INFERIOR GOOD AND GIFFEN BEHAVIOR FOR INVESTING AND BORROWING*

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August 14, 2011

Abstract

It is standard in economics to assume that assets are normal goods and demand is downward sloping in price. This view has its theoretical foundation in the classic single period model of Arrow with one risky asset and one risk free asset, where both are assumed to be held long, and preferences exhibit decreasing absolute risk aversion and increasing relative risk aversion. However when short selling is allowed, we show that the risk free asset can not only fail to be a normal good but can in fact be a Giffen good even for widely popular members of the hyperbolic absolute risk aversion (HARA) class of utility functions. Distinct regions in the price-income space are identified in which the risk free asset exhibits normal, inferior and Giffen behavior. An Example is provided in which for non-HARA preferences Giffen behavior occurs over multiple ranges of income.

*We thank the anonymous referees and Editor for their very helpful comments and suggestions.

1 Introduction

In the classic single period model with one risky asset and one risk free asset, where both are assumed to be held long, Arrow [1] shows that the risky asset is a normal good (its demand is increasing with income or wealth) if the Arrow-Pratt [1]-[12] measure of absolute risk aversion is decreasing. Arrow also proves that a sufficient condition for the income elasticity of demand for the risk free asset to be greater than one is that relative risk aversion is increasing. Aura, Diamond and Geanakoplos [2] point out that these two results together imply that both assets are normal goods.

While the assumption that the risky asset be held long is relatively harmless, the same assumption for the risk free asset is far from innocuous. Consider the case of the widely used HARA (hyperbolic absolute risk aversion)[1] utility $W(x) = -\frac{(x+a)^{-\delta}}{\delta}$, where $\delta > -1$, $a > 0$ and x denotes wealth (or end of period consumption). Optimal holdings of the risk free asset will always be both positive and negative (corresponding to different income ranges) so long as the risk preference parameter δ is above some minimum $\delta_{critical}$. Despite the fact that this utility function satisfies the Arrow requirements of decreasing absolute risk aversion and increasing relative risk aversion, when $\delta > \delta_{critical}$ the risk free asset will always be an inferior good over some income range. And it can even be a Giffen good, where corresponding to an own price increase, the asset's positive income effect swamps the negative substitution effect resulting in increased demand.

More generally, inferior good and Giffen behavior occur for other members of the HARA class and other forms of utility. If Arrow's assumption that both assets are held long is relaxed, the only member of the HARA class for which the risk free asset and risky asset are both always normal goods is the very special constant relative risk aversion (CRRA) form.[2] For a number of examples, distinct regions in the price-income space are identified in which the risk free asset exhibits normal good, inferior good and Giffen behavior. We show that when the risk free asset is an inferior or Giffen good, it can only be held short (long) if relative risk aversion is increasing (decreasing). A non-HARA example is given for which relative risk aversion is non-monotone and Giffen behavior is shown to occur over multiple income ranges. What is particularly surprising is that in contrast to much of the classic demand theory literature where very special forms of utility need to be constructed to produce Giffen behavior, in the case of financial securities it arises with perfectly standard utility functions.[3]

Given that Giffen behavior can arise with relative ease for the commonly used HARA

[1] See [7] for a description of the HARA family of utility funcitons.

[2] Following Fischer [6], it is well known that the risky asset will be an inferior good in the case of quadratic utility. But because this member of the HARA class exhibits increasing absolute risk aversion, it is rarely assumed.

[3] In their recent paper [5], Doi, Iwasa and Shimomura observe that the existing demand theory literature on Giffen behavior is void of examples based on conventional forms of utility. Indeed they too construct a specific form of utility which, although nonstandard, is argued to be well-behaved in terms of its properties.

utilities, it is natural to wonder what implications this behavior might have for equilibrium asset prices.[4] By applying a not widely known certainty result of Kohli [9], one can obtain the surprising result that in a representative agent, distribution economy, Giffen behavior of the risk free asset implies that the risky asset's equilibrium price increases with its supply.[5]

In Section 2, we consider portfolios consisting of a risk free asset and a risky asset where positive holdings of the former is not assumed. As is standard, the asset demand, or complex securities, model is embedded in a contingent claims framework. Complete markets are assumed.[6] We establish necessary and sufficient conditions for the risk free asset to be a normal good and apply these conditions to a number of different classes of utility including the HARA family. Section 3 examines when the risk free asset can be a Giffen good and provides examples for a utility in the HARA class and for one outside the class. Section 4 considers selected extensions to a two period setting. The last section contains concluding comments.

2 Risk Free Asset: Normal Good Behavior

2.1 Preliminaries

Throughout this Section and the next Section, we consider a single period setting in which a consumer with a given level of income selects asset holdings so as to maximize expected utility for end of period random consumption. In Section 4, we consider the natural extension to a two period setting where the consumer at the beginning of period 1 chooses a level of certain current period consumption c_1 as well as asset holdings the returns on which fund period 2 consumption, c_2. The notational conventions and structure of the current Section are designed to facilitate the simplest transition to the more general two period problem.

Consider a risky asset with payoff $\tilde{\xi}$, where $\tilde{\xi}$ is a random variable assuming the value ξ_{21} with probability π_{21} and ξ_{22} with the probability $\pi_{22} = 1 - \pi_{21}$. Without loss of generality, let $\xi_{21} > \xi_{22}$. It is further assumed that $\xi_{22} > 0$. Suppose there exists a risk-free asset with payoff $\xi_f > 0$. Let n and n_f denote the number of units of the risky asset and risk free asset, respectively. Throughout this paper, we assume that $\frac{E\tilde{\xi}}{p} > \frac{\xi_f}{p_f}$ which can be shown to imply that risky asset demand satisfies $n > 0$ for all I. In the current single period setting, preferences are defined over random $\tilde{c}_2$ and satisfy the standard expected

[4] We thank one of the Referees for stressing the importance of connecting our demand theory results to their equilibrium implications.

[5] For a more general analysis assuming a representative agent, exchange economy in which the implications of changing asset supplies on equilbrium asset prices and equity risk premia are examined in both one and two period settings, see Kubler, Selden and Wei [10].

[6] It should be noted that our results, Theorem (i) and (iii), extend naturally to incomplete markets. Although in general, Theorem 1(ii) does not extend, as one might expect, it does for HARA preferences where markets are effectively complete (see [4]).

utility axioms where the NM (von Neumann-Morgenstern) index $W(c_2)$ satisfies $W' > 0$ and $W'' < 0$.[7] The expected utility function $EW(\widetilde{c}_2)$ given by

$$E[W(\tilde{\xi}n + \xi_f n_f)] = \pi_{21} W(\xi_{21} n + \xi_f n_f) + \pi_{22} W(\xi_{22} n + \xi_f n_f) \tag{1}$$

is maximized with respect to n and n_f subject to the budget constraint

$$pn + p_f n_f = I, \tag{2}$$

where p and p_f are the prices of the risky and risk free assets and I is initial or date 1 income. Define the contingent claims c_{21} and c_{22} by

$$c_{21} = \xi_{21} n + \xi_f n_f, \qquad c_{22} = \xi_{22} n + \xi_f n_f. \tag{3}$$

