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Asset Prices and Default-Free Term Structure in an Equilibrium Model of Default*

We present an equilibrium production economy in which default occurs in equilibrium. The borrower chooses optimal default and consumption policies, taking into account that default is costly and the lender gains access to the technology upon default. We derive asset prices and default premia in this economy. The borrower's relative risk aversion in wealth increases with decreases in wealth due to the increased possibility of default at low wealth levels. This produces a time-varying pricing kernel and a countercyclical equity premium. We thus provide an equilibrium rationale for the default premium to influence expected asset returns.

I. Introduction

We construct an equilibrium production model of default with two agents in this paper. We lay the groundwork for linking the asset pricing with default and offer a theoretical rationale for default premium to influence asset returns. This property was assumed by Jagannathan and Wang (1996) and other scholars in the asset pricing literature. The general equilibrium production model of Cox, Ingersoll, and Ross (1985) provides our basic frame of reference. The borrower in the economy has exclusive access to the only risky production technology when the economy begins. The borrower is endowed with a limited initial endowment of the only good, which is not storable. The lender has no access to the risky technology on the initial date but is endowed with the good, which he can decide either to lend to the borrower or simply consume. Since the good is not storable, the only way for the lender to consume over time is to lend

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the good to the borrower in exchange for a stream of promised payments. To keep the focus sharply on default, we assume that the borrower offers a debt contract to the lender that promises an eternal, constant flow rate of payments to the lender.[1] The borrower faces a cost associated with default: when the promised payments are not paid, the borrower loses a fraction of his wealth plus a fixed amount to the lender. Moreover, upon default, the lender is able to access the risky technology, permitting him to become a fully utility-maximizing participant in the economy. The lender chooses his optimal strategy at time 0 also by comparing the expected utility associated with lending with the utility of consuming the good today. When the lender optimally decides to lend at time 0, the borrower can augment his endowment and determine optimally his consumption and default policies. This is the equilibrium that we study in the paper. We characterize the feasible loans and an equilibrium in which there is lending with welfare improvements to both lender and borrower. The decision to accept the loan and to default later is endogenous in the model. The borrower chooses the optimal time of default to maximize his expected lifetime utility.

Our approach has some merits and some drawbacks relative to other contributions in the literature. We contribute at a methodological level by computing allocations and prices in an economy where there is endogenous default. We use optimal stopping-time methods to do this. We offer a framework in which asset prices depend on the default premium. We derive an intertemporal capital asset pricing model (ICAPM) to make this relationship explicit. Finally, we exploit the production technology to draw some predictions about how the default-free term structure might be influenced by the probability of default. The drawbacks are the following: we exogenously impose a participation constraint on the lender and we are unable to permit some agents to default whil, at the same time, allowing other agents to be solvent. We also impose a specific debt contract as a means to augment the initial endowment, although we explore later (see fig. 1) under what circumstances borrower prefers debt to using equity. We focus our attention on allocations and prices before default, because under many realistic conditions, the economy operates with a positive probability of default but actually does not experience default for very long periods of time. We believe that this focus is reasonable: we often focus on asset prices of firms before default to examine default premium. This is one of the distinct contributions of our paper.

The subject we study in this paper has been investigated by some scholars. Two theoretical papers (Zhang 1997; Alvarez and Jermann 2000) explored the importance of default risk on asset pricing using models of endogenous solvency constraints. These models draw on the insights of Kocherlakota (1996) and Kehoe and Levine (1993) by incorporating

1. We discuss later the conditions under which equity is used instead.

participation constraints and shed some light on the risk-sharing implications of default risk. But, by construction, these models of solvency constraints eliminate the possibility of default in equilibrium. In our framework, there is default in equilibrium. In models of solvency constraints, no default-risky loan is modeled; hence, the default premium is less direct to compute. Moreover, the determination of a default-free term structure is not addressed in models of solvency constraints. Our approach also differs in a major way in the nature of risk sharing. In the model of Alvarez and Jermann (2000), default leads to autarchy with no risk-sharing possibilities. We accommodate this as a special case but in general permit the risk sharing to continue even after default. A second strand of literature, exemplified by Geanakoplos and Zame (1998), explores default in an endowment economy with two periods, wherein a durable good is used as collateral to borrow and the collateral is seized by lenders upon default. Our paper uses a production economy similar to Cox et al. (1985) and treats the case of a perishable good. Kubler and Schmedders (2001) consider an economy with a perishable good. The productive asset plays the role of collateral. They offer a computational framework to study the equilibrium properties. Zame (1993) offers a framework in which default actually helps to complete the market.[2]

Our approach allows us to shed some light on the following issues and questions:

1. What are the properties of optimal default strategies in an equilibrium model? A key result here is that there is default in equilibrium in our model. This is in sharp contrast to the results in the existing literature on equilibrium models with solvency constraints, which we review later. In addition, the optimal default boundary depends on the costs associated with default and the lender's status after default. If the lender participates in the economy as a maximizing agent after default, we show that the risk-sharing possibilities effectively reduce the cost of default and hence leads to a higher optimal default boundary. If the lender and borrower have identical preferences, default effectively leads to autarky. For most part, we focus on this latter case as it is closer to much of the equilibrium literature on defaults.
2. What is the effect of costs of default (transfer payments to the lender) on the risk aversion of the borrower? An important result in this context is that the borrower becomes much more risk averse as his wealth level drops and approaches the optimal default boundary. He reduces the optimal flow rate of consumption to reduce the likelihood of default. Eventually, when the wealth drops further and is very close to the optimal default boundary, the borrower becomes much less risk averse and starts to dissipate his wealth by increasing his optimal flow rate of consumption.

2. We thank the referee for bringing this paper to our attention.

The presence of these two regions is shown to be robust whether the borrower returns to autarky after default or is allowed to share his risk with the lender after default.

3. How does the presence of default risk affect the default-free term structure? We show that the presence of default risk induces an extra value to the risk-free asset in the economy. The borrower's increasing risk aversion as the default boundary is approached leads to a lower shadow risk-free rate. This effect is mitigated by the possibility of risk sharing with the lender after default. The term structure of default-free interest rates becomes steeper in the presence of default. In the second region the risk-free rate increases and the term structure becomes inverted. Many recent papers documented that the Treasury rates reflect a flight to a high-quality premium and argued that perhaps collateralized rates, such as the repossession rates or swap rates, are better proxies for risk-free rates at different maturity sectors. To our knowledge, this is the first paper to formally demonstrate this effect.
4. What is the relationship between default premium and asset returns? We derive a capital asset pricing model (CAPM) with default risk and show that equity premium depends on two factors: (1) the covariance of consumption with wealth, which is the standard prediction; and (2) the covariance of consumption to the household indebtedness. We characterize these factors and show two key results. First, the presence of default risk generally increases the equity premium in the economy. Second, there is a positive association between equity risk premium and default premium.

The paper is organized as follows: The next section develops the basic equilibrium model of default and motivates its construction. Section III characterizes the properties of an optimal debt contract, borrower's risk aversion in wealth, optimal consumption policies, and the default-free term structure. We also characterize the feasible regions of lending. In Section IV, we develop the CAPM with endogenous default risk. We simulate the model to present some implications for asset pricing. Here, we show how default risk influences the equity premium. We provide a theoretical rationale for the assumption that the equity premium depends on default risk. This assumption is used, for example, by Jagannathan and Wang (1996). Section V concludes. The appendix collects all our technical results.

II. The Model and the Nature of Default

We consider a production economy setting with two agents: a borrower and a lender. In the next subsection, we describe the technology and the economic environment. The subsequent sections describe the optimization problem and our notion of equilibrium in this economy.

