The Dynamics of Optimal Risk Sharing

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1. Introduction

This paper considers a risk-sharing problem in which two investors pool their resources to invest in a common risky venture. Investment returns are assumed to follow a geometric Brownian motion, and the investors' risk preferences are represented by utility functions exhibiting constant *relative* risk-aversion (CRRA). The two investors have different coefficients of relative risk-aversion and different initial wealth 2endowments. They can write a long-term insurance contract specifying a division of final output contingent on the sample path of output of the venture. The venture may end at any time with positive probability, and when it ends the two investors consume their final accumulated wealth.

To keep the analysis tractable we have stripped out of the model many features which would make it more realistic. For example, our model allows for only two investors, only one risky asset, and investors only consume at the end. In addition, we simplify the formulation of the optimal contracting problem by letting one individual, the Proposer, make a take-it-or-leave-it contract offer to the other, the Responder. Even so, the analysis of this optimal contracting problem is sufficiently complex that we are only able to approximate the optimal risk-sharing rule. For reasonable parameter configurations, however, this approximation is a good fit for the numerically determined optimal risk-sharing rule.

Optimal risk-sharing between two parties was first analyzed by Borch (1962), in the context of a reinsurance problem. He considers an optimal contract to share risk between an insurance and a reinsurance company (or between two insurance companies). While his framework is more general in many respects than the one we have just described, he only derives a necessary condition for optimal coinsurance between two risk-averse investors, the well known Borch condition.

In this paper we push the analysis further and derive explicit risk-sharing formulae that approximate the optimal risk-sharing rule. We do this by reformulating the risk-sharing problem as a recursive problem in which the Proposer offers the Responder spot contracts, each of which has three components: (i) a fixed transfer f to the Responder; (ii) a share s of spot investment returns; and (iii) a final transfer b to be paid to the Responder in the event that the venture terminates. We then derive relatively simple formulae for s and b that approximate the optimal risk-sharing spot contract. Thus, a central contribution of this paper is to derive (approximate) formulae for optimal risk-sharing for the CRRA case.

As each investor's aversion to risk and capacity to insure the other investor varies with its wealth, the optimal shares s and b vary with the underlying wealth distribution. Thus, one advantage of our recursive formulation is that it brings out explicitly the underlying dynamics of the risk-sharing problem. These dynamics can be understood as follows. Whenever the two investors engage in risk-sharing, the optimal spot contract will specify a division of total investment returns that is different from each investor's share in total wealth. As a result, the wealth distribution in the next period will be different from the wealth distribution in this period. For example, if the Responder insures the Proposer, by taking on a share of risk bigger than his share in wealth, then his wealth share will increase when there is a high investment return and decrease when there is a low investment return. Either way, the wealth distribution changes and consequently each investor's attitude towards risk and capacity to insure changes. This change in each investor's capacity to insure introduces endogenous *counterparty risk*, and forward-looking investors will take this risk into account in deciding on the optimal spot contract.[1]

To gain insight into how this counterparty risk can affect optimal risk-sharing, consider the extreme case in which the Responder is risk-neutral and the Proposer is risk-averse. It is well known that optimal risk-sharing in a one-shot insurance contracting problem in this case requires that the Responder insure the Proposer perfectly. But if the Responder were to do this repeatedly, then he would be sure to go bankrupt at some point, and then the Proposer would no longer be able to get any insurance at all. Foreseeing this, the Proposer would want to hold back from getting perfect insurance. Only when the Responder is relatively wealthy would the Proposer seek perfect insurance. When the Responder is relatively poor, the Proposer may optimally limit the amount of insurance she gets to preserve future insurance opportunities.

We are able to extend this insight to the general case, in which both investors are risk averse, by assuming that both the Proposer and the Responder are close to myopic, and by taking approximations around the myopic optimum. This approximation therefore takes the form of a *myopic benchmark* plus a *dynamic correction*.

Consider first the optimal rule for the sharing of investment risk, namely s. The myopic benchmark for s requires that the Responder take on a share in total investment risk equal to the well known ratio of the Proposer's coefficient of *absolute* risk aversion

[1] The role played by counterparty risk in our model is analogous to the role that it plays in futures markets. There, traders are required to maintain margin accounts as a way of eliminating default. Although these requirements prevent any default as such, they bear witness to the profound role played by counterparty risk, and they constrain the amount of hedging a counterparty can offer.

to the sum of both investors' coefficients of *absolute* risk aversion. As for the dynamic correction, if we denote by R and r the coefficients of relative risk aversion of the Proposer and Responder respectively, then:

1. When both investors are fairly risk tolerant, in the sense that $R, r < 2$, it is optimal for the less risk-averse investor to take on *less* risk in the dynamic-contracting problem than the myopic rule would specify. This is because the less risk averse investor is willing to take on risk on relatively unfavourable terms, so transferring more risk to that investor tends to reduce the stock of insurance available to the more risk averse investor in the future.

2. When both investors are fairly risk averse, in the sense that $R, r > 2$, it is optimal for the less risk-averse investor to take on *more* risk in the dynamic-contracting problem than the myopic rule would specify. This is because the less risk averse investor is only willing to take on risk on relatively favourable terms, so transferring more risk to that investor tends to increase the stock of insurance available to the more risk averse investor in the future.

3. When one investor (say the Responder) is fairly risk tolerant but the other investor (say the Proposer) is fairly risk averse, in other words when $R > 2 > r$, then the Proposer takes on less risk when she is relatively poor and more risk when she is relatively wealthy. This is because, when she is poor, her aversion to bearing risk outweighs her concern that the Responder may run out of money; but when she is rich, the opposite is true.

Consider next the optimal termination payment b. The myopic benchmark for b is zero. This is because there is no termination risk in the myopic limit. As for the dynamic correction, we show that: (i) if the venture has a high return then, *following* termination, the less risk averse investor compensates the more risk averse investor for the loss of a valuable investment opportunity; and (ii) if the venture has a low return then, *prior to* termination, the less risk averse investor compensates the more risk averse investor for the low returns obtained.

Although our model is highly stylized, it may be relevant to a number of applications. We have already mentioned reinsurance as one application. Insurance companies are obviously capital constrained and they rely on each other to share common risk. Our analysis sheds light on how these companies should structure their risk sharing to take account of counterparty risk. Another application, which was our initial motivation, is to portfolio- or fund-management contracts. In practice, the contract between a representative client and a fund manager often takes the simple form of a share of portfolio returns for the client equal to the client's share of investments in the fund minus a management fee, which is equal to a small percentage of the funds under management. We recognize that the main concern in portfolio management generally is the manager's incentive to run the fund in the client's best interest. Still, we believe that our analysis may be relevant if there are also dynamic risk-sharing considerations involved in the long-term relation between the client and the manager.[2]

Besides Borch (1962) and the large literature on optimal risk sharing that it has spawned (see Eeckhoudt, Gollier and Schlesinger, 2005) our paper is most closely related to the dynamic asset pricing problem with two classes of investors considered by Dumas (1989). He analyzes the equilibrium investment and consumption choices of two classes of investor with different coefficients of relative risk aversion in an otherwise standard competitive economy with aggregate shocks. Although Dumas mainly focuses on equilibrium asset pricing, his analysis proceeds via a planning problem. One key difference between his setup and ours is that he allows for ongoing consumption, while we only have consumption upon termination. Another is that we have termination risk, while he only considers an infinitely lived economy. Finally, Dumas' solution method only works in the case in which one investor has a log utility function.

There is by now accumulating evidence that consumers differ substantially in their risk preferences. Indeed, Barsky, Juster, Kimball, and Shapiro (1997), in their experimental study on risk-taking decisions, found that the behaviour of 5% of subjects was consistent with a coefficient of relative risk aversion of 33 or higher, that the behaviour of another 5% was consistent with a coefficient of 1.3 or lower, and that the median coefficient was about 7. Similarly, Guiso and Paiella (2008) and Chiappori and Paiella (2008) among others find evidence of heterogeneous risk preferences in households' actual portfolio allocations. In addition, using panel data on individual portfolio allocations between risky and riskless assets, Chiappori and Paiella (2008) are able to determine that the elasticity of the risky-asset share with respect to wealth in their sample is small and statistically insignificant, which is consistent with CRRA risk preferences. In another panel study on household

[2]For continuous-time models of portfolio-mangement contracts with moral-hazard and/or asymmetric information see Ou-Yang (2003), Cvitanic and Zhang (2007) and Cvitanic, Wan, and Zhang (2008).

portfolio choices, however, Paravisini, Rappoport and Ravina (2009) use investor fixed effects and find that the within-household elasticity of risk taking with respect to changes in household wealth is negative and quite large. They also find substantial heterogeneity in relative risk aversion in their sample, with an average coefficient of 2.85 and a median coefficient of 1.62.

The present paper is organized as follows. Section 2 describes the two investors' preferences and the investment technology. Section 3 derives the value function of the Responder under autarky. Section 4 formulates the long-run contracting problem between the Proposer and Responder. Section 5 formulates the spot-contracting problem, and Section 6 derives the associated Bellman equation of the Proposer. Section 7 establishes that any long-run contract for which the participation constraint of the Responder binds can be replicated by a flow of spot contracts for which the spot participation constraint of the Responder likewise binds. Section 8 shows that the Bellman equation of the Proposer under spot contracting can be reduced to a partial differential equation. Section 9 provides a first characterization of the optimal risk-sharing rule and termination payment in terms of the value functions of the Proposer and Responder. Section 10 uses asymptotic expansions to derive risk-sharing formulae which approximate the optimal risk-sharing rule and termination payment. Section 11 shows how the Bellman equation for the Proposer can be further reduced to a pair of ordinary differential equations on $(0, 1)$. Section 12 identifies sufficient conditions under which a solution to these equations extends continuously to $[0, 1]$. Section 13 then solves the resulting two-point boundary-value problem numerically. The numerical solutions show how well the formulae derived in Section 10 predict the qualitative shape of the optimal risk-sharing rule and termination payment, suggesting that these formulae contain most of the analytical insight into optimal risk sharing that can be obtained for our model. Section 14 offers some concluding comments.

2. Preferences and Technology

The initial wealths of the Proposer and the Responder are $W_0, w_0 > 0$. There is an exogenous termination time T, which is distributed exponentially with parameter $\beta > 0$. At any time $t \in [0, T]$ the parties have access to the same constant-stochastic-returns-to-scale investment opportunity, but they cannot consume. For an investment x, this investment opportunity yields flow returns

$$dx = x\,(\mu\, dt + \sigma\, dz),$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and z is a standard Wiener process (i.e. the shock dz at time t is normally distributed with mean 0 and variance dt and is independent of the shocks at all earlier times). Following termination, both parties consume their accumulated wealths W_{T+} and w_{T+}.[3]

The risk preferences of the Proposer and the Responder are represented by the strictly increasing and strictly concave utility functions U and u. In what follows we shall frequently assume that U and u they take the constant relative risk aversion (CRRA) form

$$U(W) = \left\{ \begin{array}{ll} \dfrac{W^{1-R}-1}{1-R} & \text{if } R \neq 1 \\ \log(W) & \text{if } R = 1 \end{array} \right\}$$

and

$$u(w) = \left\{ \begin{array}{ll} \dfrac{w^{1-r}-1}{1-r} & \text{if } r \neq 1 \\ \log(w) & \text{if } r = 1 \end{array} \right\},$$

where R and r are strictly positive scalars. However, it will sometimes be helpful to avoid imposing specific functional forms on U and u, in which case we will instead put

$$R(W) = -\frac{W\,U''(W)}{U'(W)}$$

and

$$r(w) = -\frac{w\,u''(w)}{u'(w)},$$

where R and r are now functions giving the coefficients of relative risk aversion of U and u in terms of W and w respectively.

Remark 1. *Throughout the paper, the subscript R will denote the Proposer and the subscript r will denote the Responder.*

[3] We adopt the convention that the wealths of the two parties at the beginning of period t are denoted by W_t and w_t, and that the wealths at the end of period t are denoted by W_{t+} and w_{t+}. In particular, the timepaths for W and w are continuous on the left and have limits on the right (collor). This is the opposite of the usual mathematical convention, according to which the timepaths of stochastic processes are continuous on the right and have limits on the left (corlol). However, it simplifies our exposition in several respects. See Appendix A for further details.

3. AUTARKY FOR THE RESPONDER

Consider first the case in which the Responder invests on his own. His value function $\widetilde{v} : (0, \infty) \times \{0, 1\} \to \mathbb{R}$ for this case will provide his reservation value in the bilateral contracting problems described below. It satisfies the Bellman equation

$$\widetilde{v}(w, \chi) = \left\{ \begin{array}{ll} \mathrm{E}\,[\,\widetilde{v}(w + dw, \chi + d\chi)\,] & \text{if } \chi = 0 \\ u(w) & \text{if } \chi = 1 \end{array} \right\} \tag{1}$$

where: w is the accumulated wealth of the Responder; χ is an indicator taking the value 0 if the problem has not yet terminated and the value 1 if the problem has terminated;

$$dw = w\,(\mu\,dt + \sigma\,dz);$$

and

$$d\chi = \left\{ \begin{array}{ll} 0 & \text{if the problem does not terminate} \\ 1 & \text{if the problem terminates} \end{array} \right\}.$$

Putting $v = \widetilde{v}(\cdot, 0)$ in equation (1), we obtain

$$\begin{aligned} v(w) &= \mathrm{E}\left[v(w) + v'(w)\,dw + \tfrac{1}{2}\,v''(w)\,dw^2 + (u(w) - v(w))\,d\chi\right] \\ &= v(w) + \left(v'(w)\,\mu\,w + \tfrac{1}{2}\,v''(w)\,\sigma^2\,w^2 + (u(w) - v(w))\,\beta\right) dt \end{aligned}$$

or

$$0 = \tfrac{1}{2}\,\sigma^2\,w^2\,v'' + \mu\,w\,v' + \beta\,(u(w) - v), \tag{2}$$

where we have suppressed the dependence of v on w.

If the utility u has CRRA (i.e. r is constant), then equation (2) has an explicit solution. Indeed, given that wealth follows a geometric Wiener process, it is natural to conjecture that v will take the form

$$v(w) = C_r(\rho_r\,w),$$

where ρ_r is the certainty-equivalent rate of return of the Responder. However, for our purposes, it is more convenient to work with

$$\psi_r = C_r(\rho_r)$$

than with ρ_r. We call ψ_r the normalized value function of the Responder. Exploiting the functional form for C_r, we obtain

$$v(w) = C_r(\rho_r\, w) = C_r(w) + w\, C_r'(w)\, C_r(\rho_r) = C_r(w) + w\, C_r'(w)\, \psi_r.$$

Substituting into (2) and solving for ψ_r then yields

$$\psi_r = \frac{\mu - \frac{1}{2}\, r\, \sigma^2}{\beta_r},$$

where

$$\beta_r = \beta - (1 - r)\,(\mu - \tfrac{1}{2}\, r\, \sigma^2).$$

Furthermore, it is helpful to note that

$$v'(w) = \gamma_r\, C_r'(w), \tag{3}$$

where

$$\gamma_r = \frac{\beta}{\beta_r}.$$

We call γ_r the normalized marginal value of wealth of the Responder.[4]

Since the marginal value of wealth $v'(w)$ must be positive, it follows from equation (3) that a basic requirement for our contracting problem to make sense is that $\beta_r > 0$, i.e. that:

Condition I. $\beta > (1 - r)\,(\mu - \frac{1}{2}\, r\, \sigma^2)$.

This condition captures the following simple idea. If $r < 1$, then the Responder's utility function is bounded below but unbounded above. The main problem is therefore to ensure that there is enough discounting to prevent his expected utility from becoming positively infinite; or, equivalently, that the rate of growth of wealth μ is sufficiently low relative to the other parameters that her utility does not grow excessively fast. On the other hand, if $r > 1$, then his utility function is bounded above but unbounded below. The main problem is therefore to ensure that there is enough discounting to prevent his expected utility from becoming negatively infinite; or, equivalently, that the rate of growth of wealth μ is sufficiently high relative to the other parameters that her utility does not

[4] See Appendix B for the details of the discussion in this paragraph.

fall excessively fast. Either way, the precise condition under which ρ_r is well defined turns out to be Condition I.

We shall also need the following analogue, for the Proposer, of Condition I:

Condition II. $\beta > (1-R)\,(\mu - \frac{1}{2}\,R\,\sigma^2)$.

4. THE LONG-RUN-CONTRACTING PROBLEM

Suppose that the Proposer offers the Responder a long-run contract q, according to which the two parties will pool their wealths until termination, after which the wealth pool will be shared between them. More precisely: let Ω denote the set of pairs (X, T) such that $T \in [0, \infty)$, $X : [0, \infty) \to (0, \infty)$ is continuous on $[0, T]$ and X is constant on (T, ∞); and let $q : \Omega \to \mathbb{R}$ be a bounded measurable function such that $0 < q(X, T) < X_{T+}$ for all $(X, T) \in \Omega$.[5] If the Responder accepts q then: the initial wealth pool will be

$$X_0 = W_0 + w_0; \tag{4}$$

the wealth pool will evolve according to the equation

$$dX = \left\{ \begin{array}{cc} X\,(\mu\,dt + \sigma\,dz) & \text{if } t \in [0, T] \\ 0 & \text{if } t \in (T, \infty) \end{array} \right\}; \tag{5}$$

and the final wealths of the Proposer and the Responder will be

$$W_{T+} = X_{T+} - q(X, T) \quad \text{and} \quad w_{T+} = q(X, T). \tag{6}$$

If the Responder rejects q then both parties will operate under autarky until termination.

In the long-run contracting problem, the Proposer's problem is therefore to choose q to maximize her expected utility

$$\mathrm{E}[U(W_{T+})] \tag{7}$$

subject to the dynamics (4-6) and the participation constraint of the Responder, namely

$$\mathrm{E}[u(w_{T+})] \geq v(w_0), \tag{8}$$

where v is the value function of the Responder under autarky.

[5] For further discussion of our notational conventions, see Appendix A.

