The Impact of Collateralization on Swap Rates

MICHAEL JOHANNES and SURESH SUNDARESAN*

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ABSTRACT

Interest rate swap pricing theory traditionally views swaps as a portfolio of forward contracts with net swap payments discounted at LIBOR rates. In practice, the use of marking-to-market and collateralization question this view as they introduce intermediate cash flows and alter credit characteristics. We provide a swap valuation theory under marking-to-market and costly collateral and examine the theory's empirical implications. We find evidence consistent with costly collateral using two different approaches; the first uses single-factor models and Eurodollar futures prices, and the second uses a formal term structure model and Treasury/swap data.

The traditional approach to interest rate swap valuation (Sundaresan (1991a) and Duffie and Singleton (1997)) treats a swap as a portfolio of forward contracts on the underlying floating interest rate. Under specific assumptions regarding the nature of default and the credit risk of the counterparties, Duffie and Singleton (1997) prove that swap rates are par bond rates of an issuer who remains at LIBOR quality throughout the life of the contract. This result is extremely useful for extracting zero-coupon bond prices, pricing swap derivatives, and testing spot rate models.

Despite the popularity of the traditional view, market practice brings into question some of the underlying assumptions of this approach. As the swap market rapidly grew in the late 1980s and early 1990s, an increasingly diverse group of counterparties entered the market. To mitigate their exposure to counterparty credit risk, market participants began using a number of credit enhancements to improve the credit quality of swap contracts. Without question, the most important credit enhancement is the posting of collateral in the amount of the current mark-to-market value of the swap contract (ISDA (1999)).

Due to these credit enhancements, market participants commonly view interest rate swaps as free of counterparty default risk (see, for example, Tuckman (2002)). This implies that, in contrast to the traditional approach, swap rates should be discounted not at default-risky LIBOR rates, but rather at default-free rates. Despite this observation, however, it is still common to model swap yields as par rates. Hence, an internal inconsistency arises.[1]

This paper provides a theory of swap valuation in the presence of bilateral marking-to-market (MTM) and collateralization. MTM requires that counterparties post collateral in the amount of the current mark-to-market value of the contract. This generates an important departure from the traditional theory which assumes that all cash flows exchanged between counterparties occur on the periodic swap dates. Since collateral is generally costly to post, these payments induce economic costs (benefits) to the payer (receiver). Further, since these credit enhancements are part of the swap contract, they must be accounted for in valuation (see the "Credit Support Annex" (ISDA (1994)) to the ISDA Master Swap Agreement). Due to these credit enhancements, it is also increasingly common to build models assuming that swaps are free of counterparty credit risk (Collin-Dufresne and Solnik (2001) and He (2001)) by appealing to the institutional practice of posting collateral.

Formally, we assume that counterparties post cash or Treasury securities as collateral in the mark-to-market value of the swap. Cash and Treasuries are default-free, they can easily be invested or loaned out, and they are the two most common forms of collateral (ISDA (2003)). We show that MTM and costly collateral result in intermediate cash flows that take the form of a stochastic dividend, where the dividend rate represents the cost of posting collateral. This result is reminiscent of Cox, Ingersoll, and Ross (CIR) (1981), who show that the MTM feature of futures contracts results in stochastic dividends. Futures contracts are marked-to-market daily and variation margin calls are met by cash. Because of this, it is common to assume the cost of collateral is the default-free short-term interest rate. This suggests that swaps collateralized by cash may be more reasonably thought of as a portfolio of futures contracts.[2] However, because it is common to rebate some of the interest earned on swap collateral, the collateral cost on a swap contract is net of the rebate, in contrast to a futures contract. Overall, this implies that swaps are a hybrid contract, with features of both futures (MTM) and forwards (common strike or forward price).

We model default in the swap market via an exogenous random stopping time in continuous time. Following Duffie and Singleton (1997), we use a default-adjusted short rate to model LIBOR rates, although we do not assume that default in the swap and LIBOR markets is concurrent. If collateral is costless, we show that swaps are indeed priced by discounting net swap payments at the risk-free rate as in He (2001) and Collin-Dufresne and Solnik (2001). Costless collateral is clearly counterfactual, however, as it implies that it is costless to eliminate credit risk! If collateral is costly, it enters as a negative convenience yield on the swap, altering the discount rates.

What is the directional effect of MTM and net costly collateral on swap rates? We argue that swap rates will increase. To see this, consider the swap from the perspective of the fixed receiver and assume that net collateral costs are positively related to interest rates. When floating rates fall, the swap will have a positive MTM value and the fixed receiver receives collateral. The return on invested collateral is lower due to the decreased interest rates. Conversely, when rates increase, the swap will have a negative MTM value and the fixed receiver will have to post collateral, which is now more costly due to the increased rates. Thus, intuitively, it follows that the fixed receiver will demand a higher swap rate to compensate for the acceleration of (opportunity) costs implied by MTM and costly collateral. Formally in our model, we find that under standard assumptions on net collateral costs, swap rates and swap spreads increase. These assumptions require that the net cost of collateral is positively correlated with the short rate or the spread to LIBOR.

Empirically, we find support for the presence of net costly collateral using two independent empirical approaches. First, we calibrate the CIR (1985) and Vasicek (1977) models to the Eurodollar futures curve, which provides information about the LIBOR term structure. We then compute hypothetical swap curves assuming that swaps are priced as a portfolio of futures (opportunity cost of collateral is the risk-free rate) and also as par rates (the traditional approach). Over our sample, we find that market swap rates are generally between the futures- and forwards-based swap rates, consistent with net costly collateral. More revealing, the position of market swap rates relative to the futures- and forwards-based swap rates varies substantially over time. In periods of market stress (1998 and 2000), market swap rates are almost identical to the portfolio of futures rates; in most years they are closer to the portfolio of futures rates. This is consistent with positive net collateral costs on average and high net costs in periods of market stress. The finding that on average market swap rates are between the hypothetical portfolio of futures and forwards curves is consistent with a significant positive correlation between net collateral costs and short-term rates.

Second, we specify and estimate a dynamic term structure model using Treasury and swap data to illustrate the nature and impact of costly collateral. The default-free term structure model has two factors, the short rate and a time-varying central tendency factor. The third factor is the spread between LIBOR and Treasury rates and the final factor is the net cost of collateral. Estimation results indicate that the implied net cost of collateral is generally small, but exhibits significant and interesting time-variation. In particular, net collateral costs increase around periods of market stress such as the hedge fund crises in 1998, Y2K concerns in the fall of 1999, and the bursting of the dot-com bubble in the spring of 2000. Moreover, the net cost of collateral factor is positively correlated with the short-term default-free interest rate. This reinforces the results from the Eurodollar futures market, which are consistent with net collateral costs being related the short-term default-free interest rate. Finally, the effect of costly collateral can be large, increasing swap rates at seven-years by more than 15 basis points.

The rest of the paper is organized as follows. Section I discusses the institutional features of marking-to-market and collateralization. Section II provides our swap valuation theory in both discrete and continuous time. Section III provides an empirical analysis using Eurodollar features, and Section IV characterizes costly collateral in the context of a formal term structure model. Section V concludes.

I. Institutional Features of Collateralization

In this section, we provide background on credit enhancements in swap contracts, based in large part on information from market surveys by ISDA, The International Swap and Derivatives Association. For information regarding common market practices, see ISDA (1998, 1999, 2000), Clarke (1999) or BIS (2001).

Collateralization and MTM have always been an important feature of the over-the-counter (OTC) derivatives market (Litzenberger (1992)) and their use is nearly universal. There is no precise date at which MTM and collateralization became prevalent, although there is anecdotal historical evidence that systematic collateralization began in the late 1980s and by the early-to-mid 1990s was widespread.[3] For example, Daigler and Steelman (1988) note that "there is not always a marking-to-market of collateral and there does not have to be any up-front margin" (p. 24), while Litzenberger (1992) and Brown and Smith (1993) note that it is common for lower-rated credits to be forced to post margin when entering into swaps with higher-rated counterparties. In 1994, in response to a demand for market-wide standards, ISDA introduced the Credit Support Annex (CSA) to the Master Swap Agreement, providing a legal standard for collateralization and facilitating the transfer of swap positions among diverse counterparties.

It is a common misperception that MTM and collateralization became common only after the Long-Term Capital Management (LTCM) hedge fund crisis in 1998. In fact, Lowenstein (2000) notes that LTCM both collateralized and marked their positions to market: "the banks did hold collateral, after all, and Long-Term generally settled up (in cash) at the end of each trading day, collecting on winners and paying on losers" (p. 47).

The main difference between LTCM and other counterparties is that LTCM refused to pay haircuts or initial margins that would have limited their ability to leverage.[4] In a review of collateral management during the market stress of 1997 and 1998, ISDA (1999) finds that "many institutions avoided or greatly reduced credit losses" through collateralization (p. 13). Moreover, the causes of any losses were not due to the inability of collateralization to mitigate credit risk, but instead improper implementation due to inadequate haircuts (on Russian bonds) or internal data errors (omitting certain transactions).

ISDA (2001, 2003) market surveys indicate that collateral use is widespread and increasing. ISDA (2001) finds that more than 65% of "plain vanilla derivatives, especially interest rate swaps" are collateralized according to the CSA. Discussions with market participants indicate that nearly all swaps at major investment banks are collateralized. In addition, nearly all collateral agreements are bilateral, in the sense that both counterparties post collateral if either is out-of-the money (ISDA (2003)). This is different than under unilateral agreements, whereby only the lower-rated counterparty posts collateral. In discussions with market participants, bilateral agreements are the norm for interest rate swaps. Due to the importance of collateralization, new institutions such as SWAPCLEAR have been established to mitigate credit exposure through large-scale MTM and collateralization.

ISDA (2003) reports that most of the collateral posted is in the form of USD cash (70%), US government securities (19%), or agency securities. Securities are more difficult to manage than cash as the holder must account for the risk that the value of the securities held as collateral might fall below the value of the swap at the same time the payer of collateral defaults. Thus, noncash collateral is typically subject to a haircut.

An important feature of bilateral collateralization is that interest on collateral is often rebated (see paragraph 13, item H of ISDA (1994)), in which case a short-term interest rate such as the T-bill rate is commonly paid to collateral payer. In contrast, mark-to-market gains on futures are the property of the receiver and these gains accrue interest at the short-term rate. The key to effective credit risk mitigation is frequent margin calls. ISDA's (2001) survey of market participants finds that at least 74% of survey respondents MTM at least daily.

