

Uncertain Times Ahead

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COMMENTS ARE HIGHLY APPRECIATED

Abstract

Periods of economic crisis are often accompanied or followed by periods of uncertain inflation threads. The current financial crisis is the newest example of uncertain inflation times that lie ahead. This paper extends work of Andersen, Hansen, and Sargent (2003). It derives and estimates a general equilibrium model for the real and nominal term structure of U.S. government bonds, where the investor has model misspecification doubts about the expected future path of inflation. The model distinguishes between an inflation risk premium in nominal bonds and an inflation ambiguity premium in real and nominal bonds. The empirical section estimates a special version of the model, namely with what we call log ambiguity aversion. The empirical detection error probability of the log ambiguity set-up is 36.5%, showing that the statistical distance between the worst-case and reference inflation model is very small and hard to detect with already 37 years of macro and bond data. Nevertheless, the ambiguity premium has several interesting cross-sectional and time-series implications for real bonds, nominal bonds and inflation forecasts.

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Periods of economic crisis are often accompanied or followed by periods of uncertain inflation threads. Economic recessions in 1980, 1981/1983, the Monetary Policy Experimentation in 1979/1983 are examples of economic turbulences which were accompanied or followed by excessive economic uncertainty about future inflation. The current financial crisis is the newest example of uncertain times that lie ahead. The Federal Reserve has stepped in bravely into the crisis and has applied nontraditional measures of monetary policy to ease the magnitude and shorten the length of the current financial crisis. Quantitative Easing has been one of these nontraditional measures. While some market participants have predicted a deflationary period to lie ahead, others have warned quantitative easing of the central bank will create massive inflation in the future, as seen during the period of the Monetary Policy Experimentation. In the spirit of Knight (1921), one can interpret such a period of inflation uncertainty as a period where investors have model misspecification doubts about the future path of expected inflation.¹

Inflation forecasts and fear of inflation misspecification are traded daily on the Government bond market. In order to extract inflation forecasts and worst-case inflation expectations for different time-horizons, one needs a general equilibrium bond and macro model that incorporates investors fear of model misspecification. This paper, closes this gap in the literature by specifying and estimating such a general equilibrium model for the real and nominal term structure of U.S. government bonds. The proposed model accounts for two sources of inflation premiums. The first premium is determined by the covariation of marginal utility and inflation. This is the standard inflation risk premium which is present in nominal bonds. In contrast to that, the second inflation premium accounts for model misspecification doubts about future inflation. It is present in real bonds and in nominal bonds. For real bonds it is determined as the conditional covariance between marginal utility and the amount of model misspecification doubts. For nominal bonds it is determined as the conditional covariance between nominal marginal utility and the amount of model misspecification doubts. The representative investor takes into account that the underlying inflation model could be misspecified. The greater the investor's uncertainty aversion, the more the investor is concerned about model misspecification and the higher the absolute value of the ambiguity premium in real and nominal government bond yields.

In the theoretical part of the paper, we introduce model misspecification doubts about expected inflation into a monetary version of a Cox, Ingersoll, and Ross (1985b) economy. Equilibrium prices of inflation-protected Government bonds (TIPS) and nominal bonds contain an inflation ambiguity premium. We determine the endogenous amount of inflation misspecification by solving an agent's robust control problem in the spirit of Anderson, Hansen, and Sargent (2003).

The max-min approach of the robust control problem is useful for the investor if she

¹Model misspecification doubts according to Anderson, Hansen, and Sargent (2003) and Hansen and Sargent (2007).

fears that future expected inflation might come from a set of models which are close to her reference model. The max-min solution can be divided into two separate steps. First, the investor suspects that her model for expected inflation is not the true description of future expected inflation. The investor therefore seeks an inflation model that works well across a set of models which are in a statistical sense close to her reference model and difficult to distinguish based on past data. Second, given that the agent has found her robust inflation model, she maximizes her life-time expected utility under this robust inflation model. We determine the term structure of real bonds and nominal bonds in closed-form.

In addition to a CRRA utility function with constant relative risk aversion of one, model uncertainty averse investors have an additional preference parameter which measures how strongly the investor likes or dislikes that the worst-case model deviates from the reference model. A positive but decreasing ambiguity preference parameter indicates that the investors fear of model misspecification has gone up. The investor perceives a stronger desire to protect herself against a model misspecification. It becomes therefore optimal to choose a robust inflation model which is further away from the reference inflation model.

The intuition for the market price of inflation ambiguity is the following. Times of increased inflation uncertainty are characterized by an increase in inflation volatility. Under such a scenario, the set of potential inflation models increases. Depending on the empirical correlation between inflation and consumption, it is optimal for robustness seeking investors to distort the growth rate of consumption by the latter correlation times the optimal amount of inflation misspecification. Empirical evidence and a thoroughly analysis of Piazzesi and Schneider (2006) argue for a negative correlation between inflation and consumption. Such a negative correlation induces the following economic forces in our model: If inflation becomes less predictable in uncertain times (higher inflation variance), the investor's fear about misspecification of her inflation model increases which forces her to push up her robust inflation forecast by a small positive amount. Doing this lowers at the same time her belief about future consumption growth by the former correlation times the robust inflation markup. As a result, the investor acts more cautiously by building up precautionary savings. It is optimal for a robust seeking investor to build up precautionary savings because she wants to have enough wealth in the future to smooth her consumption in case that the inflation model turns out to be worse than expected under her reference model.

The economic intuition for the impact of inflation ambiguity on the real interest rate depends crucially on the correlation of news on expected inflation and news on future consumption. A negative correlation as observed in the data and in Piazzesi and Schneider (2006) means positive news about expected inflation (increase in expected inflation) carries negative information about future consumption growth. A robust decision maker who doubts to know the true data generating process for expected inflation wants to increase her precautionary savings. This increase is optimal because compared to a rational expectations equilibrium outcome, the investor worries that expected inflation

might be higher in the future than expected under the reference model. The negative correlation with consumption indicates that under such a scenario, future consumption growth will be lower than under the reference model. A robustness seeking investor wants to build up some additional savings today to have some cushion in case the worst-case model turns out to be the true model. In our general equilibrium model, the real interest rate must fall to offset the ambiguity induced desire for a higher savings rate.

The economic components of the equilibrium nominal short-term interest rate are the real interest rate, expected inflation under the reference measure, and inflation premiums. The inflation premium is a linear function in the inflation risk premium and in the inflation ambiguity premium. The inflation risk premium is positive if inflation and marginal utility are positively correlated. A positive correlation means that positive inflation news are positive news for future marginal utility. The latter coincides with bad news for future consumption [see Piazzesi and Schneider (2006)]. In addition, and very importantly for nominal term structure modeling, the inflation ambiguity premium in the nominal short-rate is positive, contributing to a positive slope in the nominal term structure.²

The economic implication of inflation ambiguity on real bonds is best understood if one compares its equilibrium price under rational expectations with its price if the investor has model misspecification doubts. The difference of both prices is in our model entirely attributed to the conditional covariation of marginal utility and expected inflation. If expected inflation is negatively correlated with consumption the conditional covariation between marginal utility and expected inflation is positive. This induces a higher equilibrium price for a real bond under model misspecification doubts compared to the rational expectations counterpart. The mirror image of a higher equilibrium price is a lower equilibrium yield. Investors with model misspecification doubts about inflation regard the real bond as a hedge against bad consumption news and increased inflation misspecification doubts. Periods of increased inflation uncertainty coincide with periods of low consumption growth. A real bond pays out a lot in periods of low consumption growth. Since marginal utility is high in these periods, real bonds are highly valued by investors because they constitute a hedge against increased inflation ambiguity and bad consumption news.

The economic implication of inflation ambiguity on nominal bonds is very different. Nominal bonds payout in nominal units and do therefore not constitute a hedge against inflation misspecification doubts. A negative correlation between consumption and inflation makes the real payout of nominal bonds to be lowest when inflation and marginal utility are highest. The endogenous increase in the robust inflation forecast reduces the expected real payout of the nominal bond even further, which makes the nominal bond even less attractive in real units. A nominal Government bond must therefore pay a positive inflation ambiguity premium to fulfill the zero net supply equilibrium condition.

²Compare Ang and Piazzesi (2003), Joslin, Priebsch, and Singleton (2009), Piazzesi (2003) and others for empirical evidence.

In the empirical section of the paper, we estimate the general equilibrium model of the nominal and real term structure. The estimation takes the structure and parameter restrictions of the general equilibrium economy into account. The rich equilibrium restrictions that the model poses on the interest rates, market prices of risk, market price of ambiguity and expectations on macroeconomic fundamentals ensure that model parameters and states are well identified by empirical data. The model is estimated by a rich panel of bond yield and macro data at a monthly frequency for the period January 1972 to June 2009.

The overall inflation ambiguity premium in nominal bond yields is upward sloping, being close to zero for the five year nominal bond and 40 basis points for the ten-year nominal bond. The upward sloping inflation ambiguity premium in nominal bond yields is inherited from an upward sloping ambiguity premium in the robust (worst-case) inflation forecast. The ambiguity markup on five-year expected inflation has been on average 1.3 percent, while it has been on average 2.3 percent for the ten-year horizon. The positive slope means intuitively that investors fear inflation misspecifications more the longer the locked-in investment horizon.

The ambiguity premium in Tips yields is negative and downward sloping. The average inflation ambiguity premium on a five year Tips yield is -1.2 percent, while it is -2 percent for the ten-year Tips. The ambiguity premium in Tips is negative because Tips are a hedge instrument against inflation misspecification.

The cross-sectional pattern of the inflation ambiguity premium is stable across different subperiods. Real interest rates are lowered by inflation ambiguity, while robust inflation forecasts carry a positive premium. The size of the premiums vary for different subperiods. It has been highest during the Monetary Policy Experimentation period of 1979 to 1983 and lowest in the period before 1978.

All inflation forecasts (worst-case and empirical) fluctuate considerably over different subperiods. As expected, the difference between the worst-case inflation forecast and the empirical inflation forecast has been highest during the Monetary Policy Experimentation, reaching five percent for the ten-year bond. This endogenous markup sloped upwards during all periods. The model implied forecast of inflation under the empirical measure is downward sloping throughout all subperiods. This is consistent with a mean-reverting data-generating process for expected inflation.

During the current financial market turmoil ambiguity induced robust inflation expectations increased in two waves from nearly zero to a peak of 1.5 percent at the beginning of 2009, before reverting back to pre-crisis levels in summer 2009. This sharp increase during the early 2009 coincides with increasing concerns about a changing inflation model in the near future given that the money supply has been dramatically increased during the financial crisis. The reduction in the worst-case inflation forecast to pre-crisis levels, during the summer of 2009, shows that the Fed has done a good job in managing inflation expectations.

The average nominal yield curve is positive throughout all subperiods. It has been flattest during the Monetary Policy Experimentation. The ambiguity premium in the robust inflation forecast has been strongly upward sloping during that period, offsetting the downward sloping empirical inflation forecast. Intuitively, this means that if investors knew the true model for expected inflation, the nominal yield curve would have sloped downward during that period.

Besides the mounting empirical evidence of an inflation ambiguity premium in the term structure of U.S. Government bonds it is crucial to understand how close, in a statistical sense, the worst-case and the reference inflation model are. We add to the existing literature on identifying the ambiguity premium by proposing a new method [compare Anderson, Hansen, and Sargent (2003), Maenhout (2006)]. Instead of calibrating ambiguity such that the detection error probabilities are at least ten percent, we set the preference parameter for ambiguity such that the asset pricing implications of the ambiguity premium are the ambiguity analog to a coefficient of relative risk aversion of 1. We therefore like to think in terms of a "log ambiguity" aversion. The advantages for the economic interpretation are important.

First, one can interpret the empirical results of our general equilibrium model as coming from a log risk and log ambiguity general equilibrium model. The econometrician has therefore not to worry that inflation premiums in our model are pumped up by increasing the ambiguity aversion to potentially implausible values, while keeping the risk attitude at a low level. Having a log risk and log ambiguity set-up sets both aversions to very low and comparable levels.

Second, the existing asset pricing intuition of log utility carries over to log ambiguity. By that we mean that log utility induces a rather low distortion between the reference measure and the risk-neutral measure. Analogously, log ambiguity induces a small distortion between the reference model for expected inflation and the worst-case model for expected inflation. The small distortion is revealed by the sample detection error probability of 36.5% which is more conservative than a detection error probability of 10% as suggested by Anderson, Hansen, and Sargent (2003) and Maenhout (2006). Under homoscedastic Gaussian shocks a detection error probability of 50% means that an investor cannot at all distinguish between the reference model and the worst-case model. The reason is that the unconditional probability of choosing the wrong model based on the sample likelihood is 50%, which coincides with the unconditional probability of choosing the right model by pure luck. Since our model has Gaussian shocks as well, a detection error probability of 36.5% ensures that the worst-case model and the reference model are so close to each other, that the investor cannot distinguish between both models based on past 37 years of data.

Taking all theoretical and empirical results together shows that a tiny and very difficult to detect amount of model misspecification about expected inflation has severe implications on the term structure of U.S. Government bonds. Ambiguity with regard to

expected inflation is difficult to detect in the data but generates interesting and plausible empirical regularities in the cross-section and time-series of U.S. government bonds.

1 Model

We begin by describing our model as a generalized Cox, Ingersoll, and Ross (1985b) economy. Our model is a generalization in two respects. First, aggregate productivity and inflation interact in the model in an empirical consistent way. Second, we relax the assumption that the investor has perfect knowledge about the law of motion of expected inflation. There is a huge empirical literature which documents that investors have imperfect knowledge about the exact conduct of Monetary Policy [Ang, Boivin, Dong, and Loo-Kung (2009), Clarida, Gali, and Gertler (2000), Ang, Dong, and Piazzesi (2008), and others]. One important goal of U.S. monetary policy is the creation of a stable inflation process. It is therefore natural to think that if investors have imperfect knowledge about the exact conduct of monetary policy, they also have imperfect knowledge about the future path of inflation. We take this as a motivation to study a production based asset pricing model where the representative investor has model misspecification doubts about the path of future inflation expectations.