The above complex or financial securities problem is equivalent to a contingent claim optimization where

$$EW(c_{21}, c_{22}) = \pi_{21} W(c_{21}) + \pi_{22} W(c_{22}) \tag{4}$$

is maximized with respect to c_{21} and c_{22} subject to

$$p_{21} c_{21} + p_{22} c_{22} = I, \tag{5}$$

where

$$p_{21} = \frac{\xi_f p - \xi_{22} p_f}{(\xi_{21} - \xi_{22})\xi_f} > 0 \quad \text{and} \quad p_{22} = \frac{\xi_{21} p_f - \xi_f p}{(\xi_{21} - \xi_{22})\xi_f} > 0 \tag{6}$$

are the contingent claims prices. The contingent claims FOC (first order conditions) can be expressed as

$$\frac{W'(c_{21})}{W'(c_{22})} = \frac{\pi_{22}}{\pi_{21}} \frac{p_{21}}{p_{22}} =_{def} k. \tag{7}$$

Throughout we assume no arbitrage – it is easy to see that this is equivalent to

$$\frac{\xi_{21}}{p} > \frac{\xi_f}{p_f} > \frac{\xi_{22}}{p}. \tag{8}$$

It should be noted that $\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f}$ is equivalent to $c_{21} > c_{22}$ or

$$k = \frac{\pi_{22}}{\pi_{21}} \frac{p_{21}}{p_{22}} < 1. \tag{9}$$

Since we do not assume an Inada condition, a minimum level of income has to be assumed to guarantee non-negative consumption. It is easy to verify that to ensure that $c_{21}, c_{22} \geq 0$, the minimum income level is given by

$$I_{\min} = \begin{cases} p_{21} \left(W'\right)^{-1} \left(kW'(0)\right) & \left(W'(0) \neq \infty\right) \\ 0 & \left(W'(0) = \infty\right) \end{cases}. \tag{10}$$

[7] These single period NM preferences are extended in Section 4 to the two period expected utility $EW(c_1, \widetilde{c}_2) = W_1(c_1) + EW_2(\widetilde{c}_2)$ where the consumer is choosing over both c_1 and (n, n_f). The NM utility considered here can be viewed as corresponding to the two period $W_2(c_2)$.

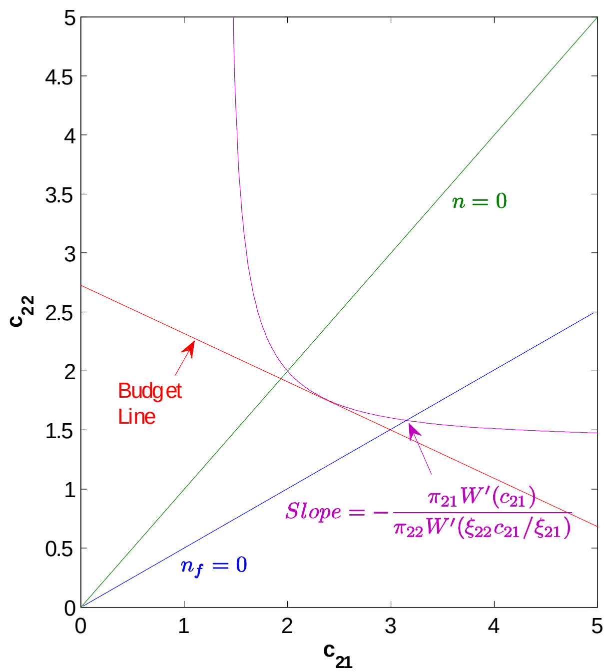


Figure 1:

This condition requires $I > I_{\min}$ to ensure that optimal contingent claims demand is in the positive orthant. The application of $I_{\min}$ will be illustrated for a number of specific utility functions below.

Although $n > 0$ is ensured by the assumption $\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f}$, the condition for $n_f > 0$ is far from free as it imposes restrictions on the consumer's preferences. Given that $\frac{\xi_{22}}{\xi_{21}}$ defines the slope of the risky asset payoff ratio in the contingent claims space, $-\frac{\pi_{21}W'(c_{21})}{\pi_{22}W'(\frac{\xi_{22}}{\xi_{21}}c_{21})}$ measures the slope of the tangent to the indifference curve at its intersection with the $n_f = 0$ ray. The following Lemma states that for $n_f > 0$, the consumer's preferences must be such that the MRS (marginal rate of substitution) at this ray is always less than the absolute value of the slope of the budget line for any c_2. See Figure 1.

Lemma 1 *The risk free asset holdings satisfy* $n_f \gtreqless 0$ *for all* I *iff* $\frac{W'(c_2)}{W'(\frac{\xi_{22}}{\xi_{21}}c_2)} \lesseqgtr k$ *for any* c_2.

In standard demand theory, a commodity is assumed to have positive demand and is said to be a normal good if its derivative with respect to income is positive. Given that Lemma 1 allows for the risk free asset to be held short, we next generalize the normal good definition to allow for borrowing.

Definition 1 *The risk free asset is said to be a normal good if and only if*[8]

$$n_f \frac{\partial n_f}{\partial I} > 0. \tag{11}$$

When $n_f > 0$, we obtain the traditional normal goods definition $\dfrac{\partial n_f}{\partial I} > 0$. If $n_f < 0$ the asset will be held short, and $\dfrac{\partial n_f}{\partial I} < 0$ indicates that as income level increases, the investor will increase the borrowing and borrowing can be viewed as a normal good. It should be noted that $-n_f \frac{\partial n_f}{\partial I}$ is the standard income effect in the Slutsky equation. If $n_f \frac{\partial n_f}{\partial I} < 0$, the risk free asset is an inferior good and the income effect will become positive. This can result in $\frac{\partial n_f}{\partial p_f}$ being positive if the positive income effect dominates the negative substitution effect.

In the analysis that follows, we will make use of the critical income level I^* which serves as the boundary along the risk free asset Engel curve dividing normal from inferior good behavior.

Definition 2 *An income level I is said to be a critical income I^* if it satisfies*

$$n_f \frac{\partial n_f}{\partial I}\bigg|_{I=I^*} = 0. \tag{12}$$

Throughout this paper we will require $I^* > I_{\min}$ to ensure that optimal consumption will be in the positive orthant. Clearly, I^* corresponds to either $n_f = 0$ or $\frac{\partial n_f}{\partial I} = 0$. Moreover, as we will see, there can exist multiple I^* values.

2.2 Normal Good Behavior: The Canonical CRRA Case

To illustrate the role of the Lemma 1 restriction on preferences, we next consider the case of CRRA utility.