A. Economic Environment

The economic environment consists of production technology, preferences, punishment mechanisms associated with default, and the manner in which equilibrium prevails in the economy. We proceed to describe each in turn.

1. Production Technology and its Access

There is a single good in the economy, and it serves as the numeraire. The production sector has a risky technology. Once an amount q_t of the good is invested in the technology at time *t,* the output evolves as follows:

$$\frac{dq_t}{q_t} = \mu dt + \sigma dz_t \tag{1}$$

where the instantaneous expected rate of return μ and the diffusion coefficient σ are exogenous positive constants.[3] The process z_t is a standard Brownian motion on the underlying probability space (Ω, Φ, Π).

Preferences and endowments. The risk-averse borrower (consumer) is endowed with an initial wealth of x_0 and has exclusive access to the risky production technology. He maximizes his lifetime discounted expected utility of consumption: $E_0[\int_0^\infty e^{-\rho t}u(c_t)dt]$, where u is his von Neumann-Morgenstern utility function and ρ is his time preference rate. In this paper, we examine a special class of utility functions whose relative risk aversion in consumption is a positive constant:[4]

$$u(c) = \frac{1}{1-A}c^{1-A}, \quad A > 0, A \neq 1 \tag{2}$$

This specification has been widely used in the theory of intertemporal consumption-portfolio selection problems, default-free term structure theory, and asset pricing.

The second agent in our model is the lender. He is restricted from participating in the production technology at time 0. He arrives at time 0 with an initial endowment. The good is not storable. Hence, the only way for the lender to consume over time is to lend to the borrower in exchange for a stream of promised future payments.

The loan contract and punishment mechanism. In our model, we assume a specific loan contract $\{\bar{C}; \alpha, K; I_0\}$ which is described next.

3. We restrict attention to a constant opportunity set to get tractable results. The generalization to a stochastic opportunity set introduces significant computational complexity.

4. We want to point out that all our major results still hold for a general von Neumann-Morgenstern utility function u, which is a strictly increasing, strictly concave $C^3(0, +\infty)$ function with $\lim_{c\to 0}u'(c) = +\infty$ and $\lim_{c\to +\infty}u'(c) = 0$ and satisfies the condition $|u(c)| \leq M(1+c)^\gamma$ for some positive constants M and γ.

The borrower can borrow an amount I_0 from the lender at time 0.[5] But this requires him to pay the lender a flow rate of $\bar{C}$ per unit time until default.[6] If the borrower decides to default at time $\tau \geq 0$, then he loses a fraction $(1 - \alpha)$ of his wealth plus a lump sum of K and the exclusive access to the risky technology.[7] The lender, in exchange for lending I_0 at time 0, derives utility by consuming the contractual debt payments $\bar{C}$ per unit time until default. During this period, the lender is outside the economy due to the participation constraint: he can neither invest in the risky technology nor trade with the borrower. Upon default, the lender collects the amount of $(1 - \alpha)W + K$ from the borrower and becomes a maximizing agent in the economy. He then has full access to the risky technology and active risk-free lending and borrowing takes place after default. There are no deadweight losses in our economy. One should note that the parameters $\{\alpha, K\}$ reflect the sharing rule of wealth between lender and borrower upon default, which is governed by the relevant bankruptcy codes applicable in the economy and the relative bargaining positions of the lender and the borrower. We have not explicitly modeled the trade-offs between equity and debt contract in this model. We briefly address this issue later to show that, under some circumstances, debt may not be the optimal contract.

B. *Optimization Problem and Equilibrium*

We now describe the equilibrium in this economy. The equilibrium after default is a standard two-person dynamic equilibrium in a production economy (similar to the one studied by Dumas 1989). Although the lender remains outside of the economy until default, he can still significantly influence the equilibrium before default through his participation in the economy after default. By letting the lender and borrower be identical, we can reduce the problem after default to autarky, which is the standard assumption imposed by Alvarez and Jermann (2000).

Given the loan contract $\{\bar{C}; \alpha, K\}$, the controls of the borrower are the amount q_t invested in the risky technology, the consumption rate c_t, and the optimal default level W^*. We define the $\{\Phi_t\}$-stopping time: $\tau = \inf\{t \geq 0 | W_t \leq W^*\}$. The wealth dynamics facing the borrower can be formally represented as

$$dW_t = [r_t(W_t - q_t) - c_t - \bar{C}]dt + \mu q_t dt + \sigma q_t dz_t \quad \text{for } 0 \leq t < \tau \quad (3)$$

5. We do not consider dynamic borrowing opportunities. This implies that the borrower has no "reputational costs" associated with default. We show that the consumer reduces the rate of consumption in poor states of the economy to stave off default. This may be interpreted as a "dynamic borrowing" action. We thank Patrick Bolton for pointing out this interpretation.

6. Given a value for $\bar{C}$, we determine I_0 endogenously. Equivalently, we can take I_0 exogenously and find the coupon level $\bar{C}$ endogenously.

7. In this sense, the risky production technology effectively serves as collateral to borrow money from the lender. This type of modeling has been used in the context of credit cycles by Kiyotaki and Moore (1997) and Krishnamurthy (1998).

where r_t is the default free interest rate. Let us denote the set of admissible controls by $A(W_0)$. The objective function facing the borrower is the expected lifetime discounted utility maximization. Formally, the borrower maximizes the value function J, which is defined as the supremum of the expected utility over the set of admissible controls:

$$J(W_0) = \sup_{A(W_0)} E_0\left[\int_0^\infty e^{-\rho t} u(c_t)dt\right] \tag{4}$$

Let us denote the optimal policy to be (c^*, q^*, W^*). The equilibrium in this economy is defined as follows.

Definition 1. An equilibrium $\{(r, c^*, q^*), I_0^*\}$ is a set of stochastic processes (r, c^*, q^*) and an initial borrowing amount I_0^* that satisfy:

(i) Market clearing condition, $q_t^* = W_t$.
(ii) Borrower's loan valuation condition,

$$I(x_0 + I_0^*) = I_0^*, \tag{5}$$

where $I(\cdot)$ is the borrower's valuation function for the loan.
(iii) Feasibility for the borrower,

$$J(x_0 + I_0^*) > J_0(x_0), \tag{6}$$

where $J_0(\cdot)$ is the borrower's valuation function in autarky.[8]
(iv) Feasibility for the lender,

$$J^L > J_0^L \tag{7}$$

where J^L is the lender's expected utility at time 0 by lending and J_0^L is the corresponding expected utility when he simply consumes the endowment.

The market clearing condition (i) implies that there is no risk-free lending or borrowing at equilibrium, as in Cox et al. (1985). The borrower's loan valuation condition (ii) is a fixed-point requirement, which says that the equilibrium level of borrowing must be such that the amount borrowed I_0^* is equal to the borrower's valuation of the loan contract at time 0. The borrower's feasibility condition (iii) states that the borrower is willing to borrow from the lender only if the loan is sufficiently attractive to him at time 0; that is, his lifetime expected utility by borrowing is

8. For the setup we have chosen (i.e., a constant opportunity set and a power utility function), Merton (1971) showed the following results for autarchy: if $k = 1/A[\rho - (1-A)(\mu - A\sigma^2/2)] > 0$, then the value function $J_0(W)$ is given by $J_0(W) = [k^{-A}/(1-A)]W^{1-A}$.

larger than the lifetime expected utility in autarky. For example, when the coupon rate $\bar{C}$ is very unfavorable to the borrower ($\bar{C}$ is very large relative to his initial endowment x_0) or the recovery rate to the lender is very favorable, then it is likely that the borrower will choose not to borrow from the lender. The lender's feasibility condition (iv) requires that the lender also has to be better off by providing such a loan contract. Otherwise, the lender simply consumes his initial endowment at time 0. For a risk-averse lender, condition (iv) is trivially satisfied, because lending is the only way for him to smooth consumption over time.