Two points should be noted. First, taken together, Conditions I and II ensure that there exists and optimal contract in the long-run contracting problem. Indeed, Condition II obviously ensures that the expected utility of the Proposer is well defined in the event of disagreement. Furthermore, the Proposer can always do at least as well in the long-run contracting problem as she can under autarky. This is because she can reproduce the autarky outcome by offering the contract

$$q(X,T) = \frac{w_0}{W_0 + w_0} X_{T+}.$$

In particular, her expected utility from the long-run contracting problem is bounded below. Finally, an upper bound for what the Proposer can hope to obtain from the long-run contracting problem is obtained if we allow her to expropriate all the wealth of the Responder. If she does this, then she will again find herself in autarky. The only difference is that she will have the larger initial wealth $W_0 + w_0$. Hence her expected utility from such a manoeuvre is well defined, again by Condition II. Analogous remarks apply to the Responder.

Second, the participation constraint of the Responder always binds in the long-run contracting problem. For, if it did not, then the Proposer could just scale down q until the participation constraint of the Responder did bind. This would have the effect of transferring a strictly positive – albeit stochastic – amount of the final total wealth from the Responder to the Proposer, and would therefore make the Proposer strictly better off.

5. THE SPOT-CONTRACTING PROBLEM

The easiest way to solve the long-run contracting problem is to show that it can be reduced to a spot contracting problem. In a spot contracting problem, the two parties start out with their initial wealths W_0 and w_0. Then, in each period $t \in [0, T]$, the Proposer offers the Responder a spot contract

$$(f, s, b) \in \mathbb{R} \times \mathbb{R} \times \left(-\tfrac{w}{W+w}, \tfrac{W}{W+w}\right).$$

If the Responder accepts then he receives:

1. a non-contingent transfer $(W + w)\, f\, dt$, which is an up-front payment for his participation in the risk-sharing arrangement;

2. a contingent transfer $(W + w)\, s\, (\mu\, dt + \sigma\, dz)$, which is his share in the total returns on investment; and
3. a contingent transfer $(W + w)\, b\, d\chi$, which is an insurance payment in the event that he loses the investment opportunity as a result of termination.

More explicitly, if the Responder accepts, then the changes in the wealths of the Proposer and the Responder are

$$\begin{aligned} dW &= (W + w)\ (-f\, dt + (1 - s)\,(\mu\, dt + \sigma\, dz) - b\, d\chi)\,, && (9) \\ dw &= (W + w)\ (f\, dt + s\,(\mu\, dt + \sigma\, dz) + b\, d\chi)\,. && (10) \end{aligned}$$

If the Responder rejects the spot contract, then both parties invest under autarky for the current period, and the changes in the wealths of the Proposer and the Responder are

$$\begin{aligned} dW &= W\,(\mu\, dt + \sigma\, dz), && (11) \\ dw &= w\,(\mu\, dt + \sigma\, dz). && (12) \end{aligned}$$

In each period $t \in (T, \infty)$, $dW = dw = 0$. Finally, at the end of period T, both parties consume their accumulated stock of wealth to obtain utilities $U(W_{T+})$ and $u(w_{T+})$.

6. THE BELLMAN EQUATION OF THE PROPOSER

In this section, we consider the case of the spot contracting problem in which the value function of the Responder is his value function under autarky, namely $\widetilde{v}$, and the spot contract offered by the Proposer is always accepted. As will become clear in the next section, this is the only case that we shall need. In this case, the Bellman equation of the Proposer can be derived as follows. Suppose that $\chi = 0$ and, for any given spot contract (f, s, b), put

$$\begin{aligned} dW^S &= (W + w)\ (-f\, dt + (1 - s)\,(\mu\, dt + \sigma\, dz) - b\, d\chi)\,, \\ dw^S &= (W + w)\ (f\, dt + s\,(\mu\, dt + \sigma\, dz) + b\, d\chi)\,, \\ dW^A &= W\,(\mu\, dt + \sigma\, dz), \\ dw^A &= w\,(\mu\, dt + \sigma\, dz). \end{aligned}$$

In other words, let dW^S and dw^S be the changes in the wealth of the Proposer and the Responder if (f, s, b) is accepted; and let dW^A and dw^A be the changes in the wealth of the Proposer and the Responder if (f, s, b) is rejected.

Further, let $A(w)$ denote the set of (f, s, b) such that:

1. the participation constraint of the Responder, namely

$$\mathrm{E}\left[\,\widetilde{v}(w + dw^S, d\chi)\,\right] \geq \mathrm{E}\left[\,\widetilde{v}(w + dw^A, d\chi)\,\right], \tag{13}$$

   is satisfied; and

2. the Bellman equation of the Responder, namely

$$\widetilde{v}(w, 0) = \mathrm{E}\left[\,\widetilde{v}(w + dw^S, d\chi)\,\right], \tag{14}$$

   is satisfied.

Then the Bellman equation of the Proposer is the equation

$$\widetilde{V}(W, w, \chi) = \left\{ \begin{array}{ll} \max\limits_{(f,s,b)\in A(w)} \mathrm{E}\left[\widetilde{V}(W + dW^S, w + dw^S, \chi + d\chi)\right] & \text{if } \chi = 0 \\ U(W) & \text{if } \chi = 1 \end{array} \right\}, \tag{15}$$

where $\widetilde{V} : (0, \infty)^2 \times \{0, 1\} \to \mathbb{R}$.

Here inequality (13) says that the Responder weakly prefers to accept (f, s, b) rather than proceed under autarky when his continuation utility is given by $\widetilde{v}$; equation (14) says that $\widetilde{v}(w, 0)$ is the expected utility to the Responder from accepting (f, s, b) when his continuation utility is given by $\widetilde{v}$; and equation (15) says that if $\chi = 0$ then $\widetilde{V}(W, w, \chi)$ is the expected utility to the Proposer from choosing the best feasible (f, s, b) when her continuation utility is given by $\widetilde{V}$, and that if $\chi = 1$ then $\widetilde{V}(W, w, \chi)$ is simply $U(W)$.

Notice that the Proposer can always ensure that the participation constraint of the Responder is satisfied by choosing $f = 0$, $s = \frac{w}{W+w}$ and $b = 0$. In other words, the Proposer can always reproduce the autarky outcome by a suitable choice of spot contract. Notice too that the only reason why dW^A does not feature explicitly in these equations is that $\widetilde{v}$ does not depend on W. Notice finally that, in the special case with which we are concerned (namely the case in which the value function of the Responder under spot contracting is simply his value function under autarky), the participation constraint holds as an equality:

Lemma 2. *The following three statements are equivalent:*

1. $(f, s, b) \in A(w)$, *i.e. both the participation constraint and the Bellman equation of the Responder hold;*
2. $\mathrm{E}\left[\,\widetilde{v}(w + dw^S, d\chi)\,\right] = \mathrm{E}\left[\,\widetilde{v}(w + dw^A, d\chi)\,\right]$, *i.e. the participation constraint of the Responder holds as an equality;*
3. $\widetilde{v}(w, 0) = \mathrm{E}\left[\,\widetilde{v}(w + dw^S, d\chi)\,\right]$, *i.e. the Bellman equation of the Responder holds.*

Proof. We show first that statement 3 implies statement 2. Indeed, since $\widetilde{v}$ is the value function of the Responder under autarky, we have

$$\widetilde{v}(w, 0) = \mathrm{E}\left[\,\widetilde{v}(w + dw^A, d\chi)\,\right]. \tag{16}$$

Combining this with statement 3 leads immediately to statement 2. We show next that statement 2 implies statement 1. Indeed, if statement 2 holds then, a fortiori, the participation constraint of the Responder must hold. On the other hand, combining statement 2 with equation (16) shows that the Bellman equation of the Responder is satisfied. That statement 1 implies statement 3 is trivial. ■

7. Replicating a Long-Run Contract

In this section, we show that any long-run contract for which the participation constraint of the Responder holds as an equality can be replicated by a flow of spot contracts for which the participation constraint of the Responder again holds as an equality. More precisely: recall that $v = \widetilde{v}(\cdot, 0)$ is the value function of the Responder under autarky prior to termination; suppose that we are given a long-run contract $q : \Omega \to (0, \infty)$ such that $\mathrm{E}[u(q(X, T))] = v(w_0)$; let $\mathcal{F}_t$ denote the information available up to the beginning of period $\min\{t,T\}$;[6,7] and put

$$m_t = \mathrm{E}\left[\, u(q(X,T)) \mid \mathcal{F}_t \,\right].$$

Then: $m_0 = v(w_0)$ by choice of q; m is a martingale; and we may apply the martingale representation theorem to show that there exist coefficients η and θ such that

$$dm = \eta\, dz + \theta\,(d\chi - \beta\, dt).$$

Here: dz and $d\chi$ are the innovations to information at time t; η and θ depend only on information available at the beginning of period t; and, by subtracting $\beta\, dt$ from $d\chi$, we ensure that $\mathrm{E}\left[dm \mid \mathcal{F}_t\,\right] = 0$. Moreover: m is continuous on $[0,T]$; m may jump at T; and m is constant and equal to $u(q(X,T))$ on (T,∞).

Next, define the certainty-equivalent wealth process c of the Responder by the formula

$$c_t = \left\{ \begin{array}{ll} v^{-1}(m_t) & \text{if } t \in [0,T] \\ u^{-1}(m_t) & \text{if } t \in (T,\infty) \end{array} \right\}.$$

In other words, let c_t be the unique solution of the equation $\widetilde{v}(c_t,\chi_t) = m_t$. Then: $c_0 = v^{-1}(m_0) = w_0$; it follows from Itô's Lemma that, for $t \in [0,T]$, we have

$$\begin{aligned} dc &= \left(\tfrac{1}{2}\,\eta^2\, g''(m_t) - \beta\,\theta\, g'(m_t)\right) dt + \eta\, g'(m_t)\, dz \\ &\quad + \left(u^{-1}(m_t+\theta) - g(m_t)\right) d\chi, \end{aligned} \tag{17}$$

where $g = v^{-1}$; and c is constant and equal to $q(X,T)$ on (T,∞).

Now, if we match the coefficients of dt, dz and $d\chi$ in equation (10) for the dynamics of the wealth of the Responder in the spot-contracting problem with the coefficients of dt, dz and $d\chi$ in equation (17) for the dynamics of the certainty-equivalent wealth of the

[6] Up to now we have largely suppressed the time subscipt. However, the argument given at the end of the section makes explicit use of two different times, namely t and $t + dt$, and it is therefore helpful to make the time subscript explicit on the four variables that are involved in that argument, namely $\mathcal{F}$, m, c and χ.

[7] The underlying stochastic drivers of our model are the standard Wiener process z and the exponentially distributed termination time T. If $t > T$, then the information available up to the beginning of time t includes the timepath of z over the interval $(T,t]$. By conditioning only on information available at the beginning of period $\min\{t,T\}$, we exclude the use of this additional stochastic information.

Responder in the long-run contracting problem, then we get

$$X\,(f+s\,\mu) = \tfrac{1}{2}\,\eta^2\,g''(m_t) - \beta\,\theta\,g'(m_t), \tag{18}$$

$$X\,s\,\sigma = \eta\,g'(m_t), \tag{19}$$

$$X\,b = u^{-1}(m_t+\theta) - g(m_t). \tag{20}$$

Solving this system of linear equations for (f, s, b) yields

$$f = \frac{\frac{1}{2}\,\sigma\,\eta^2\,g''(m_t) - (\sigma\,\beta\,\theta + \mu\,\eta)\,g'(m_t)}{X\,\sigma}, \tag{21}$$

$$s = \frac{\eta\,g'(m_t)}{X\,\sigma}, \tag{22}$$

$$b = \frac{u^{-1}(m_t+\theta) - g(m_t)}{X}. \tag{23}$$

In other words, q can be reproduced by the flow of spot contracts given by the formulae (21-23). Furthermore, for all $t \in [0, \infty)$, we have

$$\begin{aligned} \mathrm{E}\left[\,\widetilde{v}(c_t + dc, \chi_t + d\chi) \mid \mathcal{F}_t\,\right] &= \mathrm{E}\left[\,\widetilde{v}(c_{t+dt}, \chi_{t+dt}) \mid \mathcal{F}_t\,\right] \\ &= \mathrm{E}\left[\,m_{t+dt} \mid \mathcal{F}_t\,\right] \\ &= \mathrm{E}\left[\,m_t + dm \mid \mathcal{F}_t\,\right] \\ &= m_t \\ &= \widetilde{v}(c_t, \chi_t). \end{aligned}$$

In other words, the Bellman equation of the Responder holds. By Lemma 2, this is equivalent to saying that the Participation constraint of the Responder holds as an equality.

8. The Reduced Bellman Equation

In the long-run contracting problem, the Proposer can always do better by choosing a contract for which the participation constraint of the Responder holds as an equality. Furthermore, any such contract can be replicated by a flow of spot contracts for which the participation constraint of the Responder again holds as an equality. The Proposer can therefore always do at least as well in the spot-contracting problem as in the long-run contracting problem. It is therefore of considerable interest to solve the spot-contracting problem.

In this section, we make a start by showing that the Bellman equation of the Proposer under spot contracting, namely equation (15), can be reduced to a partial differential equation, namely equation (24) below. To this end: put $V = \widetilde{V}(\cdot, 0)$; denote the partial derivatives of V by V_W, V_w, V_{WW}, V_{Ww} and V_{ww}; and let $V_P = V_W - V_w$, $V_{WP} = V_{WW} - V_{Ww}$ and $V_{PP} = V_{WW} - 2\,V_{Ww} + V_{ww}$.[8] Then:

Proposition 3. *V satisfies the equation*

$$0 = \max_{(s,b)\in\mathbb{R}\times\left(-\frac{w}{W+w},\frac{W}{W+w}\right)} \Bigg\{ \mu\,(W+w)\,V_W + \tfrac{1}{2}\,\sigma^2\,(W+w)^2 \left(V_{WW} - 2\,s\,V_{WP} + s^2\,V_{PP} + \frac{V_P}{v'}\,s^2\,v'' \right) + \beta\left(U(W-(W+w)\,b) - V + \frac{V_P}{v'}\,(u(w+(W+w)\,b) - v) \right) \Bigg\}. \tag{24}$$

We shall refer to equation (24) as the reduced Bellman equation of the Proposer. The maximand in this equation involves three main terms. The first term is

$$\mu\,(W+w)\,V_W. \tag{Term 1}$$

In order to bring out the analogy with the other terms in the equation, it is helpful to separate this term into two parts. The first part is

$$(1-s)\,\mu\,(W+w)\,V_W + s\,\mu\,(W+w)\,V_w. \tag{Term 1a}$$

This is the direct benefit to the Proposer of the expected return on total wealth when it is shared according to the sharing rule s. It consists of: the Proposer's share $1-s$ in the expected return $\mu\,(W+w)$ times the shadow value V_W to the Proposer of wealth in the hands of the Proposer; plus the Responder's share s in the expected return $\mu\,(W+w)$ times the shadow value V_w to the Proposer of wealth in the hands of the Responder. The second part is

$$s\,\mu\,(W+w)\,V_P \tag{Term 1b}$$

[8] If one considers the change of variables $(W,w) \to (W+w, W-w)$, then V_P can be interpreted as the partial derivative of V with respect to $W-w$ keeping $W+w$ fixed. Analogous interpretations hold for V_{WP} and V_{PP}.

This is the indirect benefit to the Proposer of the expected return on total wealth when it is shared according to the sharing rule s. In this part: $s\,\mu\,(W+w)$ is the Responder's share in the expected return $\mu\,(W+w)$; and V_P is the shadow value to the Proposer of transfers from the Responder to the Proposer. Notice that: the shadow value of transfers $V_P = V_W - V_w$ takes into account both the impact of a transfer on the Proposer's own wealth (as measured by V_W) and the impact of a transfer on the Responder's wealth (as measured by V_w); and the impact of a transfer on the Responder's wealth must be taken into account since (by making the Responder poorer) a transfer may worsen the terms on which the Proposer can get insurance from the Responder in the future.

The second term consists of two parts. The first part (with sign reversed) is

$$-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V_{WW} - 2\,s\,V_{WP} + s^2\,V_{PP}\right). \tag{Term 2a}$$

This is the direct cost to the Proposer of the investment shocks when they are shared according to the sharing rule s. It can be written more explicitly as

$$-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\;\left((1-s)^2\,V_{WW} + 2\,s\,(1-s)\,V_{Ww} + s^2\,V_{ww}\right).$$

Notice that the Proposer cares about shocks to her own wealth, about shocks to the Responder's wealth (since these affect the terms on which she can obtain insurance) and about the correlation between the shocks to her own wealth and those to the wealth of the Responder. The second part of the second term (with sign reversed) is

$$-\tfrac{1}{2}\,\sigma^2\,(W+w)^2\,\frac{V_P}{v'}\,s^2\,v''. \tag{Term 2b}$$

This is the indirect cost to the Proposer of the shocks to W and w when they are shared according to the sharing rule s. In this part: $-\frac{1}{2}\,\sigma^2\,(W+w)^2\,s^2\,v''$ is the cost to the Responder of the shocks; v' is the shadow value to the Responder of wealth in the hands of the Responder; and V_P is the shadow value to the Proposer of transfers from the Responder to the Proposer. The cost to the Responder is initially measured in units of the Responder's utility. Dividing it by v' converts it into money terms, at which point its value to the Proposer can be found by multiplying by V_P. The second part of the second term can also be written

$$\tfrac{1}{2}\,\sigma^2\,(W+w)^2\,V_P\left(-\frac{v''}{v'}\right)s^2,$$

which emphasizes the role played by the absolute risk aversion of the Responder (namely $-\frac{v''}{v'}$). Notice that the absolute risk aversion of the Responder is endogenous (it depends on w).

The third term likewise consists of two parts. The first part (with sign reversed) is

$$\beta\,(V - U(W - (W + w)\,b)). \qquad \text{(Term 3a)}$$

This is the direct cost to the Proposer of termination when it is insured using the payment b. The second part of the third term (with sign reversed) is

$$\frac{V_P}{v'}\,\beta\,(v - u(w + (W + w)\,b)). \qquad \text{(Term 3b)}$$

This is the indirect cost to the Proposer of termination when it is insured using the payment b. In this part: $\beta\,(v - u(w + (W + w)\,b))$ is the cost to the Responder of the possibility of termination; v' is, as above, the shadow value to the Responder of wealth in the hands of the Responder; and V_P is, as before, the shadow value to the Proposer of transfers from the Responder to the Proposer.