The posting of collateral, regardless of what or how it is posted, entails a cost and, for the other counterparty, a benefit. The easiest way to see this is to first note that the receiver of collateral reduces or eliminates any losses conditional on default. Second, collateral receivers, when allowed, typically reuse or rehypothecate the collateral for other purposes. Indeed, 89% of reusable collateral is rehypothecated (ISDA (2001)). Third, even when interest is rebated, there is often a cost to posting collateral as the interest rebated is typically less than the payer's funding costs. As an example, suppose that cash is posted, the cash is invested either at general collateral (GC) repo or Federal Funds rates, and T-bill rates are refunded. Due to the well-known liquidity premium embedded in Treasuries (see Grinblatt (2001) or Longstaff (2004)), T-bill rates are lower than Federal Funds rates or GC repo rates. This difference can generate a net cost of collateral. In addition, most market participants borrow short term at rates higher than LIBOR, which generates an additional cost. Finally, another source of time-varying collateral costs is the potential for securities posted such as Treasuries to go on special, allowing the holder of special collateral to borrow at below risk-free rates. Grinblatt (2001) argues that there can be significant gains for the holders of Treasuries. Unfortunately, no information is available on interest rate rebates in the swap market.

A key to the success of MTM and collateralization are amendments to the U.S. Bankruptcy Code passed in the 1970s and 1980s that assign a special status to collateralized derivatives transactions. Unlike other creditors,[5] a derivative's counterparty receives an exemption from the Code's automatic stay provision and the 90-day preference period. These provisions prevent creditors from seizing the debtor's assets once they declare bankruptcy and may allow the court to recover any transfers from the debtor to creditors in the 90 days prior to bankruptcy. Due to this exemption, counterparties to derivatives transactions can seize any margin or collateral even though the debtor has filed for bankruptcy, which shields its assets from collection activities by other creditors. Thus, there is no concern that the debtor will have legal recourse to recover the collateral or that the creditor will have to participate in legal proceedings.[6]

II. Swap Valuation

In this section, we discuss swap valuation in the presence of MTM and costly collateral in both discrete and continuous time.

A. The Impact of MTM and Costly Collateral on Swaps: Discrete Time

To develop intuition, we first consider a simple discrete time setting to understand how MTM and costly collateral impact swap valuation. We assume that there are two periods with three dates, $t = 0, 1$, and 2. At time 1, the mark-to-market exposure of the swap is collateralized with USD cash or Treasury securities. At the end of period 2, Party A agrees to pay Party B a fixed rate and receive the floating rate. We assume it is costly to post collateral, whereas holding collateral generates a benefit. The costs and benefits are symmetric: The cost to one party equals the benefit to the other party. Let s_0 denote the fixed swap rate, $\{V_t\}_{t=0}^{2}$ be the market value of the swap contract at time t, y_1 denote the net cost (benefit) to posting (receiving) the cash collateral at time 1, and L_2 denote six-month LIBOR at time 2.

The mechanics of the swap and MTM procedure proceed as follows: at time 0, the swap rate, s_0, is set to make the market value of all future cash flows equal to zero, that is, $V_0 = 0$. At time 1, assume the market value of the seasoned swap changes, where for simplicity, $V_1 > 0$. Then Party B pays Party A \$$V_1$. At time 2, Party A receives a benefit from holding the collateral in the amount of $y_1 V_1$ and the parties net the collateral payment with the exchange of fixed and floating payments, $L_2 - s_0$.

At initiation, the market value of the swap is zero and the collateralized swap rate solves $PV_0\left[(L_2 - s_0) + V_1 y_1\right] = 0$, where PV_0 denotes the present value of the cash flows at time 0.

We intentionally do not specify what interest rate (default-free or default-risky) is used to discount the cash flows. The collateralized swap rate is different from the uncollateralized swap rate, which solves $0 = PV_0 [V_2] = PV_0 [L_2 - s_0]$, as the traditional approach ignores collateralization and treats a swap as a portfolio of forwards.

This simple example provides the intuition for the more general results in the next section and demonstrates the following implications of MTM and collateralization. First, MTM and collateralization result in a stochastic dividend, $V_1 y_1$, between contract initiation and the final period. This implies that collateralized swaps are not simply portfolios of forward contracts. The stochastic dividend result is reminiscent of Cox, Ingersoll, and Ross (1981), who demonstrate that, due to MTM, futures contracts have stochastic dividends. Second, MTM and collateralization alter the recovery characteristics in the case of default. Suppose Party B defaults on Party A and Party A keeps the collateral posted, V_1. The maximum loss is now $L_2 - s_0 + V_1$. The collateral reduces any potential losses, conditional on default, which was clearly the original intent of requiring counterparties to post collateral, and is precisely why ISDA (1999) finds that collateralization "greatly reduced credit losses" in 1998.

B. The Impact of MTM and Costly Collateral on Swaps: Continuous Time

In continuous time, we follow the extant literature and assume a reduced-form model for instantaneous LIBOR. The default risk-adjusted spot rate, R_t, is given by $R_t = r_t + \delta_t$, where r_t is the default-free short rate and δ_t is the credit risk spread to instantaneous

LIBOR. Discretely compounded LIBOR is given by $L_6(T) = 2\left[P^R(T,6)^{-1} - 1\right]$, where $P^R(T,s) = E_T^{\mathbb{Q}}\left[\exp\left(-\int_T^{T+s} R_t dt\right)\right]$ and $\mathbb{Q}$ is an equivalent martingale measure. The LIBOR rate on the swap can be either LIBOR at the time of the exchange or its value six months earlier (settled in arrears). Because the interpretation of results is cleaner with contemporaneous settlement, we often assume this case in our examples; the differences between the two rates are quite small.[7]

As a benchmark, we consider the swap valuation model in Duffie and Singleton (1997). The authors assume counterparties are default-risky, their credit risk is equal to the average credit quality of the LIBOR panel, and there is no recovery conditional on default. Together, these assumptions imply that swap payments are discounted at R_t. In the case of a single period swap, the market fixed rate, s_0^R, is given by

$$s_0^R = \frac{E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T R_t dt\right) L_6(T)\right]}{P^R(0,T)} = E_0^{\mathbb{Q}}\left[L_6(T)\right] + \frac{cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T R_t dt\right), L_6(T)\right]}{P^R(0,T)}, \quad (1)$$

where $E_0^{\mathbb{Q}}[L_6(T)]$ is the futures rate on six-month LIBOR and $L_6(T)$ can be either six-month LIBOR at time T or at time $T-6$. This result displays the close relationship between swap rates and futures rates. Since the covariance term is always negative, swap rates in the traditional approach are less than the associated futures rates.[8]

To value the collateralized swap, we assume only that default by either counterparty can be represented by a first jump time, τ, of a jump process with potentially stochastic intensity. We do not require any further assumptions on the nature of default by the counterparties. Conditional on default, we assume that there is no recovery in excess of the collateral posted. Our approach relaxes two of the assumptions in Duffie and Singleton (1997), namely, (1) that default characteristics and occurrences in the LIBOR and swap market are the same, and (2) that the counterparties are expected to remain at LIBOR quality until a default time.

We let V_t denote the market value of the swap at time t, and we assume that the counterparties are required to continuously post collateral in the MTM value of the swap, that is, they post collateral to secure their future obligations. In practice, swaps are typically marked at least daily with the option to demand additional collateral in the case of large market moves. The contract is assumed to be fully collateralized, with the amount of collateral posted at time t equal to the market value of the swap, V_t. We assume further that there is a stochastic net cost of collateral, y_t. We interpret the net cost of collateral as an instantaneous interest rate accrual on the principal of V_t. While it is common to assume that the cost of collateral when valuing futures contracts is the default-free short rate, r_t, due to the potential for rebating interest that we consider, we expect the collateral costs to be less than r_t. We also assume the cost is symmetric, which simplifies the analysis.[9] Appendix A provides a more general treatment and the details of the valuation approach.

We focus on a number of special cases that correspond to different assumptions regarding the net cost of collateral. The first case assumes that collateral is costless to post, that is, $y_t = 0$, and that the contract is fully collateralized. Since recovery is full conditional on default, the contract is default-free and the swap value is

$$V_t = E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\Phi_T\right], \tag{2}$$

where $\Phi_T = L_6(T) - s_0^r$ and $V_0 = 0$. This formally justifies the valuation approach in He (2001) and Collin-Dufresne and Solnik (2001): *If* the swap is collateralized in the MTM value of the swap *and* it is costless to post collateral, then swaps are discounted at the risk-free rate. Of course, this case is clearly counterfactual as it is not costless to post collateral. The swap rate in this case, s_0^r, is given by

$$\frac{E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right)L_6(T)\right]}{P^r(0,T)} = E_0^{\mathbb{Q}}\left[L_6(T)\right] + \frac{cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right), L_6(T)\right]}{P^r(0,T)}, \tag{3}$$

where P^r is the price of a zero-coupon bond discounted at r_t. The covariance between the risk-free discount factor and LIBOR is typically negative, since $R_t = r_t + \delta_t$. This example highlights the subtle role of MTM and costly collateral: ignoring credit risk, even though the contract is marked-to-market, it has no impact on the swap rates because collateral is costless.

Next, consider the case with net costly collateral. As we show in Appendix A, the collateralized swap value is

$$V_t = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\Phi_T + \int_t^T \exp\left(-\int_t^s r_u du\right) y_s V_s ds\right], \tag{4}$$

which is the familiar stochastic dividend yield formula, implying that

$$V_t = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T (r_s - y_s)\, ds\right)\Phi_T\right]. \tag{5}$$

Note that MTM and costly collateral *together* impact swap rates. They do not have an independent impact because MTM is a procedure and y_t measures the costs that accrue due to the procedure; thus, the two are inexorably linked. This will have important implications for our empirical work later on.

At initiation, $V_0 = 0$ and the collateralized swap rate, s_0^{r-y}, is

$$\frac{E_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T (r_s - y_s)\, ds\right) L_6(T)\right]}{P^{r-y}(0,T)} = E_0^{\mathbb{Q}}\left[L_6(T)\right] + \frac{cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T (r_s - y_s)\, ds\right), L_6(T)\right]}{P^{r-y}(0,T)}. \tag{6}$$

As in the case of costless collateral, swap contracts are again free of counterparty default risk, however, costly collateral now alters the discount rate. The impact of net costly collateral will be determined by the expected covariance between $r_t - y_t$ and $L_6(T)$. The potential impact can be significant. To see this, suppose $y_t = r_t$, which implies that swaps are priced as a portfolio of futures contracts on six-month LIBOR. The difference between futures and forwards is significant and can be large (see Sundaresan (1991b), Grinblatt and Jegadeesh (1996), and Gupta and Subrahmanyam (2000)).