Example 1 *Suppose the NM index takes the classic CRRA form*

$$W(c_2) = -\frac{1}{\delta} c_2^{-\delta}, \tag{13}$$

where $\delta > -1$. Will the risk free asset be held long or short? From the FOC for CRRA preferences, using eqn. (7), the contingent claims expansion path is given by

$$c_{22} = c_{21} k^{1/(1+\delta)} \tag{14}$$

and is linear passing through the origin with slope of $k^{1/(1+\delta)}$. Straightforward computation of the condition in Lemma 1 shows that $n_f > 0$, if and only if $k > (\frac{\xi_{22}}{\xi_{21}})^{1+\delta}$. If we define

$$\delta_{critical} = \frac{\ln k}{\ln(\xi_{22}/\xi_{21})} - 1, \tag{15}$$

[8] This same definition will be used for risky assets as well.

we have the following restriction on preferences

$$n_f \gtreqqless 0 \Leftrightarrow \delta \gtreqqless \delta_{critical}, \tag{16}$$

where k is defined in terms of the state prices p_{21} and p_{22} which in turn depend on the asset payoffs and prices following eqn. (6). Since $k < 1$, the linear expansion path will rotate clockwise as δ decreases because its slope $k^{1/(1+\delta)}$ will decline. If δ falls below the critical value given by the right hand side of eqn. (15), we have $n_f < 0$ and the expansion path will be below the $\frac{\xi_{22}}{\xi_{21}}$ (or $n_f = 0$) ray. Given that the Arrow-Pratt ([1]-[12]) relative risk aversion measure

$$\tau_R =_{def} -c_2 \frac{W''(c_2)}{W'(c_2)} = \delta + 1, \tag{17}$$

one obtains the very intuitive interpretation for eqn. (16) that if $\delta > \delta_{critical}$, the consumer is sufficiently risk averse that she will only hold the risk free asset long. Since for CRRA preferences, the expansion path is linear and pass through the origin, n_f and $\frac{\partial n_f}{\partial I}$ always have the same sign, which implies that the risk free asset is always a normal good.

2.3 Generalization of the Classic Arrow Theorems

Denoting the Arrow-Pratt ([1]-[12]) measure of absolute risk aversion by

$$\tau_A(c_2) =_{def} -\frac{W''(c_2)}{W'(c_2)}, \tag{18}$$

we next extend the Arrow [1] result to a contingent claims setting in which shorting the risk free asset is allowed.

Theorem 1 *For the contingent claims problem corresponding to eqns. (4) and (5), optimal asset demands satisfy*

(i) $\frac{\partial n}{\partial I} \gtreqqless 0$ *if* ${\tau'}_A \lesseqqgtr 0$,
(ii) $\frac{\partial n_f}{\partial I} \gtreqqless 0$ *iff* $\frac{\tau_A(c_{21})}{\tau_A(c_{22})} \gtreqqless \frac{\xi_{22}}{\xi_{21}}$,
(iii) $\frac{\partial (n_f/I)}{\partial I} \gtreqqless 0$ *if* ${\tau'}_R \gtreqqless 0$.

Remark 1 *Given that $\frac{E\tilde{\xi}}{p} > \frac{\xi_f}{p_f}$ implies $n > 0$, condition (i) of Theorem 1 coincides exactly with Arrow's result. Condition (iii) is equivalent to Arrow's second result relating to increasing relative risk aversion. To see this, note that*

$$\frac{\partial (n_f/I)}{\partial I} = \frac{I\frac{\partial n_f}{\partial I} - n_f}{I^2} \tag{19}$$

and assuming $n_f > 0$, Arrow's income elasticity result follows immediately from

$$\frac{\partial\left(n_f/I\right)}{\partial I} > 0 \Leftrightarrow \frac{\partial n_f / n_f}{\partial I / I} > 1. \tag{20}$$

Arrow's assumption that both assets are held long clearly implies that $\frac{\partial n_f}{\partial I} > 0$ and the risk free asset is a normal good as asserted by Aura, Diamond and Geanakoplos [2]. But from the application of Lemma 1 in Example 1, we see that actually for $n_f > 0$, one must assume that the consumer is sufficiently risk averse to satisfy eqn. (16). Moreover, it follows from Example 1 that it is unnecessary to assume as in [2] that $n_f > 0$ in order for both assets to be normal goods.

How should one interpret the critical $\frac{\tau_A(c_{21})}{\tau_A(c_{22})}$? It is straightforward to show that this ratio is in fact the slope of the tangent to the contingent claims expansion path at any point (c_{21}, c_{21}) along the path. Theorem 1(ii) can be viewed as requiring for n_f to be increasing with income that the tangent to the expansion path must have a slope steeper than that of the $n_f = 0$ ray defined by the risky asset payoff $\frac{\xi_{22}}{\xi_{21}}$. It should also be noted that if over an interval of income values the tangent to the expansion path has the same slope as the $c_{22} = \frac{\xi_{22}}{\xi_{21}} c_{21}$ $(n_f = 0)$ ray, then for that range of incomes n_f is invariant to changes in I. It can be shown that in the case of multiple risky assets, condition (ii) in Theorem 1 generalizes to a comparison of the angle between the tangent vector of the expansion path in the contingent claims setting and the normal vector of the $n_f = 0$ hyperplane and 90°. Hence condition (ii) is the generic result, rather than the widely quoted Arrow condition (i).

2.4 Canonical Inferior Good Case: HARA Preferences

The following example illustrates several important implications of Theorem 1 for a widely used form of HARA utility.

Example 2 *Preferences are defined by the widely used HARA form*

$$W(c_2) = -\frac{1}{\delta}(c_2 + a)^{-\delta}, \tag{21}$$

where $a > 0$, $\delta > -1$. For this utility, we have $\tau'_A < 0$ and $\tau'_R > 0$. Therefore, the risky asset is always a normal good. The expansion path is given by

$$c_{22} = k^{\frac{1}{1+\delta}}(c_{21} + a) - a. \tag{22}$$

Figure 2 illustrates expansion paths associated with different values of δ, where as is standard, each point along an expansion path has the same price ratio but different levels of income, I. (The expansion paths in the Figure are solid and the $n = 0$ and $n_f = 0$ rays are dashed.) It follows from eqn. (10) that the minimum income level $I_{\min}$ to avoid bankruptcy is given by

$$I_{\min} = \frac{a p_{21}\left(1 - k^{\frac{1}{1+\delta}}\right)}{k^{\frac{1}{1+\delta}}}. \tag{23}$$

Because $\tau'_R > 0$, it follows from the Arrow result that the risk free asset is also a normal good

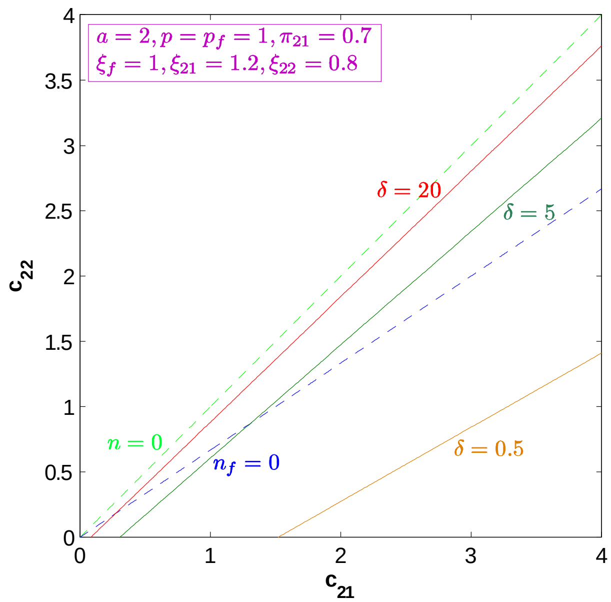


Figure 2:

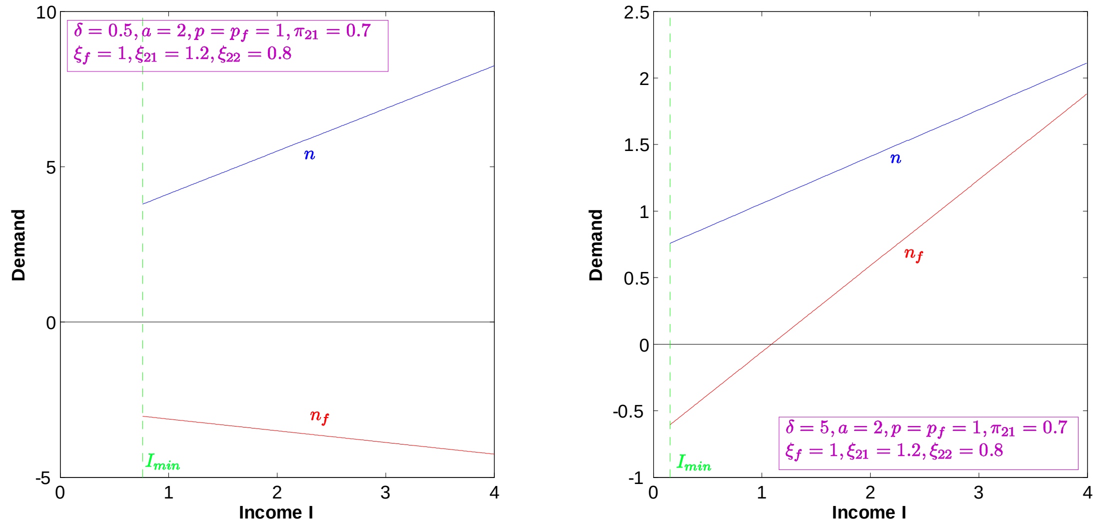
if $n_f > 0$. *However, as we show below, for the HARA utility (21) it is impossible for* $n_f > 0$ *to be satisfied for all income levels. It follows from Theorem 1 that if*

$$\frac{\xi_{22}}{\xi_{21}} > k^{\frac{1}{1+\delta}} = \frac{\tau_A(c_{21})}{\tau_A(c_{22})}, \tag{24}$$

which is equivalent to

$$\delta < \delta_{critical} = \frac{\ln k}{\ln(\xi_{22}/\xi_{21})} - 1, \tag{25}$$

we have $\frac{\partial n_f}{\partial I} < 0$. *Since* $n_f < 0$ *when* $I = 0$, *the risk free asset is a normal good for all income levels in the sense of borrowing. On the other hand if* $\delta > \delta_{critical}$, *then we have* $\frac{\partial n_f}{\partial I} > 0$. *Since* $n_f < 0$ *when* $I = 0$, *the risk free asset cannot be a normal good for all income levels. To illustrate this more explicitly, fix the parameters as follows:* $a = 2$, $p = p_f = 1$, $\xi_f = 1$, $\xi_{21} = 1.2$, $\xi_{22} = 0.8$ *and* $\pi_{21} = 0.7$. *Then* $\delta_{critical} \approx 1.09$. *We plot the asset Engel curves for* $\delta = 0.5 < \delta_{critical}$ *in Figure 3(a) and* $\delta = 5 > \delta_{critical}$ *in Figure 3(b) and indicate* $I_{\min}$ *in each case. It can be seen that when* $\delta < \delta_{critical}$, *the investor will always short the risk free asset. When the income level increases, she will borrow more, which implies that the risk free asset is a normal good in the sense of borrowing. When* $\delta > \delta_{critical}$, *the investor will only short the risk free asset at the low income levels. But since* $\frac{\partial n_f}{\partial I} > 0$ *for all the income levels, the risk free asset fails to be a normal good for the low income levels (Definition 1). Moreover, it is clear from Figure 3(b) that we can find the critical income level* I^* *such that for* $I > I^*$,



(a) $\delta < \delta_{critical}$

(b) $\delta > \delta_{critical}$

Figure 3:

$n_f > 0$ and the risk free asset becomes a normal good. To find I^*, note that in Figure 2 the expansion paths corresponding to $\delta > \delta_{critical}$ all cross the $n_f = 0$ ray. Thus for any such δ, based on the intersection point one can determine I^* as follows:

$$I^* = \frac{a(\xi_{21}p_{21} + \xi_{22}p_{22})(1 - k^{\frac{1}{1+\delta}})}{\xi_{21}k^{\frac{1}{1+\delta}} - \xi_{22}}. \tag{26}$$

In Figure 3(b) where $\delta = 5$, we have $I_{\min} = 0.30$ and $I^* = 1.09$. It can be verified that $\dfrac{\partial I^*}{\partial \delta} < 0$. Thus as the relative risk aversion parameter δ decreases, the critical income level I^* increases. When $\delta \to \delta_{critical}$ from above we have $I^* \to \infty$ and the risk free asset becomes an inferior good for virtually all levels of income.

Remark 2 *In addition to eqn. (21) and the CRRA (13), the HARA class includes negative exponential, logarithmic and quadratic utilities (e.g., [7], p.26). Each member other than the CRRA case can generate expansion paths where the risk free asset exhibits both normal and inferior good behavior over different income ranges.*[9]

[9] It should be noted that for the negative exponential case, the expansion path will always have a slope equal to 1. For quadratic utility, the expansion path always has a slope greater than 1. It follows from Theorem 1 that $\frac{\partial n_f}{\partial I} > 0$ for all the income levels. Given that $\frac{E\tilde{\xi}}{p} > \frac{\xi_f}{p_f}$, n_f will be negative at sufficiently low income levels. Therefore, the risk free asset can never be a normal good for all the income levels for these two types of utility functions.

2.5 Risk Free Asset Engel Curve Properties: Critical Role of τ'_R

We next establish an important link between τ'_R and inferior good behavior for the risk free asset and then illustrate our conclusions with a series of examples.

THEOREM 2 *Assume the general NM utility (1), and complete markets with one risk free asset and one risky asset.*

(i) If $\tau'_R > 0$, the risk free asset can become an inferior good only when $n_f < 0$.

(ii) If $\tau'_R < 0$, the risk free asset can become an inferior good only when $n_f > 0$.

(iii) If the sign of τ'_R changes over its domain, the risk free asset can become an inferior good for both $n_f < 0$ and $n_f > 0$.

REMARK 3 *In terms of Theorem 2, condition (i) is illustrated by Example 2 (above), (ii) by Examples 3 and (iii) by Example 4.*

We begin by modifying the Example 2 utility to investigate the impact of assuming decreasing rather than increasing relative risk aversion.

EXAMPLE 3 *Assume*

$$W(c_2) = -\frac{1}{\delta}(c_2 - a)^{-\delta}, \tag{27}$$

where $a > 0$, $\delta > -1$. For this utility, we have $\tau'_R < 0$. The same parameters are assumed as in Example 2. Figure 4 illustrates expansion paths associated with different values of δ. Since we require that $c_{21}, c_{22} > a$, $I_{\min} = ap_{21} + ap_{22} = \frac{ap_f}{\xi_f}$.[10] *When $\delta > \delta_{critical}$, where $\delta_{critical}$ is defined by (15), the risk free asset is a normal good. (See Figure 5(b).) When $\delta < \delta_{critical}$, $\frac{\partial n_f}{\partial I} < 0$. Since n_f starts from $\frac{a}{\xi_f}$ where $n = 0$, the risk free asset is an inferior good at low income levels and when $n_f < 0$, it becomes a normal good. (See Figure 5(a).)*

REMARK 4 *When one makes the reasonable assumption that n_f can be either negative or positive, a comparison of Examples 2 and 3 would seem to weaken Arrow's argument for assuming increasing rather than decreasing relative risk aversion. In Example 2 where $\tau'_R > 0$, if $\delta > \delta_{critical}$ the consumer with low levels of income, $I^* > I > I_{\min}$, initially shorts the risk free asset to finance investment in the risky asset. Whereas if one assumes exactly the same setting except that $\tau'_R < 0$, we see in Example 3 that for $\delta > \delta_{critical}$ the consumer initially holds the risk free asset long at low levels of income and then reduces the holdings as income increases.*[11] *For us at least, the latter case is a priori more reasonable. The property of decreasing relative risk aversion has received attention in empirical and experimental papers (e.g., Calvet and Sodini*

[10] In this case unlike the other examples, $I_{\min}$ arises from the "subsistence" requirement $c_2 > a$ rather than from the no bankruptcy requirement $c_{21}, c_{22} > 0$.