To summarize, under spot contracting, the Proposer takes into account the expected return obtained by both parties, the costs to both parties of the investment risk as mitigated by the risk-sharing rule s and the costs to both parties of the termination risk as mitigated by the risk-sharing rule b. The maximand of the Proposer involves three terms: a term in μ; a term in σ^2; and a term in β. The term in μ does not involve either of the control variables s or b; the term in σ^2 involves only s; and the term in β involves only b. The problem of optimizing s is therefore separable from the problem of optimizing b.

Proof. In view of Lemma 2, $(f, s, b) \in A(w)$ iff equation (14) holds. Putting $v = \widetilde{v}(\cdot, 0)$ in that equation, we obtain

$$v(w) = \mathrm{E}\left[v(w) + v'(w)\,\delta w + \tfrac{1}{2}\,v''(w)\,\delta w^2 + (u(w + \Delta w) - v(w))\right],$$

where

$$\begin{aligned}
\delta w &= (W + w)\,(f\,dt + s\,(\mu\,dt + \sigma\,dz)), \\
\Delta w &= (W + w)\,b\,d\chi.
\end{aligned}$$

Hence

$$v = v + \Big((W+w)\,(f+s\,\mu)\,v' + \tfrac{1}{2}\,(W+w)^2\,\sigma^2\,s^2\,v''$$
$$+ \beta\,(u(w+(W+w)\,b) - v)\Big)\,dt,$$

where we have suppressed the dependence of v and its derivatives on w, or

$$(W+w)\,(f+s\,\mu) = -\tfrac{1}{2}\,(W+w)^2\,\sigma^2\,s^2\,\frac{v''}{v'} - \beta\,\frac{u(w+(W+w)\,b) - v}{v'}. \tag{25}$$

We conclude that $A(w)$ can be characterized as the set of (f, s, b) such that equation (25) holds.

Next, putting $V = \widetilde{V}(\cdot, 0)$ in equation (15), we obtain

$$V(W,w) = \max_{(f,s,b)\in A(w)} \mathrm{E}\Big[V(W,w) + V_W(W,w)\,\delta W + V_w(W,w)\,\delta w$$
$$+ \tfrac{1}{2}\,\big(V_{WW}(W,w)\,\delta W^2 + 2\,V_{Ww}(W,w)\,\delta W\,\delta w + V_{ww}(W,w)\,\delta w^2\big)$$
$$+ (U(W+\Delta W) - V(W,w))\Big],$$

where

$$\begin{aligned} \delta W &= (W+w)\,(-f\,dt + (1-s)\,(\mu\,dt + \sigma\,dz)), \\ \Delta W &= -(W+w)\,b\,d\chi \end{aligned}$$

and δw is as above. Hence

$$V = \max_{(f,s,b)\in A(w)} \Big\{V + \Big((W+w)\,(-f+(1-s)\,\mu)\,V_W + (W+w)\,(f+s\,\mu)\,V_w$$
$$+ \tfrac{1}{2}\,(W+w)^2\,\sigma^2\,\big((1-s)^2\,V_{WW} + 2\,s\,(1-s)\,V_{Ww} + s^2\,V_{ww}\big)$$
$$+ \beta\,(U(W-(W+w)\,b) - V)\Big)\,dt\Big\},$$

where we have suppressed the dependence of V and its derivatives on (W, w), or

$$0 = \max_{(f,s,b)\in A(w)} \Big\{(W+w)\,\mu\,V_W - (W+w)\,(f+s\,\mu)\,(V_W - V_w) + \\ + \tfrac{1}{2}\,(W+w)^2\,\sigma^2\,\big((1-s)^2\,V_{WW} + 2\,s\,(1-s)\,V_{Ww} + s^2\,V_{ww}\big) \\ + \beta\,(U(W-(W+w)\,b) - V)\Big\}.$$

Using equation (25) to substitute for $(W+w)\,(f+s\,\mu)$, taking advantage of the notation V_P, V_{WP} and V_{PP} and rearranging, we obtain equation (24). ■

9. First-Order Conditions for s and b

In this section, we give a preliminary characterization of the optimal sharing rule s and the optimal termination payment b in terms of the value functions V and v of the Proposer and the Responder.

Proposition 4. *The optimal sharing rule s takes the form*

$$s = \frac{-\frac{V_{PP}}{V_P} - \frac{V_{wP}}{V_P}}{-\frac{V_{PP}}{V_P} - \frac{v''}{v'}}. \tag{26}$$

Proof. Maximizing the maximand in the reduced Bellman equation of the Proposer, namely equation (24), with respect to s boils down to maximizing the quadratic

$$V_{WW} - 2\,V_{WP}\,s + \left(V_{PP} + \frac{V_P}{v'}\,v''\right)s^2$$

with respect to s. Assuming that $V_{PP} + \frac{V_P}{v'}\,v'' > 0$,[9] this yields

$$s = \frac{V_{WP}}{V_{PP} + \frac{V_P}{v'}\,v''}.$$

Noting that $V_{WP} = V_{PP} + V_{wP}$ and dividing through by $-V_P$, we obtain the desired expression. ■

[9] It can be shown quite generally that $V_{PP} + v''\,\frac{V_P}{v'} \geq 0$, and our later analysis will confirm that this inequality is strict when both parties have constant relative risk aversion.

Expression (26) for the optimal *dynamic* sharing rule summarizes the main economic issues underlying our risk-sharing problem. In order to understand it better, it is helpful to compare it with the optimal *static* sharing rule

$$s^S = \frac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{u''}{u'}}.$$

Compared with this rule, the optimal dynamic rule exhibits three complications. First, the exogenous utility functions U and u are replaced with the endogenous value functions V and v. Second, the risk aversion of the Proposer is evaluated not with respect to her own wealth W, but instead with respect to the difference between her own wealth and that of the Responder, namely $P = W - w$.[10] Third, there is an additional term $-\frac{V_{wP}}{V_P}$ in the numerator. This term captures the idea that current changes in the Responder's wealth have implications for the price at which the Proposer will be able to obtain insurance in the future.[11]

Proposition 5. *The optimal termination payment b is the unique solution of*

$$\frac{U'(W - (W + w)\, b)}{u'(w + (W + w)\, b)} = \frac{V_P}{v'}. \tag{27}$$

Proof. Maximizing the maximand in the reduced Bellman equation of the Proposer with respect to b boils down to maximizing

$$U(W - (W + w)\, b) - V + \frac{V_P}{v'}\,(u(w + (W + w)\, b) - v)$$

with respect to b. This expression is strictly concave in b, and the first-order condition for this maximization is

$$0 = (W + w)\left(-U'(W - (W + w)\, b) + \frac{V_P}{v'}\, u'(w + (W + w)\, b)\right).$$

Rearranging, we obtain the desired equation. ■

[10] By analogy with the Proposer, one might expect the risk aversion of the Responder to be evaluated with respect to $p = w - W$. However, we have $v_p = v'$ and $v_{pp} = v''$, since v does not depend on W.

[11] The analogous term for the Responder does not occur since v does not depend on W.

The optimality condition (27) is akin to the familiar Borch condition. The optimal final transfer is set so that the ratio of the Proposer's and the Responder's marginal utility of wealth in the event that termination occurs is equal to the ratio of the Proposer's and the Responder's marginal value of transfers (in the event that termination does not occur). The close analogy between this optimality condition and the Borch condition suggests that V_P and v' can be interpreted as the welfare weights of the Proposer and the Responder in a welfare maximization problem.

10. Asymptotic Expansions

A first approach to understanding optimal risk sharing is to consider what happens when β is large, i.e. when the future is heavily discounted. More precisely, we look for approximations to V, v, s and b in the form $V^{(0)} + \frac{1}{\beta} V^{(1)}$, $v^{(0)} + \frac{1}{\beta} v^{(1)}$, $s^{(0)} + \frac{1}{\beta} s^{(1)}$ and $b^{(0)} + \frac{1}{\beta} b^{(1)}$. A striking feature of these approximations is that they give a qualitatively accurate picture of the behaviour of V, v, s and b even when β takes on much more moderate values, as is demonstrated by our numerical simulations in Section 13 below.

10.1. Myopic Terms.

We begin by identifying the myopic components of V, v, s and b, namely $V^{(0)}$, $v^{(0)}$, $s^{(0)}$ and $b^{(0)}$.

Proposition 6. *$V^{(0)} = U$, $v^{(0)} = u$, $s^{(0)} = \dfrac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{u''}{u'}}$ and $b^{(0)} = 0$.*

These expressions can be explained as follows. First, at order 0, the relationship ends immediately. The myopic value functions $V^{(0)}$ and $v^{(0)}$ are therefore simply the respective utilities U and u of consuming current wealth. Second, the myopic sharing rule $s^{(0)}$ is the familiar ratio of the Proposer's coefficient of absolute risk aversion to the sum of the two parties coefficients of absolute risk aversion. Third, the myopic termination payment $b^{(0)}$ is zero because, when β is very large, termination is essentially certain and it is not therefore possible to insure against it.

Proof. Dividing the Bellman equation of the Responder under autarky, namely (2), through by β and rearranging, we obtain

$$0 = u(w) - v + \frac{1}{\beta} \left(\frac{1}{2} \sigma^2 w^2 v'' + \mu w v' \right).$$

Hence, putting $v = v^{(0)} + \frac{1}{\beta} v^{(1)}$, denoting the first and second derivatives of $v^{(0)}$ by $v_w^{(0)}$ and $v_{ww}^{(0)}$ and rearranging, we obtain

$$0 = u(w) - v^{(0)} + \tfrac{1}{\beta}\left(\tfrac{1}{2}\,\sigma^2\, w^2\, v_{ww}^{(0)} + \mu\, w\, v_w^{(0)} - v^{(1)}\right) + \mathrm{O}\Big(\tfrac{1}{\beta^2}\Big)\,. \qquad (28)$$

Hence, equating terms of order 0, we obtain $v^{(0)} = u(w)$.

Second, dividing the Bellman equation of the Responder under spot contracting, namely (25), through by β and rearranging, we obtain

$$0 = u(w + (W + w)\, b) - v + \tfrac{1}{\beta}\left(\tfrac{1}{2}\,(W + w)^2\, \sigma^2\, s^2\, v'' + (W + w)\,(f + s\,\mu)\, v'\right)$$

Hence, putting $v = v^{(0)} + \frac{1}{\beta}\, v^{(1)}$, $f = f^{(0)} + \frac{1}{\beta}\, f^{(1)}$, $s = s^{(0)} + \frac{1}{\beta}\, s^{(1)}$ and $b = b^{(0)} + \frac{1}{\beta}\, b^{(1)}$, and equating terms of order 0, we obtain $v^{(0)} = u(w + (W + w)\, b^{(0)})$. But we have already shown that $v^{(0)} = u(w)$. It follows that $b^{(0)} = 0$.

Third, dividing the reduced Bellman equation of the Proposer, namely (24), through by β and rearranging, we obtain

$$\begin{aligned} 0 \;\; = \;\; & \max_{(s,b) \in \mathbb{R} \times \left(-\frac{w}{W+w},\, \frac{W}{W+w}\right)} \Bigg\{ U(W - (W + w)\, b) - V \\ & + \frac{V_P}{v'}\Big(u(w + (W + w)\, b) - v\Big) \;+\; \tfrac{1}{\beta}\Bigg(\mu\,(W + w)\, V_W \\ & + \tfrac{1}{2}\,\sigma^2\,(W + w)^2 \left(V_{WW} - 2\, s\, V_{WP} + s^2\, V_{PP} + \frac{V_P}{v'}\, s^2\, v''\right)\Bigg)\Bigg\}\,. \end{aligned}$$

Hence, putting $V = V^{(0)} + \frac{1}{\beta} V^{(1)}$, $v = v^{(0)} + \frac{1}{\beta} v^{(1)}$, $s = s^{(0)} + \frac{1}{\beta}\, s^{(1)}$ and $b = b^{(0)} + \frac{1}{\beta} b^{(1)}$, bearing in mind the envelope principle (which tells us that – in calculating first-order terms – we need not consider first-order variations in s and b), denoting the first derivative of $v^{(1)}$ by $v_w^{(1)}$ and rearranging, we obtain

$$0 = U(W - (W + w)\, b^{(0)}) - V^{(0)} + \frac{V_P^{(0)}}{v_w^{(0)}}\left(u(w + (W + w)\, b^{(0)}) - v^{(0)}\right)$$

$$+\tfrac{1}{\beta}\Bigg(-V^{(1)} - v^{(1)}\,\frac{V_P^{(0)}}{v_w^{(0)}} + \big(u(w+(W+w)\,b^{(0)}) - v^{(0)}\big)\,\frac{V_P^{(1)}v_w^{(0)} - V_P^{(0)}v_w^{(1)}}{\big(v_w^{(0)}\big)^2}$$

$$+\mu\,(W+w)\,V_W^{(0)}$$

$$+\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V_{WW}^{(0)} - 2\,s^{(0)}\,V_{WP}^{(0)} + \big(s^{(0)}\big)^2\,V_{PP}^{(0)} + \frac{V_P^{(0)}}{v_w^{(0)}}\,\big(s^{(0)}\big)^2\,v_{ww}^{(0)}\right)\Bigg)$$

$$+\mathrm{O}\Big(\tfrac{1}{\beta^2}\Big)\ .$$

Hence, taking advantage of the fact that $b^{(0)} = 0$ and $v^{(0)} = u$,

$$0 = U(W) - V^{(0)} + \tfrac{1}{\beta}\Bigg(-V^{(1)} - v^{(1)}\,\frac{V_P^{(0)}}{u'} + \mu\,(W+w)\,V_W^{(0)}$$

$$+\tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V_{WW}^{(0)} - 2\,s^{(0)}\,V_{WP}^{(0)} + \big(s^{(0)}\big)^2\,V_{PP}^{(0)} + \big(s^{(0)}\big)^2\,\frac{u''}{u'}\,V_P^{(0)}\right)\Bigg)$$

$$+\mathrm{O}\Big(\tfrac{1}{\beta^2}\Big)\ . \tag{29}$$

Hence, equating terms of order 0, $V^{(0)} = U(W)$.

Fourth, the first-order condition for the optimal sharing rule, namely (26), takes the form

$$s = \frac{-\frac{V_{PP}}{V_P} - \frac{V_{wP}}{V_P}}{-\frac{V_{PP}}{V_P} - \frac{v''}{v'}}.$$

Hence, putting $V = V^{(0)} + \frac{1}{\beta}\,V^{(1)}$, $v = v^{(0)} + \frac{1}{\beta}\,v^{(1)}$ and $s = s^{(0)} + \frac{1}{\beta}\,s^{(1)}$, and equating terms of order 0, we obtain:

$$s^{(0)} = \frac{-\frac{V_{PP}^{(0)}}{V_P^{(0)}} - \frac{V_{wP}^{(0)}}{V_P^{(0)}}}{-\frac{V_{PP}^{(0)}}{V_P^{(0)}} - \frac{v_{ww}^{(0)}}{v_w^{(0)}}}.$$

Finally, recalling that $V^{(0)} = U$ and $v^{(0)} = u$, we obtain the required expression. ■

10.2. Dynamic Terms: $V^{(1)}$ and $v^{(1)}$. In this section we determine the dynamic corrections $V^{(1)}$ and $v^{(1)}$ by equating the terms of order 1 in $\frac{1}{\beta}$ in the relevant equations.

Proposition 7. *We have:*

1. $V^{(1)} = \left((\mu - \frac{1}{2}\, R\, \sigma^2)\, W + \frac{1}{2}\, \dfrac{(R-r)^2\, \sigma^2}{\frac{R}{W} + \frac{r}{w}} \right) U'(W);$

2. $v^{(1)} = \left((\mu - \frac{1}{2} r\, \sigma^2)\, w \right) u'(w).$

where we have suppressed the dependence of R and r on W and w respectively.

In other words, the dynamic correction $V^{(1)}$ is composed of three elements: the risk-adjusted rate of return on the Proposer's wealth, namely

$$(\mu - \tfrac{1}{2}\, R\, \sigma^2)\, W;$$

the monetary value of the gains from sharing investment risk, namely

$$\tfrac{1}{2}\, \frac{(R-r)^2\, \sigma^2}{\frac{R}{W} + \frac{r}{w}};$$

and the marginal utility of wealth $U'(W)$. The first two elements are measured in units of wealth. Multiplying them by $U'(W)$ converts them into units of the Proposer's utility. Similarly, $v^{(1)}$ is composed of two elements: the risk-adjusted rate of return on the Responder's wealth, namely

$$(\mu - \tfrac{1}{2}\, r\, \sigma^2)\, w;$$

and the marginal utility of wealth $u'(w)$.

Notice that there is no contribution to $V^{(1)}$ reflecting the monetary value of the gains from sharing termination risk. Such a contribution would be expected to arise at order 2. However, exploring higher-order terms in the expansions is beyond the scope of this paper.[12] Also, in accordance with the bargaining positions of the two parties, the Responder does not receive any share in the gains from sharing investment risk. Finally, these formulae do not depend on the assumption that U and u are CRRA. (The formulae for $s^{(1)}$ and $b^{(1)}$ below do.)

[12] Such contributions do occur in Conditions III and IV in Section 12 below.