1.1 Domain

Time is continuous and varies over $t \in [0, \dots, \infty)$. Economic risk is represented by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q^0)$, which satisfies the usual conditions. The probability measure Q^0 describes the reference model for the economy. Our model harbors the Rational Expectations equilibrium as a special case. For intuitive reasons it is useful to treat the solution of the reference model under Q^0 as the solution to the Rational Expectations model. All expectations in the reference model are taken under Q^0 . We denote these expectations as $E[\cdot]$ instead of $E^{Q^0}[\cdot]$. The probability measure for the robust economy will be determined endogenously in equilibrium. We will denote that measure as Q . We assume all processes lie within a well defined space and are adapted to the underlying filtration.

The concept of model uncertainty describes optimal decision making and robust pricing if investors do not know the true model for expected inflation. Instead, investors want to derive optimal decision rules which are robust with regard to small perturbations in the model for expected inflation. We call a potential perturbation h and assume that all necessary technical conditions for existence and smoothness are fulfilled. Following the work of Anderson, Hansen, and Sargent (2003) we define the set of potentially misspecified models via their statistical distance, h , from the reference model Q^0 . The statistical distance, or perturbation h , is measured in terms of the likelihood ratio of two potentially

correct models. In a statistical sense, a potentially correct model, is a model whose induced probability distribution is absolutely continuous to the reference probability distribution Q^0 . The absolute continuity property means intuitively that the potential models agree with the benchmark model about "tail events". This ensures that it is difficult in a statistical sense to distinguish between the potential models and the benchmark model. The likelihood ratio of any two absolute continuous measures is given as $E_t^Q \left[\log \frac{dQ_\tau}{dQ_\tau^0} \right]$, $\tau > t$ where Q is the probability measure of a potentially misspecified economy. Our Gaussian shock to expected inflation results in a particular convenient description of such a likelihood ratio, namely, $\frac{1}{2}h^2(t)$, where h is the perturbation or distance in the mean of a Gaussian expected inflation shock under Q and the same Gaussian shock under Q^0 . More precisely, the distance between both shocks is $h = (h(t))$, where the stochastic process h is called a density generator.

In the subsequent chapters, Q denotes the equilibrium robust (worst-case) probability measure. We call its distance to the reference measure Q^0 h_w . The lower index w symbolizes that it is the shock to expected inflation that is potentially misspecified.

2 Robust Control Problem

The insight of every reasonable structural asset pricing model is that the nominal interest rate in an economy is determined by an investors optimal consumption and investment decision and by inflation. Inflation is either exogenously given [Piazzesi and Schneider (2006), Brennan and Xia (2002)] or endogenously determined [Buraschi and Jiltsov (2005), Buraschi and Jiltsov (2007)]. We derive a production based economy in the style of Cox, Ingersoll, and Ross (1985a) and Cox, Ingersoll, and Ross (1985b) with exogenous processes for total factor productivity and inflation. We extend that framework by introducing empirically meaningful interactions between expected inflation and the real side of the economy and more importantly by allowing the representative investor to face model misspecification doubts with regard to expected Inflation.

We assume that the investor understands the model and has model misspecification doubts about expected inflation. The investor wants a robust decision rule which performs well across different inflation models. Models can only differ along the dimension of expected inflation. This means that smallest perturbations in expected inflation generate a new model. The introduction of a correlation between unpredictable shocks to aggregate productivity and expected inflation channels model misspecification doubts about expected inflation to model misspecification doubts about real activity.

The following paragraph provides some intuition about the economic meaning of model misspecification doubts, or said differently about ambiguous expected inflation shocks.³ Let dW^w be the Gaussian shock to expected inflation. This shocks has a zero

³Historically, decision making under model uncertainty has been developed through two strands of

mean under the reference model. An agent who has model misspecification doubts about this shock does not know whether this shock has indeed a mean of zero or whether it has a mean of h_w .

We have assumed sufficient smoothness conditions such that Girsanov's theorem allows us to represent the shock dW^w under a probability measure Q where $dW^{Q,w}$ has mean zero:

$$dW_t^{Q,w} = dW_t^w - h_w(t)dt. \quad (2.1)$$

The process h_w characterizes the distortion between both shocks. The relative entropy of the expected inflation shock is defined as

$$\mathcal{R}_t^w(Q) = \frac{1}{2} E_t^Q \left[\int_t^\infty e^{-\rho(s-t)} h_w^2(u) du \right], \quad (2.2)$$

where ρ is the investor's subjective discount factor.

The investor does not like if there exists a non-zero distance between the worst-case model and the reference model. She therefore adds a penalty term into her utility function which dynamically lowers her utility the further away both models deviate from each other.

In addition to a coefficient of relative risk aversion, model uncertainty averse investors have an additional preference parameter θ_w which measures how strongly the investor likes or dislikes that the worst-case model deviates from the reference model. A positive but decreasing value for θ_w indicates that the investor's fear of model misspecification has gone up.⁴ The investor perceives a stronger desire to protect herself against a model misspecification. It becomes therefore optimal to choose a robust probability measure Q which is further away from the reference measure Q^0 . On the other hand, θ_w going to infinity means that the investor does not fear a potential model misspecification.

We characterize the investor's risk aversion by a time separable CRRA utility function with a relative risk aversion of 1. The desire for robustness is modeled by adding a penalty term to the CRRA utility term which contains the preference for robustness θ_w and the amount of inflation misspecification in real activity, $R_t^w(Q)$:

$$u(c_t) = \log c_t + \frac{\theta_w}{2} h_w^2(t). \quad (2.3)$$

literature. The first strand is called ambiguity aversion and follows the seminal work of Gilboa and Schmeidler (1989) and was put forward in economics and finance by Epstein and Chen and Epstein (2002), Epstein and Schneider (2003), and Epstein and Miao (2003). We review recent applications in the literature section. The second strand follows the work of Anderson, Hansen, and Sargent (2003) and Hansen and Sargent (2007) and applies robust control techniques. We review recent applications in the literature section. We use the terms ambiguity, robustness and misspecified models interchangeably.

⁴Below we derive the market price of inflation ambiguity which will move towards infinity if θ_w moves towards zero.

The robust dynamic programming problem of the investor is a generalized rational expectations problem. It is a generalization because the investor does not know the probability distribution of expected inflation. She therefore first finds a robust transition density for inflation expectations which protects her against potential model misspecifications. The degree of robustness h_w^* that the investor requires in equilibrium depends on the preference parameter for robustness θ_w . This is the first step of her two-step optimization problem. In a second step, the agent maximizes life-time utility, assuming that expected inflation moves over time according to the endogenous robust transition density.

2.1 Production Economy

We assume that the representative investor owns a firm and is endowed with an initial capital stock K_0 . That firm owns a production technology $A = (A(t))$. This technology transforms invested capital $K = (K(t))$ into an output good $Y = (Y(t))$. The production technology is linear in capital, that is $Y(t) = A(t)K(t)$. Part of the produced output is consumed, c , and the remaining part is used for gross investments. The latter includes capital depreciation, δK , and net investments dK . The exogenous process for aggregate productivity follows:

$$\frac{dA}{A} = (\mu_A + \nu_A z(t))dt + \sqrt{\sigma_{Aw} + \sigma_{bw}w(t)}(\rho_{wA}dW^w + \sqrt{1 - \rho_{wA}^2}dW^A) + \sqrt{\sigma_{au} + \sigma_{bu}u(t)}dW^{\bar{A}} \quad (2.4)$$

where $z(t)$ is the stochastic component of the expected growth rate, $w(t)$ is the stochastic component in expected inflation. The scalar σ_{bw} captures the impact that the latter has on the production technology volatility. The process $u(t)$ models the stochastic part in consumption volatility that is unrelated to expected inflation. The parameters $\mu_A, \sigma_{Aw}, \sigma_{bw}, \sigma_{au}, \sigma_{bu}$ are positive scalars. The three Gaussian shocks $dW^w, dW^A, dW^{\bar{A}}$ are pairwise orthogonal to each other. The correlation coefficient $\rho_{wA} \in [-1, 1]$ captures the correlation between shocks to expected inflation and shocks to real activity. In the data that correlation is significantly negative, implying that times of increased inflation expectations coincide with bad times for consumption growth[Piazzesi and Schneider (2006)]. The linear production technology and affine aggregate productivity process induces a one-to-one relationship between consumption volatility and aggregate productivity volatility.

The intuition for the induced consumption volatility process is the following: The variance of aggregate consumption has two components. The first component correlates with expected inflation, meaning that times of increasing expected inflation coincide with times of increased consumption variance.⁵ The second component of consumption variance captures unpredictable variations that are orthogonal to inflation, therefore entirely driven

⁵The late 1970s and early 1980s as well as the Great Moderation, starting in the mid 1980s, are a well known examples of that relationship.

by real aggregate risk. A negative correlation between expected inflation and consumption growth makes expected inflation being a carrier of bad news. Piazzesi and Schneider (2006) (PS afterwards) build a model for the nominal and real term structure which shares several of our features. Investors dislike inflation because it tends to occur in times of low consumption growth. The main distinguishing feature between PS and our model is that PS assume an investor with recursive utility whereas our investor has time-separable utility combined with the preference for robust decision making in the spirit of Anderson, Hansen, and Sargent (2003). Buraschi and Jiltsov (2005) construct a monetary production based asset pricing model with taxes where the latter are determined based on the historic purchase price instead of current nominal replacement costs. Increases in inflation lower the after-tax real return on capital and carries therefore bad news about future consumption.

We assume an exogenous process for inflation, following the tradition of Brennan and Xia (2002), Piazzesi and Schneider (2006), Lettau and Wachter (2009), and others:

$$\frac{dp}{p} = (p_0 + p_1 w(t))dt + \sqrt{\sigma_{pv} + \bar{\sigma}_{pv} v(t)} dW^p + \sqrt{\sigma_{Aw} + \sigma_{bw} w(t)} dW^w \quad (2.5)$$

where $p_0, p_1, \sigma_{pv}, \bar{\sigma}_{pv}, \sigma_{Aw}, \sigma_{bw}$ are positive scalars. The scalars p_0 and p_1 measure the constant and time-varying part of expected inflation, respectively. The strong and persistent variations of the latter are captured by $w(t)$. The variance of inflation has two orthogonal components. The first one is driven by shocks to expected inflation while the second one is characterized by nominal risk that is orthogonal to expected inflation risk. Having expected inflation affect inflation variance is not crucial for the analysis, but it is supported in the data. Cogley and Sargent (2002) find evidence that especially in the late 1970s and early 1980s inflation variance positively co-moved with long-term inflation expectations. A positive σ_{bw} accounts for that data regularity.

We group all state variables into a state vector X . In particular, $X_t = [u_t \ v_t \ w_t \ z_t]$. All state variables are assumed to follow an orthogonal system of shifted Markovian square-root processes. The components of the system are conditionally and unconditionally orthogonal to each other. We therefore characterize without loss of generality the entire system through its marginal components:

$$dX_t^{(i)} = \kappa^i (\theta^i - X_t^{(i)})dt + \sqrt{\sigma_{0,X^{(i)}} + \sigma_{1,X^{(i)}} X_t^{(i)}} dW^{X^{(i)}}, \forall i \in \{u, v, w, z\} \quad (2.6)$$

where X^i stands for the i -th component of the state vector X , $\kappa^i, \theta^i, \sigma_{0,X^i}, \sigma_{1,X^i}$ are positive scalars. Notation X^1 refers to the state variable u , whereas X^4 refers to the state variable z .

The Markovian structure of the model allows to cast the dynamic robust optimization problem into a static robust Hamilton-Jacobi-Bellman approach [Anderson, Hansen, and Sargent (2003), Hansen and Sargent (2007)]. The next section describes the economic intuition behind the solution of that optimization problem and the appendix contains details on the derivation.

2.2 Robust Hamilton-Jacobi-Bellman Solution

Let J denote the value function. The standard robust HJB equation for a log utility investor with fear about model misspecification doubts on one state variable is given by

$$\rho J = \max_c \min_{h_w} \ln c_t + \frac{\theta_w}{2} h_w^2(t) + E_t^Q[dJ] \quad (2.7)$$

$$s.t. \frac{dK}{K} = \frac{dA}{A} - \frac{c}{K} dt - \delta dt \quad (2.8)$$

where ρ is the investor's time discount factor, $\ln c_t$ is her utility she derives from choosing a consumption level c_t , $\frac{\theta_w}{2} h_w^2(t)$ is the amount of disutility that the investor perceives if she chooses an amount of inflation robustness of $h_w(t)$ and $E_t^Q[dJ]$ summarizes the amount of optimal expected continuation utility. Q indicates that the expectation is taken under the robust probability measure. The second equation states the intertemporal budget constraint. That constraint says that the net growth rate in the capital stock has to equal the growth rate in production minus consumption rate and the capital depreciation rate.

The solution to the minimization problem, coincides with the first-order condition with regard to the optimal amount of inflation misspecification. The solution reveals that the amount of protection against misspecifications of expected inflation is time-varying over the business cycle:

$$h_w = m_w \sqrt{\sigma_{Aw} + \sigma_{bw} w}, m_w \in R^+. \quad (2.9)$$

$$(2.10)$$

The parameter m_w is a function of the preference parameter for robustness, θ_w , and of the correlation between real activity and expected inflation, ρ_{wA} . Data evidence argues for a negative correlation [Piazzesi and Schneider (2006)]. A negative correlation and positive θ_w results in a positive m_w . The endogenous parameter m_w is inversely related to θ_w . A reduction of θ_w towards zero pushes m_w towards infinity. Equation (2.9) shows that this pushes the market price of inflation ambiguity towards infinity as well. This shows that although m_w and θ_w carry the same information, m_w is more meaningful from an economic point of view. An m_w of one is the model uncertainty analog of a relative risk aversion of one in the risk premium.