It is important to note the subtle differences between a futures contract and a collateralized swap. First, futures prices are reset continuously and as a consequence the value of the contract is zero. With a collateralized swap, in contrast, the swap rate remains fixed until termination of the contract (either by default or expiration). Second, with futures, the MTM gains earn r_t in a margin account, and thus $y_t = r_t$. However, as we discuss in Section II, since the net benefit of swap collateral is generally less than r_t, collateralized swap rates are different from futures rates. Thus, swaps are a hybrid contract, with features of both futures (MTM) and forwards (constant swap rate).

Is it possible to generically order the swap rates under the various assumptions regarding MTM and collateralization? There are four rates of interest: s_0^R, s_0^r, s_0^{r-y}, and the futures rates. From the swap valuation equations, we see that there is a close relationship between swap rates and futures rates, and that the covariance of the discount factors with the LIBOR rates determines the differences.

Since $cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T R_s ds\right), L_6(T)\right]$ and $cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right), L_6(T)\right]$ are negative, both s_0^r and s_0^R are less than $E_0^{\mathbb{Q}}[L_6(T)]$. If y_t is deterministic, then $s_0^{r-y} = s_0^r$. This result is related to Cox, Ingersoll, and Ross (1981), who show that futures and forwards prices are equal if interest rates are nonstochastic. Consider next the difference between the default-free swap rate, s_0^r, and the default-risky swap rate, s_0^R:

$$s_0^r - s_0^R = \frac{cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right), L_6(T)\right]}{P^r(0,T)} - \frac{cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T R_s ds\right), L_6(T)\right]}{P^R(0,T)}. \quad (7)$$

Since R_t and r_t are positively correlated,

$$cov_0^{\mathbb{Q}}\left[e^{-\int_0^T R_s ds}, L_6(T)\right] < cov_0^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}, L_6(T)\right], \quad (8)$$

and since $P^R < P^r$, this implies that $s_0^r > s_0^R$. This result is somewhat counterintuitive as it implies that eliminating counterparty credit risk actually increases swap rates. Results in the extant literature indicate that while $s_0^r > s_0^R$, the difference is rather small. For example, using the model and parameter estimates in Collin-Dufresne and Solnik (2001), the difference for ten-year swaps is about two to five basis points, depending on the state variables. In this setting, credit risk has such a small impact due to netting (as the principal is netted) and the fact that Duffie and Singleton (1997) assume that counterparty credit risk is equal to the credit risk embedded in LIBOR (which is extremely small).

If collateral is positively correlated with the default-free short rate, r_t, or the spread to LIBOR, δ_t, then (see Appendix A)

$$cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right), L_6(T)\right] < cov_0^{\mathbb{Q}}\left[\exp\left(-\int_0^T (r_s - y_s)\, ds\right), L_6(T)\right] \qquad (9)$$

and $s_0^{r-y} > s_0^r$. This shows that MTM and costly collateral further increase swap rates. In general, since we expect the net cost of collateral embedded in swap rates to be less than r_t, we have that $E_0^{\mathbb{Q}}[L_6(T)] > s_0^{r-y} > s_0^r > s_0^R$. In the next two sections, we investigate the magnitudes of the differences using futures data (Section IV) and a formal term structure model (Section V).

III. Do MTM and Costly Collateral Matter?

While MTM and collateralization are clearly contractual features of interest rate swaps, it is important to investigate whether they matter, that is, whether their presence and associated costs and benefits impact market swap rates in a meaningful manner. In this section, we use the information embedded in Eurodollar futures and market swap rates to examine this question.

As we argue in the previous sections, MTM and time-varying net costly collateral alter the discount factors and generally increase the swap rates relative to their values using the traditional approach. An obvious way to examine the impact of MTM and net costly collateral would be to construct hypothetical swap rates from refreshed LIBOR bond prices, P^R, using the par representation and to compare them to market swap rates. If market swap rates are above the hypothetical par rates, then collateral matters, that is, swaps are not discounted at R_t. Unfortunately, refreshed LIBOR bond prices are typically obtained from swap rates assuming the par representation holds, rendering this exercise circular.

To investigate the validity of the portfolio of forwards approach, we need information about the LIBOR zero-coupon term structure. The best source of this information is the futures contract on three-month LIBOR, the Eurodollar futures. Using the approach of the previous section, the futures rate at time t of a contract that expires at time $T_n > t$ is $FUT_{t,T_n} = E_t^{\mathbb{Q}} [L_3 (T_n)]$, where $L_3 (T_n)$ is three-month LIBOR. Eurodollar futures provide a "clean" piecewise view of expected LIBOR rates, and, unlike swap rates, do not require potentially controversial assumptions regarding the cost of collateral and counterparty credit risk. The Eurodollar futures market is most liquid derivatives market in the world in terms of notional dollar volume of daily transactions. The only disadvantage of Eurodollar futures is that they do not provide zero-coupon bond prices directly . Thus, we must estimate a term structure model to compute refreshed LIBOR bond prices. Unfortunately, there is no way around this issue. For robustness, we use different term structure models and are careful to compare our results to those in the literature.

We obtain daily closing prices for the Eurodollar futures contract from the Chicago Mercantile Exchange for the period January 1994 though December 2002. We discard serial month contracts, and use Wednesday-close prices of quarterly contracts for the first seven years. If Wednesday is not available, we use Thursday rates. We do not use data past seven years to avoid potential liquidity concerns on the long end of the futures curve, although, as we mention below, none of our results are sensitive to the inclusion or exclusion of the long end of the futures curve.

We use the Vasicek (1977) and Cox, Ingersoll, and Ross (1985) models and the Hull and White (1990) calibration procedure to compute convexity adjustments. It is common to use single-factor models to compute convexity adjustments for two reasons. First, they are parsimonious, with few parameters to estimate relative to multifactor models. Second, provided the models are reestimated frequently, they provide an accurate fit to the futures curve as recalibrated parameters (e.g., the long-run mean) proxy for state variables in popular multifactor models (e.g., time-varying central tendency) without introducing numerous parameters. In a comparison, the "$A_1(3)$" model considered in Collin-Dufresne, Goldstein, and Jones (2004) has 26 parameters.

For every week in our sample, we compute the parameters that provide the closest fit of the model to the observed data, that is,

$$\widehat{\Theta}_t = \arg\min \sum_{j=1}^{28} \left\| Fut_{t,T_j}(\Theta_t) - Fut_{t,T_j}^{Mar} \right\|, \tag{10}$$

where $Fut_{t,T_j}(\Theta_t)$ is the model-implied prices at time t of a futures contract expiring at time T_j, Fut^{Mar}_{t,T_j} is the market-observed futures rates, and $\|\cdot\|$ is a distance measure. We use both absolute deviations and squared deviations; the results reported below use squared deviations. All of the models considered closely fit the futures curve. To ensure smoothness of the curves, we constrain the parameters from taking extreme values.

With the calibrated models, we compute hypothetical swap rates assuming swaps are priced as a portfolio of forwards or futures. In the latter case, the portfolio of futures swap rate, $s^{FUT}_{0,T}$, on an T-year swap on six-month LIBOR with semiannual payments settled in-arrears solves

$$0 = E^{\mathbb{Q}}_0 \left[\sum\nolimits_{j=1}^{2T} \left(L_6 \left((j-1)/2 \right) - s^{FUT}_{0,T} \right) \right], \tag{11}$$

which implies that

$$s^{FUT}_{0,T} = \frac{1}{2T} \sum\nolimits_{j=1}^{2T} E^{\mathbb{Q}}_0 \left[L_6 \left((j-1)/2 \right) \right]. \tag{12}$$

Note that we take into account the fact that the dates on which swap payments are exchanged are six months later than the date on which the floating index is determined. We report the difference between the market swap rates, $s^{Mar}_{0,T}$, and the hypothetical forwards- and futures-based swap rates, $s^{For}_{0,T} - s^{Mar}_{0,T}$ and $s^{Fut}_{0,T} - s^{Mar}_{0,T}$. If swaps are priced as par rates, $s^{For}_{0,T} - s^{Mar}_{0,T}$ should be zero.

[Table I here]

We report results for five- and seven-year maturities for the two models.[10] Tables I and II provide calibration summary statistics and Figures 1 and 2 provide time series plots of $s_{0,T}^{Fut} - s_{0,T}^{Mar}$ and $s_{0,T}^{For} - s_{0,T}^{Mar}$ for the five- and seven-year maturities for both models. Throughout, we focus on results that are robust across maturities and models.

The results indicate that swaps are not always and, in fact, rarely priced as par rates. For example, focusing first on the full sample, market seven-year swap rates are about 13 (15) basis points above the portfolio of forwards swap rates for the Vasicek (CIR) model and about eight (ten) basis points below the futures rates. The same is true of the five-year swap rates, although on average the five-year swap rates are closer to the futures-based rates. In light of our theoretical arguments above, this evidence is consistent with swaps being discounted at a rate lower than LIBOR, and, in particular, MTM and time-varying net costly collateral being significantly, positively related to short interest rates.

[Table II here]

Next, note that the average position of the market swap rates vis-à-vis the portfolio of forwards and futures rates changes substantially over time. Based on year-by-year results, for example, in 1998 and 2000 the market swap rates are almost the same as the hypothetical portfolio of futures for both models and for both maturities. In 1998 there was significant market stress in the fixed income market due to the collapse of LTCM and, in 2000, interest rates were relatively high and increasing, with market stress due to the bursting of the dot-com "bubble." In contrast to these years, market swap rates were much closer to the portfolio of forwards rates for both models and both maturities in 2002, which was a year with extremely low interest rates and generally stable market conditions.

The results in these cases are intuitively consistent with the time-varying costly net collateral explanation: In periods of market stress and/or high interest rates, collateral is more costly and swap rates move closer to, and could even exceed, the portfolio of futures rates. In most other years, market swap rates are closer to the portfolio of futures rates, although not as close as in 1998 and 2000 (for both models and both maturities), and there is some minor variation across models and maturities as we discuss below.

[Figures 1 and 2 here]

Figures 1 and 2 depict the very strong time variation in the relative position of market swap rates vis-à-vis the portfolio of futures- and forwards-based swap rates. Graphically, it is apparent that market swap rates were very close to the futures-based swap rates, especially during the Mexican currency crisis in late 1994 to early 1995, the Asian currency crises in 1997, the fall of 1998, and the fall of 2000. Together, this suggests the importance of time-varying net collateral costs.

The figures also clearly illustrate the strong impact of September 11, 2001 on the results for 2001, which results in greater variation across maturities and models in this year than in others. This is not surprising for two reasons. First, 2001 is an interesting year because the yield curve was flat to slightly inverted in the early portion of the year and then extremely steeply sloped (more than 3%) post-September 11^{th}. The convexity adjustments from the two models differ in steeply sloped interest rate environments due to the state dependence in the volatility coefficient in the CIR model. Second, the fixed income markets were in general disarray after September 11 as there were major microstructure problems with trades not clearing and the Treasury intervening with a rare "snap" or same-day auction in October. Although clearly a period of market stress, short interest rates declined rapidly in 2001 as the Federal Reserve injected liquidity to stabilize the markets; this increase in liquidity would reduce the net cost of collateral and decrease market swap rates.