Proof. Equating terms of order 1 in equation (28), we obtain

$$0 = \tfrac{1}{2}\,\sigma^2\,w^2\,v^{(0)}_{ww} + \mu\,w\,v^{(0)}_{w} - v^{(1)}.$$

Hence, using the fact that $v^{(0)} = u$ and rearranging, we obtain

$$\frac{v^{(1)}}{u'} = \left(\mu + \tfrac{1}{2}\,\sigma^2\,\frac{w\,u''}{u'}\right)\,w = \left(\mu - \tfrac{1}{2}\,r\,\sigma^2\right)\,w$$

as required. Next, equating terms of order 1 in equation (29),

$$\begin{aligned} 0 \;\; &= \;\; -V^{(1)} - v^{(1)}\,\frac{V^{(0)}_P}{u'} + \mu\,(W+w)\,V^{(0)}_W \\ &\quad + \tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(V^{(0)}_{WW} - 2\,s^{(0)}\,V^{(0)}_{WP} + \left(s^{(0)}\right)^2\,V^{(0)}_{PP} + \left(s^{(0)}\right)^2\,\frac{u''}{u'}\,V^{(0)}_P\right). \end{aligned}$$

Hence, taking advantage of the fact that $V^{(0)} = U$ and rearranging,

$$\frac{V^{(1)}}{U'} = \mu\,(W+w) + \tfrac{1}{2}\,\sigma^2\,(W+w)^2\left(\left(1 - s^{(0)}\right)^2\,\frac{U''}{U'} + \left(s^{(0)}\right)^2\,\frac{u''}{u'}\right) - \frac{v^{(1)}}{u'}.$$

Finally, putting

$$\frac{v^{(1)}}{u'} = \left(\mu - \tfrac{1}{2}\,r\,\sigma^2\right)\,w, \quad s^{(0)} = \frac{-\frac{U''}{U'}}{-\frac{U''}{U'} - \frac{u''}{u'}}, \quad \frac{U''}{U'} = -\frac{R}{W} \quad \text{and} \quad \frac{u''}{u'} = -\frac{r}{w},$$

and rearranging, we obtain

$$\frac{V^{(1)}}{U'} = (\mu - \tfrac{1}{2}\,R\,\sigma^2)\,W + \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W} + \frac{r}{w}},$$

as required. ■

10.3. Dynamic Terms: $s^{(1)}$. In this section we determine the dynamic correction $s^{(1)}$ to the myopic sharing rule $s^{(0)}$. This correction is of interest for two reasons. First, it allows us to make qualitative predictions as to how the optimal risk-sharing rule s differs from the myopic sharing rule $s^{(0)}$. These predictions can then be compared with numerical simulations. Second, it offers some insight into why s differs from $s^{(0)}$ in the way that it does.

Proposition 8. *Suppose that both U and u are CRRA. Then*

$$s^{(0)} = \frac{\frac{R}{y}}{\frac{R}{y} + \frac{r}{1-y}}$$

and

$$s^{(1)} = \frac{\frac{1}{2}\,R\,r\,\sigma^2\,y^2\,(1-y)^2}{((1-y)\,R + y\,r)^5}\,(R-r)^3\big((1-y)\,R + y\,r - 2\big),$$

where $y = \frac{W}{W+w}$ is the Proposer's share in aggregate wealth.[13]

The main lessons that can be extracted from the formula for $s^{(0)}$ are as follows. First, the myopic sharing rule $s^{(0)}$ is – modulo multiplying the numerator and the denominator by $W + w$ – the ratio of the Proposer's absolute risk aversion, namely $\frac{R}{W}$, to the sum of the Proposer's and the Responder's absolute risk aversions, namely $\frac{R}{W} + \frac{r}{w}$. It is strictly decreasing in the Proposer's wealth share y. When $y = 0$, the Proposer is effectively infinitely risk averse, and $s^{(0)} = 1$. When $y = 1$, the Responder is effectively infinitely risk averse, and $s^{(0)} = 0$.

Second, notice that $s^{(0)}$ is the proportion of the investment risk on *total* wealth that the Responder bears. The proportion of the investment risk on his *own* wealth that he bears is therefore

$$\frac{W+w}{w}\,s^{(0)} = \frac{R}{(1-y)\,R + y\,r},$$

and his leverage is

$$\frac{R}{(1-y)\,R + y\,r} - 1 = \frac{y\,(R-r)}{(1-y)\,R + y\,r}.$$

If $R > r$ then his leverage is 0 when $y = 0$, and rises to $\frac{R-r}{r} > 0$ when $y = 1$. In particular, his leverage is *greater* when his wealth is *smaller*. His leverage is 0 when $y = 0$ because in that case he has all the wealth, and so risk sharing with the Proposer has a negligible impact. It is increasing in y because – from his point of view – the opportunities for risk sharing are increasing in y and, as the less risk averse party, taking advantage of these opportunities means increasing his leverage. If $R < r$, then his leverage is 0 when $y = 0$

[13] In a rare departure from our usual convention – according to which variables corresponding to the Proposer are capitalized and variables corresponding to the Responder are not – we use y to denote the Proposer's share in total wealth. This is because Y is too clumsy typographically.

and falls to $\frac{R-r}{r} < 0$ when $y = 1$. In particular, his leverage is smaller when his wealth is smaller.

Third, if $R = r$, then $s^{(0)} = 1 - y$ and $s^{(1)} = 0$. In other words, each party bears precisely the risk on their own wealth, and their wealth shares therefore remain unchanged.

Turning to the formula for the dynamic correction $s^{(1)}$, we begin with a definition:

Definition 9. *The Proposer is fairly risk tolerant if $R < 2$ and fairly risk averse if $R > 2$. Similarly, the Responder is fairly risk tolerant if $r < 2$ and fairly risk averse if $r > 2$.*

We go on to note that $s^{(1)}$ is the product of three terms, namely

$$\frac{\frac{1}{2}\,R\,r\,\sigma^2\,y^2\,(1-y)^2}{((1-y)\,R + y\,r)^5}, \quad (R-r)^3, \quad (1-y)\,R + y\,r - 2.$$

The first of these is always positive; the second has the same sign as $R - r$; and the third is affine in y. Hence, if we assume for concreteness that $R > r$, then we have three cases to consider:

The Risk-Tolerant Case When both parties are fairly risk tolerant, i.e. $2 > R > r$, then $(1-y)\,R + y\,r - 2 < 0$ for all $y \in [0, 1]$. Hence $s^{(1)} < 0$ for all $y \in (0, 1)$. This suggests that $s - s^{(0)} < 0$ for all $y \in (0, 1)$, where s is the optimal risk-sharing rule. In other words, irrespective of the distribution of wealth, it is optimal for the Proposer to transfer *less* risk to the Responder than she would under the myopic risk-sharing rule. The risk-tolerant case is illustrated in Figure 1(a).

The Risk-Averse Case When both parties are fairly risk averse, i.e. $R > r > 2$, then $(1-y)\,R + y\,r - 2 > 0$ for all $y \in [0, 1]$. Hence $s^{(1)} > 0$ for all $y \in (0, 1)$. This suggests that $s - s^{(0)} > 0$ for all $y \in (0, 1)$. In other words, irrespective of the distribution of wealth, it is optimal for the Proposer to transfer *more* risk to the Responder than she would under the myopic risk-sharing rule. The risk-averse case is illustrated in Figure 1(b).

The Mixed Case When the Proposer is fairly risk averse and the Responder is fairly risk tolerant, i.e. $R > 2 > r$, then $(1-y)\,R + y\,r - 2 > 0$ for $y \in \left[0, \frac{R-2}{R-r}\right)$ and $(1-y)\,R + y\,r - 2 < 0$ $y \in \left(\frac{R-2}{R-r}, 1\right]$. This suggests that $s - s^{(0)} > 0$ for $y \in \left(0, \frac{R-2}{R-r}\right)$ and $s - s^{(0)} < 0$ for $y \in \left(\frac{R-2}{R-r}, 1\right)$. In other words, when the Proposer has a small share in total wealth, it is optimal for her to transfer *more* risk to the Responder than she would under the myopic risk sharing rule; and, when she has a large share in total wealth, it is optimal for her to transfer *less* risk to him than she would under the myopic risk sharing rule. The mixed case is illustrated in Figure 1(c).

Figure 1 about here

The qualitative accuracy of these predictions can be demonstrated by plotting $s - s^{(0)}$, where s is the (numerically computed) optimal contract.[14] This is done in Figure 2. Figure 2(a) shows $s - s^{(0)}$ in the risk-tolerant case. This figure is very similar to Figure 1(a). The main difference is quantitative: the minimum in Figure 2(a) is somewhat lower than that in Figure 1(a). Figure 2(b) shows $s - s^{(0)}$ in the risk-averse case. This figure is very similar to Figure 1(b). The main difference is again quantitative: the maximum in Figure 2(b) is somewhat lower than that in Figure 1(b). Finally, Figure 2(c) shows $s - s^{(0)}$ in the mixed case. This figure is similar to Figure 1(c) in that the graph first rises to a positive maximum and then falls to a negative minimum. However, the balance between the left-hand hump and the right-hand hump is slightly different.

Figure 2 about here

The predictions are best understood in terms of counterparty risk. Indeed, consider the risk-tolerant case. Since $r < 2$, the Responder is willing to take on risk on relatively unfavourable terms, and the Proposer must bear in mind the possibility that he will eventually run out of wealth. This leads her to take on somewhat more risk than she would under the myopic benchmark, thereby delaying the time at which the Responder runs out of wealth. In effect, insurance is a scarce resource, and she chooses to husband it. Now consider the mixed case. The Responder is still willing to take on risk on relatively unfavourable terms, and the Proposer must still bear in mind the possibility that he will eventually run out of wealth. However, in this case the Proposer is only willing to take on the extra risk when she has a fairly large wealth share, i.e. when $y > \frac{R-2}{R-r}$. When $y < \frac{R-2}{R-r}$, the cost of bearing additional risk outweighs the benefit of husbanding insurance, and she transfers more risk to the Responder than she would under the myopic benchmark. Loosely speaking, the stock of insurance is measured by $1 - y$, and should be exploited when $y < \frac{R-2}{R-r}$ and conserved when $y > \frac{R-2}{R-r}$. Finally, consider the risk-averse

[14]See Section 13 below for more information on the numerically computed optimal contract.

case. Since $r > 2$, the Responder is only willing to take on risk on relatively favourable terms. Transferring more risk to him therefore has the indirect effect of increasing the rate of growth of his wealth and therefore the stock of insurance. The Proposer therefore does not hesitate to transfer more risk to him than she would under the myopic benchmark.

Proof of Proposition 8. Put $A = -\frac{V_{PP}}{V_P}$, $a = -\frac{v''}{v'}$ and $\theta = -\frac{V_{wP}}{V_P}$. Then

$$s = \frac{A+\theta}{A+a}$$

and

$$\begin{aligned} A^{(0)} &= \tfrac{R}{W}, & A^{(1)} &= -\tfrac{R}{W^2}\,G + \tfrac{R}{W}\,G_P - G_{PP}, \\ a^{(0)} &= \tfrac{r}{w}, & a^{(1)} &= 0, \\ \theta^{(0)} &= 0, & \theta^{(1)} &= \tfrac{R}{W}\,G_w - G_{wP}, \end{aligned}$$

where

$$G = \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W}+\frac{r}{w}}.$$

Hence, using the fact that $\theta^{(0)} = a^{(1)} = 0$ and the fact that $s^{(0)} = \frac{A^{(0)}}{A^{(0)}+a^{(0)}}$,

$$\begin{aligned} \frac{s^{(1)}}{s^{(0)}} &= \frac{A^{(1)}+\theta^{(1)}}{A^{(0)}+\theta^{(0)}} - \frac{A^{(1)}+a^{(1)}}{A^{(0)}+a^{(0)}} = \frac{A^{(1)}+\theta^{(1)}}{A^{(0)}} - \frac{A^{(1)}}{A^{(0)}+a^{(0)}} \\ &= \left(1-s^{(0)}\right)\frac{A^{(1)}}{A^{(0)}} + \frac{\theta^{(1)}}{A^{(0)}}. \end{aligned}$$

Now, using the formulae for $A^{(0)}$, $A^{(1)}$, $\theta^{(1)}$ and G, we obtain

$$\frac{A^{(1)}}{A^{(0)}} = -\frac{\frac{1}{2}\,r\,\sigma^2\,y\,(R-r)^2\,((1-y)\,R+y\,r-2)}{((1-y)\,R+y\,r)^3}$$

and

$$\frac{\theta^{(1)}}{A^{(0)}} = \frac{\frac{1}{2}\,r\,\sigma^2\,y^2\,(R-r)^2\,((1-y)\,R+y\,r-2)}{((1-y)\,R+y\,r)^3} = -y\,\frac{A^{(1)}}{A^{(0)}}.$$

We therefore get

$$\begin{aligned} s^{(1)} &= s^{(0)}\left(1-s^{(0)}-y\right)\frac{A^{(1)}}{A^{(0)}} \\ &= \frac{(1-y)\,R}{(1-y)\,R+y\,r}\left(-\frac{y\,(1-y)\,(R-r)}{(1-y)\,R+y\,r}\right)\frac{A^{(1)}}{A^{(0)}} \\ &= \frac{\frac{1}{2}\,R\,r\,\sigma^2\,y^2\,(1-y)^2\,(R-r)^3\,((1-y)\,R+y\,r-2)}{((1-y)\,R+y\,r)^5}, \end{aligned}$$

as required. ■

10.4. Dynamic Terms: $b^{(1)}$. In this section we determine the dynamic correction $b^{(1)}$ to the myopic termination payment $b^{(0)}$. Since $b^{(0)}=0$, this correction leads directly to qualitative predictions about the optimal termination payment b. The principal prediction is that the sign of b will depend on whether the investment opportunity is good or bad, in the sense that the ratio $\frac{2\mu}{\sigma^2}$ is high or low relative to the other parameters of the model. For example, suppose that $R>r$. In this case, if the value of the investment opportunity is high, then we should have $b<0$. In other words, the Responder should compensate the Proposer for the loss of the valuable investment opportunity when termination occurs. On the other hand, if the value of the investment opportunity is low, then we should have $b>0$. In other words, the Responder should compensate the Proposer for the losses that she faces while the investment is ongoing, and receives in return a payment from the Proposer when the good state (namely termination) is reached.

There is, however, an important twist to the story: for intermediate values of the ratio $\frac{2\mu}{\sigma^2}$, the sign of b should depend on the Proposer's wealth share y. In the risk-tolerant case, we predict that b will be positive for small y and negative for large y. In the risk-averse case, we predict that b will be negative for small y and positive for large y. In the mixed case, the picture is more involved. However, the most interesting possibility is that in which b will be negative for y near 0 or 1 but positive for intermediate values of y.

Proposition 10. *Suppose that both U and u are CRRA. Then*

$$b^{(0)} = 0$$

and

$$b^{(1)} = \frac{y\,(1-y)}{(1-y)\,R + y\,r}\,(R-r)\left(\tfrac{1}{2}\,\sigma^2\,B(y) - \mu\right),$$

where

$$B(y) = r\,\frac{(R-r)^2\,y^2 - 3\,R\,(R-r)\,y + R\,(2\,R-1)}{((1-y)\,R + y\,r)^2}$$

and $y = \frac{W}{W+w}$ *is the Proposer's share in aggregate wealth.*

Now, $b^{(1)}$ is the product of three terms, namely

$$\frac{y\,(1-y)}{(1-y)\,R + y\,r}, \quad R-r, \quad \tfrac{1}{2}\,\sigma^2\,B(y) - \mu.$$

The first of these is always positive; the second has the same sign as $R-r$; and the third is linear in the core parameters μ and σ^2, but depends in an apparently complicated way on y. Fortunately, this complexity is more apparent than real: if we differentiate B with respect to y, then we obtain a formula that is highly reminiscent of the formula for $s^{(1)}$, namely

$$B'(y) = \frac{R\,r\,(R-r)}{((1-y)\,R + y\,r)^3}((1-y)\,R + y\,r - 2).$$

Assuming for concreteness that $R > r$, we therefore arrive at the three same cases that we encountered in the context of our discussion of $s^{(1)}$, namely:

The Risk-Tolerant Case If $2 > R > r$, then we have $B' < 0$ for all $y \in [0,1]$. There are therefore three subcases to consider, namely

$$\tfrac{2\mu}{\sigma^2} < B(1), \quad \tfrac{2\mu}{\sigma^2} \in (B(1), B(0)), \quad \tfrac{2\mu}{\sigma^2} > B(0).$$

In the first subcase, $b^{(1)} > 0$ for all $y \in (0,1)$; in the second, there exists $\overline{y} \in (0,1)$ such that $b^{(1)} > 0$ for $y \in (0,\overline{y})$ and $b^{(1)} < 0$ for $y \in (\overline{y},1)$; in the third, $b^{(1)} < 0$ for all $y \in (0,1)$. This case is illustrated in Figure 3(a).

The Risk-Averse Case If $R > r > 2$, then we have $B' > 0$ for all $y \in [0,1]$. There are therefore again three subcases to consider, namely

$$\tfrac{2\mu}{\sigma^2} < B(0), \quad \tfrac{2\mu}{\sigma^2} \in (B(0), B(1)), \quad \tfrac{2\mu}{\sigma^2} > B(1).$$

In the first subcase, $b^{(1)} > 0$ for all $y \in (0,1)$; in the second, there exists $\overline{y} \in (0,1)$ such that $b^{(1)} < 0$ for $y \in (0,\overline{y})$ and $b^{(1)} > 0$ for $y \in (\overline{y},1)$; in the third, $b^{(1)} < 0$ for all $y \in (0,1)$. This case is illustrated in Figure 3(b).

The Mixed Case If $R > 2 > r$, then B is inverse-U shaped: $B' > 0$ for $y \in \left[0, \frac{R-2}{R-r}\right)$ and $B' < 0$ for $y \in \left(\frac{R-2}{R-r}, 0\right]$. Putting

$$\overline{B} = \max\{B(y) \mid y \in [0,1]\} = \tfrac{1}{4}\, r\,(4+R),$$

there are therefore four subcases to consider, namely

$$\tfrac{2\mu}{\sigma^2} < \min\{B(0),B(1)\}, \qquad \tfrac{2\mu}{\sigma^2} \in \left(\min\{B(0),B(1)\},\max\{B(0),B(1)\}\right),$$
$$\tfrac{2\mu}{\sigma^2} \in \left(\max\{B(0),B(1)\},\overline{B}\right), \qquad \tfrac{2\mu}{\sigma^2} > \overline{B}.$$

This case is illustrated in Figure 3(c).[15]

Figure 3 about here

Since $b^{(0)} = 0$, these observations concerning $b^{(1)}$ translate directly into predictions about the optimal termination payment b. These predictions are remarkable at three levels. First, they are qualitatively correct: every case and subcase identified above occurs. Second, they are quantitatively correct: they can be used to find specific choices of the parameters for which these cases and subcases occur. Third, they are correct for realistic values of β, and not just for the large values of β for which the expansions are theoretically valid.[16]

Proof of Proposition 10. The first-order condition for b, namely (27), takes the

[15] The second subcase of the mixed case divides into two subsubcases depending on whether $B(0) < B(1)$ or $B(0) > B(1)$. In this connection, it is worth noting that $B(0) < B(1)$ iff the harmonic mean of R and r exceeds 2. This is another instance where the outcome depends on whether mean risk aversion lies above or below 2.