Let's put some more intuition around the market price of inflation ambiguity, h_w . If inflation volatility goes up by $\sqrt{\sigma_{bw} \Delta_w}$, the set of potential inflation models increases by $m_w \sqrt{\sigma_{bw} \Delta_w}$, $m_w \in R^+$. Since expected inflation is negatively correlated with aggregate productivity, the ambiguity distorted growth rate on the latter changes by $\rho_{wA} m_w \sigma_{bw} \Delta_w < 0$. In economic terms this means that if expected inflation increases, inflation itself becomes less predictable (higher variance in inflation). The investor's fear about the misspecification of her inflation model increases which forces her to push up her robust inflation forecast by $m_w \sigma_{bw} \Delta_w$. Doing this lowers at the same time her belief about future consumption growth by ρ_{wA} times the robust inflation markup. As a result, the investor acts

more cautiously by building up precautionary savings, because she wants to have enough wealth in the future to smooth her consumption if the inflation model turns out to be worse than in her reference model.

The solution of the second stage problem in the robust optimization problem is a straight forward extension of CIR setup. The solution of that together with the equilibrium dynamics of the real and nominal stochastic discount factor are summarized in the proposition:

Proposition 1 *The representative investor optimally allocates a constant fraction of her wealth to consumption $c_t = \rho K_t, \forall t > 0$. The endogenous real and nominal interest rates ensure that the remaining part of her wealth at time t is invested into the firm. The equilibrium process for her capital stock is*

$$\frac{dK}{K} = \frac{dA}{A} - (\rho + \delta)dt. \quad (2.11)$$

The equilibrium real stochastic discount factor $\xi^r = (\xi^r(t))$ solves

$$\begin{aligned} -\frac{d\xi^r}{\xi^r} = & \left(E_t\left[\frac{dA}{A}\right] - \delta - \left\langle \frac{dA}{dA}, \frac{dA}{dA} \right\rangle + \rho_{wA}m_w(\sigma_{Aw} + \sigma_{bw}w(t)) \right) dt \\ & + \left(\frac{dA}{A} - E_t\left[\frac{dA}{A}\right] \right). \end{aligned} \quad (2.12)$$

The equilibrium real interest rate (real market price of risk) coincides with the drift (volatility) of the negative growth rate of the real stochastic discount factor. The equilibrium nominal stochastic discount factor $\xi^n = (\xi^n(t))$ solves

$$\begin{aligned} -\frac{d\xi^n}{\xi^n} = & \left(r(t) + E_t\left[\frac{dp}{p}\right] + h_w\sqrt{\sigma_{Aw} + \sigma_{bw}w(t)} - \left\langle \frac{dp}{p}, \frac{dp}{p} \right\rangle + \left\langle \frac{d\xi^r}{\xi^r}, \frac{dp}{p} \right\rangle \right) \\ & + \left(\frac{dA}{A} - E_t\left[\frac{dA}{A}\right] + \frac{dp}{p} - E_t\left[\frac{dp}{p}\right] \right). \end{aligned} \quad (2.13)$$

The nominal interest rate (nominal market price of risk) coincides with the drift (volatility) of the nominal stochastic discount factor.

Proposition 1 states that is optimal for the robust seeking investor to consume at every period t and every state $\omega \in \Omega$ a constant fraction of her capital stock. The remaining part of her wealth is invested into the firm. The equilibrium expected growth rate of her capital stock equals the expected growth rate of aggregate productivity minus the gross consumption rate (rate of consumption plus capital depreciation). The robust forecast of the instantaneous increase in the aggregate capital growth rate is changed by $\rho_{wA}m_w(\sigma_{Aw} + \sigma_{bw}w)$. Since ρ_{wA} is negative in the data, an inflation ambiguity averse investor reduces her prospects for the real economy.

The following subsections explain in more detail the economics behind the equilibrium real and nominal interest rate, as well as the real and nominal market prices of risk and market prices of ambiguity. Moreover, we will derive the term structure of real and nominal bond yields. This allows the empirical analysis of the cross-section of the inflation ambiguity premium for real and for nominal bonds.

2.2.1 Real Stochastic Discount Factor

Proposition 1 states the equilibrium real interest rate. Plugging the assumed model dynamics into Proposition 1 verifies that the real interest rate is given by:

$$r(t) = \mu_A + \nu_A z(t) - \delta - (\sigma_{Aw} + \sigma_{au}) - \sigma_{bw}w(t) - \sigma_{bu}u(t) + \rho_{wA}m_w(\sigma_{Aw} + \sigma_{bw}w(t)) \quad (2.14)$$

The real interest rate coincides with the expected growth rate of the economy minus consumption risk induced precautionary savings and minus inflation uncertainty induced precautionary savings. The last term is due to the preference of the investor to be robust with regard to potential inflation misspecifications. The latter term is the only difference between the real risk-free rate in an otherwise equal rational expectations equilibrium. The impact of inflation ambiguity on the real interest rate depends crucially on the correlation of expected inflation and consumption. A negative correlation as observed in the data means positive news about expected inflation (increase in expected inflation) carries negative information about future consumption growth. A robust decision maker who doubts to know the true data generating process for expected inflation wants to increase her precautionary savings. This increase is optimal because compared to a rational expectations equilibrium outcome, the investor worries that expected inflation might be higher in the future than expected under the reference model. A negative correlation between expected inflation and consumption growth tells the investor that under the worst-case inflation model, future consumption growth will be lower than under the reference model. A robustness seeking investor wants to build up some additional savings today to have some cushion in case the worst-case inflation model turns out to be the true model. It is the nature of a general equilibrium model, that the real interest rate must fall to offset the ambiguity induced desire for a higher savings rate.

The real interest rate in an economy with a misspecified inflation process has two precautionary savings terms. The first, $\langle \frac{dA}{dA}, \frac{dA}{dA} \rangle$ coincides with the rational expectations counter part under log utility. The second one, $\rho_{wA}m_w(\sigma_{Aw} + \sigma_{bw}w(t))$, arises because an underestimated inflation growth rate leads to lower than expected consumption growth rates in the future.

It is intuitive that as long as $\rho_{MA} \neq 0$ ⁶, the real interest rate differs from the rational expectations counterpart. In the set-up where $m_w = \frac{1}{\rho_{wA}}$, a one percent increase in consumption volatility and a one percent increase in the volatility of inflation expectations

⁶That event happens with probability one.

lower the real interest rate by one percent. The latter arises because of inflation ambiguity, whereas the first arises because of logarithmic consumption risk aversion. This has several interesting economic implications. First, as long as $|m_w| < \frac{1}{|\rho_{wA}|}$ logarithmic consumption risk aversion induces a higher precautionary savings term than fears of inflation misspecification. Second, one can model the risk attitude of the investor independently of the attitude towards model misspecification. Increasing the coefficient of relative risk aversion increases the risk component of pre-cautionary savings linearly, while it leaves the pre-cautionary savings motive induced by model-misspecification doubts unchanged. Third, although concerns for model-misspecification and risk aversion seem observationally equivalent in terms of the observed real interest rate, the fear of model-misspecification is in practice constrained by detection-error probabilities. The higher one chooses m_w the further apart are the benchmark and the worst-case model for expected inflation, which makes it easier for the decision maker to discriminate between both models. This limits the size of premia that is induced by model uncertainty, whereas in general one can set risk aversion to infinity. Fourth, $m_w = 1$ implies uncertainty adjustments for inflation misspecification that are lower than for any CRRA risk adjustment with coefficient of relative risk aversion that is equal to $|\rho_{wA}|$ or higher.⁷

Proposition 1 also summarizes the market price of real risk. We collect them into a vector MPR_t^r . That vector has dimension 3×1 :

$$MPR^r = \begin{pmatrix} \rho_{wA} \sqrt{\sigma_{Aw} + \sigma_{bw}w} \\ \sqrt{1 - \rho_{wA}^2} \sqrt{\sigma_{Aw} + \sigma_{bw}w} \\ \sqrt{\sigma_{au} + \sigma_{bu}u} \end{pmatrix}. \quad (2.15)$$

The first component of the previous equation measures the risk premium for risky fluctuations in expected inflation, the second component measures the required compensation for fluctuations in dW^A and the last component measures fluctuations in $dW^{\bar{A}}$. In the time-series, the first two market prices of risk track changes in expected inflation, whereas the last market price of risk tracks changes in the real component of stochastic volatility in GDP. A negative correlation ρ_{wA} in the data implies that the real market price for changes in expected inflation is negative. Its absolute value is highest in times of high expected inflation.

2.2.2 Nominal Stochastic Discount Factor

U.S. nominal Government bonds are nominal assets. To analyze these bonds it is therefore necessary to derive the nominal values for the short-term interest rate and for nominal

⁷The empirical section estimates the model with setting $mu_w = 1$. The resulting detection-error probability is 36.5%. This confirms empirically that the worst-case model is extremely close to the reference model.

market prices of risk. Proposition 1 states these values. We copy the expression for reader's convenience:

$$R(t) = r(t) + E_t\left[\frac{dp}{p}\right] + h_w\sqrt{\sigma_{Aw} + \sigma_{bw}w(t)} - \left\langle \frac{dp}{p}, \frac{dp}{p} \right\rangle + \left\langle \frac{d\xi^r}{\xi^r}, \frac{dp}{p} \right\rangle. \quad (2.16)$$

The nominal short-term interest rate has several components. These are the real interest rate $r(t)$, the rational expectations inflation forecast $E_t[\frac{dp}{p}]$, the inflation ambiguity premium $h_w\sqrt{\sigma_{Aw} + \sigma_{bw}w(t)}$, precautionary savings due to inflation risk $\left\langle \frac{dp}{p}, \frac{dp}{p} \right\rangle$, and the inflation risk premium $\left\langle \frac{d\xi^r}{\xi^r}, \frac{dp}{p} \right\rangle$. Previous research has focused on understanding the inflation risk premium [Piazzesi and Schneider (2006), Buraschi and Jiltsov (2005) and Wachter (2006), and others]. In this paper, we focus on the inflation ambiguity premium, instead.

The above expression for the equilibrium nominal interest rate summarizes several economic intuitive relationships. First, the short-term nominal interest rate is a linear function in the real interest rate, expected inflation and inflation premiums. The inflation premium is a linear function in the inflation risk premium and in the inflation ambiguity premium. The inflation risk premium $\left\langle \frac{d\xi^r}{\xi^r}, \frac{dp}{p} \right\rangle$ is positive, if inflation and marginal utility are positively correlated. A positive correlation means that positive inflation news are positive news for marginal utility, which coincides with bad news for future consumption. A negative correlation in our model between consumption and inflation $\rho_{wA} < 0$ captures the positive inflation risk premium [see Piazzesi and Schneider (2006)]. Relaxing the assumption that the representative investors has no model misspecification doubts about inflation induces an additional premium. Importantly for nominal term structure modeling, this inflation ambiguity premium is positive, contributing to a positive slope in the nominal term structure.⁸

We denote the market price of nominal risk as $MPR^n = (MPR^n(t))$. It has two components, the real market price of risk and inflation volatility:

$$MPR^s = \begin{pmatrix} (\rho_{wA} + 1)\sqrt{\sigma_{Aw} + \sigma_{bw}w} \\ \sqrt{1 - \rho_{wA}^2}\sqrt{\sigma_{Aw} + \sigma_{bw}w} \\ \sqrt{\sigma_{au} + \sigma_{bu}u} \\ \sqrt{\sigma_{pv} + \sigma_{pv}v} \end{pmatrix}. \quad (2.17)$$

The first component of the previous equation measures the risk premium for risky fluctuations in expected inflation, the second and third component measures the required compensation for fluctuations in the two real output shocks (dW^A and $W^{\bar{A}}$). The last

⁸Compare Ang and Piazzesi (2003), Joslin, Priebsch, and Singleton (2009)), Piazzesi (2003) and others for empirical evidence.

component states the market price of risk for the nominal part of inflation shocks. In the time-series, the first two market prices of risk track changes in expected inflation, whereas the last two market price of risk track changes in the real component of stochastic volatility in GDP and the nominal component of stochastic volatility in inflation, respectively.

Our model has two channels through which inflation ambiguity affects the nominal short-rate. Both channels work into opposite directions. First, inflation ambiguity lowers the real interest rate and therefore ceteris paribus the nominal interest rate. At the same time, inflation expectations go up which ceteris paribus increases the nominal short-rate. Which channel is stronger? Below we show that there is a natural relation that ensures that the overall ambiguity premium on the nominal risk-free rate is positive.

2.2.3 Ambiguity in the Nominal Short Rate

We plug in the expression for the real interest rate into nominal short-term interest rate:

$$\begin{aligned}
R(t) = & \underbrace{E_t\left[\frac{dA}{A}\right] - \left\langle \frac{dA}{A}, \frac{dA}{dA} \right\rangle}_{\text{RealInterestRateReferenceMeasure}} + \underbrace{E_t\left[\frac{dp}{p}\right] - \left\langle \frac{dp}{p}, \frac{dp}{p} \right\rangle + \left\langle \frac{d\xi^r}{\xi^r}, \frac{dp}{p} \right\rangle}_{\text{NominalInterestRateReferenceMeasure}} \\
& + \underbrace{\rho_{wA} m_w(\sigma_{Aw} + \sigma_{bw} w(t))}_{\text{AmbiguityPremiumRealInterestRate}} \\
& + \underbrace{m_w(\sigma_{Aw} + \sigma_{bw} w(t))}_{\text{AmbiguityPremiumExpectedInflation}} . \tag{2.18}
\end{aligned}$$

The last equation reveals that the nominal short-term interest rate in an economy where the model for expected inflation is potentially misspecified consists of two components. The first component is the nominal short-term interest rate in the rational expectations economy. That component coincides with the first line on the right hand side. It comprises the real interest rate plus inflation expectations (reference measure) minus inflation pre-cautionary savings and plus the inflation risk premium. The second component is the ambiguity premium. The ambiguity premium has two components as well, the first one reduces the nominal interest rate because fear of inflation misspecification urges the investor to increase her precautionary savings which reduces the real interest rate. The second part of the ambiguity premium is positive and arises from the equilibrium increase in the inflation forecast. This is optimal because an increase in expected inflation coincides with a reduction of the expected consumption path. The correlation between consumption and expected inflation is bounded by 1 which means that the overall inflation ambiguity premium on the nominal short-term interest rate is positive. The event that the overall inflation ambiguity premium on the nominal short-term interest rate is zero (or negative) is a zero probability event, because it only occurs if $\rho_{wA} \leq -1$.