[11] It should be noted that $\delta_{critical}$ can take on a range of values based on different assumptions of the underlying parameters such as asset returns returns, probabilities and prices.

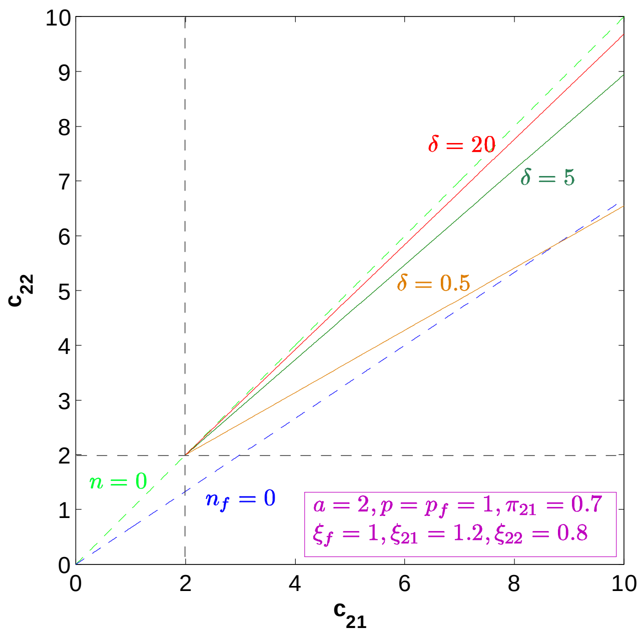
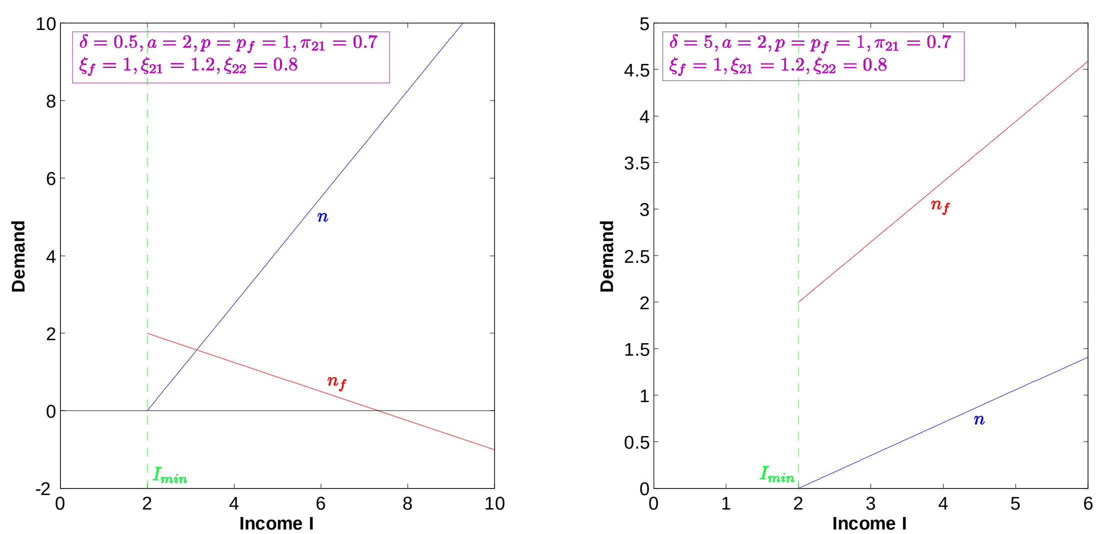


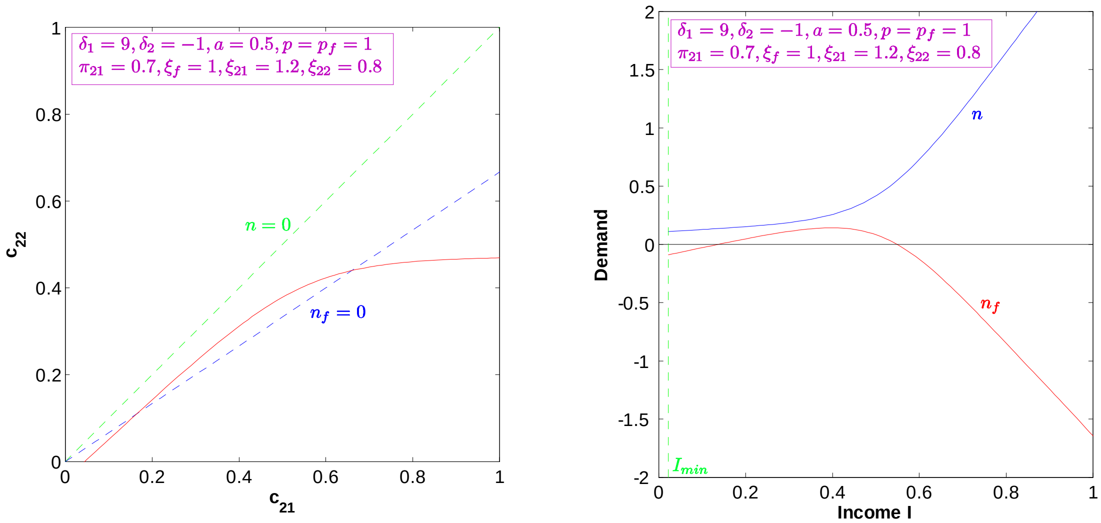
Figure 4:



(a) $\delta < \delta_{critical}$

(b) $\delta > \delta_{critical}$

Figure 5:



(a) Expansion Path

(b) Engel Curve

Figure 6:

[3]) challenging the Arrow assumption of increasing relative risk aversion. It should also be noted that Meyer and Meyer [11] have recently proposed employing a multiperiod version of (27) as an alternative to the standard (internal) habit formation representation in try to resolve the equity premium puzzle. Both the standard NM habit utility and (27) exhibit decreasing relative risk aversion.

The next Example, based on non-HARA utility functions, illustrates that when the sign of τ'_R varies, the sign of $\frac{\partial n_f}{\partial I}$ can vary in both the $n_f > 0$ and $n_f < 0$ regions.