[16] The three parameters β, μ and σ^2 are not independent of one another: one of them can be scaled out of the problem. For the present purposes, it is convenient to scale out σ^2. In this way we arrive at the dimensionless parameter $\frac{\beta}{\sigma^2}$. Our asymptotic expansions are premised on the assumption that this parameter is large. In all of the three baseline parameter constellations used in Section 13 below, we have $\sigma^2 = 0.0225$ and therefore $\frac{\beta}{\sigma^2} = 2.22$. This is certainly not large. However, it is still greater than 1, and is not therefore small either.

form

$$\frac{U'(W-(W+w)\,b)}{u'(w+(W+w)\,b)}=\frac{V_P}{v'}.$$

Hence, using logarithmic differentiation and taking advantage of the fact that $b^{(0)}=0$,

$$-\frac{(W+w)\,U''(W)\,b^{(1)}}{U'(W)}-\frac{(W+w)\,u''(w)\,b^{(1)}}{u'(w)}=\frac{V_P^{(1)}}{V_P^{(0)}}-\frac{(v^{(1)})'}{(v^{(0)})'}.$$

Now

$$\begin{aligned} V_P^{(0)} &= U'(W), & V_P^{(1)} &= \left(-\tfrac{R}{W}\,(I+G)+(I+G)_P\right)U'(W),\\ v_w^{(0)} &= u'(w), & v_w^{(1)} &= \left(-\tfrac{r}{w}\,i+i_w\right)u'(w), \end{aligned}$$

where

$$I=(\mu-\tfrac{1}{2}\,R\,\sigma^2)\,W,\quad i=(\mu-\tfrac{1}{2}r\,\sigma^2)\,w\quad\text{and}\quad G=\tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W}+\frac{r}{w}}$$

denote the investment return of the Proposer, the investment return of the Responder and the monetary value of the gains from trade respectively. Hence

$$(W+w)\left(\frac{R}{W}+\frac{r}{w}\right)b^{(1)}=\left(-\frac{R}{W}\,(I+G)+(I+G)_P\right)-\left(-\frac{r}{w}\,i+i_w\right)$$

or

$$(W+w)\,b^{(1)}=-s^{(0)}\left((I+G)-\tfrac{W}{R}\,(I+G)_P\right)+\left(1-s^{(0)}\right)\left(i-\tfrac{w}{r}\,i_w\right).$$

In other words, when termination occurs: the Responder pays the Proposer a fraction $s^{(0)}$ of the total loss $I+G$ to the Proposer from termination; and the Proposer pays the Responder a fraction $1-s^{(0)}$ of the total loss i to the Responder from termination. These payments are, however, offset by terms reflecting the opportunity cost of buying the termination insurance ex ante. Finally, substituting for I, i and G and collecting terms in μ and σ^2 yields the required formula for $b^{(1)}$. ■

11. Analytical Results

In the previous section we derived a number of detailed, but approximate, results about the optimal contract. In the present section we derive a number of exact results, and we introduce – and demonstrate the importance of – the concept of total endogenous risk aversion.

Perhaps the most significant result is the finding that, if the Proposer is more risk averse than the Responder (in the sense that $R > r$) then she will always take on less than her share in the risk (which under autarky would be $\frac{W}{W+w}$).[17] This result is derived from a conceptually important formula linking the rule for sharing investment risk to the gains from sharing investment risk.[18] Another significant result shows that total endogenous risk aversion and the gains from sharing investment risk are always both strictly positive.[19] The strict positivity of total endogenous risk aversion ensures that the leverage of both parties is never infinite. In particular, the two parties never exploit the Wiener noise to engage in endogenous betting. The strict positivity of the gains from sharing investment risk ensures that the leverage of both parties is never zero. We also show that the gains from sharing termination risk are positive, but not strictly so.[20] This highlights the fact that the strict positivity of total endogenous risk aversion and of the gains from sharing investment risk is not something that can be taken for granted. Finally, we establish a conceptually important formula linking the rule for sharing investment risk to the marginal value of wealth.

One final contribution of this section is to lay the groundwork for the numerical simulations reported in Section 13 below. However, knowledge of this groundwork is not essential for an understanding of the simulations themselves. The reader who is anxious to see the numerical results (and how they compare with the approximations), but who is not interested in the analytical results, may therefore wish to skip the remainder of this section, review Conditions III and IV in Section 12, and then proceed to Section 13.

11.1. Some Normalizations. In order to derive our analytical results, we shall need several normalizations. The broad idea is to normalize both the Proposer's value function V and the control variables s and b with respect to the wealth of both the Proposer and the Responder. However, we shall also need the notation Φ and ϕ (explained below) to handle the algebra associated with the termination payment.

It will be helpful to begin with the Proposer's problem under autarky. By analogy with the Responder's problem under autarky, we see at once that the value function of the Proposer under autarky takes the form

$$C_R(\rho_R W) = C_R(W) + W\, C_R'(W)\, \psi_R,$$

[17]See Proposition 15.
[18]See equation 44.
[19]See Proposition 13.
[20]See Proposition 16.

where ρ_R is the certainty-equivalent rate of return of the Proposer under autarky,

$$\psi_R = C_R(\rho_R) = \frac{\mu - \frac{1}{2}\,R\,\sigma^2}{\beta_R}$$

is the normalized value function of the Proposer under autarky and

$$\beta_R = \beta - (1 - R)\,(\mu - \tfrac{1}{2}\,R\,\sigma^2).$$

Moreover the marginal value of wealth of the Proposer under autarky takes the form

$$\gamma_R\,C_R'(W),$$

where

$$\gamma_R = \frac{\beta}{\beta_R}$$

is the normalized marginal value of wealth of the Proposer under autarky.

By analogy with this problem, it is natural to look for a solution to the Proposer's problem under bilateral contracting in the form

$$V(W,w) = C_R(\rho(y)\,W) = C_R(W) + W\,C_R'(W)\,\psi(y),$$

where

$$\psi(y) = C_R(\rho(y))$$

is the normalized value function of the Proposer.[21] Furthermore it is helpful to note that the marginal value of transfers then takes the form

$$V_P(W,w) = \gamma(y)\,C_R'(W), \tag{30}$$

where $V_P(W,w)$ is (by definition) $V_W(W,w) - V_w(W,w)$, and

$$\gamma(y) = 1 + (1 - R)\,\psi(y) + y\,\psi'(y)$$

is the normalized marginal value of transfers.

Proposition 11. *Let $\lambda(W,w)$ be the Lagrange multiplier on the participation constraint*

[21] It is clear on economic grounds that $\psi' < 0$. We shall not, however, need this result below.

in the long-run contracting problem when the Proposer has initial wealth W and the Responder has initial wealth w. Then $\gamma(y) = \frac{v'(w)}{U'(W)}\lambda(W,w)$. In particular, $\gamma(y) > 0$ for all $y \in (0,1)$.

Proof. For the proof, we shall use the notation from Section 4. In particular, we shall denote the initial wealths by W_0 and w_0. In this notation, the Lagrangean for the long-run contacting problem takes the form

$$\mathrm{E}[U(X_{T+} - q(X,T))] - \lambda(W_0, w_0)\,(v(w_0) - \mathrm{E}[u(q(X,T))])\,,$$

where we have not made explicit the dependence of q on W_0, w_0. Now, the first-order condition for the choice of q is

$$-U'(X_{T+} - q) + \lambda\, u'(q) = 0 \quad \text{a.s.}, \tag{31}$$

where we have suppressed the dependence of q on (X,T) and λ on (W_0, w_0), and the participation constraint takes the form

$$\mathrm{E}[u(q)] = v(w_0). \tag{32}$$

(Recall that it was shown in Section 4 that the participation constraint must bind.)

Now, fixing $X_0 = W_0 + w_0$ and varying W_0, we may differentiate (31) and (32) to obtain

$$(U''(X_{T+} - q) + \lambda\, u''(q))\, q_P + u'(q)\, \lambda_P = 0 \quad \text{a.s.} \tag{33}$$

and

$$\mathrm{E}[u'(q)\, q_P] = -v'(w_0), \tag{34}$$

Making q_P the subject of (33), we obtain

$$q_P = -\frac{u'(q)}{U''(X_{T+} - q) + \lambda\, u''(q)}\,\lambda_P.$$

Hence, substituting in (34) and exploiting the fact that λ_P is deterministic,

$$\lambda_P = -\frac{v'(w_0)}{\mathrm{E}\left[-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}\right]}$$

and

$$q_P = -\frac{-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}}{\mathrm{E}\Big[-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}\Big]}\,\frac{v'(w_0)}{u'(q)}. \tag{35}$$

Next, since $V(W_0, w_0) = \mathrm{E}[U(X_{T+} - q)]$, we have

$$V_P(W_0, w_0) = \mathrm{E}[-U'(X_{T+} - q)\, q_P] = \mathrm{E}[-\lambda\, u'(q)\, q_P]$$

(where the second equality follows from (31))

$$= \mathrm{E}\left[\lambda\, u'(q)\,\frac{-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}}{\mathrm{E}\Big[-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}\Big]}\,\frac{v'(w_0)}{u'(q)}\right]$$

(by (35))

$$= \lambda\, v'(w_0)\,\mathrm{E}\left[\frac{-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}}{\mathrm{E}\Big[-\frac{u'(q)^2}{U''(X_{T+}-q)+\lambda\, u''(q)}\Big]}\right] = \lambda\, v'(w_0).$$

Next, translating (30) into the notation of the current proof, we have

$$V_P(W_0, w_0) = \gamma(y_0)\, C_R'(W_0).$$

Hence

$$\gamma(y_0) = \frac{v'(w_0)}{C_R'(W_0)}\,\lambda.$$

Finally, (31) yields

$$\lambda = \frac{U'(X_{T+} - q)}{u'(q)} > 0.$$

This completes the proof. ■

Working in terms of ψ has the desirable effect of normalizing with respect to the wealth of the Proposer, but it does not normalize with respect to the wealth of the Responder. To see this, note that the gain to the Proposer from trade with the Responder is measured by the difference $\psi(y) - \psi_R$. This difference will be non-negligible when the wealth share of the Proposer is small, precisely because we have already normalized for the wealth of the Proposer. However, it will be negligible when the wealth share of the Proposer is large, since then the Responder has too little wealth to offer much in the way of gains from trade. The second normalization that we need can be achieved by introducing

$$\zeta(y) = \frac{\psi(y) - \psi_R}{(1-y)\,\gamma_R}.$$

The factor $1 - y$ in the denominator of this formula normalizes the gain from trade with respect to the wealth of the Responder. The factor γ_R serves to simplify the algebra which follows. We shall refer to ζ as the normalized gain from sharing risk.

We shall also need corresponding normalizations for the contractual variables s and b, namely

$$z = \frac{s - (1-y)}{y\,(1-y)} \quad \text{and} \quad g = \frac{b}{y\,(1-y)}.$$

From an economic perspective, z is simply the ratio of the Responder's leverage to the Principal's share in total wealth. Indeed, the total investment risk borne by the Responder is $s\,(W + w)$, and his leverage is therefore

$$\frac{s\,(W+w) - w}{w} = \frac{s - (1-y)}{1-y}.$$

From a mathematical perspective, notice that the Responder's contractual share in the total investment risk, namely s, differs from his autarky share, namely $1 - y$. This difference will be small when y is near 0, since then the Responder has to bear almost all the risk (because he owns almost all the wealth). It will also be small when y is near 1, since then the Responder bears almost none of the risk (because he owns almost none of the wealth). Dividing through by $y\,(1-y)$ therefore normalizes the departure of the Responder's contractual share from his autarky share. Similarly, the proportion b of total wealth transferred between the two parties in the event of termination is small when y is near 0 or 1. It is therefore helpful to normalize b so that it is measured relative to the Proposer's wealth when y is near 0 and the Responder's wealth when y is near 1. We shall refer to z as the normalized rule for sharing investment risk, and to g as the normalized rule for sharing termination risk.

Next, recall from the discussion following Proposition 3 that the total benefit from termination risk is captured by the term

$$\beta\left(U(W - (W+w)\,b) - V + \frac{V_P}{v'}\,(u(w + (W+w)\,b) - v)\right)$$

in the reduced Bellman equation of the Proposer (namely (24)). Taking advantage of the fact that $V = C_R(\rho W)$, $v = C_r(\rho_r w)$, $V_P = \gamma C_R'(W)$ and $v' = \gamma_r C_r'(w)$, this term can be rewritten in the form

$$\beta (W + w) C_R'(W) y (1 - y) \left(\Phi(g, y, \gamma) - \frac{\psi}{1 - y} - \frac{\gamma}{\gamma_r} \frac{\psi_r}{y} \right), \tag{36}$$

where the function Φ is defined in two steps.[22] The first step is to define Φ in the interior of wealth space. More precisely, we put

$$\Phi(g, y, \gamma) = \frac{C_R(1 - (1 - y) g)}{1 - y} + \frac{\gamma}{\gamma_r} \frac{C_r(1 + y g)}{y}$$

for all $(y, \gamma) \in (0, 1) \times (0, \infty)$ and all $g \in (-y^{-1}, (1 - y)^{-1})$. The second step is to extend this definition to the boundary of wealth space. More precisely, since $C_r(1) = 0$ and $C_r'(1) = 1$, Φ extends continuously to the case $(y, \gamma) \in \{0\} \times (0, \infty)$ and $g \in (-\infty, 1)$ by means of the formula

$$\Phi(g, 0, \gamma) = C_R(1 - g) + \frac{\gamma}{\gamma_r} g.$$

Similarly, since $C_R(1) = 0$ and $C_R'(1) = 1$, Φ extends continuously to the case $(y, \gamma) \in \{1\} \times (0, \infty)$ and $g \in (-1, \infty)$ by means of the formula

$$\Phi(g, 1, \gamma) = -g + \frac{\gamma}{\gamma_r} C_r(1 + g).$$

Finally, we shall also need the function ϕ given by the formula

$$\phi(y, \gamma) = \max_{g \in (-y^{-1}, (1-y)^{-1})} \{\Phi(g, y, \gamma)\}.$$

for all $(y, \gamma) \in (0, 1) \times (0, \infty)$, and by the formulae

$$\phi(0, \gamma) = \max_{g \in (-\infty, 1)} \{\Phi(g, 0, \gamma)\}$$

[22] Expression (36) shows that the total benefit from termination risk can be broken down into three components: the benefit from sharing termination risk, as captured by the term involving $\Phi(g, y, \gamma)$; the cost to the Proposer of losing the investment opportunity, as captured by the term involving ψ; and the cost to the Responder of losing the investment opportunity, as captured by the term involving ψ_r.

and

$$\phi(1,\gamma) = \max_{g\in(-1,\infty)} \{\Phi(g,1,\gamma)\}$$

for all $\gamma \in (0,\infty)$.

We shall refer to ϕ as the normalized gain from sharing termination risk. Also, since the normalized gain from sharing risk ζ is made up of the normalized gain from sharing termination risk and the normalized gain from sharing investment risk, we refer to $\zeta - \phi$ as the normalized gain from sharing investment risk.

11.2. Total Endogenous Risk Aversion the Gains from Sharing Investment Risk. In this section, we establish two crucial properties of our model: total endogenous risk aversion is strictly positive; and the gains from sharing investment risk are strictly positive. The main tool for establishing these results is the following lemma, which gives the equations obtained when we substitute for V and its derivatives in terms of γ, γ', ζ and ζ', and for s and b in terms of z and g.

Lemma 12. *The reduced Bellman equation of the Proposer, namely equation (24), can be written equivalently as a pair of one-dimensional equations for γ and ζ, namely*

$$\begin{aligned} 0 \;=\; \max_{(z,g)\in\mathbb{R}\times(-y^{-1},(1-y)^{-1})} &\Bigg\{ \frac{\beta}{\gamma\,\sigma^2}\left(\Phi(g,y,\gamma)-\zeta\right) \\ &+ (R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \Bigg\} \end{aligned} \tag{37}$$

and

$$y\,(1-y)\,\zeta' = \frac{\gamma-\gamma_R}{\gamma_R} - ((1-R)-(2-R)\,y)\,\zeta, \tag{38}$$

where we have suppressed the dependence of γ and ζ on y.

Notice that choosing g reduces to maximizing $\Phi(g,y,\gamma)$ with respect to g, and that choosing z reduces to maximizing

$$(R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \tag{39}$$

with respect to z. Also, the maximand (39) for z involves two terms: a linear incentive and a quadratic penalty. The coefficient governing the linear incentive is $R-r$. Hence the more the risk aversion of the Proposer exceeds that of the Responder, the greater will be the Responder's normalized leverage z. The coefficient governing the quadratic penalty is $(1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma}$. The multiplicative factor $\frac{W+w}{W\,w}$ aside, this is total endogenous risk aversion.[23] Hence it is total endogenous risk aversion that puts a brake on leverage.

Proof. See Appendix C. ■

We can now establish the main result of this section:

Proposition 13. *Suppose that $R \neq r$. Then, for all $y \in [0,1]$, we have*

1. $(1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma} > 0$ *and*
2. $\zeta - \phi > 0$,

where we have suppressed the dependence of ϕ on y and γ.

In other words, both total endogenous risk aversion and the normalized gain from sharing investment risk are strictly positive. The proof of this result proceeds in four steps. The first step shows that total endogenous risk aversion must be non-negative, because otherwise the two parties would use the Wiener noise to construct bets, and thus arbitrage away the infinite gains from trade implicit in strictly negative total risk aversion. The second step shows that the gain from sharing investment risk must be non-negative, because otherwise the two parties would simply avoid all sharing of investment risk. Building on the first step, the third step shows that, if total endogenous risk aversion were zero, then infinite sharing of risk would take place. Building on the second step, the fourth step shows that, if the gain from sharing investment risk were zero, then a small amount of sharing of investment risk would in fact be desirable.