Although the following idea is not supported in the data, it helps to shed further light on how ambiguity affects the nominal short rate: What happens to the nominal interest rate if inflation expectations were positively correlated with consumption? In such an environment, positive inflation news would be positive consumption news. The real interest rate would fall in equilibrium because of an increased precautionary savings term. That channel would be the same as in the case of a negative correlation. On the other hand, it would be optimal for the robustness seeking investor to lower her inflation forecast, because a lower inflation forecast coincides with a lower consumption growth forecast. For such an investor that behavior is optimal because her ultimate goal is the generation of a more cautious expected consumption path. The reduction in the inflation forecast would amplify the reduction of the equilibrium nominal interest rate. The ambiguity premium would be negative for real bonds and for nominal bonds. It is this inflation ambiguity channel, that depends crucially on the correlation of consumption with expected inflation. In the data that correlation is negative but not -1 . That explains why such an investor's inflation premium is higher than the negative ambiguity premium in the real interest rate.

We now show how one can use the term structure of Government bonds to learn about the cross-section of the ambiguity premium. We start the analysis with inflation-protected Government bonds. Their price coincides with the risk-adjusted expected value of receiving one unit of the numeraire in the future, discounted by the real short-term interest rate. The real short-term interest rate contains only one component of the ambiguity premium. Inflation-protected Government bond yields for different time to maturities provide therefore information of how inflation ambiguity affects the future expected real short rate for different future time horizons. We continue with analyzing the term structure of nominal Government bonds. Like the nominal short-rate, this term structure contains both components of the inflation ambiguity premium.

2.3 Term Structure of Inflation-Protected Government Bonds

The price of an inflation-protected Government bond is the closest empirical counterpart of a real and default-free bond. The price of such a bond with maturity in τ periods contains not only valuable information about investor's view of where the real economy stands in τ periods from now, but it also contains an assessment of how risky and uncertain the development of the real economy is. The yield of such a real bond aggregates this information into one number. An observed drop in such a yield can occur because the investor expects real aggregate activity to fall, or precautionary savings went up, the risk premium went down, the ambiguity premium went up, or a combination of these. A ceteris paribus increase in the amount of ambiguity leads to a drop in real yields because of the negative correlation between inflation and consumption.

We denote the price at time t of a τ -maturity real bond with $B_t(\tau)$. Its equilibrium

price is determined by the Euler equation:

$$B_t(\tau) = E_t^Q \left[e^{-\rho\tau} \frac{u_c(c(t+\tau))}{u_c(c(t))} \right] \quad (2.19)$$

where Q denotes the robust probability measure and u_c denotes the partial derivative of the utility function u with regard to consumption. We rewrite such a "robust" Euler equation into the standard Euler equation. We call the latter standard because it denotes the expectation under the benchmark measure. A simple measure transformation recovers:

$$B_t(\tau) = E_t^{Q^0} \left[e^{-\rho\tau} \frac{u_c(c(t+\tau))}{u_c(c(t))} \frac{dQ}{dQ^0} \right] \quad (2.20)$$

where the exponential martingale $\frac{dQ}{dQ^0}$ stands for the entropy (distance) between the worst-case and the reference measure.⁹

The difference between the equilibrium price of an inflation-protected bond under rational expectations and under model misspecification doubts is entirely attributed to the conditional covariation of marginal utility and the amount of model misspecification doubts. If expected inflation is negatively correlated with consumption the conditional covariation between marginal utility and expected inflation is positive. This induces a higher equilibrium price for a real bond under model misspecification doubts compared to rational expectations. The mirror image of a higher equilibrium price is a lower equilibrium yield. Investors with model misspecification doubts about inflation regard the real bond as a hedge against bad consumption news and against inflation misspecification doubts. Periods of increased inflation uncertainty coincide with periods of low consumption growth. A real bond pays out a lot in such periods and since the investor's marginal utility is high in these periods, real bonds constitute a hedge against increased inflation ambiguity and bad consumption news. Mathematically, this follows from (2.20)

$$B_t(\tau) = E_t^{Q^0} \left[e^{-\rho\tau} \frac{u_c(c(t+\tau))}{u_c(c(t))} \right] + e^{-\rho\tau} cov_t \left(\frac{u_c(c(t+\tau))}{u_c(c(t))}, \frac{dQ}{dQ^0} \right). \quad (2.21)$$

The first term on the right hand side is the real bond price under rational expectations and the second term summarizes the inflation ambiguity premium for inflation-protected bond prices.

The solution to the Euler equation exists in closed-form. The model can be cast into a special version of an affine model in the spirit of Duffie, Pan, and Singleton (2000) and is given by:

$$B_t(\tau) = e^{A^r(\tau) + B^{r'}(u_t w_t z_t)'} \quad (2.22)$$

⁹ $\frac{dQ}{dQ^0} = e^{-\frac{1}{2} \int_0^T h_w^2(s) ds + \int_0^T h_w(s) dW^w}$

where functions A^r and B^r depend on structural parameters of the economy. The appendix contains the derivation.

The yield of the real bond is defined by $y_t^r(\tau) = -\frac{1}{\tau} \ln B_t(\tau)$. It is affine in the state variables:

$$y_t^r(\tau) = -\frac{1}{\tau} \left(A^r(\tau) + B^{r'}(\tau)(u_t w_t z_t)' \right). \quad (2.23)$$

2.4 Term Structure of Nominal Government Bonds

The price of a nominal Government bond with maturity in τ periods contains not only valuable information about investor's view of where the real economy and inflation stands in τ periods from now, but it also contains an assessment of how risky and uncertain the development of the real economy and inflation are. The yield of such a nominal bond aggregates this information into one number. An observed drop in the nominal yield can occur because the investor expects a drop in the real yield, or precautionary savings went up due to higher inflation risk, or the inflation risk premium went down, or the inflation ambiguity premium fell, or a combination of these economic forces. If the economy moves into a state in which the investor perceives her inflation model to be potentially more misspecified, she increases her robust inflation forecast, which leads to an increase in nominal government bond yields.

We denote the price at time t of a τ -maturity nominal bond with $N_t(\tau)$. Its equilibrium price is determined by the Euler equation:

$$N_t(\tau) = E_t^Q \left[e^{-\rho\tau} \frac{u_c(c(t+\tau))}{u_c(c(t))} \frac{p_t}{p_{t+\tau}} \$1 \right] \quad (2.24)$$

where Q denotes the robust probability measure and u_c denotes the partial derivative of the utility function u with regard to consumption. We rewrite such a "robust" Euler equation into the standard Euler equation. We call the latter standard because it denotes the expectation under the benchmark measure. A simple measure transformation recovers:

$$N_t(\tau) = E_t^{Q^0} \left[e^{-\rho\tau} \frac{u_c(c(t+\tau))}{u_c(c(t))} \frac{dQ}{dQ^0} \frac{p_t}{p_{t+\tau}} \$1 \right] \quad (2.25)$$

The difference between the equilibrium price of a nominal bond under rational expectations and under model misspecification doubts is entirely attributed to the conditional covariation of nominal marginal utility with the amount of model misspecification doubts:

$$N_t(\tau) = E_t^{Q^0} \left[e^{-\rho\tau} \frac{u_c(c(t+\tau))}{u_c(c(t))} \frac{p_t}{p_{t+\tau}} \right] + e^{-\rho\tau} cov_t \left(\frac{u_c(c(t+\tau))}{u_c(c(t))} \frac{p_t}{p_{t+\tau}}, \frac{dQ}{dQ^0} \right). \quad (2.26)$$

The first term on the right hand side coincides with the rational expectations price of a nominal bond. The second term coincides with the total inflation ambiguity premium in nominal bond prices if the investor has model misspecification doubts about inflation.

The solution to the Euler equation exists in closed-form. The model can be cast into a special version of an affine model in the spirit of Duffie, Pan, and Singleton (2000) and is given by:

$$N_t(\tau) = e^{A^{\$}(\tau) + B^{\$'} X_t} \quad (2.27)$$

where functions $A^{\$}$ and $B^{\$}$ depend on structural parameters of the economy. The appendix contains the details.

The yield of the nominal bond is defined by $y_t^{\$}(\tau) = -\frac{1}{\tau} \ln N_t(\tau)$. It is affine in the state variables:

$$y_t^{\$}(\tau) = -\frac{1}{\tau} \left(A^{\$}(\tau) + B^{\$'}(\tau) X_t' \right). \quad (2.28)$$

3 Data and Econometric Methodology

The goal of this section is to briefly describe the data and the econometric methodology used for estimating the model.

3.1 Data

We estimate the general equilibrium model of the nominal and real term structure at once. The estimation takes the structure and parameter restrictions of the general equilibrium economy into account. The rich equilibrium restrictions that the model poses on the interest rates, market prices of risk, market price of ambiguity and expectations on macroeconomic fundamentals ensure that model parameters and states are well identified by empirical data.

The model is estimated at a monthly frequency and the sample period is January 1972 to June 2009. We use continuously compounded nominal U.S. government bond yields of maturities 1, 2, 3, 4, 5, 6, 7, 8, 9,10 years. The nominal bond yields are taken from the Board of Governors of the Federal Reserve System which provides these bond yields based on the work of Gurkaynak, Sack, and Wright (2007). The time-series of the Federal funds rate is obtained from the St. Louis Fed. We use continuously compounded yields of U.S. Treasury Inflation-Protected Securities (TIPS) of maturities 5,6,7,8,9,10 years, for the time horizon January 2003 to June 2009. This data is provided by the Board of Governors of the Federal Reserve System and is based on the work of Gurkaynak, Sack, and Wright (2010). We use only Tips with maturities of five years and longer to exclude

potential misspecifications that might arise from the one quarter inflation indexation lag. Although Tips data is available since the late 1990s, we restrict the data set to start in 2003, because it is known that Tips were relatively illiquid until the early 2000.

We take the time-series for real GDP growth and the GDP implicit price deflator from the St. Louis Fed FRED data base. This data is only available on a quarterly frequency. We use cubic splines to interpolate the data to monthly frequency. We use one-year ahead forecast data on GDP implicit inflation and GDP growth provided by the Survey of Professional Forecasters. The time-series for one-year ahead GDP growth forecasts is available only from the third quarter of 1981. We therefore use it from 1982 onwards. The forecast data is only available on a quarterly frequency. We use splines to interpolate the data to a monthly frequency.

3.2 Estimation and Identification of Model Parameters and Model State Variables

The rich panel allows a precise mapping of the general equilibrium model to the data. The month-on-month growth rate of the price level and GDP ensures that the model fits the fundamental dynamics well. The inclusion of the forecast data helps to identify the latent state variables. The GDP one-year ahead growth rate forecast helps especially to identify the z state, which is the time-varying part of the model implied GDP growth rate. The GDP implicit one-year ahead inflation forecast helps especially to identify w , the time-varying part of the model implied inflation growth rate. The 78×6 Tips panel helps to identify the states u, w, z . Equation (2.21) demonstrates the close link between the price of a real bond and the expected GDP growth rate forecast. The panel of Tips and the time-series of the expected GDP growth forecast help therefore to identify the inflation ambiguity component in Tips prices. The 450×10 panel of nominal bond yields helps to identify all state variables and in particular v, w . Equation (2.26) shows the close connection between the price of a nominal bond and the price of a real bond, the expected GDP growth rate forecast and the expected inflation forecast. Since we include all of these components into the estimation, the estimation allows us to identify the inflation ambiguity component present in nominal bond yields. The appendix contains details about the general equilibrium implied analytical solutions to the macro growth rates, macro forecasts and bond yields.

The state variables are latent. The inclusion of forward-looking data, such as real bond yields, nominal bond yields and expected GDP growth and expected inflation ensure a precise estimate of these states. We estimate the model by Quasi Maximum Likelihood (QML). We use simulated annealing to maximize the model implied likelihood over the entire parameter space. For each parameter bundle, we assume that the 1,3,5,10 year nominal bond yields are observed without measurement error. This provides an estimate of the state variables. This is a standard filtering procedure in empirical affine term

structure modeling[see Chen and Scott (1993), Duffee (2002), Buraschi and Jiltsov (2005) and others].

We assume that the measurement error shocks are conditionally joint normal distributed and orthogonal to the shocks to the filtered states. To reduce the parameter space we fix the following parameters ex-ante: the capital depreciation rate δ is set to 0.1, the subjective time-discount factor ρ is set to 0.03. We restrict the impact of ambiguity by setting the m_w parameter to 1. From an asset pricing point of view this is the counterpart to having a relative risk aversion of 1 (log utility). One can therefore interpret the model as a logarithmic ambiguity model. Risk and ambiguity enter both with a weighting of one.¹⁰ Setting $m_w = 1$ and using the equilibrium relationship between m_w and the ambiguity preference parameter θ_w in equation (2.9) allows to back out the latter. Since it does not carry an intuitive economic interpretation, we prefer to think in terms of m_w instead.

4 Empirical Results

We report parameter estimates of the general equilibrium model in Table 1. The second column shows that the time-varying component of expected inflation, w , is the most persistent state variable, having a half-life of 27 years.¹¹ The least persistent state variable is the time-varying component of expected GDP growth, z , with a half-life of 0.2 years.

The log ambiguity model fits the data very well. The annualized mean pricing error for nominal yields and real yields, is 17 and 19 basis points, respectively. This is an excellent fit, given that we fit also GDP growth, inflation and one-year ahead inflation and GDP forecast. The fit to the term structure is even more remarkable if one takes the rich general equilibrium restrictions into account that the model posts on interest rates, market prices of risk and market prices of ambiguity.