Critical to the characterization of our equilibrium with default is the optimal default boundary W^*. We show in the appendix that the value function satisfies the following properties:

(i) $J(\cdot)$ is strictly increasing and strictly concave.
(ii) $J(\cdot)$ is continuous on $[W^*, \infty)$ with $J(W^*) = J_B(\alpha W^* - K)$, where $J_B(\cdot)$ is the borrower's valuation function after default.
(iii) (Smooth pasting condition) $\lim_{W \to W^{*}+} J'(W) = [\partial J_B(\alpha W^* - K)]/\partial W^*$.
(iv) (Dynamic programming principle)

$$J(W_0) = \sup_{\mathcal{A}(W_0)} E_0\left[\int_0^{\tau} e^{-\rho t} u(c_t)dt + e^{-\rho\tau} J_B(\alpha W^* - K)\right]. \quad (8)$$

We also show in the appendix that, for any $t < \tau$, the value function J(·) is the unique $C^2(W^*, +\infty)$ solution of the Bellman equation:

$$\rho J = \tfrac{1}{2}\sigma^2 W^2 J_{WW} + (\mu W - \bar{C})J_W + \max_{c \geq 0}[u(c) - cJ_W] \quad (W > W^*) \quad (9)$$

with boundary condition $J(W^*) = J_B(\alpha W^* - K)$ and $\lim_{W \to W^{*}+} J'(W) = [\partial J_B(\alpha W^* - K)]/\partial W^*$. And the optimal policy c_t^* is given by

$$c^*(W) = (u')^{-1}[J_W(W)]. \quad (10)$$

The approach to solve this problem is by backward induction. We first solve the two-person general equilibrium after default. An example of such a two-person general equilibrium is the one studied by Dumas (1989), in which the borrower and lender are both risk averse but different (as in Dumas 1989, one agent has a power utility function and the other has a log utility). The wealth-sharing rule is obtained by maximizing the welfare function (which is a weighted sum of the utilities of the two agents), as in Dumas (1989). Since we know at default the share of wealth of borrower and lender, we can precisely compute the constant weight λ^* (used in the welfare function) for a given default boundary.

Using this weight we determine the value function $J_B(\alpha W^* - K)$ of the borrower upon default. The value function of the borrower reflects the risk-sharing possibilities after default. In particular, the risk aversion of lender influences the value function of the borrower. We then use this value function of the borrower as the boundary condition to search for the optimal default boundary of the borrower as explained in the appendix.[9] We designed and implemented a finite-difference scheme to numerically solve such a free-boundary problem. The formal analysis that leads to the determination of the optimal default boundary is presented in the appendix. There, we also present and discuss the technical results that characterize the properties of the equilibrium. These results show that the economy which we study has a well-defined equilibrium and the value function and its derivatives converge to their counterparts in a general equilibrium model with identical consumers. We also present, in detail in the appendix, the numerical procedure we use in the paper to compute the equilibrium.

We now proceed directly to illustrate our numerical results in the next few sections. In view of the computational complexity, we focus simply on a baseline case where the borrower and the lender are identical. For an active risk-sharing lender who is different from the borrower, the results are qualitatively similar to the baseline case.[10]

III. Optimal Default, Time-Varying Risk Aversion, and Term Structure

We examine a baseline case where we set the borrower's subjective discount factor $\rho = 0.05$ and the risk aversion parameter $A = 2.0$. We also assume for the baseline case that the lender is identical to the borrower. At time 0, the borrower is endowed with 1 unit of consumption good: $x_0 = 1.0$. For the risky technology, we assume the instantaneous expected rate of return $\mu = 0.10$ and the diffusion coefficient $\sigma^2 = 0.02$. We further assume that the sharing rule between borrower and lender upon default is $\alpha = 0.25$ and $K = 0.05$. In the following context, we first describe the optimal coupon rate for the borrower under the baseline setting. Given that such an optimal contract is sustainable by both borrower and lender,

9. It should be emphasized that the computational burden associated with solving this problem by backward induction is nontrivial: we have to solve the two-person general equilibrium model after default for every wealth level to determine the optimal default boundary.

10. When the lender's utility is log, and thus different from the borrower's, the risk-aversion results are somewhat muted. In general, for a lender who is different from the borrower, active trading takes place between the borrower and the lender after default. This leads to welfare gains to both the lender and the borrower. As a consequence, the borrower's effective cost of default is reduced and his optimal policy before default is different. One would expect that, for a different lender, the borrower's relative risk aversion before default becomes lower and the default boundary W^* becomes higher.

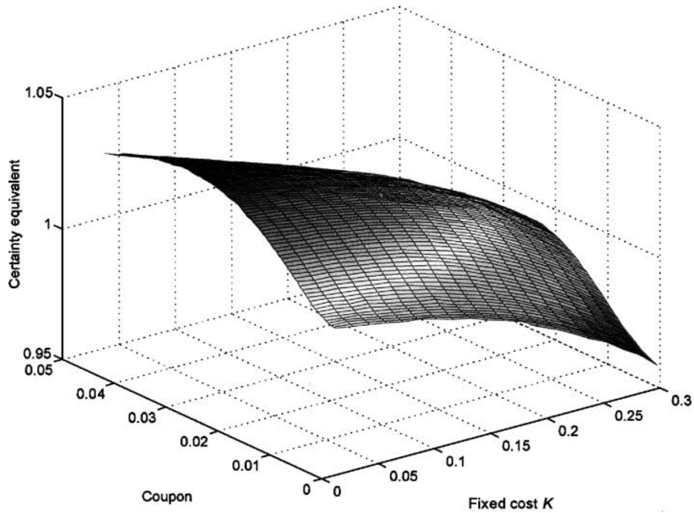

FIG. 1.

we next characterize the property of the equilibrium for our model under such a contract.

A. Lending and Risk Sharing

We first characterize the feasibility of a certain loan contract for the borrower. Note that the borrower is willing to take on the loan only if his lifetime expected utility by borrowing is larger than the lifetime expected utility in autarky (condition [iii] of the equilibrium). To examine how much utility he can gain by taking on the loan, we define the relative certainty equivalence as

$$\mathrm{CE}(x_0) = \frac{J_0^{-1}[J(x_0 + I^*)]}{x_0}, \tag{11}$$

which measures the normalized utility change for the borrower with an initial wealth level x_0. Relying on the concavity of the value functions J and J_0, the relative certainty equivalence CE$(\cdot)$ is a well-defined continuous function. A borrower is willing to accept a loan contract if and only if his relative certainty equivalence CE > 1.

Figure 1 shows the borrower's relative certainty equivalence for different coupon rate $\bar{C}$ and lump sum cost K. For our baseline setting $K = 0.05$, the relative certainty equivalence reaches the maximum level when coupon rate $\bar{C} = 0.027$. Typically, the relative certainty equivalence is smaller for a borrower with a higher lump sum cost K. When the lump sum cost is very high (e.g., $K > 0.26$), the maximum level of certainty equivalence is smaller than 1. In this situation, the borrower prefers to

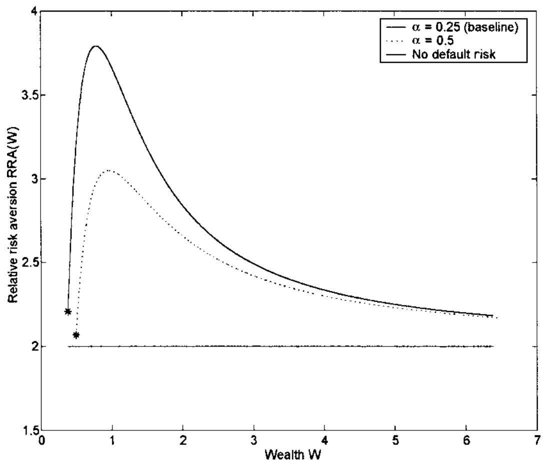

FIG. 2.

remaining autarky and does not borrow from the lender. Equity is the preferred mechanism for any scale expansion under these circumstances. In these situations, the two agents can become equity holders in the expanded production opportunity set.