Proof. Since the maximand in equation (37) is additively separable in g and z, equation (37) can be written equivalently as

$$0 = \max_{z \in \mathbb{R}} \left\{ -\frac{\beta}{\gamma\,\sigma^2}\,(\zeta-\phi) + (R-r)\,z - \tfrac{1}{2}\left((1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \right\}.$$

[23] The endogenous absolute risk aversion of the Proposer is $-\frac{V_{PP}}{V_P}$ and the endogenous absolute risk aversion of the Responder is $-\frac{v''}{v'}$. Hence total endogenous risk aversion is

$$-\frac{V_{PP}}{V_P} - \frac{v''}{v'} = \frac{W+w}{W\,w}\left((1-y)\,R+y\,r-\frac{y\,(1-y)\,\gamma'}{\gamma}\right).$$

Cf. Section 11.5 below. Proposition 13 below shows that this is strictly positive.

However, as it stands, this equation is not fully precise: z can take any real value, and therefore the coefficients of the equation are unbounded. To obtain a precise version of the equation, we need to normalize by dividing through by $1+z^2$.[24] Cf. Krylov [9]. Doing so yields

$$\begin{aligned} 0 &= \sup_{z\in\mathbb{R}} \Bigg\{ -\frac{\beta}{\gamma\,\sigma^2}\,(\zeta-\phi)\,\frac{1}{1+z^2} + (R-r)\,\frac{z}{1+z^2} \\ &\quad - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right)\frac{z^2}{1+z^2}\Bigg\}. \end{aligned}$$

In particular, the objective must be non-positive for all $z\in\mathbb{R}$. Letting $z\to\infty$ therefore yields

$$-\tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) \le 0; \tag{40}$$

and putting $z=0$ yields

$$-\frac{\beta}{\gamma\,\sigma^2}\,(\zeta-\phi) \le 0. \tag{41}$$

These are the weak versions of the two inequalities that we aim to establish. It therefore remains only to show that neither (40) nor (41) can hold as an equality.

We begin with (40). Suppose for a contradiction that $(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma} = 0$. Then we must have

$$\frac{1}{1+z^2}\left(-\frac{\beta}{\gamma\,\sigma^2}\,(\zeta-\phi) + (R-r)\,z\right) \le 0$$

for all $z\in\mathbb{R}$. However, since $R\neq r$, the expression in parentheses is a non-constant affine function of z. It must therefore be strictly positive for some choice of z. This contradiction establishes that inequality (40) must be strict.

We turn now to (41). Suppose for a contradiction that $\zeta-\phi=0$. Then we must have

$$\frac{1}{1+z^2}\left((R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right)z^2\right) \le 0$$

for all $z\in\mathbb{R}$. However: the expression in the outer parentheses is quadratic in z; it is 0 at $z=0$; and, since $R\neq r$, it has a non-zero slope at $z=0$. It must therefore be strictly

[24] Note that the normalization should in principle be applied consistently throughout the paper. However, we have suppressed it for expositional convenience. We make it explicit here since this is the one place where it plays an important role.

positive for some choice of z. This contradiction establishes that inequality (41) must be strict. ■

11.3. The Rule for Sharing Investment Risk. In this section, we show that the Responder takes on more than his share of investment risk when he is less risk averse (in the sense that $r < R$), and less than his share of investment risk when he is more risk averse (in the sense that $r > R$). The main tool for establishing these results is the following lemma, which takes advantage of the fact that total endogenous risk aversion is strictly positive to eliminate z from (37) and (38).

Lemma 14. *The reduced Bellman equation of the Proposer, namely equation (24), can be written equivalently as a pair of one-dimensional equations for γ and ζ, namely*

$$y\,(1-y)\,\gamma' = \left((1-y)\,R + y\,r - \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2\,\gamma}{\beta\,(\zeta-\phi)}\right)\gamma \tag{42}$$

and

$$y\,(1-y)\,\zeta' = \frac{\gamma-\gamma_R}{\gamma_R} - ((1-R)-(2-R)\,y)\,\zeta. \tag{43}$$

Moreover the optimal rule for sharing investment risk takes the form

$$z = \frac{\beta\,(\zeta-\phi)}{\frac{1}{2}\,(R-r)\,\sigma^2\,\gamma}. \tag{44}$$

We refer to the pair of equations (42-43) as the one-dimensional Bellman equation of the Proposer.

Proof. See Appendix D. ■

We can now establish the main result of this section:

Proposition 15. *1. If $R > r$ then $z > 0$ for all $y \in [0,1]$.*

2. If $R < r$ then $z < 0$ for all $y \in [0,1]$.

In other words: if the Responder is less risk averse than the Proposer, then he takes on more than his share of investment risk; and, if he is more risk averse than the Proposer, then he takes on less than his share of investment risk.

Proof. Proposition 13 shows that $\zeta - \phi > 0$ for all $y \in [0,1]$. With this information in hand, both parts of the Proposition follow directly from equation (44). ■

11.4. The Gains from Sharing Termination Risk. The following Proposition shows that the gains from sharing termination risk are non-negative:

Proposition 16. *For all $y \in [0,1]$, we have $\phi \geq 0$.*

Proof. For all $(y,\gamma) \in [0,1] \times (0,\infty)$: $\phi(y,\gamma)$ is the maximum of $\Phi(g,y,\gamma)$ over the appropriate set of g; 0 is always one of the possible values for g; and $\Phi(0,y,\gamma) = 0$. It follows that $\phi(y,\gamma) \geq \Phi(0,y,\gamma) = 0$ for all $(y,\gamma) \in (0,1) \times (0,\infty)$. ■

Remark 17. *It is not possible to show that the gains from sharing termination risk are always strictly positive, i.e. that $\phi > 0$. Indeed, $\phi = 0$ iff $g = 0$. Moreover, on the basis of the asymptotic expansions given in Section 10 above, we would expect g to take the value 0 for some $y \in (0,1)$ whenever*

$$\min\{B(0), B(1)\} < \tfrac{2\mu}{\sigma^2} < \max\{B(0), B(1)\},$$

where B is the function given in Proposition 10. This prediction is borne out by Figure 7 below.

11.5. The Rule for Sharing Investment Risk Again. Our normalizations also give rise to a striking formula for the difference between the optimal and myopic rules for sharing investment risk. Among other things, this formula establishes a conceptual connection between the dynamic adjustment $s - s^{(0)}$ to the myopic sharing rule $s^{(0)}$ and the normalized marginal value of transfers γ.

Proposition 18. *We have*

$$s - s^{(0)} = \frac{(R - r)\,\frac{\gamma'}{\gamma}}{\left(\frac{R}{y} + \frac{r}{1-y}\right)\left(\frac{R}{y} + \frac{r}{1-y} - \frac{\gamma'}{\gamma}\right)}. \tag{45}$$

In particular:

1. *If* $R > r$, *then* $s - s^{(0)} > 0$ *iff* $\gamma' > 0$.

2. *If* $R < r$, *then* $s - s^{(0)} > 0$ *iff* $\gamma' < 0$.

What is remarkable about this formula is that it shows that the effect of the dynamics on the optimal rule for sharing investment risk can be understood entirely in terms of the single quantity $\frac{\gamma'}{\gamma}$. In particular: if the Proposer is *more* risk averse than the Responder, then she will transfer more risk to him than under the myopic benchmark iff γ is *increasing* in her wealth share y; and, if she is *less* risk averse than him, then she will transfer more risk to him than under the myopic benchmark iff γ is *decreasing* in y.

Proof. In Proposition 4, we gave a general formula for the optimal rule for sharing investment risk, namely

$$s = \frac{-\frac{V_{PP}}{V_P} - \frac{V_{Pw}}{V_P}}{-\frac{V_{PP}}{V_P} - \frac{v''}{v'}}. \tag{46}$$

Now, using the formulae for the derivatives of V given in the proof of Lemma 12 in Appendix C, we obtain

$$-\frac{V_{PP}}{V_P} = \frac{1}{W+w}\left(\frac{R}{y} - \frac{\gamma'}{\gamma}\right) \tag{47}$$

and

$$-\frac{V_{Pw}}{V_P} = -\frac{V_{WP} - V_{PP}}{V_P} = \frac{1}{W+w}\, y\, \frac{\gamma'}{\gamma}. \tag{48}$$

Moreover, we know that

$$-\frac{v''}{v'} = \frac{r}{w} = \frac{1}{W+w}\, \frac{r}{1-y}. \tag{49}$$

Using (47-49) to substitute for $-\frac{V_{PP}}{V_P}$, $-\frac{V_{wP}}{V_P}$ and $-\frac{v''}{v'}$ in (46), and simplifying, we obtain

$$s = \frac{\left(\frac{R}{y} - \frac{\gamma'}{\gamma}\right) + y\,\frac{\gamma'}{\gamma}}{\left(\frac{R}{y} - \frac{\gamma'}{\gamma}\right) + \frac{r}{1-y}}. \tag{50}$$

Furthermore, Proposition 8 tells us that

$$s^{(0)} = \frac{\frac{R}{y}}{\frac{R}{y} + \frac{r}{1-y}}. \tag{51}$$

Subtracting (51) from (50) and simplifying yields (45). Finally, Proposition 13 tells us that $\frac{R}{y} + \frac{r}{1-y} - \frac{\gamma'}{\gamma} > 0$. Hence the sign of $s - s^{(0)}$ is determined by the sign of $(R - r)\,\frac{\gamma'}{\gamma}$. ■

12. Small and Large Wealth Shares

The previous section makes clear that most of the economic interest of our model centres on the wealth share of the Proposer, namely y. It is also easy to see that the limiting cases of our model, in which y goes to 0 and 1, are economically meaningful. When y is close to 0, the Proposer is small relative to the Responder. This means that she can get essentially unlimited insurance from him. This insurance is not free: the Responder still has to be compensated for bearing the extra risk involved. Rather, it is as if the Proposer becomes a price taker at the most favourable rate that the Responder is willing to offer. Similarly, when y is close to 1, the Proposer is large relative to the Responder. She can nonetheless make a gain by sharing some risk to him. This gain is small relative to her own wealth, but non-negligible relative to his, and has a non-negligible impact on his leverage and on the ratio of his post-termination wealth to his pre-termination wealth. In the present section, we identify the conditions under which these two limiting cases make sense. These conditions also underpin the numerical analysis of Section 13 below.

When $y = 0$, the Responder is effectively operating under autarky. His leverage is 0, and the ratio of his post-termination wealth to his pre-termination wealth is 1. No condition beyond Condition I is therefore required to ensure that the problem is well behaved from his point of view, nor can this condition be relaxed. The Proposer, on the other hand, derives considerable gains from the sharing of both investment and termination risk. These extra gains can only make her better off. There are therefore two possibilities. First, if $R < 1$, then: her utility function is unbounded above; the extra gains from risk sharing make it harder to ensure that her utility is bounded above; and Condition II therefore needs to be strengthened in order to ensure that the problem is well behaved from her point of view. Second, if $R > 1$, then: her utility function is unbounded below; the extra gains from risk sharing make it easier to ensure that her utility is bounded below; and Condition II can therefore be weakened while still ensuring that the problem is well behaved from her point of view. Either way, the new condition required is:

Condition III. $\beta > (1-R)\left((\mu - \frac{1}{2}\,R\,\sigma^2) + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{R} + \beta_r\right)$.[25]

The difference between Condition III and Condition II is that, in Condition II, the certainty-equivalent rate of return of the Proposer, namely $\mu - \frac{1}{2}\,R\,\sigma^2$, derives entirely from investment. By contrast, in Condition III, the certainty-equivalent rate of return of the Proposer has three components: (i) $\mu - \frac{1}{2}\,R\,\sigma^2$, which derives from investment; (ii) $\frac{1}{2}\,\frac{(R-r)^2\sigma^2}{R}$, which derives from sharing investment risk; and (iii) β_r, which derives from sharing termination risk.[26]

A detailed derivation of Condition III is given in Appendix E. As a corollary of this derivation, we obtain explicit formulae for $g(0)$, $z(0)$, $\gamma(0)$, $\zeta(0)$ and $\phi(0, \gamma(0))$. See that appendix for details.

When $y = 1$, the situation is analogous. This time it is the Proposer who is effectively operating under autarky. Her leverage is 0, and the ratio of her post-termination wealth to her pre-termination wealth is 1. No condition beyond Condition II is therefore required to ensure that the problem is well behaved from her point of view. The Responder, on the other hand, takes full advantage of the gains from trading with his very large counterparty. In order to ensure that these gains do not lead him to take on infinite leverage, or to accept a ratio of post-termination wealth to pre-termination wealth of 0, the following condition is required:

Condition IV. $\beta > (1-r)\left((\mu - \frac{1}{2}\,r\,\sigma^2) + \frac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{r} + \beta_R\right)$.[27]

[25] Recall that $\beta_r = (1-r)\left(\mu - \frac{1}{2}\,r\,\sigma^2\right)$.

[26] According to Proposition 7, the first-order approximation to the monetary value that the Proposer attaches to the relationship with the Responder is

$$(\mu - \tfrac{1}{2}\,R\,\sigma^2)\,W + \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{\frac{R}{W} + \frac{r}{w}}.$$

Normalizing this expression with respect to the Proposer's wealth, i.e. dividing through by W, yields

$$(\mu - \tfrac{1}{2}\,R\,\sigma^2) + \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2\,(1-y)}{(1-y)\,R + y\,r};$$

and putting $y = 0$ then yields

$$(\mu - \tfrac{1}{2}\,R\,\sigma^2) + \tfrac{1}{2}\,\frac{(R-r)^2\,\sigma^2}{R}.$$

This confirms the interpretation of the first two terms in the text. In order to confirm the interpretation of the third term, one would need to look at the second-order approximation to the monetary value that the Proposer attaches to the relationship with the Responder. This is because, unlike the myopic rule for sharing investment risk $s^{(0)}$, the myopic rule for sharing termination risk $b^{(0)}$ vanishes. Hence there is no first-order gain to the Proposer from sharing termination risk.

[27] Recall that $\beta_R = (1-R)\left(\mu - \frac{1}{2}\,R\,\sigma^2\right)$.

This condition, required to control the behaviour of the Responder, is completely analogous to Condition III, which is required to control the behaviour of the Proposer. It should be compared with Condition I.

The derivation of Condition IV is similar (but not identical) to that of Condition III. The details are given in Appendix F. As a corollary of the derivation, we obtain explicit formulae for $g(1)$, $z(1)$, $\gamma(1)$, $\zeta(1)$ and $\phi(1, \gamma(1))$.

We conclude the current section by explaining exactly how to combine Conditions I-IV in order to ensure that our problem is well behaved in the limiting cases in which y goes to 0 and 1. From an intuitive point of view, it is clear that: Condition III is stronger than Condition II when $R < 1$ (because the extra gains from risk sharing could make the Proposer's certainty-equivalent rate of return ρ too large); and Condition III is weaker than Condition II when $R > 1$ (because the extra gains from risk sharing help to ensure that the Proposer's certainty-equivalent rate of return ρ is not too small). The relevant composite condition from the point of view of the Proposer is therefore:

Condition V. $$\beta > \left\{ \begin{array}{cc} (1-R)\left(\mu - \frac{1}{2}\,R\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{R} + \beta_r\right) & \textit{if}\, R \le 1 \\ (1-R)\left(\mu - \frac{1}{2}\,R\,\sigma^2\right) & \textit{if}\, R \ge 1 \end{array} \right\}.$$

Similarly, the relevant composite condition from the point of view of the Responder is:

Condition VI. $$\beta > \left\{ \begin{array}{cc} (1-r)\left(\mu - \frac{1}{2}\,r\,\sigma^2 + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{r} + \beta_R\right) & \textit{if}\, r \le 1 \\ (1-r)\left(\mu - \frac{1}{2}\,r\,\sigma^2\right) & \textit{if}\, r \ge 1 \end{array} \right\}.$$

If both these conditions hold, then our problem is well behaved in the following senses:[28]

1. The mapping
$$y \mapsto \gamma(y) \in (0, \infty) \text{ for } y \in [0, 1]$$
is continuous. In particular, γ is bounded away from 0 and ∞.

[28] The relationship between Conditions V and VI on the one hand, and Conditions I-IV on the other, involves a subtlety. It is easy to see that the former imply the latter when either $R \geq 1$ or $r \geq 1$. For example, if $R \geq 1$, then Condition V reduces to Condition II. But Condition II is equivalent to saying that $\beta_R > 0$. In the light of this, Condition VI reduces to the stronger of Condition I and Condition IV. In particular Condition I holds, and we can go through an analogous chain of reasoning to conclude that Condition II implies Condition III. When both $R < 1$ and $r < 1$, it is not immediately obvious that Conditions V and VI (which reduce to Conditions III and IV respectively) together imply Conditions I and II. This is, however, the case.

2. From this it follows that the mappings

$$y \mapsto g(y) = \operatorname{argmax} \Phi(\cdot, y, \gamma(y)) \in \left\{ \begin{array}{ll} (-\infty, 1) & \text{if } y = 0 \\ (-y^{-1}, (1-y)^{-1}) & \text{if } y \in (0,1) \\ (-1, \infty) & \text{if } y = 1 \end{array} \right\}$$

and

$$y \mapsto \phi(y, \gamma(y)) = \max \Phi(\cdot, y, \gamma(y)) \in [0, \infty)$$

are well defined and continuous. In particular, $g(0) = g(0+)$ and $g(1) = g(1-)$.

3. The mapping

$$y \mapsto \zeta(y) \in (\phi(y, \gamma(y)), \infty) \text{ for } y \in [0,1]$$

is continuous. In particular, ζ is bounded away from 0 and ∞.

4. From this it follows that the mapping

$$y \mapsto z(y) = \frac{\beta\,(\zeta(y) - \phi(y, \gamma(y)))}{\frac{1}{2}\,(R - r)\,\sigma^2\,\gamma(y)}$$

is well defined and continuous. In particular, $z(0) = z(0+)$, $z(1) = z(1-)$, z is bounded away from 0 and ∞ if $R > r$ and z is bounded away from 0 and $-\infty$ if $R < r$.

