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The principal performance measures in an inventory system involve key characteristics of the system's inventory position, i.e., the total inventory the firm is economically committed to, as well as the average order size or order frequency. As to the former, the focus among operation managers is on the maximum inventory (position), the average inventory and the minimum inventory, the latter being related to the so-called safety stock concept. Financial analysts and macroeconomists pay particular attention to the sales/inventory ratio, also referred to as the inventory turnover.

We derive general conditions under which monotonicity of the above key performance measures can be established within a (single-item) inventory system governed by an optimal (r, q) or (r, nq) policy. When the sample paths of the leadtime demand process are step functions, we refer to the model as the discrete model and the long-run average cost is of the form:

$$c(r, q|t) = \frac{dK + \sum_{y=r+1}^{r+q} G(y|t)}{q}. \tag{1}$$

(When the sample paths are continuous functions, or the so-called continuous model, we have a similar expression with the sum replaced by an integral.) d and K represent the long-run average demand rate and the fixed cost incurred for every order batch of size q respectively. All other model primitives $t \in T$ impact the long-run average cost exclusively via the so-called instantaneous expected cost function $G(y|t)$.

The fixed cost K impacts only the first term in the numerator of (1). Zheng (1992) already showed that the optimal reorder level r^* is decreasing while the optimal order size q^* and the optimal order-up-to level $R^* \equiv r^* + q^*$ are increasing in this parameter. In contrast, the average demand rate d impacts both terms in the numerator of the long-run average cost function and the net monotonicity effect on the optimal policy parameters is therefore, sometimes, ambiguous. We establish our monotonicity properties with respect to all other general model primitives $t \in T$, merely requiring that the space T be endowed with a partial order $\preceq$. As such, t may be a cost parameter, or a parameter of the demand or leadtime distribution. Alternatively, t may represent the distribution of a random variable or a complete stochastic process, or a cost rate function.

Our first main result is that the optimal reorder level r^* and the optimal order-up-to level R^* are decreasing (increasing) in t whenever the function $G(y|t)$ is supermodular (submodular) in (y, t), that is, any of the difference functions $G(y_2|t) - G(y_1|t)$, with $y_1 < y_2$, is increasing (decreasing) in t. Thus, the monotonicity patterns of r^* and R^* are identical in the continuous model and the discrete model and the general conditions under which they are obtained are identical as well. As to the remaining policy parameter q^*, i.e., the optimal order quantity, here the monotonicity patterns that can be expected, differ themselves, between the continuous model and the discrete model . In the continuous model, q^* can often be guaranteed to be monotone in various model parameters. In the discrete model, occasional unit increases (decreases) between stretches where $q^*(t)$ is decreasing (increasing) can not be excluded. This gives rise to a new monotonicity property which we refer to as *rough monotonicity*: an integer valued function is roughly decreasing (increasing) if the step function does not exhibit any pair of consecutive increases (decreases). In addition, more restricted conditions for the structure of the instantaneous expected cost function $G(y|t)$ are necessary and sufficient to obtain the monotonicity property of q^* in the continuous model and rough monotonicity in the discrete model. These more restricted conditions relate to settings where t represents a (scalar) parameter.

The most frequently used model in which the long-run average cost of an (r, q) or (r, nq) policy is given by (1) or its continuous counterpart has the following assumptions: the item is obtained at a given price per unit; inventory costs are accrued at a rate which is a convex increasing function of the inventory level; stockouts are backlogged where backlogging costs are, again, accrued at a rate which is a convex increasing function of the backlog size; leadtimes are generated by a so-called exogenous and sequential process, ensuring that consecutive orders do not cross and the leadtimes are independent of the demand process. We refer to this as the *standard inventory model.*

Our general results imply, in particular, that r^* and R^* are decreasing in the item's purchase price, assuming that the inventory carrying cost rate function increases monotonically with the purchase price. Similarly, r^* and R^* are decreasing in other parameters on which the marginal inventory carrying cost rate function depends monotonically, for example, the physical maintenance and warehousing cost per unit of inventory, or more generally, when the marginal holding cost rate function is replaced by a pointwise larger one. In contrast, r^* and R^* are both *increasing* when the marginal backlogging cost rate function is replaced by a pointwise larger one. As a final application for the standard inventory model, compare two leadtime demand processes such that the leadtime demand distribution under the first process is stochastically smaller than that under the second process. (Dominance of the steady-state leadtime demand distribution may arise because of a change of the demand process, a stochastic enlargement of the leadtime distribution, or both.) We show that r^* and R^* are always smaller under the first process compared to the latter. As far as q^* is concerned, our general results imply, for example, monotonicity with respect to the purchase price, holding/backorder cost rates, and in the case of normal leadtime demands, their mean and standard deviation. Similarly, if the demand process is a Brownian Motion and leadtimes are fixed, q^* is increasing in the drift, volatility and the leadtime. Sufficient conditions for (rough) monotonicity can often be stated in terms of broadly applicable properties of the cdf of the leadtime demand distribution such as log-concavity.

To our knowledge, other than Zheng (1992) only two papers have addressed monotonicity properties in systems that are governed by general (r, q) policies. Song and Zipkin (1996b) address systems with i.i.d. leadtimes, assuming the optimal reorder level $r(q)$ is selected in conjunction with any exogenously specified order quantity q. Since under i.i.d. leadtimes the long-run average cost can only be approximated, these authors investigate the impact of increased leadtime variability on the average backlogging/inventory size by conducting a numerical study. As in this paper, Song et al. (2010) consider systems governed by the globally optimal (r, q)-policy. They show that both r^* and R^* decrease when the steady-state leadtime or leadtime demand distribution becomes stochastically larger. We obtain this as a special corollary of our general monotonicity result under submodular instantaneous expected cost functions $G(y|t)$. They also show that a stochastically smaller leadtime or leadtime demand is not guaranteed to result in a lower average cost, while the less *variable* leadtime or leadtime demand distribution does. The remainder of Song et al. (2010) establishes monotonicity properties of the policy parameters under increased (leadtime) demand variability, but only under certain conditions on the model parameters.

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