The relationship between the proportional cost $1 - \alpha$ and the relative certainty equivalence CE is similar to the relationship between the lump sum cost K and the relative certainty equivalence. For the sake of brevity, this result is not presented in the paper.

B. *Relative Risk Aversion in Wealth*

In this section, we characterize the behavior of the indirect value function. In particular, we plot in figure 2, the relative risk aversion in wealth (RRA) of the borrower as measured using his value function in our economy. Note that the relative risk aversion in wealth for the general equilibrium economy with no default under our hypothesized assumptions is simply a constant given by $A = 2.0$. In figure 2, we plot the wealth along the x-axis and the relative risk aversion in wealth along the y-axis. We find that the relative risk aversion increases in a significant manner as the wealth level drops from ∞ to $\hat{W}_{\mathrm{RRAmax}} = 0.782$, where the relative risk aversion in wealth reaches its peak $\mathrm{RRA}_{\max} = 3.8$. A further decrease of wealth leads to a reduction in RRA until it reaches the default boundary $W^* = 0.381$. Thus, there are two regions in this economy. In one region, the relative risk aversion increases with decreases in wealth. As the wealth drops, the probability of default increases and the borrower becomes more risk averse in this region. We call this region "flight to quality." This is a metaphor for the borrower's implicit preference for less risky assets and his aversion for the more risky assets. We show later that, in the flight-to-quality region, the borrower's shadow risk-free rates falls with decreases in wealth. The second region where the borrower's relative risk aversion decreases with decreases in wealth is a manifestation of the overinvestment distortions in our economy. In this region, he has an implicit preference for risky asset. We show later that, in this region, the borrower increases his rate of consumption and thus dissipates the collateral. Hence, we call this region the "collateral-dissipation" region. Under this set of parameters, the initial augmented wealth for the borrower is $W_0 = 1.557$. The corresponding relative risk aversion coefficient at time 0 is $\mathrm{RRA}_0 = 3.14$. The relative importance of these two regions depends on the magnitude of the lump sum costs K. This is discussed further later in the paper.

In figure 2, we also plot the effect of the recovery rate parameter α on relative risk aversion. Note that, upon default, the borrower keeps a fraction α of his wealth. Naturally, as α increases, the recovery rate on the loan falls. We found that, when the recovery rate increases (i.e., α decreases), two effects occur. First, the optimal default boundary decreases; the borrower is more careful about defaulting the loan, which implies that the optimal default boundary W^* is decreasing as α decreases. Simultaneously, the borrower becomes more risk averse and the relative risk aversion RRA increases.

Time-varying risk aversion plays an important role in the asset pricing literature. Campbell and Cochrane (1999) show that models of habit formation which produce time-varying risk aversion can help explain aggregate stock market behavior. We show that time-varying risk aversion may also arise due to the presence of default. Our model implies that pronounced increases in risk aversion may result in economies where risk-sharing possibilities after default are limited and the costs of default are high.

C. Equilibrium Default-Free Interest Rate

The instantaneous default-free instantaneous interest rate in our model is given by $r(W) = \mu - WJ_{WW\sigma^2}/J_W$. In figure 3, we plot the instantaneous risk-free rate as a function of wealth.

Default risk has two striking effects on the equilibrium risk-free rate. First, when the borrower is in the flight-to-quality region, the equilibrium risk-free rate is always below the one given by the default-free economy, which equals a constant: $R = \mu - A\sigma^2$. At wealth levels close to $\hat{W}_{r_{\max}} = \hat{W}_{\mathrm{RRA}_{\max}} = 0.782$, the equilibrium interest rate is well below the level implied in an economy with no default risk. In the illustration in figure 3, the maximum difference is about 358 basis points at a wealth level $\hat{W}_{r_{\max}} = 0.782$. Note that the presence of default risk has important pricing consequences for the default-free interest rates in this flight-to-quality region. These rates display a cyclical behavior: when the economy's wealth decreases, the real risk-free rates go down; and when the economy's wealth increases, the real risk-free rates increase. A further decrease in wealth leads the economy to the region of collateral dissipation and the

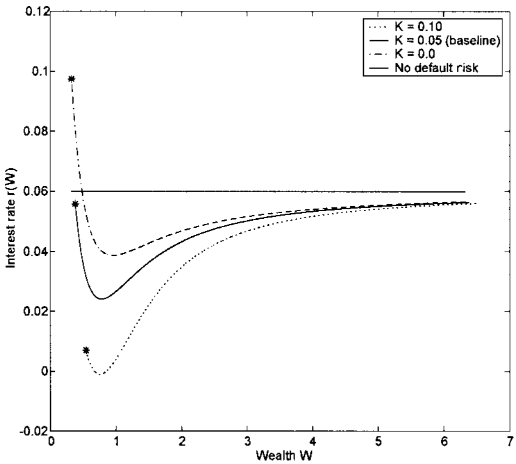

FIG. 3.

interest rate begins to rise as the wealth decreases. The region of collateral dissipation depends on K. For very high wealth levels, that is, as $W \to \infty$, the risk-free rate approaches the level given by the model with no default risk.

In figure 3, we also plot the effect of the lump sum cost K on equilibrium default-free interest rates. As K increases, the interest rates fall and the region where collateral is dissipated becomes smaller. Thus, we find that the overinvestment distortions are mitigated by the lump sum costs of default as opposed to the proportional costs of default. The intuition for this is the following: with proportional costs, the borrower loses more when the wealth level at which he defaults is high. So he has an incentive to consume more when default is imminent. This way he leaves less collateral to the lender. With a lump sum cost of default, this incentive is sharply curbed.

D. Default-Free Term Structure

We now characterize the default-free term structure in this economy.[11] In a standard equilibrium setting, the term structure is flat, as the yield to maturity for a zero-coupon bond $R(t,T) = -[\ln P(t,T)]/(T-t)$ is simply a constant equal to the instantaneous interest rate $\mu - A\sigma^2$. However, in our model, the shape of the term structure is wealth dependent and exhibits a rich pattern, as shown in figure 4.

11. Let us denote $P(t, T)$ as the price at time t for a zero coupon bond that pays 1 unit consumption good at time T. $P(t, T)$ satisfies the following partial differential equation (PDE) with boundary conditions $P(W^*, \tau) = e^{-R(T-\tau)}, P(W, T) = 1$:

$$-r(W)P + [r(W)W - c^*(W) - \bar{C}]P_W + \tfrac{1}{2}\sigma^2 W^2 P_{WW} + P_t = 0. \qquad (12)$$

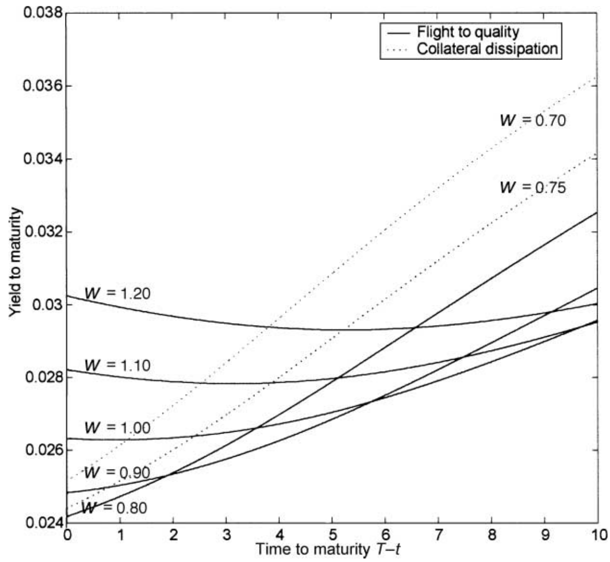

FIG. 4.