The goal of this chapter is to explain the most important cross-sectional characteristics of the estimated inflation ambiguity premium. The analysis focuses on the entire sample and on economic meaningful subperiods. That section is followed by highlighting the most important time-series properties of the estimated inflation ambiguity premium. We focus especially on the inflation ambiguity premium implications for real yields and for the robust inflation forecasts.

4.1 The Cross-Section of Inflation Ambiguity

Figure 1 presents the estimated model implied inflation ambiguity premium and its components for medium- and long-term nominal bonds. The solid line plots the inflation

¹⁰Compare equation (2.14).

¹¹The half-life is defined as $\frac{\ln(2)}{\kappa}$.

ambiguity premium of nominal bond yields. The blue dotted curve plots the ambiguity markup in inflation expectations. The red dashed line depicts the inflation ambiguity premium inherent in Tips yields. The overall inflation ambiguity premium in nominal bond yields is upward sloping, being close to zero for the five year nominal bond and 40 basis points for the ten-year nominal bond. A positive ambiguity premium in nominal bond yields means intuitively that nominal bonds are not a hedge against inflation misspecification doubts. This is very intuitive because assume that a nominal bond price has been determined with an expected inflation of 5 percent. If the inflation model changes and expected inflation becomes 10 percent, the real payoff of the bond drops by 5 percent. Since inflation and consumption are negatively correlated, a shift in the inflation model towards the worst-case one coincides with bad news about future consumption. This means that the real payoff of the nominal bond is low in times of increased inflation misspecification doubts which coincides with times of high marginal utility. The nominal bond has therefore to offer a positive premium for inflation misspecification doubts.

The upward sloping inflation ambiguity premium in nominal bond yields is inherited from an upward sloping ambiguity premium in the robust (worst-case) inflation forecast. The ambiguity markup on five-year ahead robust inflation expectations has been on average 1.3 percent, while it has been on average 2.3 percent for the ten-year ahead horizon. The positive slope means intuitively that investors fear inflation misspecifications more the longer the locked-in investment horizon. If the investor buys a ten year nominal bond she locks-in into a fixed ten-year inflation forecast. While inflation has been quite low over the last ten years, it has been skyrocketing in the late 1970s and early 1980s. Investors therefore know that they do not know the true long-term inflation model. This goes hand in hand with the absent ability to forecast long-term inflation precisely. For an ambiguity averse investor it is optimal in such a situation to distort her inflation forecast upwards in order to be protected against an unfavorable inflation model that might arise in the future.

The ambiguity premium in Tips yields is negative and downward sloping. The average inflation ambiguity premium on a five year Tips yield is -1.2 percent, while it is -2 percent for the ten-year Tips. The ambiguity premium in Tips is negative because Tips are a hedge instrument against inflation misspecification. Intuitively, assume that the model for expected inflation changes in such a way that long-term inflation expectations go up by 5 percent. Assuming perfect inflation indexation in Tips yields means that the real payoff of Tips bonds is not affected at all by the increase in expected inflation. On the other hand, under this scenario the investor would be faced by lower GDP growth expectations, because inflation expectations and consumption are negatively correlated in the data. If GDP growth less than previously expected, the price of Tips bonds goes up, and equivalently its yield goes down.¹² That is an important economic insight because it shows that Tips bonds pay out well in bad times. An ambiguity-averse investor is willing

¹²See equation (2.20) for details.

to pay a premium for that hedge which means that the ambiguity premium is negative.

4.1.1 Cross-Section Evidence from different Sub-Periods

Figure 2 extends the analysis of the inflation ambiguity components to different subperiods. We construct the subperiods as follows. We estimate the model over the entire sample 1972 to 2009. The estimated parameters are held fixed for all subperiods. The state variables for a given subperiod are averaged. The various panels represent a different subperiod. From top left to bottom right this is the period right before the Monetary Experimentation (1972-1979), the period of the Monetary Policy Experimentation (1979-1983), the time period of the Great Moderation (1984-2007), the first four years of Fed Chairman Ben Bernanke (2005.6-2009.6), the first two years of the financial crisis (2007.5-2009.6), and the entire sample 1972-2009. The green $-*$ line summarizes the overall inflation ambiguity premium. The solid blue line graphs the inflation ambiguity premium in the robust bond implied inflation forecast and the red $-.$ line plots the inflation ambiguity premium in real yields. The cross-sectional pattern of the inflation ambiguity premium is stable across different subperiods. Real interest rates are lowered by inflation ambiguity, while robust inflation forecasts carry a positive premium. The size of the premium varies for different subperiods. It has been highest during the Monetary Policy Experimentation period of the 1979 to 1983 and lowest before that. The total inflation ambiguity premium has been upward sloping, positive and small. The latter makes it difficult to detect that premium. The inflation ambiguity premium has remained stable during the current financial market turmoil.

Figure 3 presents inflation forecasts in nominal bond yields over the same sub-periods than above. The purple $*$ line summarizes the bond implied inflation forecast. That measure coincides with the term structure of robust inflation forecasts (robust measure). This robust inflation forecast has two components, the inflation ambiguity premium which is presented in the red $-.-$ graph and the empirical inflation forecast (under the empirical measure) which is plotted as the green $-*$ graph. All inflation forecasts fluctuate considerably over different subperiods. As expected, the difference between the worst-case inflation forecast and the empirical inflation forecast has been highest during the Monetary Policy Experimentation, reaching five percent for the ten-year bond. This endogenous markup sloped upwards during all periods. The ambiguity premium in the inflation forecast dropped slightly during the current financial crisis, supporting the view that the Federal Reserve has done a good job in controlling, communicating and managing inflation expectations. The model implied forecast of inflation under the empirical measure is downward sloping throughout all subperiods. This is consistent with a mean-reverting data-generating process for inflation.

Figure 4 presents the nominal yield curve and its components over the same subperiods as above. The solid black line represents the average nominal yield curve. The green $-*$ line represents the inflation forecast (empirical measure). The red $-.$ graph represents

the inflation ambiguity premium in the inflation forecast. Adding up the previous two lines gives the bond model implied robust inflation forecast (robust measure). The pink * graph represents the estimated Tips yield curve.¹³ The solid blue "star" line represents the inflation risk premium. The average nominal yield curve is positive throughout all sub-periods. It has been flattest during the Monetary Policy Experimentation. The ambiguity premium in the robust inflation forecast has been strongly upward sloping during that period, offsetting the downward sloping empirical inflation forecast. Intuitively, this means that if investors knew the true model for expected inflation, the nominal yield curve would have sloped downward during that period. The painful and recent experience of double digit and volatile inflation realizations has increased investor's fear of model misspecification doubts about the expected inflation model. The increased misspecification doubt led to the positive and upward sloping inflation ambiguity premium which made the term structure of the robust inflation forecast to be pretty flat.

During the current financial market crisis, the nominal yield curve has sloped upwards, while the inflation risk premium and empirical inflation forecast have remained flat. The real yield curve and the inflation ambiguity premium have sloped upwards, giving the nominal yield curve its upward sloping shape. The empirical ten-year inflation forecast is 1.2 percent, while the inflation ambiguity premium is 1.3 percent. Taking these together says that investors have priced nominal bonds during the crisis with a robust inflation ten-year inflation forecast of 2.5 percent.

4.2 Time-Series Properties of the Inflation Ambiguity Premium

Figure 5 contrasts the model implied total inflation ambiguity premium in the nominal short-rate with the nominal short-rate itself. We use the federal funds rate from the data as the nominal short-rate. The top panel plots the inflation ambiguity premium. The lower panel plots the federal funds rate, its 95% confidence interval and the federal funds rate in an otherwise equal rational expectations equilibrium. The black solid line corresponds to the federal funds rate, as observed in the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied federal funds rate under the reference model. The rational expectations and worst-case model for the federal funds rate are so close to each other, that it is difficult to detect the difference by eye-bolling. In the next section, we use detection-error probabilities to show formally that the worst-case model is statistically very close to the reference model.

Looking closer at the top panel of Figure 5 reveals that the inflation ambiguity premium is positive for the nominal short rate. During the last recessions the ambiguity premium has increased at the beginning of the recessions and fallen afterwards. The inflation ambiguity premium in the nominal short rate has been 10 basis points during

¹³Since the estimation contains only Tips yields from 2003 onwards, we assume that the fitted yield curve of Tips has remained unchanged during the unobserved time-period.

the 1970s, 70 basis points during the Monetary Policy Experimentation period of the early 1980s and steadily declining afterwards. The peak of 80 basis points has been achieved right at the beginning of the 1981 recession.¹⁴

Figure 6 plots the model implied total inflation ambiguity premium in the ten-year nominal yield with the ten-year nominal yield itself. The top panel plots the inflation ambiguity premium. The lower panel plots the the ten-year nominal yield, its 95% confidence interval and the ten-year nominal yield under the reference measure. The black solid line corresponds to the ten-year nominal yield, as observed in the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied ten-year nominal yield under the reference model. The top panel reveals that the total ambiguity premium in the ten-year nominal yield is positive. It has risen several months before the official start of the last NBER recessions and fallen afterwards. The ten-year ambiguity premium has been 30 basis points during the 1970s, 130 basis points during the Monetary Policy Experimentation period of the early 1980s and steadily declining afterwards. The peak of 170 basis points has been achieved right at the beginning of the 1981 recession.

The volatile variations in the ten-year ambiguity premium seem to suggest that the worst-case and rational expectations ten-year yield are far apart. The lower panel of Figure 6 shows that the opposite is true. Over the entire sample, one is not able to distinguish between both models.¹⁵ Even during the peak of the Monetary Policy Experimentation period, both models lie well within the 95% confidence interval of the observed ten-year nominal yield.

Figure 5 and Figure 6 show that there is a small but significant amount of ambiguity in nominal bond yields. The amount of model misspecification in bond yields lies well within even the most conservative confidence intervals of observed nominal yields.

Figure 7 decomposes the inflation ambiguity premium of the nominal short-rate into its two components. The panel on the top shows the overall inflation ambiguity premium, which coincides with the upper panel in Figure 5. The middle panel of Figure 7 shows how inflation ambiguity has lowered the real short rate over the past forty years. The mean reduction in the real short rate due to inflation ambiguity has been 60 basis points. Consistent with the explanations in the theoretical section, periods of uncertain future inflation are periods of higher precautionary savings. Before the period of the Monetary Policy Experimentation, inflation uncertainty induced precautionary savings have reduced the real interest rate by 10 basis points. The increase in precautionary savings due to higher uncertainty of the inflation model during the Monetary Policy Experimentation has led to a reduction of the real risk free rate of -3 percent. With the beginning of the

¹⁴While our study is the first to analyze the inflation ambiguity premium, recent research has focussed on the inflation risk premium. See Buraschi and Jiltsov (2005), Wachter (2006), Piazzesi and Schneider (2006). We do not focus on that premium here. We just mention that the inflation risk premium is positive and time-varying in our model, as well. Figure 4 plots the inflation risk premium together with the other yield curve components for different subperiods.

¹⁵The next section applies formally detection-error probabilities and confirms the visual inspection.

Great Moderation, that premium has become smaller, fluctuating around 50 basis points in the 1990s and early 2000. Fear of inflation misspecification doubts in the recessions of 1980, 1981/1983 and 1990/1991 has led to an increase in precautionary savings and to a reduction in the real interest rate already several months before the recessions officially started.

The lower panel of Figure 7 plots the ambiguity premium in very short-term inflation expectations. The mean value has been 75 basis points. Highest ambiguity markups on short-term inflation expectations were required during the Monetary Policy Experimentation of the early 1980s. The premium reached 3.5 percent at the beginning of the 1981 recession. The markup on expected inflation reduced persistently during the Great Moderation and fluctuated around 30 basis points during the mid 1990s and early 2000. During the 2001 recession that inflation ambiguity premium has been essentially zero, while it started to increase at the end of the 2001 recession. In the aftermath of 9/11, the Fed lowered the nominal short-term interest rate below the inflation rate. The increase in the ambiguity induced markup in inflation expectations indicates that investors were concerned that the inflation model might change in the very near future.

During the current financial market turmoil ambiguity induced inflation expectations increased in two waves from nearly zero to 1.5 percent at the beginning of 2009. This sharp increase coincides with increasing concerns about a changing inflation model in the future given that the money supply has been dramatically increased during the financial crisis. For ambiguity averse investors it is optimal to increase the worst-case inflation forecast today, even if current inflation rates are low. The increase in ambiguity induced precautionary savings has lowered the short-term real interest rate by up to -1.2 percent. The net effect of inflation ambiguity in the short-term nominal interest rate has been an increase from zero to 0.3 percent at the beginning of 2009.

4.3 Detection Error Probabilities

Anderson, Hansen, and Sargent (2003) and Maenhout (2006) show that one can use Detection Error Probabilities to judge how far apart from a statistical point of view the reference model is from the worst-case model. Intuitively, detection error probabilities summarize in one number how likely it is after observing the data that one model has a higher likelihood although the data has been generated by another model.

There are two cases that need to be considered. The first case corresponds to the scenario where the reference model is the true DGP and the data set with length T produces a higher likelihood for the worst-case model compared to the reference model. Under such a scenario, the econometrician would choose the worst-case model although the data has been generated by the reference model. The second case coincides with the scenario that the worst-case model is the true DGP and the data set with length T produces a higher likelihood for the reference model compared to the worst-case model.

Such an empirical finding would urge the the econometrician to choose the reference model over the the worst-case model, although the data has been generated by the worst-case model.