EXAMPLE 4 *Assume*

$$W(c_2) = -\frac{1}{\delta_1}(c_2 + a)^{-\delta_1} + c_2. \tag{28}$$

where $a > 0$, $\delta_1 > -1$. We have $\tau'_A < 0$ and τ'_R doesn't have a definite sign. An expansion path and Engel curves are illustrated in Figure 6. It can be seen that the consumer shorts the risk free asset at low income levels and again beginning at intermediate income levels. But $\frac{\partial n_f}{\partial I}$ is positive at low income levels and negative at high income level. At intermediate income levels, the consumer is long the risk free asset but n_f is not monotone in I. It can be also seen from Figure 6(b) that there are three I^* for this utility. Two correspond to $n_f = 0$ and one corresponds to $\frac{\partial n_f}{\partial I} = 0$.[12]

[12] As suggested by this Example, the utility (28) may offer interesting potential in providing a partial response to Hart's [8] query as to what assumptions are required to get meaningful comparative statics findings for the risk free asset when the utility function does not exhibit the portfolio separation property implied by the HARA class.

3 Giffen Good Behavior

3.1 Law of Demand Violations and Risk Free Asset Giffen Behavior

Given our findings that the risk free asset can become an inferior good, it is natural to wonder if it can also be a Giffen good, i.e., the risk free asset can violate the (own good) LOD (Law of Demand) $\frac{\partial n_f}{\partial p_f} > 0$. To see that shorting is allowed by this definition, note first that if $n_f < 0$, when the price p_f increases, the effective cost of borrowing $\frac{\xi_f}{p_f}$ will decrease. And if the consumer responds to the decreased cost of borrowing by the increasing borrowing when $n_f < 0$, then borrowing satisfies the LOD. On the other hand, if the consumer reduces the borrowing, when the cost decreases the risk free asset becomes a Giffen good.

It is well known that if one of the goods is a Giffen good, then the LOD is violated. However, violation of the LOD is only a necessary and not a sufficient condition for Giffen good behavior. Moreover, Quah ([13], Proposition 1) has shown that the LOD is violated in the financial securities setting, if and only if it is also violated in the contingent claims setting. Since in the contingent claims setting, both contingent claims commodities are normal goods, one may wonder whether Giffen good behavior can ever occur in the complex security setting. In the next Subsection, we provide two examples illustrating that the risk free asset can indeed be a Giffen good.[13]

3.2 Giffen Behavior: Income and Price Regions

Next we show for both the HARA utility used in Example 3 and the non-HARA utility in Example 4 that by choosing the appropriate parameter values, the risk free asset can become a Giffen good. Also for the former, we characterize regions in the (p_f, I) parameter space corresponding to normal, inferior and Giffen behavior.

First it should be noted that for Example 5 since the demand function is linear in income

$$n_f = \alpha(p_f) + \beta(p_f)I, \tag{29}$$

one can compute the break-even income for Giffen behavior I^G by solving the equation $\frac{\partial n_f}{\partial p_f} = 0$ as follows

$$I^G = -\frac{\alpha'(p_f)}{\beta'(p_f)}. \tag{30}$$

EXAMPLE 5 *Assume the following specialization of the Example 3 HARA utility*

$$W(c_2) = -\frac{(c_2 - a)^{-\delta}}{\delta}, \tag{31}$$

[13] Due to the equivalence of the LOD between the contingent claims setting and complex security setting, these examples show that the LOD can be violated even when both goods are normal.

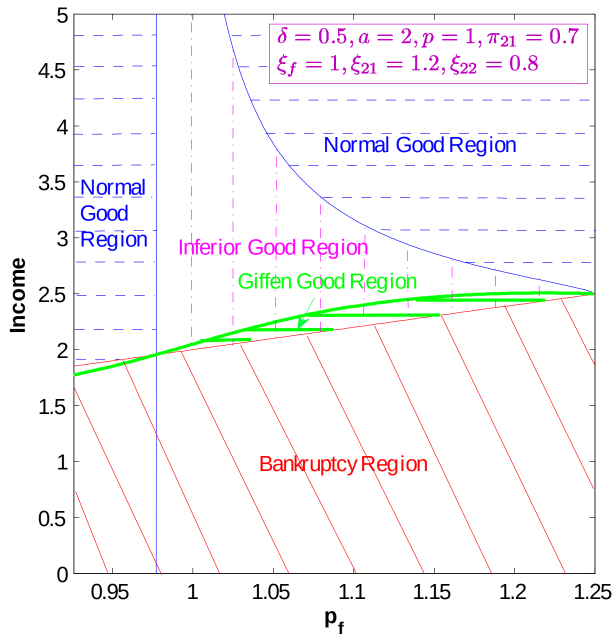


Figure 7:

where $a > 0$.[14] *Assume the parameters $a = 2$, $p = 1$, $\xi_f = 1$, $\xi_{21} = 1.2$, $\xi_{22} = 0.8$ and $\pi_{21} = 0.7$. When $\delta < \delta_{critical}$, the risk free asset will exhibit regions of normal good, inferior good and Giffen good. See Figure 7. We have $I_{\min} = \frac{ap_f}{\xi_f}$. It is clear from Figure 7 that there are two separate normal good regions. To understand why, note that*

$$\frac{\partial p_{21}}{\partial p_f} < 0 \quad \textit{and} \quad \frac{\partial p_{22}}{\partial p_f} > 0 \qquad \Rightarrow \qquad \frac{\partial k}{\partial p_f} < 0. \tag{32}$$

Therefore, for $\delta_{critical} = \frac{\ln k}{\ln(\xi_{22}/\xi_{21})} - 1$, we have

$$\frac{\partial \delta_{critical}}{\partial p_f} = \frac{1}{k \ln(\xi_{22}/\xi_{21})} \frac{\partial k}{\partial p_f} > 0. \tag{33}$$

When p_f is small, $\delta_{critical} < \delta$ and the risk free asset is always a normal good. When p_f is large enough such that $\delta_{critical} > \delta$, I^* becomes positive, implying inferior good behavior when $I < I^*$ and normal good behavior when $I > I^*$. In terms of the Figure, as p_f approaches 0.98 from above, $\delta_{critical} \to \delta$ and $I^* \to \infty$.

Given that in Example 5 the contingent claims are normal goods, it is natural to ask why the risk free asset can be a Giffen good. As in the typical potato Giffen story, the consumer's income is only slightly above $I_{\min}$ and c_{22} is close to the subsistence level a. It is clear from Figure 5(a) that most of her income is invested in the risk free rather than

[14] It can be shown that when $a < 0$, the risk free asset can exhibit Giffen good behavior for appropriate parameters.

the risky asset. In this case, safety from starvation is provided by the risk free asset rather than potatoes. Now if p_f increases, the return on the asset $\frac{\xi_f}{p_f}$ falls and the substitution effect tends to drive the consumer to reduce her holdings of the risk free asset. But if she does, since $\frac{\xi_{22}}{p} < \frac{\xi_f}{p_f}$, c_{22} will decline[15] and she will face a greater risk of starvation and hence the associated income effect outweighs the substitution effect leading her to actually increases her demand for risk free asset. In other cases, the risk free asset can be a Giffen good at income levels not necessarily close to $I_{\min}$ and the impact of the price change on the consumer's risk aversion and desire for safety are the keys to explaining the behavior.

Quite surprisingly, as we show next for the Example 5 (and 3) HARA utility, it is possible to find a region in the price, income space where the risk free asset exhibits Giffen behavior for any value of the risk aversion parameter δ and for any distribution of the risky asset's returns.