A few additional issues require discussion. First, there are a couple of periods during which market swap rates were slightly above the portfolio of futures rates, notably, in 1995, 1998, and 2000. For example, in 1995, the five-year swap rates were slightly above the futures rates, but the seven-year rates were below but still closer to the futures rates. It is important to note that in such instances (a) the magnitude is quite small, at only a few basis points, (b) the effect is short lived, and (c) the effect is concentrated at the five-year maturity. Our Gaussian model formally allows for the swap rates to exceed the futures rates, if the expectations of net collateral costs are greater than the risk-free rate. We find this unlikely, especially given the fact that the swap rates are below the portfolio of futures rates for the other maturities. There are also brief periods, for example, in the fall of 2001, when the slope of the yield curve was at record levels and swap rates were below the portfolio of forwards rates.

We cannot expect our simple models to capture every movement in futures and swap rates, and the periods above are likely due to model misspecification, market segmentation, or temporary mispricings. The latter two explanations are quite plausible given that the swap rates were above the portfolio of futures for a relatively short time and only by a few basis points. As Grinblatt and Jegadeesh (1996) note, these mispricings can exist as it is not possible to directly arbitrage the futures and forward/swap market precisely because the futures and forwards generally have different expiration dates. Grinblatt and Jegadeesh (1996) also find that mispricing occasionally occurs and discuss the limits of arbitrage in the context of the Eurodollar futures market. Second, it is likely that collateral use and the liquidity of longer-dated Eurodollar futures increased over the first part of the sample, especially in 1994 and 1995, which could impact the results in the first two years of our sample.

Third, the convexity adjustments we report are generated from the Vasicek and CIR models. Since different models generate different convexity adjustments, it is important to ensure our convexity adjustments are reasonable. Our main reference in this regard is Gupta and Subrahmanyam (2000), who quantify the convexity adjustments in swaps. They compute convexity adjustments from a number of different models and, in conclusion, argue that "The results using any of the models suggest that the convexity adjustment can be very large for long-dated contracts. For a ten-year futures contract, our calculations suggest that this adjustment is on the order of $80 - 100$ basis points, which translates into a convexity adjustment of about $35 - 40$ basis points for a ten-year swap. Even a conservative estimate of the bias for a five-year USD swap is about $12 - 16$ basis points (p. 269)."

Our results are consistent with their results. As a comparison, for the Vasicek model for five-year and ten-year swaps, our total convexity adjustment is about 14 (12.3+1.4) and 31 (14.1+16.9) basis points over our whole sample, which is at the lower end of the range that Gupta and Subrahmanyam (2000) considered reasonable. Also, and again consistent with Gupta and Subrahmanyam (2000), we find that the CIR adjustments are generally larger than the Vasicek adjustments. The differences between the models generally occur in extreme periods, for example, when the term structure slope is very high. One alternative approach would be to subtract, for example, 12 to 16 basis points from the fitted five-year futures rates to get the portfolio of forwards rates. As is clear from Figures 1 and 2, market swap rates would again be closer to the futures rate and the time-variation is similar. As a final check, we obtain convexity adjustments from a proprietary model used at a major investment bank for 2002 and find that these adjustments are in the upper end of the range mentioned by Gupta and Subrahmanyam (2000). This is further evidence that our convexity adjustments are reasonable.

To conclude, the evidence indicates that swaps are priced not as par rates, but instead, swaps are almost always significantly greater than par rates. Moreover, the relative position changes substantially over time, consistent with our time-varying collateral cost argument. Are there alternative explanations that would generate this effect? The only alternative that we are aware of is the liquidity explanation of Grinblatt (2001), which relies on the convenience yield generated by holding Treasury securities. The convenience yield is generated by the repo specialness of on-the-run Treasuries. Grinblatt (2001) and Duffie and Singleton (1997) incorporate this liquidity factor and argue that it results in an adjusted discount rate for swap payments of $\widetilde{R}_t = r_t - l_t + \delta_t$, where l_t is the liquidity cost. Since $\widetilde{R}_t < R_t$, this would be consistent with market swap rates lying above those implied by the par representation. However, our results indicate that swaps are often priced close to and statistically indistinguishable from a portfolio of futures. In this case, the liquidity-based argument would imply that $\widetilde{R}_t = 0$ or that $l_t = r_t + \delta_t$. This is implausibly high for a liquidity proxy. For example, consider a five-year swap. The benchmark five-year Treasury note on which the convenience yield would accrue was auctioned monthly in the 1990s. Thus, at most, the on-the-run specialness would accrue for a maximum of one month and therefore is likely to be a minor component of swap rates.

IV. Characterizing the Net Cost of Collateral

Given the results in the previous section and our modeling framework from Section II, the next issue is to characterize the net cost of collateral. Evidence from the previous section indicates that swaps are priced between a portfolio of forwards and futures, and the relative position varies over time. In our context, that means that y_t is related to the short-term default-free interest rate and the impact is time-varying.

The purpose of this section is to give an illustrative feel for the time series properties of the net cost of collateral process in the context of a term structure model. A term structure model allows us to impose a model of net costly collateral on the data and therefore to quantify its impact on swap rates and provide estimates of the state variable, y_t. With this state variable, we can examine, for example, whether the net cost of collateral implied from the model increases in periods of crisis or in high interest rate environments as one's intuition might suggest. In the context of the term structure model, we can also quantify the impact of y_t on swap rates. It is important to document that time-varying net collateral costs are quantitatively important. He (2001) and Liu, Longstaff, and Mandel (2001) provide more general models for explaining swap spreads.

We follow the literature and use a Gaussian term structure model, which we describe in detail in Appendix B. To address costly collateral, we must model both the default-free term structure and the LIBOR/swap term structure. To model the short rate, we use the two-factor model from Collin-Dufresne and Solnik (2001), where the state variables are the default-free short rate, r_t, and the long-run central tendency factor, θ_t, which captures the slope of the default-free yield curve. In addition to r_t and θ_t, the swap curve is influenced by δ_t and y_t. Since costly collateral only matters in so far as it is correlated with other term structure variables, we allow for general interactions between δ_t, y_t, and the other term structure variables. Specifically, we are interested in interactions between net collateral costs and the risk-free rate, which we measure by the parameter $\kappa_{y,r}$.

Our goal in using this model is to estimate the impact of net costly collateral in a formal setting and to examine the correlation of costly collateral with other term structure variables, specifically, δ_t and r_t. Given the large number of state variables, we try to economize on the parameters, in part because the most general and flexible specifications have a very large number of parameters. For example, Collin-Dufresne, Goldstein, and Jones (2004) estimate a flexible three-factor affine model ($A_1(3)$) that has 26 parameters, about half of which are insignificant. Thus, we choose the simplest default-free model and a parsimonious specification for δ_t and y_t.

It is important to recognize that all statements are model dependent and, as Dai and Singleton (2000) note, different models and parameterizations can be observationally equivalent (in terms of the ability to price swaps), although the factors will be rotated.[11] In general, this suggests that we should take care in interpreting δ_t and y_t as their levels and scales can be arbitrarily altered by rotation in different model specifications. However, two points should minimize these concerns. First, in the case of δ_t, we constrain the long-run mean of δ_t to equal the in-sample average of the three-month LIBOR-Treasury (TED) spread, which ensures that δ_t can be safely identified as the spread between LIBOR and Treasuries. Second, our two-stage estimation procedure identifies r_t and θ_t exclusively from Treasury prices, which ensures that y_t is identified solely from the swap curve and that y_t

does not alter the estimates of r_t or θ_t.

We note that there could be other factors not in our model which are also important for understanding the impact and character of net collateral costs. For example, Duffie and Singleton (1997) and Grinblatt (2001) argue that repo specialness of Treasuries may be important for pricing both Treasuries and swaps. This repo specialness, which closely related to the on-the-run/off-the-run Treasury spreads, is important for understanding net collateral costs as Treasuries are commonly posted as collateral. Similarly, net collateral costs should increase in periods of market stress and thus a model incorporating a factor such as flight-to-quality would have interesting implications for net collateral costs. Since collateral only matters in so far as it is correlated with other systematic factors, our results will be limited to identifying the systematic relationship between collateral and r_t, δ_t, and θ_t.

[Table II here]

We estimate the model using a two-stage maximum likelihood procedure. Table III provides summary statistics of the data we use; and details of the estimation approach are in Appendix B. Table IV provides maximum likelihood estimates, which are largely consistent with prior studies. For example, κ_r, σ_r, κ_r, λ_r, $\rho_{r,\theta}$, κ_δ, and σ_δ are all qualitatively similar to Collin-Dufresne and Solnik (2001), although there are slight differences due to different specifications and data periods. Regarding the off-diagonal terms, we are primarily interested in $\kappa_{y,r}$, which is positive and strongly significant and captures the positive relationship between net collateral costs and the default-free short-term rate. This is consistent with the results in the previous section that document that swaps are generally priced above a hypothetical portfolio of forwards, which would occur if net collateral costs were positively related to r_t. Table V summarizes the pricing errors. The pricing errors are similar to, but the average RMSEs are slightly smaller than, those in Collin-Dufresne and Solnik (2001), which is not a surprise as we have an additional factor.

[Tables III, IV and V here]

The implied states are very highly correlated with their analogs in the Treasury and LIBOR/swap data. For example, the in-sample means for r_t and δ_t are nearly identical to sample means of the T-bill and TED spread. The correlation between the δ_t and the TED spread is almost 98%. The correlation between the δ_t and r_t is 32.2%. While standard structural models would imply that this spread should be negative, our positive correlation is not a surprise as the TED spread and the three-month T-bill rate were strongly positively correlated over the same time period. The in-sample mean of y_t is 69 basis points, which is plausible for a net cost of collateral when interest is rebated. The correlation between y_t and r_t is 38.9%, which, along with the positive and significant $\kappa_{y,r}$, points to the close relationship between the net cost of collateral and the default-free short-term interest rate. This is broadly consistent with the findings in the previous section.