In conclusion we emphasize that, in order to obtain existence of an optimal contract in either the long-run contracting problem or the spot-contracting problem, it is necessary and sufficient that Conditions I and II hold. However, if these are the only conditions imposed, then certain pathologies can occur. For example, if Condition II holds but Condition V fails, then $\gamma \uparrow \infty$ as $y \downarrow 0$.[29] Similarly, if Condition I holds but Condition VI fails, then $\gamma \downarrow 0$ as $y \uparrow 1$.[30] Conditions V and VI eliminate these pathologies at the cost of raising β slightly. They also simplify the numerical simulations presented in the next section.

[29] This in turn implies, for example, that the ratio of the post-termination wealth of the Proposer to the pre-termination wealth of the Proposer will go to zero as $y \downarrow 0$.

[30] This in turn implies, for example, that the ratio of the post-termination wealth of the Responder to the pre-termination wealth of the Responder will go to zero as $y \uparrow 1$.

13. NUMERICAL SOLUTIONS

The asymptotic expansions in $\frac{1}{\beta}$ have yielded approximations for the optimal risk-sharing rule s and termination payment b when β is large. Do these approximations tell us most of what we need to know about the general case, or do new phenomena arise when β is no longer large? To answer this question, we need to compute numerical solutions for the optimal contract, and to compare them with the approximations. In this section we therefore solve the system (42-43) numerically, and use the solutions to compute the optimal contract. The numerical solutions are obtained using the MatLab program *bvp4c*. We then compare the optimal contract with the predictions obtained from the asymptotic expansions. It turns out that the asymptotic expansions offer a remarkably accurate guide to the shapes of the optimal risk-sharing rule and the termination payment, and to how this shape varies with changes in the parameter values, even when β is no longer large. In sum, we find that the expansions do give a robust picture of the optimal contract.

On the basis of the asymptotic expansions, we would expect to see three different cases: a risk-tolerant case, in which both parties have a low coefficient of relative risk aversion; a risk-averse case, in which they both have a high coefficient of relative risk aversion; and a mixed case, in which the Proposer has a high and the Responder a low coefficient of relative risk aversion. On the basis of the asymptotic expansions, we would also expect to see the following boundaries between the three cases: the risk-tolerant case should appear iff $2 > R > r$, the risk-averse case iff $R > r > 2$, and the mixed case iff $R > 2 > r$.[31]

The expansions predict further that: in the risk tolerant case, the (numerically computed) optimal risk-sharing rule s will be lower than the (analytically calculated) myopic sharing rule $s^{(0)}$ (i.e. that $s - s^{(0)} < 0$ for all y); in the risk averse case, s will be higher than $s^{(0)}$; and, in the mixed case, s will be higher than $s^{(0)}$ for low y and lower than $s^{(0)}$ for high y.

They also make a rich set of predictions about the termination payment b. We shall explore this last set of predictions in more detail below. For the moment, we simply note one general prediction, namely that $b < 0$ for all y provided that the investment opportunity is sufficiently valuable (i.e. provided that $\frac{2\mu}{\sigma^2}$ is sufficiently large).

In order to make a start on exploring the predictions made by the asymptotic expan-

[31] There are actually six cases in all, since the same three cases also arise when the Proposer is less risk averse than the Responder, $R < r$. The asymptotic expansions give accurate predictions for this case too. We do not report the corresponding numerical solutions for the sake of brevity.

sions, we use the three numerical examples shown in Table 1. These examples correspond to the three main cases identified above. In all three examples: $\sigma = 0.15$, which is in line with estimates of the volatility of the US stock market; and $\beta = 0.05$, which is loosely calibrated on estimates of subjective discount rates. We have chosen the coefficients of relative risk aversion R and r from within the range that has been found in empirical studies, to ensure that the risk preferences of the Proposer and Responder are sufficiently different to generate meaningful risk sharing and therefore gains from trade.[32] The final parameter μ is then chosen in such a way that Conditions V and VI are satisfied. For instance, in the risk-tolerant example, the cost of risk is low so that μ must not be too large; and, in the risk-averse example, the cost of risk is high so that μ must not be too small.

Example	R	r	μ	σ	β
Risk-Tolerant	1	0.5	0.025	0.15	0.05
Risk-Averse	10	2.5	0.12	0.15	0.05
Mixed	8	1.3	0.10	0.15	0.05

Table 1: Parameter values for the three main examples

Figure 4 plots the analytical formula for $\frac{s^{(0)}}{1-y}$ and the numerical solution for $\frac{s}{1-y}$ as a function of y in the three examples. Note that $\frac{s^{(0)}}{1-y}$ is the fraction of the investment risk on his own wealth that the Responder bears under the myopic contract, and $\frac{s^{(0)}}{1-y} - 1$ is his leverage. The functions $\frac{s}{1-y}$ and $\frac{s}{1-y} - 1$ under the optimal contract are defined the same way. In all three examples, both $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$ increase from 1 to $\frac{R}{r}$ as y increases from 0 to 1. They take the value 1 when $y = 0$, because in that case the Responder has all the wealth, and so risk sharing with the Proposer has a negligible impact on his leverage. They are increasing in y because – from the point of view of the Responder – the opportunities for risk sharing are increasing in y and, as the more risk tolerant party, taking advantage of these opportunities means increasing his leverage. Notice that the leverage of the Responder can be quite substantial when $y = 1$: in the risk-tolerant example he is 100% levered when $y = 1$; in the risk-averse example he is 300% levered; and in the mixed example he is about 515% levered.

Figure 4 about here

[32] For examples of such studies, see Barsky, Juster, Kimball, and Shapiro (1997), Guiso and Paiella (2008), Chiappori and Paiella (2008) and Paravisini, Rappoport and Ravina (2009).

The relationship between $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$ is exactly as predicted by the dynamic correction $\frac{1}{\beta}\frac{s^{(1)}}{1-y}$: in the risk-tolerant example, the Responder takes on less risk than he would under the myopic benchmark, i.e. $\frac{s}{1-y} < \frac{s^{(0)}}{1-y}$; in the risk-averse example, the Responder takes on more risk than he would under the myopic benchmark, i.e. $\frac{s}{1-y} > \frac{s^{(0)}}{1-y}$; and, in the mixed example, the Responder takes on more risk when his wealth is high and less risk when his wealth is low, i.e. $\frac{s}{1-y} > \frac{s^{(0)}}{1-y}$ when y is low and $\frac{s}{1-y} < \frac{s^{(0)}}{1-y}$ when y is high.

The size of the difference $\frac{s}{1-y} - \frac{s^{(0)}}{1-y}$ in the Responder's leverage under respectively the optimal contract and the myopic benchmark can be seen more clearly in Figure 5. In the risk-tolerant example, this difference troughs for y around 0.92, with a value of about -0.07. At this point the myopic benchmark is around 1.85, so the difference is about -3.7% of the benchmark. A somewhat larger effect (reflecting the greater difference in risk aversion) is obtained for the risk-averse example: the difference peaks for y around 0.79, with a value of about 0.13. At this point the myopic benchmark is around 2.44, so the difference is about 5.5% of the benchmark. The largest effects are obtained for the mixed example. In this example, the difference first peaks for y around 0.70, with a value of about 0.21, and then troughs for y around 0.98, with a value of about -0.38. So, in the mixed example, the extreme differences are about 8.7% and -6.7% of the benchmark respectively.

Figure 5 about here

Turning next to the termination payment, $\frac{b}{1-y}$ is the fraction of the Responder's wealth that the Proposer pays to the Responder on termination. Figure 6 shows that, in all three examples, this fraction is decreasing in y. It is zero at $y = 0$ because the Responder has all the wealth, and any payment from the Proposer to the Responder is therefore negligible relative to the Responder's wealth. It is negative for $y > 0$ because the investment opportunity is valuable, and it is therefore the less risk-averse Responder who compensates the Proposer when it terminates. What is perhaps most striking is the sheer size of the payments made by the Responder: in the risk-tolerant example, he pays out about 62% of his wealth when $y = 1$; in the risk-averse example, he pays out about 26%; and in the mixed example, he pays out almost 94%.[33] The magnitude of these payments underlines

[33] According to equation (72) of Appendix F, we have

$$g(1) = C_r^{-1}\left(\frac{1}{r\,\beta_r}\left(\beta_r - \beta_R - \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2}{r}\right)\right) - 1.$$

The numerical solutions for $\frac{b}{1-y}$ at $y = 1$ are in excellent agreement with this formula.

the importance of sharing termination risk.

Figure 6 about here

In sum, these predictions obtained from the expansions are remarkably accurate even though the β we use is actually rather small. In particular, these simulations (and numerous unreported ones) precisely confirm the borderlines between the cases obtained from the expansions.

The other much finer predictions of the asymptotic expansions for $\frac{b}{1-y}$, to which we now turn, prove to be just as accurate. Let

$$B(0) = 2\,r - \frac{r}{R}, \quad B(1) = R + r - \frac{R}{r} \quad \text{and} \quad \overline{B} = \tfrac{1}{4}\,r\,(4+R).$$

Then the finer predictions for $\frac{b}{1-y}$ from the asymptotic expansions are as follows:

Risk-Tolerant Case

1. $\frac{b}{1-y} < 0$ for all $y \in (0,1)$ if $\frac{2\mu}{\sigma^2} > B(0)$;
2. $\frac{b}{1-y} > 0$ for all $y \in (0,1)$ if $\frac{2\mu}{\sigma^2} < B(1)$; and,
3. $\frac{b}{1-y} > 0$ for low y and $\frac{b}{1-y} < 0$ for high y if $\frac{2\mu}{\sigma^2} \in (B(1), B(0))$.

Risk-Averse Case

1. $\frac{b}{1-y} < 0$ for all $y \in (0,1)$ if $\frac{2\mu}{\sigma^2} > B(1)$;
2. $\frac{b}{1-y} > 0$ for all $y \in (0,1)$ if $\frac{2\mu}{\sigma^2} < B(0)$; and,
3. $\frac{b}{1-y} < 0$ for low y and $\frac{b}{1-y} > 0$ for high y if $\frac{2\mu}{\sigma^2} \in (B(0), B(1))$

Mixed Case

1. $\frac{b}{1-y} < 0$ for all $y \in (0,1)$ if $\frac{2\mu}{\sigma^2} > \overline{B}$;
2. $\frac{b}{1-y} > 0$ for all $y \in (0,1)$ if $\frac{2\mu}{\sigma^2} < \min\{B(0), B(1)\}$; and,
3. $b < 0$ for low y, $b > 0$ for intermediate y and $b < 0$ for high y if $\frac{2\mu}{\sigma^2} \in \big(\max\{B(0), B(1)\}, \overline{B}\big)$.[34]

[34] We omit the subcase in which $\frac{2\mu}{\sigma^2} \in (\min\{B(0), B(1)\}, \max\{B(0), B(1)\})$ since it divides into two subsubcases depending on whether $B(0) < B(1)$ or $B(0) > B(1)$.

In the context of our three examples, these predictions take the numerical form shown in Table 2.

Example	$\frac{b}{1-y} < 0$	$\frac{b}{1-y} > 0$	$\frac{b}{1-y}$ intersecting 0
Risk-Tolerant	$\frac{2\mu}{\sigma^2} > 0.498$ or (replacing σ) $\mu > 0.0056$	$\frac{2\mu}{\sigma^2} < -0.498$ or $\mu < -0.0056$	$\frac{b}{1-y} > 0$ for low y, and $\frac{b}{1-y} < 0$ for high y when $-0.0056 < \mu < 0.0056$
Risk-Averse	$\frac{2\mu}{\sigma^2} > 8.533$ or $\mu > 0.096$	$\frac{2\mu}{\sigma^2} < 4.711$ or $\mu < 0.053$	$\frac{b}{1-y} < 0$ for low y, and $\frac{b}{1-y} > 0$ for high y when $0.053 < \mu < 0.096$
Mixed	$\frac{2\mu}{\sigma^2} > 3.90$ or $\mu > 0.044$	$\frac{2\mu}{\sigma^2} < 2.4$ or $\mu < 0.027$	$\frac{b}{1-y} < 0$ for low and high y, $\frac{b}{1-y} > 0$ for intermediate y when $0.035 < \mu < 0.044$

Table 2: Predictions for $\frac{b}{1-y}$

Amazingly, Figure 7 bears out all these extremely fine predictions. Figure 7(a) plots $\frac{b}{1-y}$ for $\mu \in \{-0.01, 0, 0.01\}$ in the risk-tolerant example; Figure 7(b) plots $\frac{b}{1-y}$ for $\mu \in \{0.03, 0.08, 0.12\}$ in the risk-averse example (In this figure, β has to be chosen appropriately in order to ensure that Conditions V and VI are satisfied.); and, Figure 7(c) plots $\frac{b}{1-y}$ for $\mu \in \{0.02, 0.04, 0.05\}$ in the mixed example (In this figure, β must again be chosen appropriately in order to ensure that Conditions V and VI are satisfied.)

Figure 7 about here

What is particularly remarkable about Figure 7 is the way in which the asymptotic expansions give a detailed guide as to what patterns to expect, and precise suggestions for the parameter values that will give rise to those patterns.

14. CONCLUSION

In this paper we have analyzed an optimal risk-sharing problem in which two parties invest in a common constant-returns-to-scale risky asset. The two parties have different coefficients of relative risk aversion, and they start with different wealth endowments. We have taken out many interesting features from the model to keep the analysis tractable. In particular, we have only allowed for consumption at the end, and we have only considered an extreme bargaining situation in which one of the parties can make take-it-or-leave-it offers to the other. Within this model we have, however, been able to push the characterization of optimal risk-sharing quite far.

For example, we have used asymptotic expansions to obtain approximations to the optimal risk-sharing rules. These approximations capture in a transparent way the main trade-offs that the contracting parties face. Moreover numerical simulations confirm that the picture that they generate is qualitatively (and sometimes quantitatively) accurate.

The approximations can be decomposed into a *myopic benchmark* and a *dynamic correction*. In the case of the optimal rule s for the Responder's share in investment risk, the myopic benchmark $s^{(0)}$ is the classical ratio of the Proposer's absolute risk aversion to the sum of the Proposer's and the Responder's absolute risk aversions, namely

$$\frac{\frac{R}{W}}{\frac{R}{W}+\frac{r}{w}},$$

where R and r are the coefficients of relative risk aversion of the Proposer and the Responder, and W and w are their wealths. This formula captures the basic aspects of risk sharing. For example, the wealthier or the less risk averse the Proposer, the less the investment risk taken on by the Responder.

The myopic benchmark does not, however, capture counterparty risk. For example, if the Responder is risk neutral (i.e. if $r = 0$), then it predicts that the Responder will take on all the investment risk. However, if the Responder took on all the investment risk, then he would run out of wealth in finite time. The Proposer would thereafter not be able to obtain any insurance. In other words, the Responder's insurance capacity is finite, and the Proposer should take this into account by adjusting the risk-sharing rule to conserve it as it begins to run low.

These ideas are captured by the dynamic correction $\frac{1}{\beta}\, s^{(1)}$, which can be written in the form

$$\frac{\frac{1}{2}\, R\, r\, \sigma^2\, y^2\, (1-y)^2}{\beta\, ((1-y)\, R + y\, r)^5}\, (R-r)^3\, ((1-y)\, R + y\, r - 2)\,,$$

where β is the hazard rate of termination, σ is the volatility of investment returns and $y = \frac{W}{W+w}$ is the Proposer's share in total wealth. This formula for the dynamic correction is the product of three terms. The first is always positive, so the sign of the dynamic correction is determined by two considerations: whether the Proposer is more risk averse than the Responder, in the sense that $R > r$; and whether the average risk aversion of the two parties is large, in the sense that $(1-y)\,R + y\,r > 2$. (As each party's coefficient of relative risk aversion is weighted by the other party's share in total wealth, it is the risk aversion of the poorer party that matters most in this inequality.)

The three key predictions from this formula for the dynamic correction are then as follows. First, if both investors are fairly risk tolerant (in the sense that $R, r < 2$), then the investor who is more risk averse takes on a *larger* share of total investment risk than she would under the myopic benchmark. Indeed, the investor who is less risk averse is willing to take on risk on relatively *unfavourable* terms. So the more risk he takes on, the sooner he will run out of wealth. The optimal dynamic contract therefore transfers *less* risk to him than the optimal myopic contract.

Second, if both investors are fairly risk averse (in the sense that $R, r > 2$), then the investor who is more risk averse takes on a *smaller* share of total investment risk than she would under the myopic benchmark. This is because the investor who is less risk averse is only willing to take on risk on relatively *favourable* terms. So taking on more risk actually delays the time at which he will run out of wealth. The optimal dynamic contract therefore transfers *more* risk to him than the optimal myopic contract.

Third, if one investor is fairly risk averse and the other is fairly risk tolerant (in the sense that $R > 2 > r$ or $r > 2 > R$), then the investor who is more risk averse takes on a *smaller* share of total investment risk when her wealth is *small* and a *larger* share when her wealth is *large*. This is because, while she would like to reduce the amount of risk transferred to her risk-tolerant counterparty, the cost of bearing the extra risk herself is too high when her wealth is low.

In sum, the approximations to the optimal risk-sharing rule s we have derived capture in a relatively simple way the trade-off between getting more insurance coverage today and preserving future insurance options. Moreover, these rules are explicit and easy to apply.

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A. Notational Conventions

We adopt the convention that the timepaths of all variables in our model are continuous on the left and have limits on the right (collor). Furthermore, we denote the values of (for example) W and w at the beginning of period t by W_t and w_t; and we denote the values of W and w at the end of period t by W_{t+} and w_{t+}. One of the advantages of this convention is that it enables us to let W and w, which we sometimes use as a shorthand for W_t and w_t, denote the values of the Proposer's wealth and the Responder's wealth at the beginning of period t; and to let $W + dW$ and $w + dw$ denote the values of these variables at the end of period t. This is particularly useful when writing down and working with the Bellman equations (1) and (15), and when discussing the replication of long-run contracts in Section 7.

In practice, the two exogenous stochastic drivers of our model are z and χ. The timepaths of z are continuous, because z is a standard Wiener process. The timepaths of χ are single-step step functions, which are 0 at and to the left of T and 1 to the right of T. In particular, $\chi_T = 0$ and $\chi_{T+} = 1$.