For any ambiguity preference parameter θ_w , the probability of making an error (detection error), $\epsilon_T(\theta)$, conditional on a data set with length T is defined as

$$\epsilon_T(\theta_w) = \frac{1}{2} (Pr(\ln Q_T > \ln Q_T^0 | Q^0, \mathcal{F}_0) + Pr(\ln Q_T^0 > \ln Q_T | Q, \mathcal{F}_0)). \quad (4.1)$$

The first term on the right hand side stands for the probability that the data rejects the Q^0 model although it is the true model. The last term stands for the probability that the data implied likelihood rejects the worst-case model although the data is sampled from Q . Since the investor does not know in which model she is in, she puts equal probabilities to both models. An increase in the preference parameter θ_w leads the agent to optimally choose a worst-case model Q which is further apart from the reference model Q^0 . Through the lenses of detection error probabilities this means that $\epsilon_T(\theta_w)$ gets smaller, having a lower bound at zero. That means that both models are easier to distinguish from one another. On the contrary, two Gaussian models which are not distinguishable from the data have a detection error probability of 50%.

Anderson, Hansen, and Sargent (2003) set θ such that $\epsilon_T(\theta)$ is bigger than 10%. In our model, we suggest an additional way of estimating ambiguity. We set θ_w to a value such that m_w equals 1. For asset pricing this corresponds to the ambiguity analog of a CRRA risk aversion of 1. We estimate the model with fixing $m_w = 1$ and calculate the maximum likelihood implied detection error probability. In our sample that probability is 36.5%. That number says that although our sample has 450 monthly observations, the investor has a 36.5% chance of choosing the wrong model based on the empirical likelihood. This is more conservative than having a 10% detection error probability as in Andersen et al. It further shows that past data has not been enough to distinguish between our reference model and our worst-case model.

Taken all results together shows that a small amount of ambiguity with regard to expected inflation is difficult to detect in data but generates interesting and plausible empirical regularities in the cross-section and time-series of U.S. government bonds.

5 Related Literature

This paper sheds light on how fear of inflation misspecification affects the term structure of real and nominal Government bonds. Recent empirical models of the term structure find that business cycle factors and inflation factors describe changes in nominal yields or nominal bond risk premia (Ang and Piazzesi (2003), Ang, Piazzesi, and Wei (2006), Duffee (2007), Joslin, Priebsch, and Singleton (2009), among others). Gürkaynak, Sack,

and Swanson (2005) and Ang, Bekaert, and Wei (2008) show that changes in inflation expectations matter a lot for understanding variations in nominal bond yields. Our general equilibrium model extends this line of research by developing and estimating a term structure model for real and nominal bonds where investors face model uncertainty with regard to inflation expectations.

Empirical macroeconomic research documents that investors have imperfect knowledge about the exact conduct of Monetary Policy [Ang, Boivin, Dong, and Loo-Kung (2009), Clarida, Gali, and Gertler (2000), Ang, Dong, and Piazzesi (2008), Cogley and Sargent (2002), and others]. One important goal of U.S. monetary policy is the creation of a stable inflation process. It is therefore natural to think that if investors have imperfect knowledge about the exact conduct of monetary policy, they also have imperfect knowledge about the future path of inflation. We take this as a motivation to extend the work of Anderson, Hansen, and Sargent (2003) into term structure implications of inflation misspecification doubts.

Early contributions of managing model misspecification doubts with robust control tools have been introduced by Anderson, Hansen, and Sargent (2003), Cagetti, Hansen, Sargent, and Williams (2002), Hansen and Sargent (2007), Hansen and Sargent (2005), Hansen, Sargent, Turmuhambetova, and Williams (2005), Maenhout (2004), and Maenhout (2006). All of these papers have focused on model misspecification doubts about the real economy. Inflation expectations and model misspecification doubts about expected inflation has not been analyzed in a robust control framework.¹⁶ We extend that literature by proposing a simple max-min general equilibrium model which endogenously derives market prices of inflation risk and market prices of inflation ambiguity. The endogenously derived market prices of risk and ambiguity in our paper are stated in Proposition 1 and equation (2.9). We further extend the literature by showing how one can directly control the asset pricing implications of ambiguity. Current research uses detection error-probabilities to calibrate the amount of ambiguity [see Anderson, Hansen, and Sargent (2003), Maenhout (2006)]. We propose to fix the impact of ambiguity ex-ante, in the same way as the modeler fixes the risk aversion impact on asset prices via the coefficient of relative risk aversion. We achieve this by setting $m_w \equiv 1$. Equation (2.14) shows that the resulting ambiguity implications for spot financial assets are of similar magnitude than that of having logarithmic risk aversion. We therefore call this special case "logarithmic model misspecification doubts". We confirm with error-detection probabilities that the resulting worst-case model is hardly distinguishable, in a statistical sense, from the reference model.

In contrast to our paper, most current research on model uncertainty focuses on equity pricing and portfolio allocation. Garlappi, Uppal, and Wang (2007) analyze a portfolio

¹⁶An alternative tool for handling model misspecification doubts is based on multiple recursive priors. Early contributions of that work are Gilboa and Schmeidler (1989), Epstein and Wang (1994), Epstein and Schneider (2003), Anderson, Hansen, and Sargent (2003), Chen and Epstein (2002), Epstein and Miao (2003).

selection problem where an investor accounts for uncertainty about the estimated expected returns. Uppal and Wang (2003) study an intertemporal portfolio choice problem where an investor faces an ambiguous return distribution of her stock portfolio. They find that ambiguity can lead to underdiversified portfolios relative to the mean-variance portfolio. Liu, Pan, and Wang (2005) introduce ambiguity aversion about the jump probability in the return of stocks and show that this helps to explain the implied volatility smirk of stock options. Drechsler (2009) introduces ambiguity aversion into a long-run risk set-up and shows that model uncertainty can help to account for the variance premium and option skew of equity index options. Sbuelz and Trojani (2002), Maenhout (2004), Leippold, Trojani, and Vanini (2008) apply ambiguity aversion to the equity premium puzzle and/or excess volatility puzzle. Dow and Werlang (1992), Trojani and Vanini (2004) and Cao, Wang, and Zhang (2005) study the impact of ambiguity aversion on the limited stock market participation. Miao and Wang (2009) study the impact of ambiguity aversion on the optimal exercise decision. Chen, Ju, and Miao (2009) study a consumption and portfolio choice problem when the investor is exposed to two potentially misspecified stock returns models, one with return predictability and the other one with i.i.d returns.

Recently, some papers have focused on ambiguity and the term structure of real bonds. Kleshechelski and Vincent (2009) and Gagliardini, Porchia, and Trojani (2009) study how model uncertainty with regard to the real economy affects the real term structure. The focus of our analysis and our model structure is very different. We focus on the nominal term structure and we take empirical evidence for the importance of inflation expectations for nominal bond pricing as given and allow the investor to have model misspecification doubts about expected inflation only. Our work therefore extends existing research by analyzing the effect of inflation ambiguity on Government bonds.

Our model adds to the general equilibrium term structure research, which so far has neglected inflation ambiguity. Piazzesi and Schneider (2006) build a model for the nominal and real term structure which shares several of our features. Investors dislike inflation because it tends to occur in times of low consumption growth. The main distinguishing feature between theirs and our model is that Piazzesi and Schneider (2006) assume an investor with recursive utility whereas our investor has time-separable utility combined with the preference for robust decision making in the spirit of Anderson, Hansen, and Sargent (2003). Buraschi and Jiltsov (2005) construct a monetary production based asset pricing model with taxes where the latter are determined based on the historic purchase price instead of current nominal replacement costs. A rise in inflation lowers the after-tax real return on capital and carries therefore bad news about future consumption. Whereas these papers focus on the inflation risk premium, we focus on the inflation ambiguity premium. Buraschi and Jiltsov (2007) and Wachter (2006) construct a term structure model for nominal bonds and show that habit formation can help to capture movements in the nominal yield curve. Rudebusch and Swanson (2008) construct a DSGE model with habit formation in consumption and labor market frictions to study the nominal bond risk premium.

6 Conclusion

This paper has derived and estimated a general equilibrium model for the real and nominal term structure of U.S. government bonds which accounts for two sources of inflation premiums. The first premium is the inflation risk premium, while the second premium is the inflation ambiguity premium. The model is an extension of a monetary Cox, Ingersoll, and Ross (1985b) economy where the representative investor model misspecification doubts about expected inflation. The equilibrium price of real bonds is higher than under rational expectations because real bonds constitute a hedge against inflation misspecification doubts. On the other hand, the price of nominal bonds falls because under inflation misspecification doubts, the robust seeking investor expects nominal bonds to payout even less in real units than under rational expectations.

In the empirical section of the paper, we use a rich panel of U.S. macro and bond data to estimate a special form of the general equilibrium model. The estimated model is a special case, because we fix the ambiguity preference parameter such that the asset pricing implication are the analog to a log utility risk aversion investor. The resulting detection error probability for the sample is 36.5%. Such a high error probability indicates that log ambiguity aversion cannot be rejected by the econometrician's data. Instead, the reference model and the endogenous worst-case inflation model are statistically so close to each other, that the unconditional probability of choosing the wrong model, even after observing 37 years of data, is 36.5%. That is mounting evidence that the estimated overall inflation ambiguity premium on a nominal U.S. government bond yield accounts at least for 40 basis points. The paper has explained that the implications for worst-case inflation forecast and real yields is even higher. The estimation has also shown that the inflation ambiguity premium has been rather stable for different subperiods and highest during periods of high monetary policy uncertainty.

Taking all theoretical and empirical results together, we conclude that a tiny and very difficult to detect amount of model misspecification about expected inflation has severe implications on the term structure of U.S. Government bonds. Ambiguity with regard to expected inflation is difficult to detect in the data but generates interesting and plausible empirical regularities in the cross-section and time-series of U.S. government bonds. The results are important for academics and policy makers. The analysis suggests that it might be very fruitful for future research as well as policy making to take the evidence of this paper and to expand it further.

A Appendix

A.1 Proposition 1

The derivation of the optimal consumption rule, real and nominal stochastic discount factor is standard in the literature. Examples for the first can be found in Cox, Ingersoll, and Ross (1985a), Duffie (2001)

A.2 Bond Pricing

The equilibrium model can be cast into Duffie, Pan, and Singleton (2000) form. Let RN denote the risk-neutral probability measure. The price of the real bond is given by

$$B_t(\tau) = E_t^{RN} \left[e^{-\int_t^{t+\tau} r(u) du} \right] \quad (\text{A.1})$$

where r is given in Proposition 1. Applying Feynman-Kac theorem reveals that the solution to the stochastic problem is given by

$$B_t(\tau) = e^{A^r(\tau) + B^{r'}(\tau)(u_t \ w_t \ z_t)'} \quad (\text{A.2})$$

where the loadings are deterministic functions of the parameter of the economy.

The price of the nominal bond is given by

$$N_t(\tau) = E_t^{RN} \left[e^{-\int_t^{t+\tau} R(u) du} \right] \quad (\text{A.3})$$

where r is given in Proposition 1. Applying Feynman-Kac theorem reveals that the solution to the stochastic problem is given by

$$N_t(\tau) = e^{A^{\mathbb{S}}(\tau) + B^{\mathbb{S}'}(\tau)(u_t \ v_t \ w_t \ z_t)'} \quad (\text{A.4})$$

where the loadings are deterministic functions of the parameter of the economy.

A.3 Macroeconomic Forecasts

The analytical solution to the macroeconomic forecasts follows directly from an application of Feynman-Kac. We sketch the solution for consumption growth. The solution for inflation follows analogously.

Define $F(\tau) = E_t[c_{t+\tau}]$. The latter can be re-written as $c_t E_t[e^{\ln \frac{c_{t+\tau}}{c_t}}]$. Guess that the solution to F follows an exponential function that is affine in the underlying states. Apply Feynman-Kac theorem, i.e. $F_\tau = \mathcal{A}F$, where \mathcal{A} is the second order differential operator

applied to F and F_τ stands of for the first derivative of F with regard to τ . Matching coefficients reveals that the guess is correct and that $E_t[c_{t+\tau}] = e^{\alpha(\tau)+\beta(\tau)z_t}$, where the loadings are deterministic functions of the parameter of the economy.

The same exercise applied to $F(\tau) = E_t[p_{t+\tau}]$ reveals that $E_t[p_{t+\tau}] = e^{\alpha_1(\tau)+\beta_1(\tau)w_t}$, where the loadings are deterministic functions of the parameters of the economy.

A.4 Detection Error Probability

The derivation of the detection-error probabilities follows directly from Maenhout (2006).

Table 1: PARAMETER ESTIMATES (Standard Error)

Panel A: State Variables

Drift, Volatility				
	κ	θ	σ_0	σ_1
u_t	0.189086 (<1.0e-4)	0.608258 (2e-4)	0.00348689 (<1.0e-4)	0.0406041 (<1.0e-4)
v_t	0.641549 (3.6e-3)	0.432117 (3.9e-4)	0.0406041 (1.6e-3)	0.0390363 (3e-3)
w_t	0.0254606 (5e-4)	0.166858 (2.6e-3)	3.2618e-05 (<1.0e-4)	0.0911743 (<1.0e-4)
z_t	3.2736 (8e-2)	0.0996513 (<1.0e-4)	4.20842e-06 (<1.0e-4)	0.0162083 (<1.0e-4)

Panel B: Growth and Inflation

μ_A	0.33949 (<1.0e-4)
ν_A	1.22225 (1.8e-4)
ρ_{wA}	-0.801438 (1.7e-4)
σ_{au}	0.00132425 (1e-2)
σ_{bu}	0.165111 (<1.0e-4)
p_0	0.0869924 (0.5)
p_1	0.00159694 (<1.0e-4)
σ_{pv}	0.00249956 (<1.0e-4)
$\bar{\sigma}_{pv}$	0.0679067 (<1.0e-4)

Table 2: Yield Curve Pricing Errors

		Nominal Yields					
maturity in years		2	4	6	7	8	9
in basis points, annualized		4	9.7	10.3	10.4	11.1	12.7
		Real Yields					
maturity in years		5	6	7	8	9	10
in basis points, annualized		15.1	16.56	18.4	20.4	22.8	26.1

Figure 1: Cross-Section of Inflation Ambiguity Premium , 1972.1 - 2009.6

This figure presents the estimated model implied inflation ambiguity premium and its components for medium- and long-term bonds. The solid line plots the inflation ambiguity premium of nominal bond yields. The blue dotted curve plots the ambiguity markup in inflation expectations. The red dashed line depicts the inflation ambiguity premium inherent in Tips yields.