[Figure 3 here]

Figure 3 provides times series of the implied states. The state variables r_t, θ_t, and δ_t, are as expected, but the net cost of collateral has an interesting time-variation. Net collateral costs increase dramatically around periods of market stress such as the hedge fund crises in 1998, Y2K concerns in fall 1999, and the bursting of the dot-com bubble in spring of 2000. This also is consistent with economic intuition: The cost of posting collateral increases dramatically during periods of market stress. Further support for this hypothesis comes from summary statistics of the TED and ten-year swap spread. The TED spread is, on average, 35 basis points over the sample period and the ten-year swap spread (ten-year swap rate minus ten-year Treasury rate) is 60 basis points. In addition to the large magnitude of the swap spread relative to the TED spread, the correlation between the ten-year swap spread and the TED spread is 39%, which is a smaller magnitude than one would expect from standard models such as those in He (2001). Accordingly, researchers have typically turned to additional factors such as liquidity or a large credit or liquidity risk premium (Liu, Longstaff, and Mandel (2001)). In our model, the additional factor is the net cost of collateral, which is positively related to the short-term interest rates. This feature squares nicely with the high observed correlation (41%) between the ten-year swap spread and the three-month Treasury rate.

[Figure 4 here]

Finally, our model implies that net swap payments should be discounted at $r_t - y_t$ instead of $r_t + \delta_t$. We now examine the quantitative implications of our model for swap curves. The top panel of Figure 4 plots the swap curves discounted at r_t, R_t, and $r_t - y_t$ and the bottom panel plots the differences between the par curve (discounted at R_t) and the two credit-adjusted curves. Each of these curves is computed assuming the state variables are equal to their in-sample means.

Note first the small difference between the par curve and the curve discounted at r_t. He (2001, p. 15) notes a similar result. The small magnitude is in part model driven as our model (and that in He (2001)) assumes that the shocks between r_t and δ_t are independent and there is at best a modest, model-implied relationship between r_t and δ_t (as we note in the previous paragraph). Alternatively, using the parameters in Collin-Dufresne and Solnik (2001), the difference between the par curve and the curve discounted at r_t is in the range of three basis points at the ten-year maturity (with states evaluated at in-sample means). This is also consistent with prior findings (Sun, Sundaresan, and Wang (1993) and Duffie and Huang (1996)) who found that the credit risk component of swaps is rather small, on the order of a couple of basis points.

The main results indicate that the quantitative impact of collateral on swap rates is significant. To compare our results here with those in Section IV, note that by inspection Figure 4 implies that the difference between the seven-year par (portfolio of forwards) and seven-year collateralized rates is about 15 to 20 basis points. Recall that Section IV used Eurodollar futures data up to seven years and the results indicated that market rates were 13 (15) basis points higher than hypothetical forward rates in the Vasicek (CIR) model at the seven-year maturity. These results are remarkably close, despite the fact that in this section we do not use futures data. Taken together, our results point to a strong relationship between net collateral costs and r_t. Overall, the two experiments provide qualitatively and quantitatively similar conclusions.

A few caveats are in order. First, given the difficulties in identifying all of the parameters in dynamic term structure models, we use the longest possible time series, from 1990 to 2002. Unlike the results in the previous section that are based on year-by-year calculations, we assume here that the parameters are constant throughout this period and that collateral is priced throughout. As we mention in Section I, it is likely that the impact of collateral changed over this period of time, especially during the early portion, and thus should be taken into account when interpreting our results. Second, our results, like virtually all in fixed income, are model- and specification-dependent. Other variables (e.g., flight-to-quality), time-varying parameters, or more complicated parameterizations could impact the results.

V. Conclusion

In this paper, we examine theoretical and empirical implications of MTM and collateralization on swap rates. Theoretically, we show that collateralized swaps are free of counterparty default risk and that costly collateral enters as a convenience yield, altering the discounting of net swap payments. Empirically, we find broadly consistent evidence from two independent sources of information, the Eurodollar futures market and the Treasury/LIBOR/swap term structure, which points to the importance of costly collateral. Often, swaps are priced close to portfolios of futures rather than portfolios of forwards discounted at the instantaneous LIBOR rates.

Two related issues require further analysis. First and foremost, net cost of collateral, liquidity, and default are clearly related. For example, while we follow He (2001) and Collin-Dufresne and Solnik (2001) and use Treasuries for the default-free curve, it might be useful to use alternatives such as the repo rates or Federal Funds rates, that may more accurately capture the default-free rate. This would allow us to separately model the liquidity and flight-to-quality components of Treasuries and the default component in LIBOR. Given these components, we could analyze the relative contributions and relationships among liquidity, default and costly collateral. Second, it would be particularly interesting to understand the theoretical determinants of costly collateral. If there are market participants with differing credit profiles, whose collateral costs matter? Santos and Scheinkman (2001) develop a model that could be extended to handle this issue. Third, in this paper we characterize the impact of collateral on swap rates, but nearly all OTC derivatives are collateralized and marked-to-market. Like swaps, it is common to discount OTC derivatives using the LIBOR curve. It would be interesting to analyze the impact of net costly collateral on these other derivative contracts.

Appendix A: Pricing Collateralized Swaps in Continuous Time

This appendix provides the details of our continuous time swap valuation approach with MTM and net costly collateral. The general valuation approach uses the intuition of the discrete time model combined with a formal treatment of default. If τ is a random time indicating default, we define $\mathbf{1}_{[\tau>T]} = 1$ if there is no default by time T (see Bielecki and Rutkowski (2002) for formal definitions). We do not require any further assumptions on the nature of default by the counterparties.

We assume C_t is posted as collateral and that y_t is the interest rate accrual on this collateral. The value, V_t, of a collateralized swap at time $t < \tau$ is then given by

$$V_t = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\Phi_T\mathbf{1}_{[\tau>T]} + \exp\left(-\int_t^\tau r_s ds\right)C_\tau\mathbf{1}_{[\tau\leq T]}\right] + \tag{A1}$$
$$E_t^{\mathbb{Q}}\left[\mathbf{1}_{[\tau>T]}\int_t^T exp\left(-\int_t^s r_u du\right)y_s C_s ds + \mathbf{1}_{[\tau\leq T]}\int_t^\tau \exp\left(-\int_t^s r_u du\right)y_s C_s ds\right].$$

The first term, $\exp\left(-\int_t^T r_s ds\right)\Phi_T\mathbf{1}_{[\tau>T]}$, is the usual discounted net swap payment assuming that there is no default prior to expiration. The second term is the discounted value of the collateral that is posted and recovered at the time of default. We assume that there is no recovery in excess of collateral, although it is easy to incorporate various assumptions regarding recovery (e.g., Duffie and Singleton (1999)). The third term is the discounted value of the accrued interest on the collateral, assuming no default. At each point in time, there is a net benefit of $y_s C_s$, which accrues to the holder of the collateral and is appropriately discounted back. The final term is the discounted value of the net interest rate accruals on the collateral, conditional on a default.

There are a number of special cases of interest. First, consider the case in which it is costless to post and maintain collateral, that is, $y_s = 0$. Here, the market value of a swap struck at s_0^r, V_t, is given by the solution of

$$V_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s ds \right) \Phi_T \mathbf{1}_{[\tau > T]} + \exp\left(-\int_t^{\tau} r_s ds \right) C_{\tau} \mathbf{1}_{[\tau \leq T]} \right], \tag{A2}$$

where $\Phi_T = L_6(T) - s_0^r$ and $V_0 = 0$. The first term in the expectation is the present value of the cash flows conditional on no default and the second component is the present value of the amount received conditional on a default occurring at time $\tau \leq T$, C_τ. We assume that the amount posted in collateral is equal to the MTM value of the swap, $C_t = V_t$, which implies the swap price process solves (for $t < \tau$)

$$V_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s ds \right) \Phi_T \mathbf{1}_{[\tau > T]} + \exp\left(-\int_t^{\tau} r_s ds \right) V_{\tau} \mathbf{1}_{[\tau \leq T]} \right]. \tag{A3}$$

Since recovery is full conditional on default, a collateralized swap contract is simply a contract with a random termination time. The law of iterated expectations implies that

$$V_t = E_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \Phi_T \right], \tag{A4}$$

which is the same value as a claim paying Φ_T with no default (as recovery is full).

Next, consider the case with time-varying net costly collateral. Under full collateralization, the collateralized swap value is

$$V_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T r_s ds \right) \Phi_T + \int_t^T \exp\left(-\int_t^s r_u du \right) y_s V_s ds \right], \tag{A5}$$

which is the familiar stochastic dividend yield formula, implying that

$$V_t = E_t^{\mathbb{Q}} \left[\exp\left(-\int_t^T (r_s - y_s)\, ds \right) \Phi_T \right]. \tag{A6}$$

This is similar to the futures valuation approach in Cox, Ingersoll, and Ross (1981). At initiation, $V_0 = 0$ and the collateralized swap rate, s_0^{r-y}, is

$$\frac{E_0^{\mathbb{Q}} \left[\exp\left(\int_0^T (y_s - r_s)\, ds \right) L_6(T) \right]}{P^{r-y}(0,T)} = E_0^{\mathbb{Q}} \left[L_6(T) \right] + \frac{cov_0^{\mathbb{Q}} \left[\exp\left(\int_0^T (y_s - r_s)\, ds \right), L_6(T) \right]}{P^{r-y}(0,T)}. \tag{A7}$$

Finally, at the end of Section II.B, we provide an ordering of swap rates. To sign the difference between s_0^r and s_0^{r-y}, we note that if

$$cov_0^{\mathbb{Q}} \left[\exp\left(-\int_0^T r_s ds \right), L_6(T) \right] < cov_0^{\mathbb{Q}} \left[\exp\left(-\int_0^T (r_s - y_s)\, ds \right), L_6(T) \right], \tag{A8}$$

then $E_0^{\mathbb{Q}} \left[L_6(T) \right] > s_0^{r-y} > s_0^r > s_0^R$. To understand this condition, consider a first-order approximation to the exponential $e^x = 1 + x$. This implies that the required condition is

$$\begin{aligned} cov_0^{\mathbb{Q}} \left[1 - \int_0^T r_s ds, L_6(T) \right] &< cov_0^{\mathbb{Q}} \left[1 - \int_0^T r_s ds + \int_0^T y_s ds, L_6(T) \right] \\ &= cov_0^{\mathbb{Q}} \left[1 - \int_0^T r_s ds, L_6(T) \right] + cov_0^{Q} \left[\int_0^T y_s ds, L_6(T) \right]. \end{aligned} \tag{A9}$$

Thus, if $cov_0^{\mathbb{Q}} \left[\int_0^T y_s ds, L_6(T) \right] > 0$, swap rates in the presence of costly collateral are higher than those assuming no default (or costless collateral). This condition holds if, for example, net collateral costs are positively correlated to either the default-free short rate, r_t, or the instantaneous TED spread, δ_t.

Appendix B: Term Structure Model Specification

This appendix provides the details of our dynamic term structure model. To evaluate the role of costly collateral, we need a model that characterizes the risk-free term structure and the LIBOR and swap market. We use a multifactor Vasicek (1977) style model with conditionally Gaussian factors, which is common for modeling swap rates (see, for example, Grinblatt (2001), Collin-Dufresne and Solnik (2001), He (2001), Liu, Longstaff, and Mandel (2001). Our goal in building a model is to have the simplest possible specification that allows us to characterize costly collateral.