PROPOSITION 1 *Assume the NM HARA utility in eqn. (31). Then for any* $a > 0$, $\delta > -1, \pi_{21}, \pi_{22}$ *and* $\xi_{21}, \xi_{22} > 0$, *there exists an income level* I *and a specific range of* p_f *such that the risk free asset becomes a Giffen good.*

REMARK 5 *The intuition for this result can be seen in terms of Figure 7. When* $\delta_{critical} = \delta$, *which corresponds to the vertical at* $p_f \approx 0.98$, *the* I^G *curve and the* $I_{\min}$ *line will always intersect at a point on the vertical. At this point, the slope of the* I^G *curve is greater than the* $I_{\min}$ *line. Hence, there will always be a Giffen region as shown in the Figure to the right of the intersection point on the vertical.*

We conclude this Section by showing that for non-HARA preferences, Giffen good behavior can occur over multiple regions of income.

EXAMPLE 6 *Assume*

$$W(c_2) = -\frac{(c_2+0.2)^{-9}}{9} + c_2, \tag{34}$$

Figure 8 illustrates that there exist two regions of income where $\frac{\partial n_f}{\partial p_f} > 0$. *Giffen behavior for the intermediate income range* $3.93 < I < 4.94$ *is clear from Figure 8(a). The lower range is shown in the magnified view in Figure 8(b).*

4 Two Period Setting

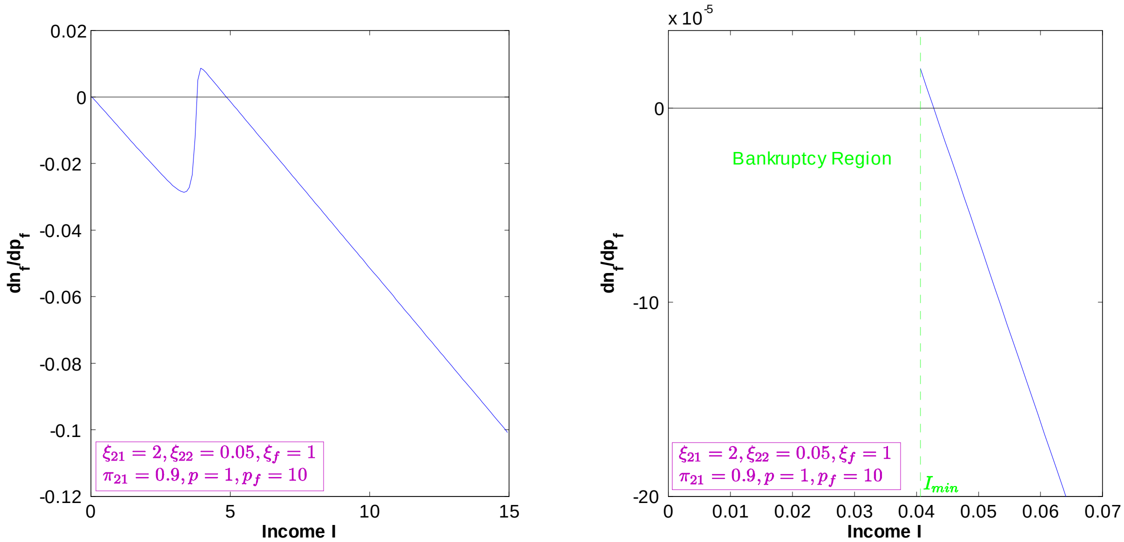
In this Section, we will extend our analysis to a two period setting. Consider maximizing

$$EW(c_1, \widetilde{c}_2) = W_1(c_1) + \pi_{21} W_2(c_{21}) + \pi_{22} W_2(c_{22}) \tag{35}$$

[15] Noticing that

$$\triangle c_{22} = \xi_{22} \triangle n + \xi_f \triangle n_f = p_f \triangle n_f \left(\frac{\xi_f}{p_f} - \frac{\xi_{22}}{p} \right),$$

if $\triangle n_f < 0$, we have $\triangle c_{22} < 0$.



(a) (b)

Figure 8:

with respect to c_1, c_{21} and c_{22} subject to

$$p_1 c_1 + p_{21} c_{21} + p_{22} c_{22} = I. \tag{36}$$

Because $W(c_1, c_2)$ is additively separable, it is clear that conditional on a fixed c_1 one can optimize $\pi_{21} W_2(c_{21}) + \pi_{22} W_2(c_{22})$ for c_{21} and c_{22} independent of $W_1(c_1)$ resulting in the first order condition for the conditional optimization

$$\frac{W_2'(c_{21})}{W_2'(c_{22})} = k, \tag{37}$$

which is identical to that obtained in the single period case. Given the conditionally optimal c_{21} and c_{22} demands, (35)-(36) can then be solved for optimal c_1. It should be stressed that if we go from the single period $W(c_2)$ in Section 2 to the current two period $W(c_1, c_2) = W_1(c_1) + W_2(c_2)$ then all of the Lemmas, Theorems and Corollaries in Section 2 continue to hold except that the condition for $c_1 > 0$ must be added to the no bankruptcy restriction. The reason is as follows. The comparative statics results in Section 2 are based on a comparison of $\partial c_{21}/\partial I$ and $\partial c_{22}/\partial I$, which can be obtained from differentiation of the first order condition and the budget constraint. As argued above, the first order condition remains the same in the two period setting. Although the budget constraint changes, this change does not affect our comparison results.

Although we can find the Giffen good behavior for the risk free asset in the two period setting when choose the appropriate parameters, it is important to note that the above argument does not imply that the risk free asset is a Giffen good in the two period case whenever it is in the one period case. To see this, note that the price of the risk free asset affects demand both directly and indirectly

$$\frac{dn_f}{dp_f} = \frac{\partial n_f(p_f, c_1)}{\partial p_f} + \frac{\partial n_f(p_f, c_1)}{\partial c_1}\frac{\partial c_1}{\partial p_f}. \tag{38}$$

The first term on the right hand side is the direct effect of p_f on n_f through the conditional portfolio optimization while the second term is the impact through optimal period one consumption. Suppose that for the income after period one consumption, $I - p_1 c_1$, the risk free asset is a Giffen good, then $\frac{\partial n_f(p_f,c_1)}{\partial p_f} > 0$. This Giffen behavior is reinforced (diminished) depending on whether the second term is positive (negative). It is straightforward to show that $\frac{\partial n_f(p_f,c_1)}{\partial c_1}$ has the same sign as $-\frac{\partial n_f(p_f,c_1)}{\partial I}$ which is positive if the risk free asset is a Giffen good. But since $\frac{\partial c_1}{\partial p_f}$ can be positive or negative, the sign of $\frac{dn_f}{dp_f}$ is uncertain.

5 Concluding Comments

In this paper, Arrow's seminal single period results on the relation between asset demand, risk aversion and income are extended by dropping his restrictive assumption that the risk free asset is held long. When shorting is allowed, even for well-behaved utility functions satisfying Arrow's assumptions that $\tau'_A < 0$ and $\tau'_R > 0$, the risk free asset can not only become an inferior good but also a Giffen good. The sign of τ'_R plays a critical role in determining whether inferior and Giffen behavior occurs when the risk free asset is held long or short. We investigate when Giffen good behavior occurs and the relation between its occurrence in one and two period settings. In addition to providing general results, we illustrate them with numerous examples based on HARA and non-HARA utility functions.