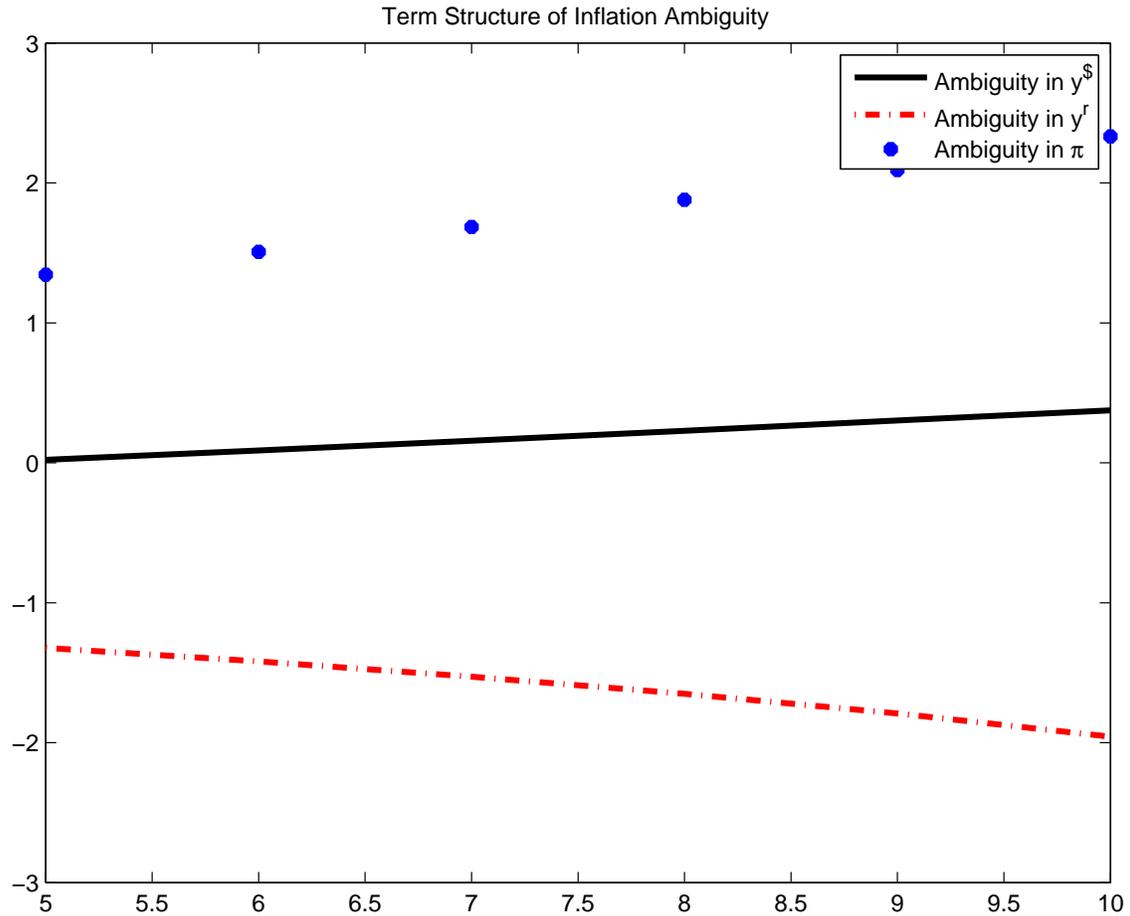


Figure 2: **Components of Inflation Ambiguity Premium over Different Sup-Periods**

This figure presents the model implied components of the inflation ambiguity premium in nominal bond yields for several time-periods. The model is estimated over the entire sample 1972 to 2009 and sample averages are used to construct the graphs. The different sample periods are the periods (from top left to bottom right): before the Monetary Experimentation (1972-1979), during the Monetary Policy Experimentation (1979-1983), the Great Moderation (1984-2007), the first four years of Fed Chairman Bernanke’s term (2005.6-2009.6), the financial crisis (2007.5-2009.6), and the entire sample 1972-2009. The green $-*$ line summarizes the overall inflation ambiguity premium. The solid blue line graphs the inflation ambiguity premium in the robust bond implied inflation forecast. The red $-.$ line plots the inflation ambiguity premium in the Tips term structure.

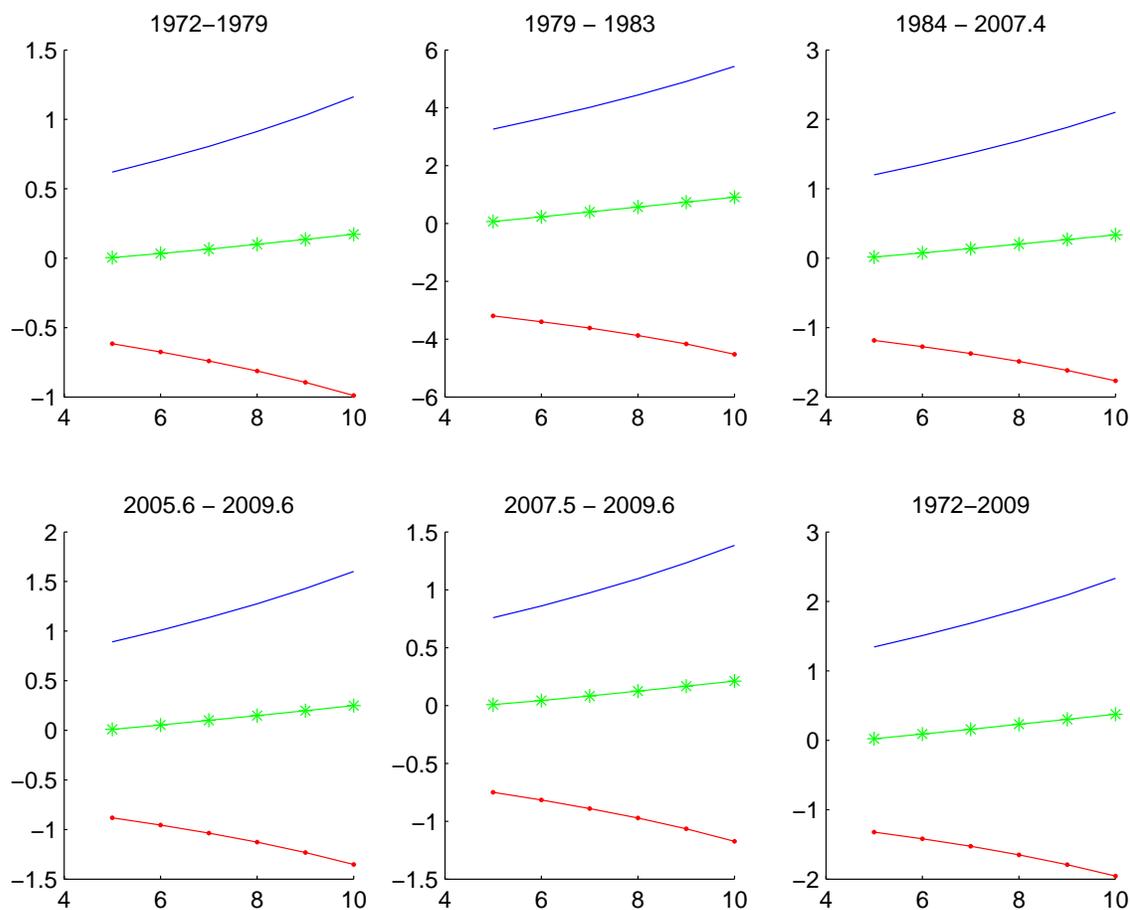


Figure 3: US Government Bond implied Inflation Forecast and its Components over Different Sup-Periods

This figure presents the model implied components of the inflation forecast in nominal bond yields for several time-periods. The model is estimated over the entire sample 1972 to 2009 and sample averages are used to construct the graphs. The different sample periods are the periods (from top left to bottom right): before the Monetary Experimentation (1972-1979), during the Monetary Policy Experimentation (1979-1983), the Great Moderation (1984-2007), the first four years of Fed Chairman Bernanke's term (2005.6-2009.6), the financial crisis (2007.5-2009.6), and the entire sample 1972-2009.

The purple * line summarizes the bond implied inflation forecast. That measure coincides with the term structure of robust inflation forecasts (robust measure). This robust inflation forecast has two components, the inflation ambiguity premium which is presented in the red -.- graph and the empirical inflation forecast (under the empirical measure) which is plotted as the green -* graph.

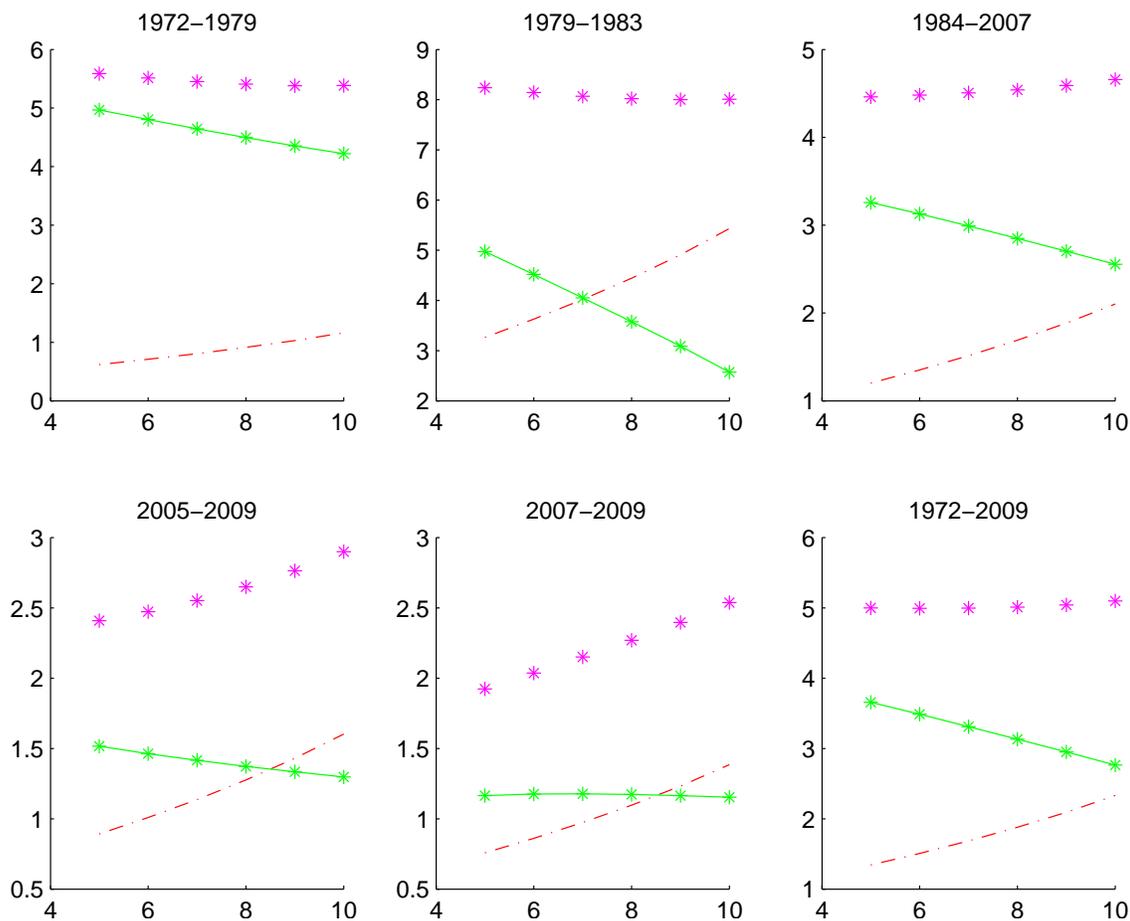


Figure 4: Components of Nominal Yield Curve , 1972.1 - 2009.6

This figure presents the average model implied nominal yield curve and its components for several time-periods. The model is estimated over the entire sample 1972 to 2009 and sample averages are used to construct the graphs. The different sample periods are the periods (from top left to bottom right): before the Monetary Experimentation (1972-1979), during the Monetary Policy Experimentation (1979-1983), the Great Moderation (1984-2007), the first four years of Fed Chairman Bernanke’s term (2005.6-2009.6), the financial crisis (2007.5-2009.6), and the entire sample 1972-2009.

The solid black line represents the average nominal yield curve. The green $-*$ line represents the inflation forecast (empirical measure). The red $-.$ graph represents the inflation ambiguity premium in the inflation forecast. Adding up the previous two lines gives the bond model implied robust inflation forecast (robust measure). The pink $*$ graph represents the estimated Tips yield curve. The solid blue "star" line represents the inflation risk premium.