The default-free term structure is given by the two-factor Gaussian model

$$
\begin{aligned}
dr_t &= k_r\left(\theta_t - r_r\right)dt + \sigma_r dW_t^r\left(\mathbb{P}\right) \qquad \text{(B1)}\\
d\theta_t &= k_\theta\left(\theta_\theta - \theta_t\right)dt + \sigma_\theta dW_t^\theta\left(\mathbb{P}\right),
\end{aligned}
$$

where θ_t is the central tendency factor. We assume a constant market price of interest rate risk, λ_r, so that the drift of r_t, under an equivalent martingale measure $\mathbb{Q}$, is given by $\left[k_r\left(\theta_t - r_r\right) - \lambda_r\right]$. We originally included a constant market price of risk for θ_t, but the estimate is insignificant (as in Collin-Dufresne and Solnik (2001)), and thus we set this coefficient to zero. We assume the Brownian motions are correlated with constant correlation $\rho_{r,\theta}$. To simplify estimation and identification, we follow Duffie, Pedersen, and Singleton (2002) and constrain θ_θ to be equal to the sample average of the three-month T-bill rate.

Since time-varying net collateral costs only matter if they are correlated with other term structure variables, we assume that y_t and δ_t are correlated in levels and their evolution depends on the level of the short-rate as shown below:

$$d\delta_t = \left[\kappa_\delta\left(\theta_\delta - \delta_t\right) + \kappa_{r,\delta}(r_t - \theta_\theta) + \kappa_{\delta,y}\left(y_t - \theta_y\right) - \lambda_\delta\right] dt + \sigma_\delta dW_t^\delta\left(\mathbb{Q}\right) \tag{B2}$$

$$dy_t = \left[\kappa_y\left(\theta_y - y_t\right) + \kappa_{y,r}(r_t - \theta_\theta) + \kappa_{y,\theta}(\theta_t - \theta_\theta) - \lambda_y\right] dt + \sigma_y dW_t^y\left(\mathbb{Q}\right).$$

Their dynamics under $\mathbb{P}$ are the same without the constant risk premium parameters. We de-mean the "off-diagonal" terms, which aids in identification and ensures that θ_y and θ_δ are interpreted as long-run means. We originally included an off-diagonal term, $\kappa_{y,\delta}$, but it is not significant and we set this coefficient to zero. As it is not possible to separately identify correlations and off-diagonal terms, we assume that W_t^δ and W_t^y are independent.

To estimate the model, we follow Duffie, Pedersen, and Singleton (2002) and use a two-stage maximum likelihood procedure. In the first stage, we estimate the two-factor default-free term structure using Treasuries. The second stage uses LIBOR and swap rates to estimate the parameters indexing δ_t and y_t. We fit the three-month and seven-year Treasury rates without error and the three-, five-, and ten-year rates with error. In using Treasuries for the default-free curve, we follow Collin-Dufresne and Solnik (2001) and Liu, Longstaff, and Mandel (2001) and use Treasury rates to extract information about the default-free term structure. One could alternatively use either term Federal Funds or general collateral repo rates, although these series are short dated and seriously polluted by microstructure noise (e.g., settlement Wednesdays).[12]

In the second stage, we take the first-stage parameters and state variables as given and estimate a two-factor model for the LIBOR/swap market. We fit the three-month LIBOR rate and seven-year swap rates without error and the three-, five-, and ten-year rates with error. The two-step procedure sacrifices asymptotic statistical efficiency. The informational loss is measured by the information contained in the LIBOR/swap curve regarding the default-free parameter estimates and is likely to be small.

References

Bank for International Settlements, 2001, Collateral in wholesale financial markets: Recent trends, risk management and market dynamics, Prepared for the Committee on the Global Financial System.

Bielecki, Tomasz, and Marek Rutkowski, 2002, *Credit Risk: Modeling, Valuation and Hedging,* (Springer, New York).

Brown, Keith, and Donald Smith, 1993, Default risk and innovations in the design of interest rate swaps, *Financial Management* 22, 94-105.

Brown, Keith, and Donald Smith, 1995, *Interest Rate and Currency Swaps: A Tutorial*, The Research Foundation of the Institute of Chartered Financial Analysts, (Charlottesville, Virginia).

Clarke, Michael, 1999, An introduction to the legal aspects of collateralization, Technical report, JP Morgan.

Collin-Dufresne, Pierre, Robert Goldstein and Chris Jones, 2004, Identification and estimation of 'maximal' affine term structure models: An application to stochastic volatility, Working paper, Carnegie Mellon.

Collin-Dufresne, Pierre, and Bruno Solnik, 2001, On the term structure of default premia in the swap and LIBOR markets, *Journal of Finance* 56, 1095-1115.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1981, The relation between forward prices and futures prices, *Journal of Financial Economics* 9, 321-346.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385-408.

Dai, Qiang, and Kenneth Singleton, 2000, Specification Analysis of Affine Term Structure Models, *Journal of Finance* 55, 1943-1978.

Daigler, Robert, and Donald Steelman, 1988, Interest rate swaps and financial institutions, Working paper, Kidder Peabody, New York.

Duffie, Darrell, and Ming Huang, 1996, Swap rates and credit quality, *Journal of Finance* 51, 921-950.

Duffie, Darrell, and Kenneth Singleton, 1997, An econometric model of the term structure of interest rate swap yields, *Journal of Finance* 52, 1287-1323.

Duffie, Darrell, and Kenneth Singleton, 1999, Modeling term structure models of defaultable bonds, *Review of Financial Studies* 12, 687-720.

Duffie, Darrell, Lasse Pedersen, and Kenneth Singleton, 2002, Modeling sovereign yield spreads: A case study of Russian debt, *Journal of Finance* 55, 119-159.

Grinblatt, Mark, 2001, An analytic solution for interest rate swap spreads, *International Review of Finance* 2, 113-149.

Grinblatt, Mark, and Narasimhan Jegadeesh, 1996, The relative pricing of Eurodollar futures and forward contracts, *Journal of Finance* 51, 1499-1522.

Gupta, Anurag, and Marti Subrahmanyam, 2000, An empirical examination of the convexity bias in the pricing of interest rate swaps, *Journal of Financial Economics* 55*, 239-279.*

He, Hua, 2001, Modeling term structures of swap spreads, Working paper, Yale University.

Hull, John, and Alan White, 1990, Pricing interest rate sensitive securities, *Review of Financial Studies* 3, 573-592

International Swaps and Derivatives Association, Inc., 1994, ISDA Credit Support Annex.

International Swaps and Derivatives Association, Inc., 1998, Guidelines for Collateral Practitioners.

International Swaps and Derivatives Association, Inc., 1999, ISDA 1999 Collateral Review.

International Swaps and Derivatives Association, Inc., 2000, ISDA Collateral Survey 2000.

International Swaps and Derivatives Association, Inc., 2001, ISDA Margin Survey 2001.

International Swaps and Derivatives Association, Inc., 2003, ISDA Margin Survey 2003.

Jarrow, Robert, and Gary Oldfield, 1981, Forward contracts and futures contracts, *Journal of Financial Economics* 9, 373-382.

Litzenberger, Robert, 1992, Swaps: Plain and fanciful, *Journal of Finance* 42, 403-417.

Liu, Jun, Francis Longstaff, and Ravit Mandell, 2001, The Market Price of Credit Risk, Working paper, UCLA.

Longstaff, Francis, 2004, The Flight to Liquidity Premium in U.S. Treasury Bond Prices, *Journal of Business* 77, 511-526.

Lowenstein, Roger, 2000, *When Genius Failed: The Rise and Fall of Long-Term Capital Management* (Random House, New York).

Muelbroek, Lisa, 1992, A comparison of forward and futures prices of an interest rate-sensitive financial asset, *Journal of Finance* 47*, 381-396.*

Richard, Scott, and Suresh Sundaresan, 1981, A Continuous time equilibrium model of forward prices and futures prices in a multigood economy, *Journal of Financial Economics* 9, 347-371.

Santos, Tano, and Jose Scheinkman, 2001, Competition among Exchanges, *Quarterly Journal of Economics* 116, 1027-1061.

Sun, Tong-Sheng, Suresh Sundaresan, and Ching Wang, 1993, Interest rate swaps, an empirical investigation, *Journal of Financial Economics* 34, 79-99.

Sundaresan, Suresh, 1991a, The valuation of swaps, in Sarkhis Khoury, ed.: *Recent Developments in International Banking and Finance* (Elsevier North-Holland, New York).

Sundaresan, Suresh, 1991b, Futures prices on yields, forward prices and implied forward prices from the term structure, *Journal of Financial and Quantitative Analysis* 26, 409-424.

Tuckman, Bruce, 2002, *Fixed Income Securities: Tools for Today's Markets*, 2nd Edition (John Wiley & Sons, New York).

Vasicek, Oldrich, 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177-188.

Notes

[1]For example, Tuckman (2002) notes that "it becomes clear that there is an internal inconsistency in the convention for pricing swaps... On the one hand, it is assumed that there is no risk of counterparty default; The cashflows are assumed to be as specified in the swap agreement... On the other hand, all cashflows are discounted at LIBOR or swap rates" (p. 390).

[2]Cox, Ingersoll, and Ross (1981), Richard and Sundaresan (1981), and Jarrow and Oldfeld (1981) provide theoretical arguments for the differences and, in the case of interest rate sensitive securities, Sundaresan (1991b), Muelbroek (1992), and Grinblatt and Jegadeesh (1996) examine the empirical evidence.

[3]As an alternative to collateralized swaps, in the late 1980s a number of banks offered a contract known as a "mark-to-market swap" in which the fixed swap rate was adjusted on each swap date (see Brown and Smith (1993, 1995)). This contract, which fundamentally changes the nature of the contract (the fixed rate is no longer fixed), never became popular. We thank Keith Brown for pointing this contract out to us and for helpful discussions on the evolution of the swap market.

[4]When LTCM was going under, John Meriwether apparently said that if the firm survived, "I guess we would call ourselves 'No-Haircut Capital Management'" (Lowenstein

(2000), p. 177).

[5]Technically, this statement is not completely correct. In unusual cases, certain other creditors are exempt from the automatic stay provisions. These include government agents (e.g., local police departments) exercising their regulatory powers and certain Federal agencies (e.g., Housing and Urban Development) with monetary claims against the debtor. These exemptions rarely, if ever, arise in the context of the bankruptcy of a party to a derivatives transaction. We thank Ed Morrison for pointing this out.