In the flight-to-quality region, the default-free term structure becomes more expensive as the wealth goes down and the curve gets steeper. The impact of the default risk on default-free term structure is quite subtle: it arises from the implied relative risk-aversion in wealth and the optimal default boundary. We find that our model with a constant opportunity set is able to deliver fairly rich shapes of default-free term structure once default risk is admitted. The default risk (which is priced as a factor through the new equilibrium default-free instantaneous rate) permeates through the pricing of the yield curve.

Recently in the United States economy, several practitioners have attributed the fall in the Treasury interest rates and the increase in the slope of the Treasury term structure to the increased risk aversion concerning the potential for costly defaults in the telecommunications sector of the economy. Our model's implications are certainly consistent with this view.

E. Optimal Consumption

In the absence of default risk, the general equilibrium model implies that the optimal consumption is given by kW. Moreover, the elasticity of optimal consumption with respect to wealth is a constant. In figure 5a, we plot the normalized optimal consumption $C(W)/kW$; and in figure 5b, the normalized elasticity of optimal consumption with respect to wealth. It is useful to note that the consumption elasticity in our model is the ratio of the RRA in wealth to the RRA in consumption. Since the RRA in

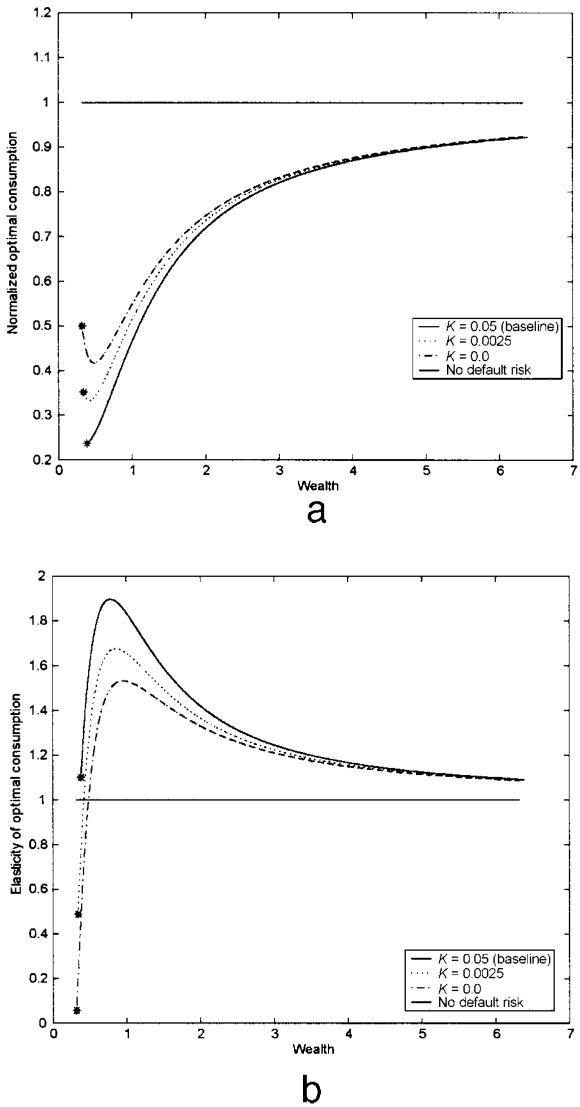

FIG. 5.—(a) Normalized optimal consumption. (b) Normalized elasticity of optimal consumption with respect to wealth.

wealth has already been characterized in figure 2, it is easy to interpret figure 5.

In our model, the optimal consumption rate and the elasticity of consumption depend on how close the economy is to defaulting. In particular, they depend on in which region the wealth level falls. As wealth approaches infinity, our economy approaches that of an equilibrium model with no default. In this region, the consumption and elasticity is close to the baseline case. In the flight-to-quality region, as the wealth decreases, the rate of consumption falls and the wealth elasticity of consumption rises. The elasticity is generally higher than what is implied by the base case for moderate to high levels of wealth. Our prediction is that the wealth elasticity of consumption increases as the economy approaches the default boundary. This is when the economy behaves with greater caution to avoid default. Our results in this context are in conformity with the evidence reported by Olney (1999) concerning the consumption data in the United States in the Great Depression. Olney (1999) reports that, prior to the Great Depression, the bankruptcy code favored the sellers of consumer durables on installment credit to the households. She argues that the households tried to avoid default by curtailing their consumption, which in turn precipitated the depression. Our model predicts that the consumption elasticity is higher at lower wealth levels when the default probability is high. Note that, when the lump sum costs are lower, the collateral-dissipation region or the overinvestment region increases.[12]

IV. Equity Premium

In this section, we discuss the equity premium in the economy. The possibility that the equity premium may be related to default risk has been recognized by many scholars in empirical asset pricing. Papers by Chen (1991), Fama and French (1989), Keim and Stambaugh (1986), and Ferson and Campbell (1991) show that the market risk premium is time varying and varies over the business cycle. Stock and Watson (1989) and Bernanke (1990) stress the superior ability of proxies of default premium to forecast business cycles. Jagannathan and Wang (1996) assumed that the conditional equity risk premium is a linear function of the default premium in the economy. Their conditional CAPM (which also takes into account the returns from human capital) can explain the cross-sectional variations in equity returns more successfully. Empirical evidence also suggests that default risk proxies, such as the junk bond spreads over default-free security yields, are useful in explaining the returns on stocks and default-free bonds. Chen, Roll, and Ross (1986) present evidence that the spread on high-yield bonds explains the returns on stocks. All these papers suggest that the existence of default risk may affect the equity returns.

The value of equity in our economy is the wealth net of the market value of borrowing at any time. We denote this by $E(W) = W - I(W)$, which can be viewed as a contingent claim with continuous payout c^*.

12. We note that, when the lender has a logarithmic utility and thus is distinct from the borrower, the borrower's consumption policy is less conservative due to the possibility of more risk-sharing after default.

The risk premium in the underlying Cox et al. (1985) setting is simply a constant: $\mu - A\sigma^2$. However, it is wealth dependent in our model. The presence of default risk in our model has a strong effect on the equity risk premium. As the probability of default begins to increase, the agent becomes more risk averse and consumes less to avoid the costliness of default. Such a behavior causes the equity risk premium to be systematically higher in our model when the wealth level in the region of flight to quality.

Since $I(\cdot) \in C^2(W^*, \infty)$,we apply the Ito's lemma to $E(W)$ and write the stochastic differential equation governing the movement of E(·) as

$$\frac{dE}{E} = (\mu_E - c^*)dt + V_E dz_t, \tag{13}$$

where the instantaneous rate of return on equity is given by $\mu_E = [(1-I_W)(\mu W - \bar{C}) + I_W c^* - \tfrac{1}{2} I_{WW}\sigma^2 W^2]/E$. Following theorem 2 of Cox et al. (1985), we state the following lemma without proof.