Appendix

A Proof of Lemma 1

From the definition of n_f, we have

$$n_f > 0 \Leftrightarrow \frac{c_{22}}{c_{21}} > \frac{\xi_{22}}{\xi_{21}}. \tag{39}$$

Using the first order condition, we will obtain

$$n_f > 0 \Leftrightarrow W'(c_{21}) < kW'(\frac{\xi_{22}}{\xi_{21}}c_{21}) \tag{40}$$

for any $c_{21} > 0$. Since a similar argument can be applied to the other cases, we can conclude

$$n_f \gtreqless 0 \Leftrightarrow W'(c_2) \lesseqgtr kW'(\frac{\xi_{22}}{\xi_{21}}c_2) \text{ for any } c_2 > 0. \tag{41}$$

B Proof of Theorem 1

Differentiating the FOC with respect to the income I yields $W''(c_{21})\frac{\partial c_{21}}{\partial I} = kW''(c_{22})\frac{\partial c_{22}}{\partial I}$. Differentiating the budget constraint with respect to the income I, we obtain $p_{21}\frac{\partial c_{21}}{\partial I} + p_{22}\frac{\partial c_{22}}{\partial I} = 1$. Combining the two equations above yields

$$\frac{\partial c_{21}}{\partial I} = \frac{1}{p_{21} + p_{22}\frac{\tau_A(c_{21})}{\tau_A(c_{22})}} \quad \text{and} \quad \frac{\partial c_{22}}{\partial I} = \frac{\frac{\tau_A(c_{21})}{\tau_A(c_{22})}}{p_{21} + p_{22}\frac{\tau_A(c_{21})}{\tau_A(c_{22})}}, \tag{42}$$

where we have used $\frac{W''(c_{21})}{kW''(c_{22})} = \frac{\tau_A(c_{21})}{\tau_A(c_{22})}$. Noticing that

$$\frac{\partial n}{\partial I} = \frac{\frac{\partial c_{21}}{\partial I} - \frac{\partial c_{22}}{\partial I}}{\xi_{21} - \xi_{22}} = \frac{\left(p_{21} + p_{22}\frac{\tau_A(c_{21})}{\tau_A(c_{22})}\right)^{-1}}{(\xi_{21} - \xi_{22})\xi_f}\left(1 - \frac{\tau_A(c_{21})}{\tau_A(c_{22})}\right) \tag{43}$$

and $c_{21} > c_{22}$, we have

$$\tau'_A \gtreqless 0 \Leftrightarrow \frac{\partial n}{\partial I} \lesseqgtr 0. \tag{44}$$

Also notice that

$$\frac{\partial n_f}{\partial I} = \frac{\xi_{21}\frac{\partial c_{22}}{\partial I} - \xi_{22}\frac{\partial c_{21}}{\partial I}}{(\xi_{21} - \xi_{22})\xi_f} = \frac{\left(p_{21} + p_{22}\frac{\tau_A(c_{21})}{\tau_A(c_{22})}\right)^{-1}}{(\xi_{21} - \xi_{22})\xi_f}\left(\xi_{21} - \xi_{22}\frac{\tau_A(c_{22})}{\tau_A(c_{21})}\right). \tag{45}$$

Therefore, we have

$$\xi_{21}\tau_A(c_{21}) \gtreqless \xi_{22}\tau_A(c_{22}) \Leftrightarrow \frac{\partial n_f}{\partial I} \gtreqless 0. \tag{46}$$

Finally, since

$$\frac{n_f}{I} = \frac{1}{p\frac{n}{n_f} + p_f}, \tag{47}$$

we have

$$\frac{\partial(n_f/I)}{\partial I} \gtreqless 0 \Leftrightarrow \frac{\partial(n_f/n)}{\partial I} \gtreqless 0 \Leftrightarrow n\frac{\partial n_f}{\partial I} - n_f\frac{\partial n}{\partial I} \gtreqless 0. \tag{48}$$

Noticing that

$$n\frac{\partial n_f}{\partial I} - n_f\frac{\partial n}{\partial I} = \frac{\left(p_{21} + p_{22}\frac{\tau_A(c_{21})}{\tau_A(c_{22})}\right)^{-1} c_{21}}{(\xi_{21} - \xi_{22})\xi_f}\left(1 - \frac{\tau_R(c_{22})}{\tau_R(c_{21})}\right) \tag{49}$$

and $c_{21} > c_{22} > 0$, we have

$$\tau'_R \gtreqless 0 \Leftrightarrow \frac{\partial s}{\partial I} \gtreqless 0 \Leftrightarrow \frac{\partial(n_f/I)}{\partial I} \gtreqless 0. \tag{50}$$

C Proof of Theorem 2

If $\tau_R' > 0$, it follows from Theorem 1(iii) that

$$\frac{\partial(n_f/I)}{\partial I} > 0 \Leftrightarrow \frac{\partial n_f}{\partial I} > \frac{n_f}{I}. \tag{51}$$

If $n_f > 0$, then the risk free asset is a normal good. Therefore, the risk free asset can become an inferior good only when $n_f < 0$. The other cases can be proved similarly.

D Proof of Proposition 1

It can be verified that the optimal demand for the risk free asset is given by

$$n_f = \alpha(p_f) + \beta(p_f)I, \tag{52}$$

where

$$\alpha(p_f) = \frac{a}{\xi_f} - \frac{ap_f}{\xi_f}\beta(p_f) \quad \text{and} \quad \beta(p_f) = \frac{\xi_{21}k^{\frac{1}{1+\delta}} - \xi_{22}}{\left(p_{21} + p_{22}k^{\frac{1}{1+\delta}}\right)(\xi_{21} - \xi_{22})\,\xi_f}. \tag{53}$$

Since

$$\alpha'(p_f) = -\left(\frac{a\beta(p_f)}{\xi_f} + \frac{ap_f\beta'(p_f)}{\xi_f}\right), \tag{54}$$

It follows from eqn. (30) that

$$I^G = -\frac{\alpha'(p_f)}{\beta'(p_f)} = \frac{a\beta(p_f)}{\xi_f\beta'(p_f)} + \frac{ap_f}{\xi_f}. \tag{55}$$

And we can also calculate that

$$\frac{\partial I^G}{\partial p_f} = \frac{a\beta'(p_f)}{\xi_f\beta'(p_f)} + \frac{a\beta(p_f)}{\xi_f}\frac{\partial\frac{1}{\beta'(p_f)}}{\partial p_f} + \frac{a}{\xi_f} = \frac{2a}{\xi_f} + \frac{a\beta(p_f)}{\xi_f}\frac{\partial\frac{1}{\beta'(p_f)}}{\partial p_f}. \tag{56}$$

It follows from eqn. (15) that when $\delta = \delta_{critical}$,

$$\xi_{21}k^{\frac{1}{1+\delta}} - \xi_{22} = 0 \Rightarrow \beta(p_f) = 0. \tag{57}$$

Therefore, we have

$$I^G = \frac{ap_f}{\xi_f} = I_{\min} \qquad \text{and} \qquad \frac{\partial I^G}{\partial p_f} = \frac{2a}{\xi_f} > \frac{a}{\xi_f} = \frac{\partial I_{\min}}{\partial p_f}. \tag{58}$$

Denoting the critical p_f corresponding to $\delta = \delta_{critical}$ as p_f^c, due to the continuity of I^G and $I_{\min}$, we can conclude that there exists an $\epsilon > 0$ such that if $p_f \in \left(p_f^c, p_f^c + \epsilon\right)$, we have $I^G > I_{\min}$, implying that the risk free asset is a Giffen good.

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