[6]The Financial Institutions Reform, Recovery, and Enforcement Act of 1989 also implies that collateral posted by commercial banks (regulated entities outside of the U.S. Bankruptcy Code) can be seized.

[7]Sundaresan (1991a) finds that the difference between the fixed swap rate when the floating payments are settled in-arrears and those settled contemporaneously is a fraction of a basis point. In a slightly different setting, Duffie and Huang (1996) argue that the differences are negligible.

[8]This well-known positive difference between futures and forward rates is typically called the "convexity" correction. See, for example, Sundaresan (1991b), Grinblatt and Jegadeesh (1996), and Gupta and Subramanyam (2000).

[9]At first glance, the assumption of common collateral costs is not consistent with the

fact that a very diverse group of counterparties participates in the swap market, and that these counterparties presumably have very different funding costs. This assumption is not unique to swaps, as the same assumption is commonly used to price futures, another market populated by diverse counterparties with different costs of posting collateral. In a formal model, Santos and Scheinkman (2001) study this problem and argue that there is nothing puzzling about counterparties with different credit quality trading at a common price. Intuitively, those with high collateral costs enter into fewer contracts, thereby adjusting through quantities rather than prices. In their model, the collateral costs that are reflected in market prices are those of the marginal market participant.

[10]For the results using the first seven years of futures data, the results hold, and are even stronger, at the ten-year maturity. The results are also qualitatively and quantitatively unchanged if the whole futures curve is used for estimation. We thank the referee for encouraging us to investigate these issues.

[11]We thank Ken Singleton for discussions regarding the specification and the interpretation of the factors.

[12]Even if clean series for the these variables were available, our results would not likely change. The reasoning is as follows. Standard models indicate that swap spreads are properly amortized present discounted values of short-term spreads. In our case, the short-term spread is LIBOR-T-bill, which is about 35 basis points on average. As short-term

spreads are highly volatile and rapidly mean-reverting, these models imply the same for swap spreads. However, as He (2001) and Liu, Longstaff, and Mandel (2001) note, swap spreads tend to be persistent and much larger (60-70 basis points) than the present value of the short-term spreads. If instead we used the LIBOR-GC repo short-term spread, the problem would be even worse: This series is also rapidly mean-reverting, but it has a mean of only about 15 basis points, in which case, our collateral factor would likely play an even greater role.

Table I

Vasicek Calibration Results

For each Wednesday from 1994 to 2002, we calibrate the Vasicek (1977) model to fit the Eurodollar futures curve. Given the calibrated model, we compute hypothetical swap rates assuming swaps are priced via the par representation (portfolio of forwards) and as a portfolio of futures contracts. The columns marked forwards (futures) give the difference between the hypothetical swap priced as a portfolio of forwards (futures) and the market swap rate. The standard errors are in parentheses.

	5 Year		7 Year	
Time Period	Forwards	Futures	Forwards	Futures
1994-2002	-12.3 (0.4)	1.4 (0.3)	-12.7 (0.4)	8.4 (0.3)
1994	-8.8 (0.7)	7.2 (0.7)	-11.2 (1.1)	12.6 (0.7)
1995	-14.6 (0.8)	-2.5 (0.9)	-12.5 (0.7)	6.7 (0.7)
1996	-18.8 (0.8)	3.4 (0.9)	-24.0 (1.0)	10.3 (0.7)
1997	-13.3 (0.6)	0.3 (0.5)	-15.9 (0.9)	5.7 (0.5)
1998	-14.5 (0.6)	-6.0 (0.6)	-13.8 (0.5)	-0.2 (0.5)
1999	-6.1 (0.7)	3.2 (0.7)	-6.4 (0.6)	8.6 (0.6)
2000	-12.5 (1.1)	-3.6 (1.1)	-11.5 (0.9)	2.4 (0.9)
2001	-15.1 (0.1)	-1.1 (1.0)	-10.9 (0.9)	8.6 (1.0)
2002	-7.6 (1.6)	11.8 (1.5)	-7.9 (1.4)	20.8 (1.0)

Table II

Cox, Ingersoll, and Ross Calibration Results

For each Wednesday from 1994 to 2002, we calibrate the Cox, Ingersoll, and Ross (1985) model to fit the Eurodollar futures curve. Given the calibrated model, we compute hypothetical swap rates assuming that swaps are priced via the par representation (portfolio of forwards) and as a portfolio of futures contracts. The columns marked forwards (futures) give the difference between the hypothetical swap priced as a portfolio of forwards (futures) and the market swap rate. The standard errors are in parenthesis.

	5 Year		7 Year	
Time Period	Forwards	Futures	Forwards	Futures
1994-2002	-12.0 (0.4)	3.5 (0.5)	-14.7 (0.4)	10.0 (0.5)
1994	-14.4 (0.1)	7.7 (0.7)	-19.6 (0.9)	13.1 (0.7)
1995	-16.0 (0.7)	-1.6 (0.8)	-16.6 (0.7)	7.4 (0.7)
1996	-13.3 (0.9)	3.4 (0.9)	-16.3 (0.9)	10.2 (0.7)
1997	-12.5 (0.5)	1.2 (0.5)	-16.3 (0.4)	6.4 (0.5)
1998	-17.3 (0.5)	-5.8 (0.5)	-18.9 (0.5)	0.1 (0.5)
1999	-8.7 (0.6)	4.5 (0.8)	-12.3 (0.5)	9.7 (0.7)
2000	-17.0 (1.0)	-3.1 (1.1)	-19.8 (0.8)	2.9 (0.9)
2001	-5.6 (1.9)	5.6 (2.0)	-3.4 (1.7)	13.8 (1.7)
2002	-2.9 (1.6)	19.4 (1.9)	-8.6 (1.2)	26.5 (1.4)

Table III

Treasury and LIBOR/Swap Summary Statistics

Summary statistics of interest rate data used for estimation. All series are sampled weekly, on Wednesdays, from January 1990 to December 2002.

	Treasury		LIBOR/Swap	
	mean	std	mean	std
3-month	4.750	1.510	5.103	1.612
3-year	5.696	1.325	6.152	1.359
5-year	6.022	1.212	6.531	1.252
7-year	6.256	1.150	6.758	1.206
10-year	6.362	1.138	6.968	1.156

Table IV

Estimation Results

Two-stage maximum likelihood estimates obtained using weekly Treasury and LIBOR/swap market data from 1/2/1990 to 10/29/2002. We calculate the standard errors using the outer-product of the scores.

Parameter	Estimate	S.E.	Parameter	Estimate	S.E.
$\kappa_\theta \times 10$	8.323	0.621	$\lambda_\delta \times 10^2$	-1.447	0.353
$\theta_\theta \times 10^2$	4.65	-	$\theta_y \times 10^3$	6.862	1.916
$\sigma_\theta \times 10$	1.549	0.228	$\sigma_y \times 10^2$	1.019	0.174
$\kappa_r \times 10$	9.069	1.027	$\lambda_y \times 10^3$	6.668	1.781
$\sigma_r \times 10^3$	8.135	0.131	$\kappa_y \times 10^2$	4.527	1.499
$\lambda_r \times 10^2$	-5.275	0.376	$\kappa_{y,r} \times 10$	1.141	0.234
$\rho_{r,\theta} \times 10$	-3.627	0.383	$\kappa_{y,\theta} \times 10^2$	-3.830	2.243
κ_δ	1.669	0.147	$\kappa_{\delta,y} \times 10$	8.038	1.558
$\sigma_\delta \times 10^3$	9.116	1.740	$\kappa_{\delta,r} \times 10$	-1.633	0.436

Table IV

Pricing Errors

Mean and root-mean squared errors (RMSE) in basis points for the Treasury and swap yields that are fitted with error. Three-month Treasury and LIBOR and seven-year Treasury and swap rates are fit perfectly in the two-stage maximum likelihood procedure.

		3-Year	5-Year	10-Year
Treasuries	Mean	-0.254	-0.261	0.133
	RMSE	4.66	3.15	2.78
Swaps	Mean	0.351	0.161	0.045
	RMSE	4.60	2.25	2.97

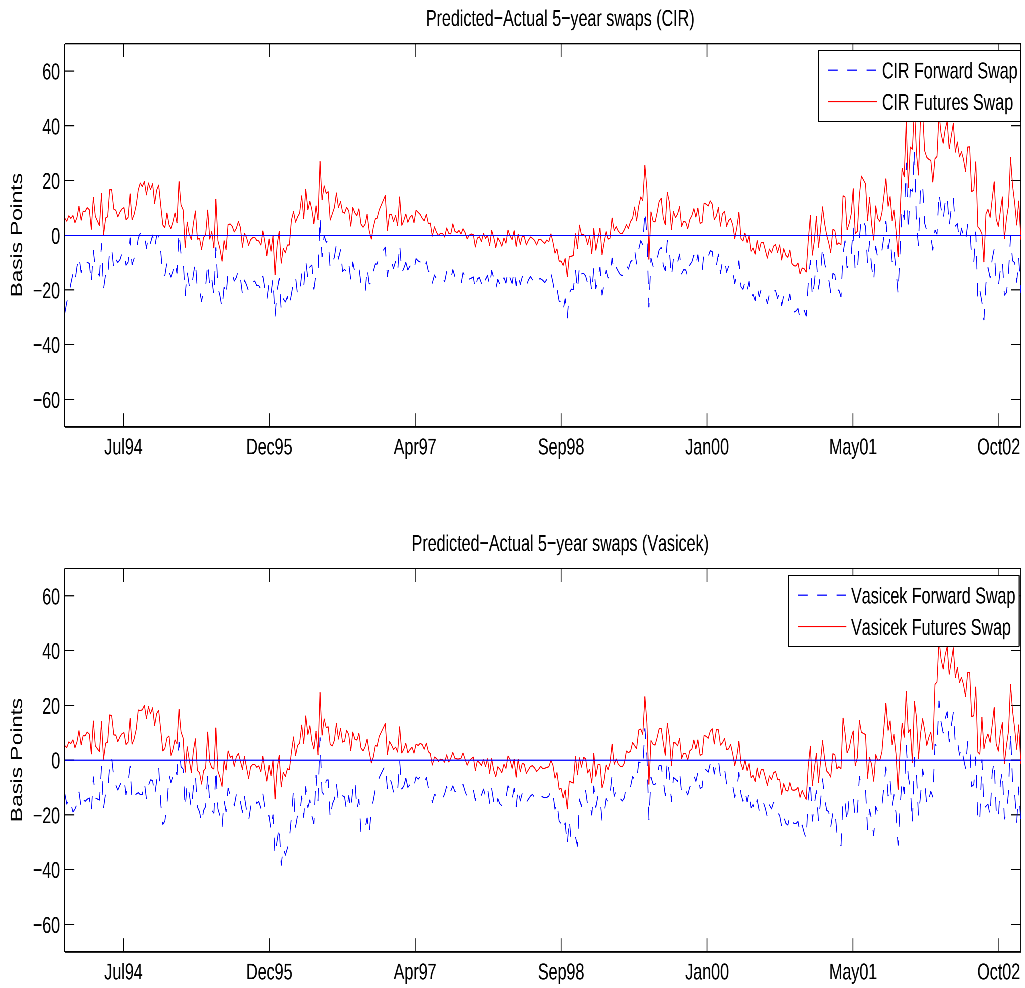

Figure 1. Dynamics of five-year swap rates. This figure provides weekly time series of the differences between the hypothetical futures (solid line) and forwards (dashed line) and market swap rates for the Vasicek and CIR models at the five-year maturity.