LEMMA 1. The instantaneous risk premium $\mu_E(W) - r(W)$satisfies a version of CAPM:

$$\mu_E - r = \frac{1}{E}\left[\frac{-u_{cc}(c^*)}{u_c(c^*)}\right](\mathrm{COV}c^*, W) - \frac{1}{E}\left[\frac{-u_{cc}(c^*)}{u_c(c^*)}\right](\mathrm{COV}c^*, I), \tag{14}$$

where $(\mathrm{COV}c^*, W)$ denotes the instantaneous covariance between optimal consumption and wealth.

The CAPM says that the equity premium depends on the covariance of consumption with wealth and the covariance of consumption with risky debt value. If the latter covariance is negative, the equity premium is higher, ceteris paribus.

We investigate the implications of our ICAPM in two ways: first, we explore how the two covariances influence the equity premium. This is reported in figure 6.

Note that the second covariance term, which captures the covariance of consumption with the household debt is never positive (after incorporating the negative sign). In the limit, when wealth increases to infinity, this covariance term vanishes. This covariance becomes more negative as the wealth goes down. On the other hand, since consumption is influenced by default, the first covariance term actually increases more than the decrease in the second covariance term as the wealth goes down, thereby causing a net increase in equity premium. This result is stable for a number of parameter configurations in the flight-to-quality region. In figure 7, we relate the default premium to the equity premium as both are simultaneously set in our economy.

Note, in figure 7, that as wealth increases, both the default premium and the equity premium decline in our economy, although the default

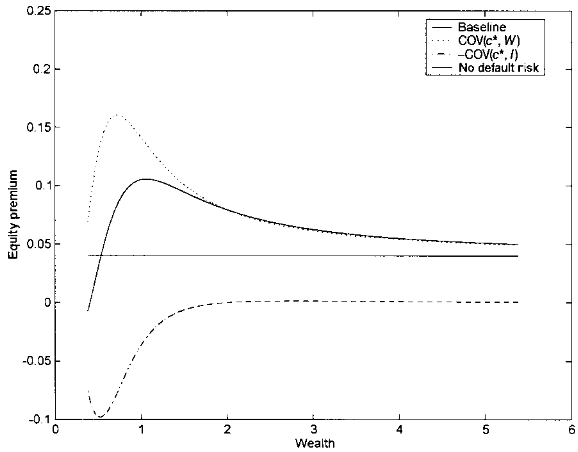

FIG. 6.

premium declines much more rapidly than the equity premium. As the wealth declines, the premia rise slowly at first, then much more sharply. We thus provide a framework that accounts for the comovement of default and equity premia.[13]

In a paper, Lettau and Ludvigson (2001) present evidence that allowing for time variation in risk premia may be essential to the success of conditional consumption CAPM. The source of such variations may come from such factors as habit formation, labor earnings, or as in this paper, default risk. All these approaches deliver a variation in risk aversion that is countercyclical: the risk aversion is high in recession and low in

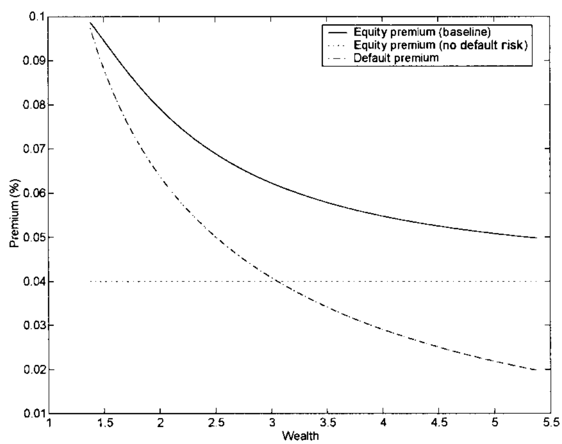

FIG. 7.

13. In the collateral-dissipation region, the effects are different for the reasons discussed earlier.

booms. Thus, we have three competing alternative drivers to the time variation in risk premia. Future empirical work can test to what extent these drivers are useful in understanding the time variation in equity premia.

V. Conclusion

We presented an equilibrium production model of default. This model extends the general equilibrium production model of Cox et al. (1985) to a case where there are two agents and presents an equilibrium in which default occurs with a positive probability. The model allows one to determine endogenously the optimal default boundary, optimal consumption, risk-free term structure, and the default premium. A key implication of our model is that the risk aversion in wealth of the borrower displays time variation through endogenous wealth dependency. Our model predicts that there are two (endogenously determined) regions. In one region, the risk aversion increases with decreases in wealth. In the other region, the risk aversion decreases with decreases in wealth. The model permits the borrower to be a lifetime expected utility maximizer. The lender is initially subjected to a participation constraint, which is removed upon default by the borrower, when he becomes a utility-maximizing, a risk-sharing player in the economy.

The model can be extended in many ways. We chose to model the lender through a participation constraint before default. Alternatively, the lender could participate in the economy throughout the time period as a utility maximizer with access to either the risky or risk-free asset or both. This extension takes the model closer to a truly general equilibrium analysis with default. We also focused on the simple case of static borrowing. The case with dynamic borrowing opportunities is a natural extension to our framework.

Appendix

Characterization of the Equilibrium

To characterize this competitive economy, we first look at the planning problem with the same physical production opportunities but no default-free borrowing and lending. In this situation, the wealth process before default is as follows:

$$dW_t = [\mu W_t - c_t - \bar{C}]dt + \sigma W_t dz_t \quad \text{for } 0 \leq t < \tau \tag{A1}$$

and the central planner seeks to maximize the corresponding value function $\tilde{J}$:

$$\tilde{J}(W_0) = \sup_{\tilde{A}(W_0)} E_0\left[\int_0^\infty e^{-\rho t}u(c_t)dt\right] \tag{A2}$$

where $\tilde{A}(W_0)$ is the corresponding set of admissible controls.

It is evident that, if $J = \tilde{J}$ and $r = \rho(L\tilde{J}_W/\tilde{J}_W)$, then the solution to the original competitive equilibrium are exactly equivalent to this simple planning problem.[14] So, in the following context, we characterize the planner's dynamic programming problem (A2). For notational simplicity, we do not distinguish the variables in the planning economy and the competitive economy in the following context.

LEMMA 2.

(i) $J(\cdot)$ is strictly increasing and strictly concave.

(ii) $J(\cdot)$ is continuous on $[W^*, \infty)$ with $J(W^*) = J_B(\alpha W^* - K)$, where $J_B(\cdot)$ is the borrower's valuation function after default.

(iii) (Smooth pasting condition) $\lim_{W \to W^{*}+} J'(W) = [\partial J_B(\alpha W^* - K)]/\partial W^*$.

(iv) (Dynamic programming principle)

$$J(W_0) = \sup_{A(W_0)} E_0\left[\int_0^{\tau} e^{-\rho t} u(c_t)dt + e^{-\rho\tau} J_B(\alpha W^* - K)\right] \tag{A3}$$

Proof. Principles (i) and (ii) follow from Zariphopoulou (1994) proposition 2.1. For (iii), see Dumas (1991), who provides an extensive discussion of "smooth pasting" or "super contact" conditions. The dynamic programming principle (iv) is presented with proof in Fleming and Soner (1993). Q.E.D.

The next lemma is a key result, which we use to characterize the value function and the optimal default boundary.