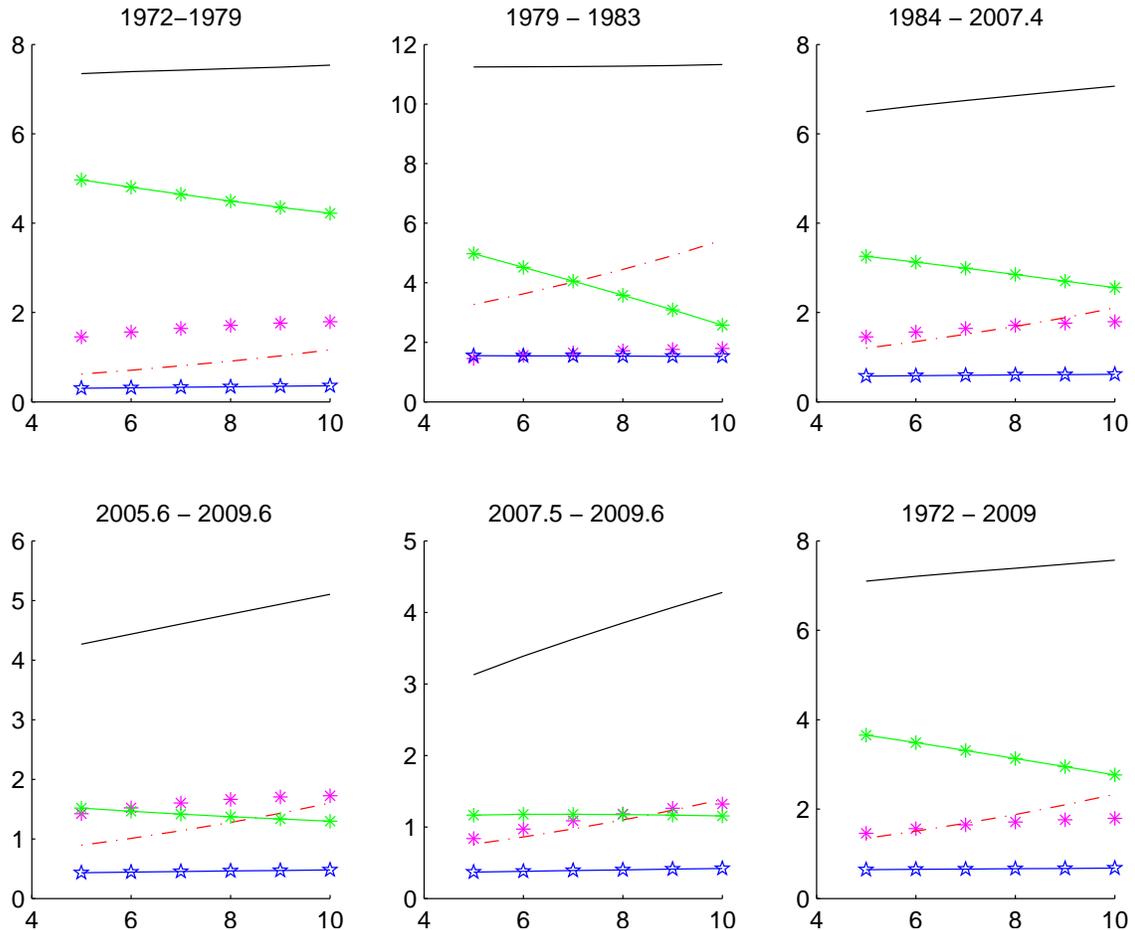


Figure 5: Nominal Short Rate with 95% Confidence Interval and Total Inflation Ambiguity Premium, 1972.1 - 2009.6

This figure contrast the model implied total inflation ambiguity premium in the nominal short-rate with the nominal short-rate itself. We use the federal funds rate from the data as the nominal short-rate. The top panel plots the inflation ambiguity premium. The lower panel plots the federal funds rate, its 95% confidence interval and the federal funds rate without ambiguity premium. The black solid line corresponds to the federal funds rate, as observed in the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied federal funds rate under the reference model. The reference and worst-case model for the federal funds rate are so close to each other, that it is difficult to detect the difference by eye-bolling.

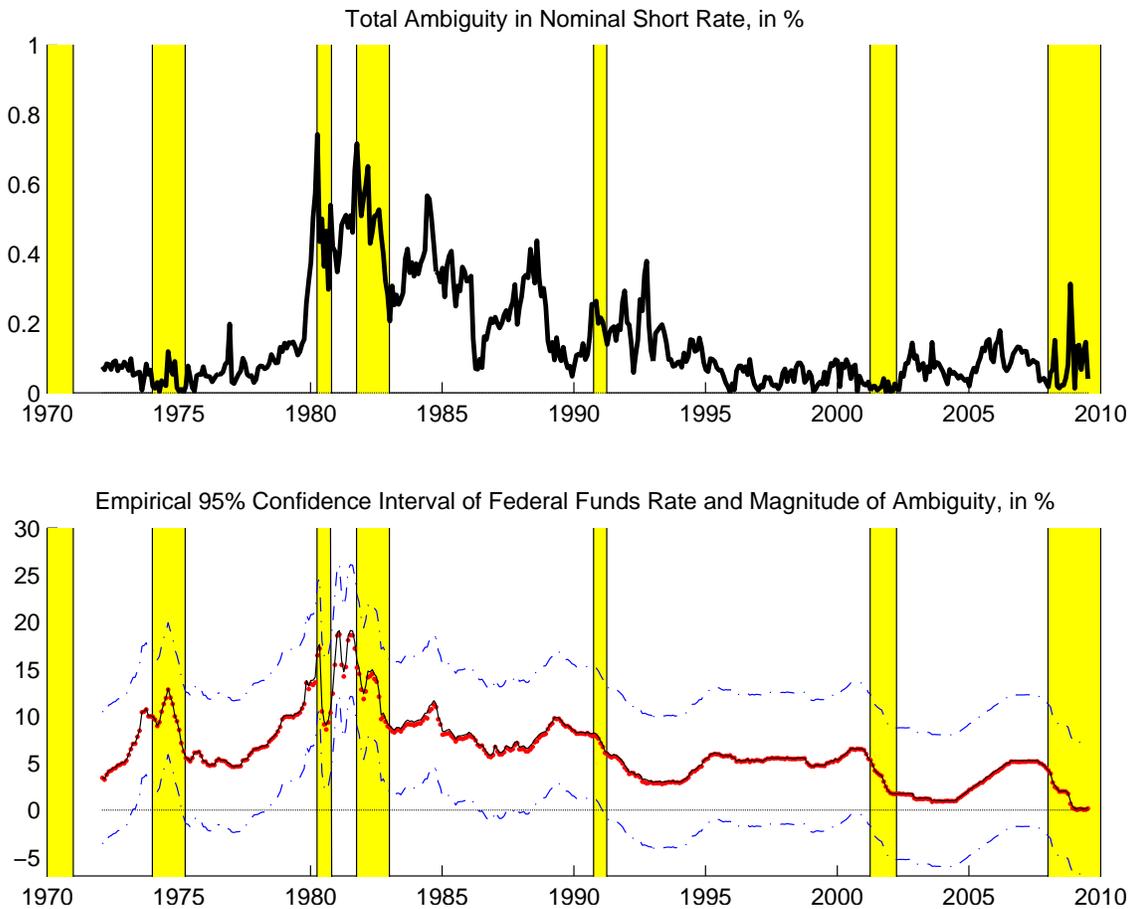


Figure 6: **Ten-Year Nominal Yield with 95% Confidence Interval and Total Inflation Ambiguity Premium, 1972.1 - 2009.6**

This figure contrast the model implied total inflation ambiguity premium in the ten-year nominal yield with the ten-year nominal yield itself. The top panel plots the inflation ambiguity premium. The lower panel plots the the ten-year nominal yield, its 95% confidence interval and the ten-year nominal yield under the reference measure. The black solid line corresponds to the ten-year nominal yield, as observed in the data. The dashed blue lines mark its 95% confidence interval. The red dotted line plots the model implied ten-year nominal yield under the reference model. The reference and worst-case model for the federal funds rate are so close to each other, that it is difficult to detect the difference by eye-bolling.

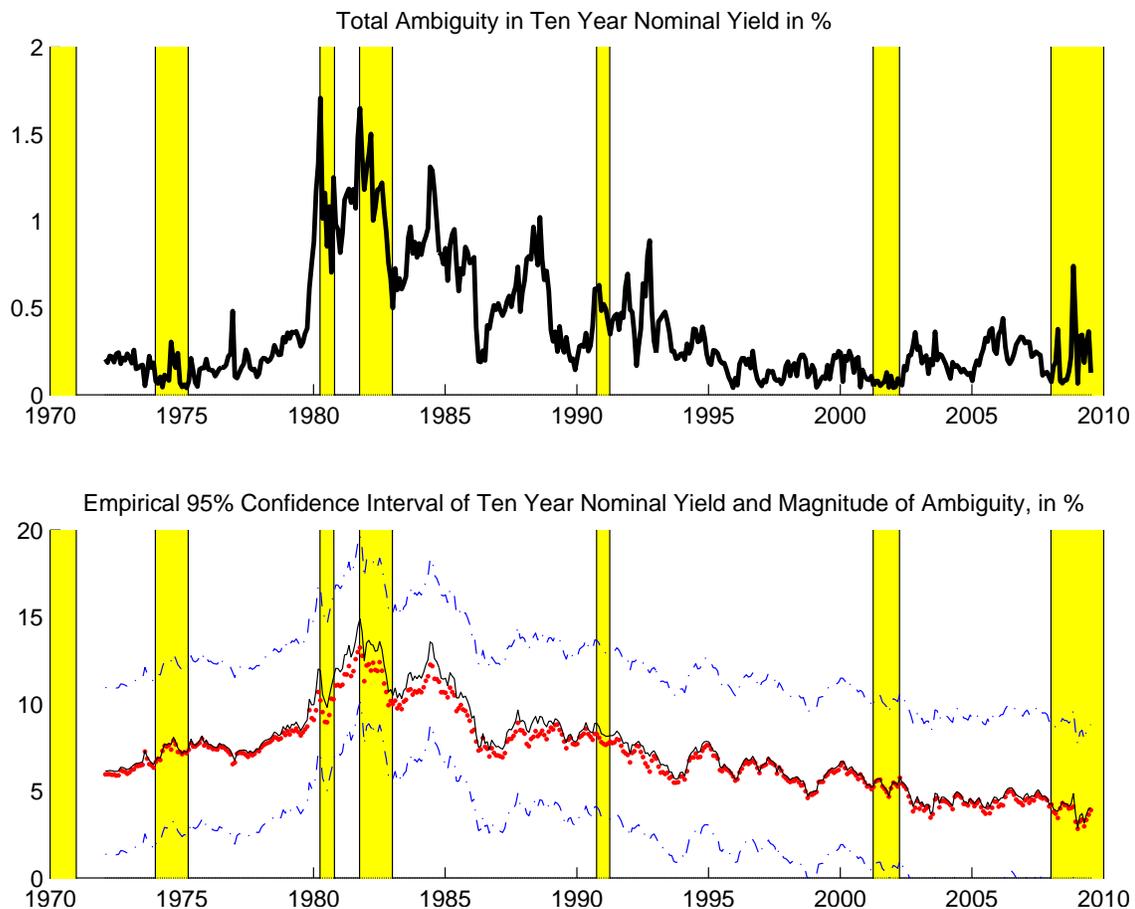
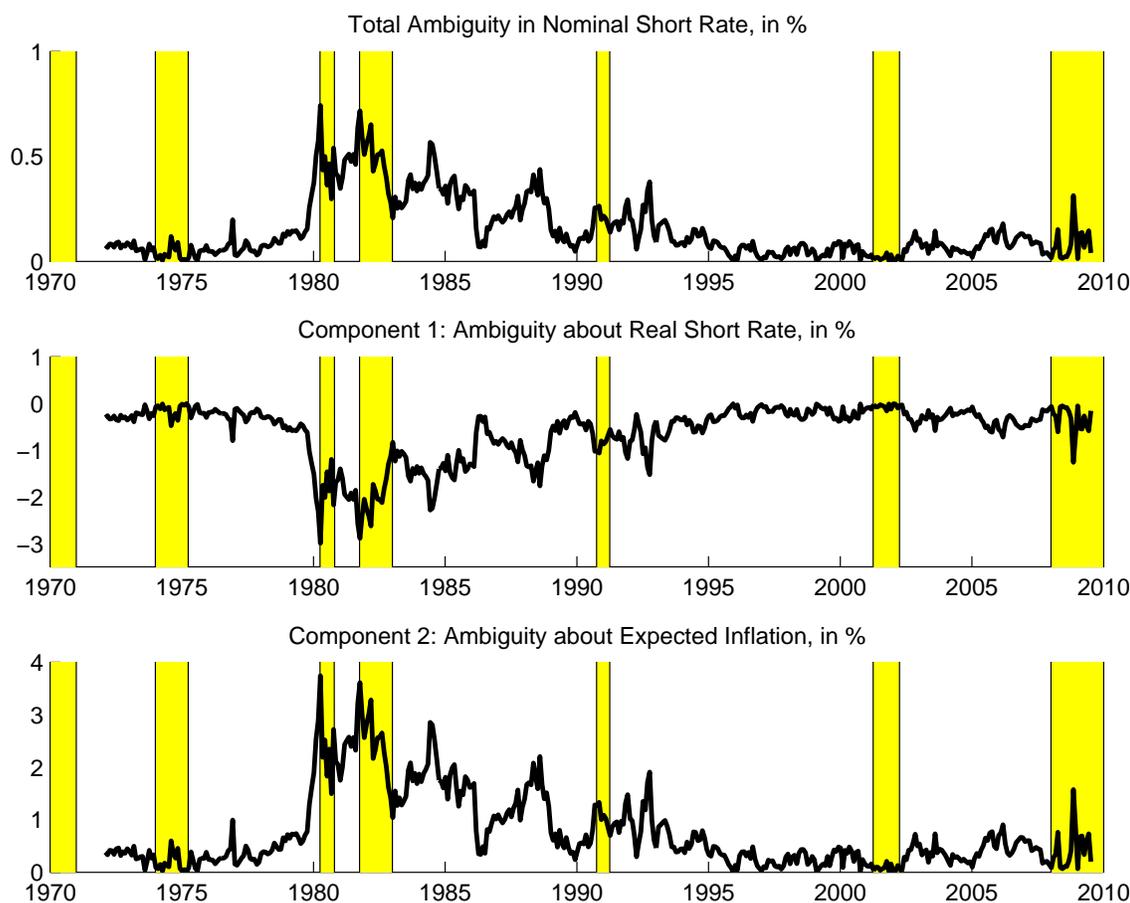


Figure 7: Components of Ambiguity in the Nominal Short Rate , 1972.1 - 2009.6

This figure presents the model implied inflation ambiguity components in the nominal short-rate. Inflation ambiguity enters the nominal short-rate through two channels. First, it lowers the real short-rate (2nd panel) and second, it increases the robust inflation forecast (3rd panel). The net effect is depicted in the top panel.



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