts that $(\mathbf{D} - r - q\mathbf{U})$ is convex in (q, r) and that $\max(f(\cdot), 0)$ is convex for any convex function $f(\cdot)$, we clearly see that $b(q, r, \mathbf{D}, \mathbf{U})$ is indeed convex in (q, r). Therefore, $B(q, r)$ is a convex function of (q, r). It is worth noting that the convexity of $B(q, r)$ implies convexity of the long-run average holding and backlogging costs in case these costs are proportional with the inventory and backlog size, respectively.

To obtain an expression for $A(q, r)$, the average stockouts per unit time, we make an additional assumption that the demand process is Poisson and we use the Poisson Arrivals See Time Average property (PASTA) to express $A(q, r)$ as

$$A(q, r) = \lambda \Pr\{\mathbf{D} > \mathbf{I}\} = \lambda E_{\mathbf{I}}[1 - H(\mathbf{I})], \qquad (3)$$

where λ is the mean demand rate and $H(\cdot)$ is the probability distribution function of $\mathbf{D}$. But now a difficulty arises: For discrete demands such as Poisson, the assumption made earlier that the inventory position $\mathbf{I}$ is a uniform *continuous* random variable is no longer valid. However, following a long tradition of approximating discrete variables by continuous ones, as in Zipkin, see below, we approximate $\mathbf{I}$ in (3) by a continuous random variable uniformly distributed on $(r, r + q]$. We then

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substitute **I** by $r + q\mathbf{U}$ with **U** being uniform on (0, 1] and rewrite (3) as

$$A(q, r) = \lambda E_{\mathbf{U}}[a(q, r, \mathbf{U})], \tag{4}$$

where $a(q, r, \mathbf{U}) = 1 - H(r + q\mathbf{U})$. We note that the above expression for $A(q, r)$ is identical to the one used in Zipkin.

For $A(q, r)$ in (4) to be convex, it is sufficient to ensure that $a(q, r, \mathbf{U})$ is convex in (q, r) for any fixed value of **U**. To establish the latter condition, we impose a restriction on $H(\cdot)$ on the relevant range of r: Specifically, we assume that $1 - H(t)$ is convex, or equivalently, $H(t)$ is concave, for $t \geq r$. This additional restriction is implied by the assumption made in Zipkin that the probability density function of **D** is nonincreasing for $t \geq r$. It should be mentioned that under Poisson demands, the concavity of $H(\cdot)$ is guaranteed for nonnegative safety stock and fixed leadtimes or stochastic leadtimes that are independent of the number and size of outstanding orders.

References

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