LEMMA 3. For any $t < \tau$, the value function J$(\cdot)$is the unique $C^2(W^*, +\infty)$ solution of the Bellman equation:

$$\rho J = \tfrac{1}{2}\sigma^2 W^2 J_{WW} + (\mu W - \bar{C})J_W + \max_{c \geq 0}[u(c) - cJ_W] \quad (W > W^*) \tag{A4}$$

with boundary condition $J(W^*) = J_B(\alpha W^* - K)$ and $\lim_{W \to W^*{}_+} J'(W) = [\partial J_B(\alpha W^* - K)]/\partial W^*$. And the optimal policy c_t^* is given by

$$c^*(W) = (u')^{-1}[J_W(W)] \tag{A5}$$

Proof. Equation (A4) is uniformly elliptic and hence has a unique smooth $C^2(W^*, +\infty)$ solution (see Krylov 1987).[15] Applying the verification theorem (Fleming and Rishel 1975) leads to our lemma. Q.E.D.

Unlike the standard Cox et al. (1985) single-agent economy, there is no closed form solution to the Hamilton-Jacobi-Bellman (HJB) equation (A4). This is due to the presence of borrowing and lending, in particular, the nonhomogeneous term $\bar{C}J_W$ in equation (A4). We designed and implemented a finite-difference scheme to numerically solve such a free-boundary problem. Since the value function is $C^2(\hat{W}, +\infty)$ smooth, the convergence of our numerical scheme directly follows the consistency and stability of the theory of finite-difference schemes (see Strikerda 1989). A description of our procedure is outlined in the next section.

14. Here, we assume the existence of an interior equilibrium. The statement follows theorem 1 of Cox et al. (1985).

15. A one-dimensional differential equation is said to be uniformly elliptic if the coefficient of the second-order derivative A_{22} satisfies $0 < \alpha_1([a, b]) \leq a_{22} \leq \alpha_2([a, b])$ for any interval $[a, b]$ where α_1 and α_2 are two constants depending only on $[a, b]$.

Although we cannot get an explicit solution to (A4), intuition suggests that the economy with active lending will converge to a standard Cox et al. (1985) single-agent economy when W is very large. We formally state the following limiting results under a special case, when the borrower is identical to the lender. Under this situation, the borrower's valuation function after default J_B simply is his valuation function in autarky J_0.

PROPOSITION 4.

(i) The $\lim_{W\to\infty} J(W) = J_0(W) = [k^{-A}/(1-A)]W^{1-A}$.

(ii) The $\lim_{W\to\infty} J_W(W) = (1-A)[k^{-A}/(1-A)]W^{-A}$.

(iii) The $\lim_{W\to\infty} J_{WW}(W) = (1-A)(-A)[K^{-A}/(1-\mathrm{i})]W^{-A-1}$.

Proof. In (i), we first observe that $J(W) \le J_0(W)$, hence $\lim_{W\to\infty} J(W) \le J_0(W)$. So, we need to show that $\lim_{W\to\infty} J(W) \ge J_0(W)$. Since W^* is optimally chosen by the borrower, we have $\mathrm{J}(W) \ge \tilde{J}(W)$, where $\tilde{J}(W)$ is the solution to HJB equation (A4) with $W^* = 0$. So, it is sufficient to prove that $\lim_{W\to\infty} \tilde{J}(W) = J_0(W)$. Let $\upsilon^{(\varphi)}(W) = \varphi^{1-A}\tilde{J}(W/\varphi)$, for all $W \ge 0, \varphi > 0$. Then, we have $\lim_{W\to 0} \upsilon^{(\varphi)}(W) = u(0)/\rho$ uniformly in φ. Moreover, we can see that $\upsilon^{(\varphi)}$ is the unique $C^2(0,+\infty)$ solution, which satisfies

$$\rho\upsilon^{(\varphi)} = \frac{A}{1-A}(\upsilon_W^{(\varphi)})^{1-\frac{1}{A}} + uW\upsilon_W^{(\varphi)} - \bar{C}\varphi\upsilon_W^{(\varphi)} + \tfrac{1}{2}\sigma^2 W^2 \upsilon_{WW}^{(\varphi)} \tag{A6}$$

with $\lim\limits_{W\to 0} \upsilon^{(\varphi)}(W) = \frac{u(0)}{\rho}$.

Note that $\upsilon^{(\varphi)}$ can be interpreted as the value function for a borrower with coupon rate $\bar{C}\varphi$ and default level $W^* = 0$, it is obvious that $\upsilon^{(\varphi)}$ preseveres all the properties of J and $\upsilon^{(\varphi)} = J_0$. Hence, $\upsilon^{(\varphi)}$ is locally uniformly bounded. Moreover, $\upsilon_W^{(\varphi)}$ is also locally uniformly bounded since $\upsilon^{(\varphi)}$ is concave and locally Lipschitz. So there exists a subsequence $\upsilon^{(\varphi_n)}$ that converges to a function $\bar{J}$ locally uniformly on $(0, \infty)$. To show that $\bar{J}$ coincides with J_0, we need the stability properties of viscosity solutions. We record the following lemma from Lions (1983).

LEMMA 5. Let $\varepsilon > 0, F^\varepsilon$ be a continuous function from $R^+ \times R \times R \times R$ to R and J^ε be viscosity solution of $F^\varepsilon(W, J^\varepsilon, J^{\varepsilon'}, J^{\varepsilon''}) = 0$ in $[0,\infty)$. We assume that F^ε converges locally uniformly on $R^+ \times R \times R \times R$ to some function F and J^ε converges locally uniformly on $[0, \infty)$ to some function J. Then, J is a viscosity solution of $F(W, J, J', J'') = 0$ in $[0,\infty)$.

So, according to lemma 5, $\bar{J}$ is the unique viscosity solution of (A4). On the other hand, the value function J_0 is also a viscosity solution to (A4). Therefore, $\bar{J} = \lim_{\varphi_n\to 0}\upsilon^{(\varphi_n)} = J_0$, which leads to $\lim_{W\to\infty}\tilde{J}(W) = J_0(W)$. Hence, $J_0(W) \le \lim_{W\to\infty} J(W) \le J_0(W)$. The result directly follows the fact that J_W is also locally uniformly bounded.

Taking limit on both sides of the HJB equation (A4), the convergences of J_{WW} is straightforward. Q.E.D.

The convergence of J, J_W, J_{WW} implies that not only the asymptotic behavior of value function J converges to that of J_0, but also all the interesting variables, which depend up to second derivative of J, converge to those variables in the standard Cox et al. (1985) economy. For example, the shadow default-free (instantaneous) interest rate $r(W)$ and the optimal consumption policy satisfy

$$\lim_{W\to\infty} r(W) = \mu - A\sigma^2,$$

$$\lim_{W\to\infty}\frac{c(W)}{W}=k. \tag{A7}$$

The result is important in the sense that it provides a formal proof that the economy we model approaches the classic general equilibrium production economy with no default as the wealth approaches infinity.

We also show that, in the limit as the economy approaches the default boundary, the consumption policy and the default-free interest rates can be solved in closed form when the borrower is identical to the lender.