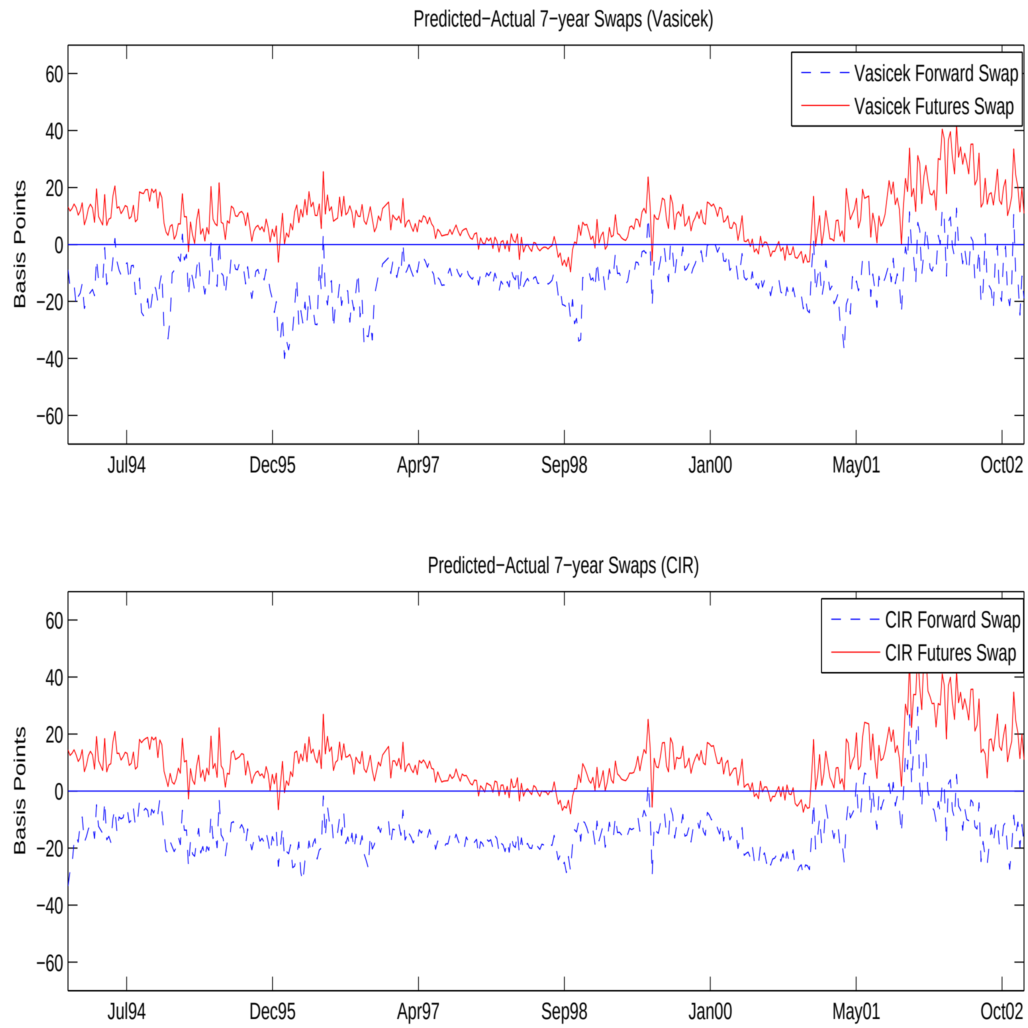

Figure 2. Dynamics of seven-year swap rates. This figure provides weekly time series of the differences between the hypothetical futures (solid line) and forwards (dashed line) and market swap rates for the Vasicek and CIR models at the seven-year maturity.

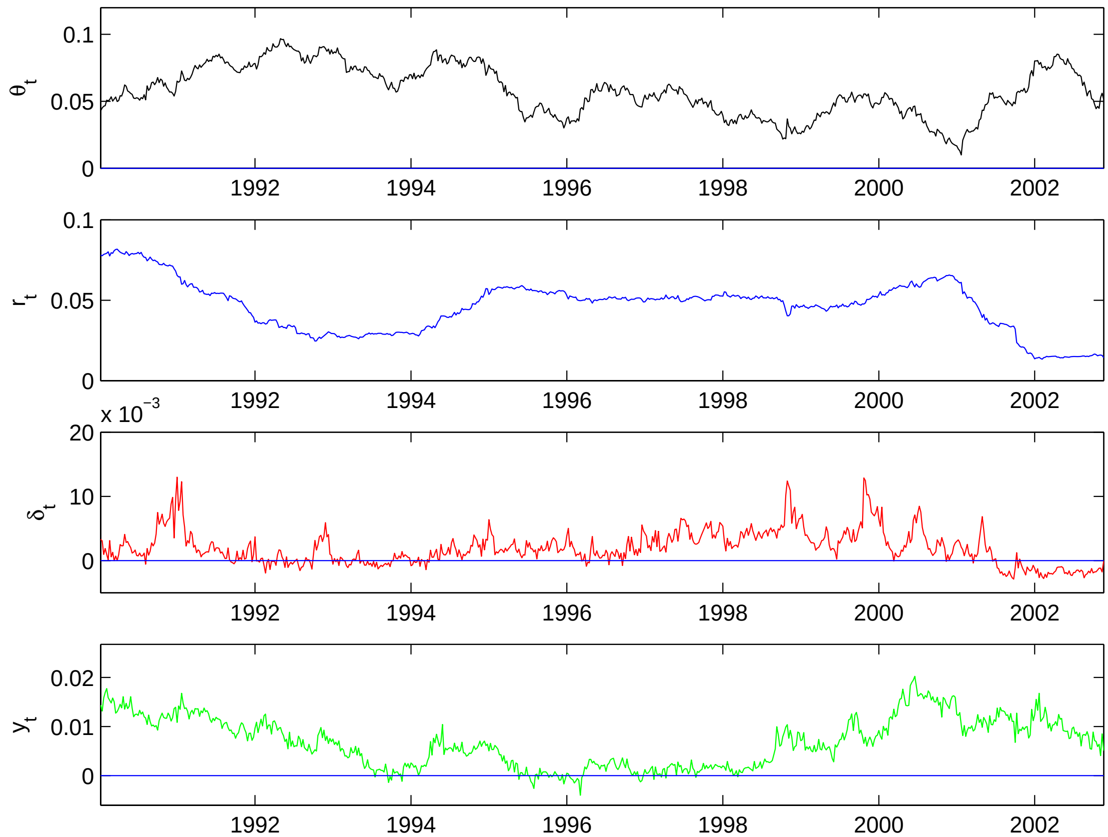

Figure 3. State variable dynamics. This figure provides time series of the inverted factors: the central tendency (top panel), the short rate (second panel), the instantaneous spread from Treasuries to LIBOR (third panel), and the net cost of collateral process (bottom panel).

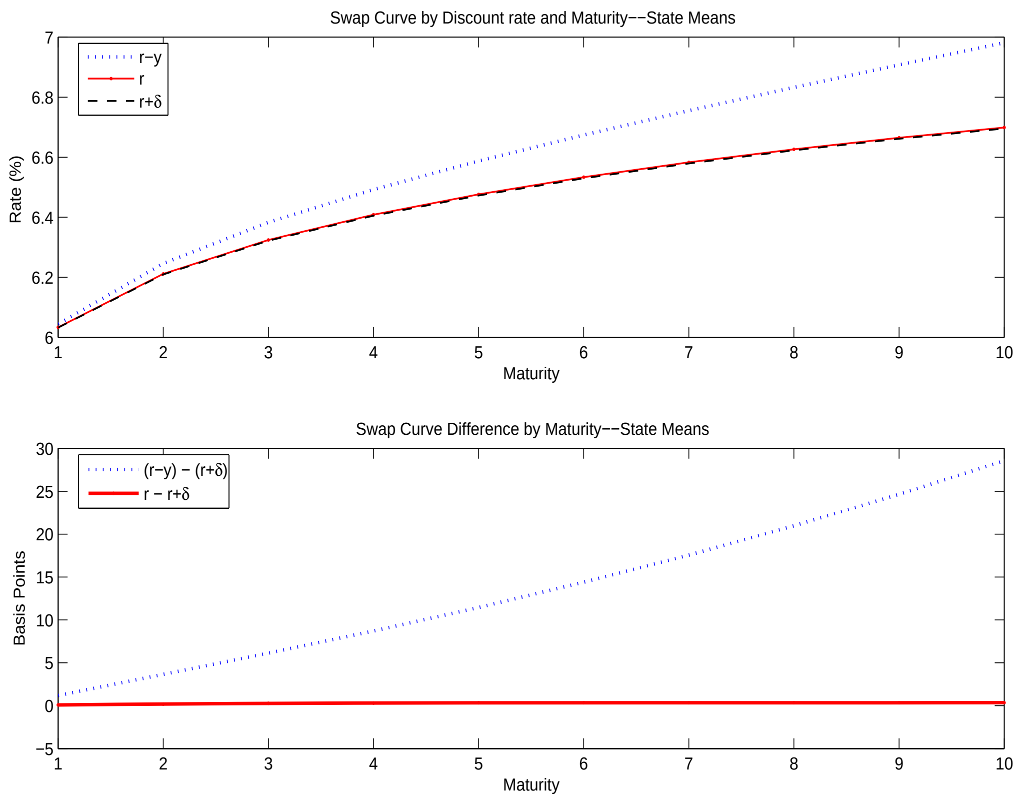

Figure 4. The impact of collaterization on swap curves. The top panel provides various swap curves. The dashed line gives the par swap curve (discounted at $R_t = r_t + \delta_t$), the solid line gives the default-free swap curve (discounted at r_t), and the dotted line gives the collateralized swap curve (discounted at $r_t - y_t$) using the average values for the state variables. The bottom panel displays the difference between collateralized swap rates, the swap rates implied by the par representation, and the swap rates discounted at r_t.