From the boundary condition $J(W^*) = J_0(\alpha W^* - K)$ and the "smooth pasting" condition $\lim_{W\to W^{*}+} J'(W) = [\partial J_0(\alpha W^* - K)]/\partial W^*$, we can characterize the limiting behavior of $r(W)$ and $c(W)$ when wealth level is close to default:

$$\lim_{W\to W^{*+}} r(W) = \frac{2\rho}{1-A}\frac{W-K/\alpha}{W} - \mu + \frac{2\bar{C}}{W} - \frac{2A}{1-A}\alpha^{1-\frac{1}{A}}k\frac{W-K/\alpha}{W},$$

$$\lim_{W\to W^{*+}} c(W) = \alpha^{1-\frac{1}{A}}k\left(W-\frac{K}{\alpha}\right). \tag{A8}$$

With the value function and the optimal consumption rule determined, we can specify the borrower's valuation of the loan. When $W \le W^*$, the borrower defaults and the value of the loan simply is that left over upon default: $(1-\alpha)W - K$. When $W > W^*$, the future payments for the loan can be summarized as $\bar{C}, s < \tau$; and $(1-\alpha)W^* - K, s = \tau$. Given the smoothness of the value function J, the borrower's value for such a loan at time t can be expressed as the expectation of the product of its future payoff, a time-discount factor $e^{-\rho(s-t)}$ and a risk adjustment factor $[J_W(W_s, s)]/[J_W(W_t, t)]$. The existence and uniqueness of the valuation is guaranteed by the dynamic completeness of the market. In particular, when $W > W^*$, the borrower values the loan at time 0 by

$$I(W) = E_t\left[\int_t^{\tau} e^{-\rho s}\frac{J_W(W_s,s)}{J_W(W,t)}\bar{C}ds + e^{-\rho\tau}\frac{J_W(W^*,\tau)}{J_W(W,t)}\{(1-\alpha)W^* - K\}\right]. \tag{A9}$$

It can be shown that I(·) also satisfies the following ordinary differential equation (ODE) for W=W*:

$$-r(W)I + (r(W)W - c^*(W) - \bar{C})I_W + \tfrac{1}{2}\sigma^2W^2I_{WW} + \bar{C} = 0. \tag{A10}$$

From standard differential equation theory, for example, Krylov (1987), we know that the uniformly elliptic ODE in (A10) has a unique $C^2(W^*, \infty)$ class solution $I(\cdot)$. Combining this with fixed-point equation (7), we are able to determine the equilibrium borrowing amount at time 0, $I_0^*(x_0 \mid \bar{C}, \alpha, K)$ for a specific choice of ($\bar{C}$,α, K). The following theorem provides a formal proof for the existence of such a fixed point I_0^*.

THEOREM 6. For any level of initial endowment x_0, there always exists a I_0^* associated with ($\bar{C}$,α, K), where $\bar{C} \ge 0, 0 < \alpha \le 1$, and $0 \le K/\alpha < x_0$ such that I_0^* satisfies equation (7).

Proof. First of all, note that, for all $W \geq W^*$, the function $I(\cdot)$ satisfy ODE (A10), which is uniformly elliptic; hence, $I(\cdot)$ is continuous on $[W^*, \infty)$. For $K/\alpha < W < W^*$, we have $I(\cdot) = (1-\alpha)(\cdot - K/\alpha)$, which is continuous. So, $I(\cdot)$ is continuous on $(K/\alpha, \infty)$, which implies that $I(\cdot + x_0)$ is continuous on $(0, \infty)$ for $x_0 > K/\alpha$. Second, $I(\cdot)$ also satisfy the expectation form (A9); hence, $I(\cdot) > 0$. Also, $I(\cdot)$ satisfies the boundary condition $\lim_{W\to\infty} I(W) = \bar{C}/R$.

Now, we define function $g(x) = I(x_0 + x) - x$. Noting $g(\cdot)$ is also continuous and we have

$$\left\{ \begin{array}{l} g(0) = I(x_0) > 0 \\ g(\infty) < 0 \end{array} \right\},$$

the existence of such a fixed point I_0^* immediately follows. Q.E.D.

Numerical Solution to Equation (A4)

In this section, we describe the numerical procedure to solve the HJB equation (A4). The approach is backward induction: we first solve the two-person general equilibrium after default to get the borrower's value function upon default, then we enter it as the boundary condition to solve equation (A4).

For our baseline setting where the borrower and the lender are identical, the solution to the two-person general equilibrium problem after default is trivial: the borrower's valuation function after default J_B simply is his valuation function in autarky J_0. For the case when the lender has a logarithmic utility, there is no closed-form solution for J_B. Following Dumas (1989), we first solve the problem of the central planner, who maximizes the welfare function, which is a weighted average (with constant weight λ) of each individual's utility function. The welfare optima specify the wealth-sharing rule between the two agents. Because we also know the share of wealth of the borrower and the lender upon default, we can determine the constant weight λ^* for a given default boundary W^* through the fixed-point requirement (as equation {18} in Dumas 1989). Using this particular weight λ^*, we then determine the borrower's value function upon default $J_B(\alpha W^* - K)$ associated with such a default boundary W^*.

Once we have determined the borrower's valuation function upon default, we use a finite-difference scheme analogous to policy iteration to solve the HJB equation (A4). First of all, for a fixed critical default boundary W^*, we introduce a discrete grid $\{W_0, W_1, W_2, \ldots, W_N\}$. The low boundary W_0 is set to W^* and the upper boundary W_N is an artificially chosen large number; hence, the grid size h is $(W_N - W^*)/N$. A finite-difference approximation for J_W and J_{WW} is

$$J_{iW} = \frac{J_{i+1} - J_{i-1}}{2h},$$

$$J_{iWW} = \frac{J_{i+1} - 2J_i + J_{i-1}}{h^2}, \quad i = 1, \ldots, N-1. \tag{A11}$$

We impose two Dirichlet boundary conditions: $J(W_0) = J_B(\alpha W^* - K)$ and $J_N = [K^{-A}/(1-A)]W_N^{1-A}$. The second one comes from the asymptotic property of

J. An alternative Neumann boundary condition, $J_W(W_N) = 0$, also is applied to check the robustness of our result. We conclude that these two boundary conditions lead to exactly identical result except for very large wealth level close to upper boundary W_N.

We adopt the following "policy iteration" algorithm to solve the nonlinear equation (A4):

Step (0). First we guess an initial $J_i^{(0)}$. For example, we can take the standard Cox et al. (1985) value function $J_0(W)$ as the initial form of $J(W)$; that is, $J_i^{(0)} = [k^{-A}/(1-A)]W_i^{1-A}$, $i = 1, \ldots, N-1$. Hence, the initial policy $C_i^{(0)}$ is given by $(u')^{-1}(J_{iW}^{(0)}) = [(J_{i+1}^{(0)} - J_{i-1}^{(0)})/2h]^{-1/2}$, $i = 1, \ldots, N-1$.

Step (k). Let $J^{(k-1)}$ denote the solution of kth step of the iterative procedure and $C^{(k-1)}$ the corresponding optimal policy, where $C^{(k-1)} = (u')^{-1}(J_{iW}^{(k-1)})$. Then, $J^{(k)}$ is computed as a solution of the tridiagonal system:

$$\rho J_i^{(k)} + \left(\frac{J_{i+1}^{(k)} - J_{i-1}^{(k)}}{2h}\right)(\bar{C} + C_i^{(k-1)} - \mu W_i) - \tfrac{1}{2}\sigma^2 W_i^2 \left(\frac{J_{i+1}^{(k)} - 2J_i^{(k)} + J_{i-1}^{(k)}}{h^2}\right)$$
$$= u(C_i^{(k-1)}),\ i = 1, \ldots, N-1,$$
$$J_0^{(k)} = J_B(\alpha W^* - K), J_N^{(k)} = kW_N^{1-A}.$$

The iteration procedure is repeated until $\max_i |J_i^{(k)} - J_i^{(k-1)}| < \varepsilon$, where ε is the desired tolerance level.

After finishing the "policy iteration," we compute the error for "smooth pasting condition":

$$\text{error}(W^*) = \left|\frac{J_1 - J_0}{h} - \frac{\partial J_B(\alpha W^* - K)}{\partial W^*}\right|. \tag{A12}$$

The optimal default level W^* is determined by a line search for the minimum of error (W^*). The grid size h is chosen small enough that the finite-difference scheme is no longer sensitive to h.

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