The three most important endogenous variables are W, w and $X = W + w$. Both W and w are continuous on $[0, T]$ and constant on (T, ∞), with a single jump at T corresponding to the jump in the exogenous driver χ. Because the jumps in W and w result from a transfer between the two parties, namely the termination payment, the timepaths of X are continuous. Moreover X is constant on (T, ∞).

In view of our convention, the timepath of a variable over the stochastic interval $[0, T]$ tells us the value of the variable at the beginning of every period $t \in [0, T]$ and – by taking limits on the right – the value of the variable at the end of every period $t \in [0, T)$. (Notice that, if $T = 0$, then the interval $[0, T)$ is empty.) In particular, it tells us the initial value of the variable, i.e. the value at the beginning of period 0, but not the final value of the variable, i.e. the value at the end of period T. (If we adopted the usual convention, namely that the timepaths of variables are continuous on the right with limits on the left (corlol), then the timepath of a variable over the stochastic interval $[0, T]$ would tell us the final value of the variable at the end of period T but not the initial value of the variable at the beginning of time 0. This is distinctly odd!) We therefore need to supply the final value. We do this by requiring that the variable be defined but constant on (T, ∞), and by interpreting the value there as the final value. This convention has the advantage that the value at the end of period T can – like the values at the end of any other period

$t \in [0,T)$ – be found by taking the limit on the right.

In particular, the information conveyed by the pair (X,T) – on which the long-run contract q depends – consists of: (i) T, which is the termination time; (ii) the restriction of X to the interval $[0,T]$, which tells us the value X_t of X at the beginning of each period $t \in [0,T]$ and the value X_{t+} of X at the end of each period $t \in [0,T)$; and (iii) the value of X on the interval (T,∞), which tells us the value X_{T+} of X at the end of each period T. The final payment $q(X,T)$ can depend on all this information.

B. THE BELLMAN EQUATION OF THE RESPONDER

As noted in the text, we have

$$v(w) = C_r(\rho_r\, w) = C_r(w) + w\, C_r'(w)\, C_r(\rho_r) = C_r(w) + w\, C_r'(w)\, \psi_r.$$

Hence

$$u(w) - v(w) = C_r(w) - (C_r(w) + w\, C_r'(w)\, \psi_r) = -w\, C_r'(w)\, \psi_r.$$

Next, from the formula $v(w) = C_r(\rho_r\, w)$, we obtain

$$v'(w) = \rho_r\, C_r'(\rho_r\, w) = \rho_r\, C_r'(\rho_r)\, C_r'(w) = (1 + (1-r)\, \psi_r)\, C_r'(w),$$

and thence

$$v''(w) = (1 + (1-r)\, \psi_r)\, C_r''(w) = -\frac{r}{w}\, v'(w).$$

Next, using these expressions for $u(w) - v(w)$, $v'(w)$ and $v''(w)$, we can substitute for v in equation (2). This yields

$$\begin{aligned} 0 &= \tfrac{1}{2}\sigma^2\, w^2\, v'' + \mu\, w\, v' + \beta\, (u(w) - v) \\ &= \left(\mu - \tfrac{1}{2}\, r\, \sigma^2\right) w\, v' - \beta\, w\, C_r'(w)\, \psi_r \\ &= \left(\left(\mu - \tfrac{1}{2}\, r\, \sigma^2\right) (1 + (1-r)\, \psi_r) - \beta\, \psi_r\right) w\, C_r'(w) \\ &= \left(\mu - \tfrac{1}{2}\, r\, \sigma^2 - \beta_r\, \psi_r\right) w\, C_r'(w), \end{aligned}$$

where $\beta_r = \beta - (1-r)\,(\mu - \frac{1}{2}\, r\, \sigma^2)$. Dividing through by $w\, C_r'(w)$ then yields

$$\psi_r = \frac{\mu - \frac{1}{2}\, r\, \sigma^2}{\beta_r}.$$

Finally, it follows directly from this last formula that $1 + (1 - r)\,\psi_r = \frac{\beta}{\beta_r}$. Hence $v'(w) = \frac{\beta}{\beta_r}\,C_r'(w)$.

C. Proof of Lemma 12

Elementary calculations show that

$$\begin{aligned}
V_W &= (\gamma - y\,(\gamma - 1 - (1-R)\,\psi))\,C_R'(W),\\
V_P &= \gamma\,C_R'(W),\\
V_{WW} &= \left((1-y)^2\,\gamma' - \frac{R}{y}\,\gamma + R\,y\,(\gamma - 1 - (1-R)\,\psi)\right)\frac{C_R'(W)}{W+w},\\
V_{WP} &= \left((1-y)\,\gamma' - \frac{R}{y}\,\gamma\right)\frac{C_R'(W)}{W+w},\\
V_{PP} &= \left(\gamma' - \frac{R}{y}\,\gamma\right)\frac{C_R'(W)}{W+w}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\frac{U(W - (W+w)\,b) - V}{(W+w)\,C_R'(W)} &= \frac{W}{W+w}\,\frac{C_R(W - (W+w)\,b) - C_R(\rho\,W)}{W\,C_R'(W)}\\
&= \frac{W}{W+w}\left(C_R\!\left(1 - \frac{W+w}{W}\,b\right) - C_R(\rho)\right)\\
&= y\,(C_R(1 - (1-y)\,g) - \psi)
\end{aligned}$$

and

$$\begin{aligned}
\frac{V_P}{v'}\,\frac{u(w + b\,(W+w)) - v}{(W+w)\,C_R'(W)} &= \frac{\gamma\,C_R'(W)}{\gamma_r\,C_r'(w)}\,\frac{C_r(w + (W+w)\,b) - C_r(\rho_r\,w)}{(W+w)\,C_R'(W)}\\
&= \frac{\gamma}{\gamma_r}\,\frac{w}{W+w}\,\frac{C_r(w + (W+w)\,b) - C_r(\rho_r\,w)}{w\,C_r'(w)}\\
&= \frac{\gamma}{\gamma_r}\,\frac{w}{W+w}\left(C_r\!\left(1 + \frac{W+w}{w}\,b\right) - C_r(\rho_r)\right)\\
&= \frac{\gamma}{\gamma_r}\,(1-y)\,(C_r(1 + y\,g) - \psi_r)\,.
\end{aligned}$$

Substituting into equation (24), dividing through by $(W+w)\,C_R'(W)$, taking advantage of the notation Φ, putting $s = (1-y)\,(1+y\,z)$, collecting terms in β, z and z^2, dividing through by $\gamma\,\sigma^2\,y\,(1-y)$ and rearranging therefore yields equation (37). Finally, we have

$$\gamma = 1 + (1-R)\,\psi + y\,\psi'$$

and

$$\psi = \psi_R + (1-y)\,\gamma_R\,\zeta.$$

Differentiating the latter equation to get ψ' in terms of ζ', substituting in the former equation and rearranging, we obtain equation (38).

D. PROOF OF LEMMA 14

As noted in the proof of Proposition 13, equation (37) can be written equivalently as

$$0 = \max_{z\in\mathbb{R}} \left\{ -\frac{\beta}{\gamma\,\sigma^2}\,(\zeta-\phi) + (R-r)\,z - \tfrac{1}{2}\left((1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}\right) z^2 \right\}.$$

Since the maximand is quadratic in z, and since the coefficient of z^2 is strictly negative by Proposition 13, we can calculate the maximum explicitly. Doing so yields

$$0 = -\frac{\beta}{\gamma\,\sigma^2}\,(\zeta-\phi) + \frac{\frac{1}{2}\,(R-r)^2}{(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}}. \tag{52}$$

Rearranging this equation, we obtain (42). Furthermore, the maximum is attained when

$$z = \frac{R-r}{(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}}. \tag{53}$$

Eliminating the expression $(1-y)\,R + y\,r - \frac{y\,(1-y)\,\gamma'}{\gamma}$ from (52) and (53), we obtain (44). Finally, (43) is the same as (38).

E. THE BOUNDARY AT $y = 0$

Putting $y = 0$ in equation (42) and rearranging yields

$$\zeta(0) - \phi(0,\gamma(0)) = \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2}{R\,\beta}\,\gamma(0). \tag{54}$$

Similarly, putting $y = 0$ in equation (43) and rearranging yields

$$1 + (1 - R)\,\zeta(0) = \frac{\gamma(0)}{\gamma_R}. \tag{55}$$

We also have

$$\phi(0, \gamma(0)) = \max_{g \in (-\infty, 1)} \left\{ C_R(1 - g) + \frac{\gamma(0)}{\gamma_r}\, g \right\} \tag{56}$$

Now, if we denote the maximizer in equation (56) by $g(0)$, then the first-order condition yields

$$\frac{\gamma_r}{\gamma(0)}\, C_R'(1 - g(0)) = 1, \tag{57}$$

and equation (65) itself becomes

$$\phi(0, \gamma(0)) = C_R(1 - g(0)) + \frac{\gamma(0)}{\gamma_r}\, g(0). \tag{58}$$

Hence

$$\begin{aligned} 1 + (1 - R)\,\phi(0, \gamma(0)) &= 1 + (1 - R)\,C_R(1 - g(0)) + (1 - R)\,\frac{\gamma(0)}{\gamma_r}\, g(0) \\ &= (1 - g(0))\,C_R'(1 - g(0)) + (1 - R)\,\frac{\gamma(0)}{\gamma_r}\, g(0) \\ &= (1 - g(0))\,\frac{\gamma(0)}{\gamma_r} + (1 - R)\,\frac{\gamma(0)}{\gamma_r}\, g(0) \\ &= \frac{\gamma(0)}{\gamma_r}\,(1 - R\, g(0)) \end{aligned} \tag{59}$$

(from (58), from the properties of C_R, from (57) and simplifying respectively).

Next, we note that (54,55,59) is a system of three linear equations in the three unknowns

$$\frac{1 + (1 - R)\,\zeta(0)}{\gamma(0)}, \quad \frac{1 + (1 - R)\,\phi(0, \gamma(0))}{\gamma(0)} \quad \text{and} \quad g(0).$$

Solving this system, we obtain

$$\frac{1+(1-R)\,\zeta(0)}{\gamma(0)} = \frac{1}{\gamma_R}, \tag{60}$$

$$\frac{1+(1-R)\,\phi(0,\gamma(0))}{\gamma(0)} = \frac{1}{\gamma_R} - (1-R)\frac{\frac{1}{2}(R-r)^2\sigma^2}{R\,\beta}, \tag{61}$$

$$g(0) = \frac{\gamma_r}{R}\left(\frac{1}{\gamma_r} - \frac{1}{\gamma_R} + (1-R)\,\frac{(R-r)^2\,\sigma^2}{2\,R\,\beta}\right). \tag{62}$$

Putting $\gamma_R = \frac{\beta}{\beta_R}$, $\gamma_r = \frac{\beta}{\beta_r}$ and then $\beta_R = \beta-(1-R)\,(\mu-\frac{1}{2}\,R\,\sigma^2)$ in (62), and rearranging, it follows that $1-g(0)>0$ iff $\beta > (1-R)\left((\mu-\frac{1}{2}\,R\,\sigma^2)+\frac{1}{2}\,\frac{(R-r)^2\sigma^2}{R}+\beta_r\right)$, which is Condition III.

Next, if $R \neq 1$, then we can obtain explicit expressions for all of the quantities $z(0)$, $g(0)$, $\gamma(0)$, $\zeta(0)$ and $\phi(0,\gamma(0))$. First, putting $y=0$ in equation (44), and then applying (54), yields

$$z(0) = \frac{\beta\,(\zeta(0)-\phi(0,\gamma(0)))}{\frac{1}{2}\,(R-r)\,\sigma^2\,\gamma(0)} = \frac{R-r}{R}.$$

Second, we note that we already have an explicit expression for $g(0)$, namely (62). Third, we can recover $\gamma(0)$ directly from (57) in the form

$$\gamma(0) = \gamma_r\,C_R'(1-g(0)),$$

where we have not substituted in for $g(0)$. Fourth, (60) and (61) yield

$$\zeta(0) = \frac{\gamma(0)-\gamma_R}{(1-R)\,\gamma_R}$$

and

$$\phi(0,\gamma(0)) = \frac{\gamma(0)-\gamma_R}{(1-R)\,\gamma_R} - \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2}{R\,\beta}\,\gamma(0),$$

where we have not substituted in for $\gamma(0)$.

Finally, if $R=1$, then much simpler formulae obtain, namely $z(0)=1-r$, $g(0)=1-\gamma_r$, $\gamma(0)=1$, $\zeta(0)=\frac{1-\gamma_r}{\gamma_r}+\frac{\frac{1}{2}\,(1-r)^2\,\sigma^2}{\beta}+\log(\gamma_r)$ and $\phi(0,\gamma(0))=\frac{1-\gamma_r}{\gamma_r}+\log(\gamma_r)$.

F. The Boundary at $y = 1$

Putting $y = 1$ in equation (42) and rearranging yields

$$\zeta(1) - \phi(1, \gamma(1)) = \frac{\frac{1}{2}(R-r)^2 \sigma^2}{r\,\beta}\,\gamma(1). \tag{63}$$

Similarly, putting $y = 1$ in equation (43) and rearranging yields

$$\frac{\zeta(1) - 1}{\gamma(1)} = -\frac{1}{\gamma_R}. \tag{64}$$

We also have

$$\phi(1, \gamma(1)) = \max_{g \in (-1, \infty)} \left\{ -g + \frac{\gamma(1)}{\gamma_r}\, C_r(1+g) \right\}. \tag{65}$$

Now, if we denote the maximizer in equation (65) by $g(1)$, then the first-order condition yields

$$\frac{\gamma(1)}{\gamma_r}\, C_r'(1 + g(1)) = 1, \tag{66}$$

and equation (65) itself becomes

$$\phi(1, \gamma(1)) = -g(1) + \frac{\gamma(1)}{\gamma_r}\, C_r(1 + g(1)). \tag{67}$$

Hence

$$\begin{aligned}
\frac{\gamma(1)}{\gamma_r} + (1-r)\,\phi(1,\gamma(1)) &= -(1-r)\,g(1) + \frac{\gamma(1)}{\gamma_r}\Big(1 + (1-r)\,C_r(1+g(1))\Big) \\
&= -(1-r)\,g(1) + \frac{\gamma(1)}{\gamma_r}(1+g(1))\,C_r'(1+g(1)) \\
&= -(1-r)\,g(1) + (1+g(1)) \\
&= 1 + r\,g(1)
\end{aligned}$$

(from (67), from the properties of C_R, from (66) and simplifying respectively), or

$$\frac{1}{\gamma_r} + (1-r)\,\frac{\phi(1,\gamma(1)) - 1}{\gamma(1)} = r\,\frac{1 + g(1)}{\gamma(1)} \tag{68}$$

on rearranging.

Next, we note that (63,64,68) is a system of three linear equations in the three unknowns

$$\frac{\zeta(1)-1}{\gamma(1)}, \quad \frac{\phi(1,\gamma(1))-1}{\gamma(1)} \quad \text{and} \quad \frac{1+g(1)}{\gamma(1)}.$$

Solving this system, we obtain

$$\frac{\zeta(1)-1}{\gamma(1)} = -\frac{1}{\gamma_R}, \tag{69}$$

$$\frac{\phi(1,\gamma(1))-1}{\gamma(1)} = -\frac{1}{\gamma_R} - \frac{\frac{1}{2}(R-r)^2\sigma^2}{r\,\beta}, \tag{70}$$

$$\frac{1+g(1)}{\gamma(1)} = \frac{1}{r}\left(\frac{1}{\gamma_r} - (1-r)\left(\frac{1}{\gamma_R} + \frac{\frac{1}{2}(R-r)^2\sigma^2}{r\,\beta}\right)\right). \tag{71}$$

Furthermore, putting $\gamma_R = \frac{\beta}{\beta_R}$, $\gamma_r = \frac{\beta}{\beta_r}$ and then $\beta_r = \beta - (1-r)\,(\mu - \frac{1}{2}\,r\,\sigma^2)$ in (71), and rearranging, it follows that $1+g(1) > 0$ iff $\beta > (1-r)\,\left((\mu - \frac{1}{2}\,r\,\sigma^2) + \frac{1}{2}\,\frac{(R-r)^2\sigma^2}{r} + \beta_R\right)$, which is Condition IV.

Next, we can obtain explicit expressions for all of the quantities $z(1)$, $g(1)$, $\gamma(1)$, $\zeta(1)$ and $\phi(1,\gamma(1))$. First, putting $y=1$ in equation (44), and then applying (54), yields

$$z(1) = \frac{\beta\,(\zeta(1) - \phi(1,\gamma(1)))}{\frac{1}{2}(R-r)\,\sigma^2\,\gamma(1)} = \frac{R-r}{r}.$$

Second, (67) can be rearranged to yield

$$\frac{1}{\gamma_r}\,C_r(1+g(1)) = \frac{1+g(1)}{\gamma(1)} + \frac{\phi(1,\gamma(1))-1}{\gamma(1)}.$$

Hence

$$\begin{aligned} g(1) &= C_r^{-1}\!\left(\gamma_r\left(\frac{1+g(1)}{\gamma(1)} + \frac{\phi(1,\gamma(1))-1}{\gamma(1)}\right)\right) - 1 \\ &= C_r^{-1}\!\left(\frac{\gamma_r}{r}\left(\frac{1}{\gamma_r} - \frac{1}{\gamma_R} - \frac{\frac{1}{2}(R-r)^2\sigma^2}{r\,\beta}\right)\right) - 1. \end{aligned} \tag{72}$$

Third, (66) yields

$$\gamma(1) = \frac{\gamma_r}{C_r'(1+g(1))},$$

where we have not substituted in for $g(1)$. Fourth, (69) and (70) yield

$$\zeta(1) = \frac{\gamma_R - \gamma(1)}{\gamma_R}$$

and

$$\phi(1, \gamma(1)) = \gamma(1) \left(\frac{1}{\gamma(1)} - \frac{1}{\gamma_R} - \frac{\frac{1}{2}\,(R-r)^2\,\sigma^2}{r\,\beta} \right),$$

where we have not substituted in for $\gamma(1)$.

Figure 1: $\frac{1}{\beta} s^{(1)}$

Figure 1(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\sigma = 0.15$, $\beta = 0.05$)

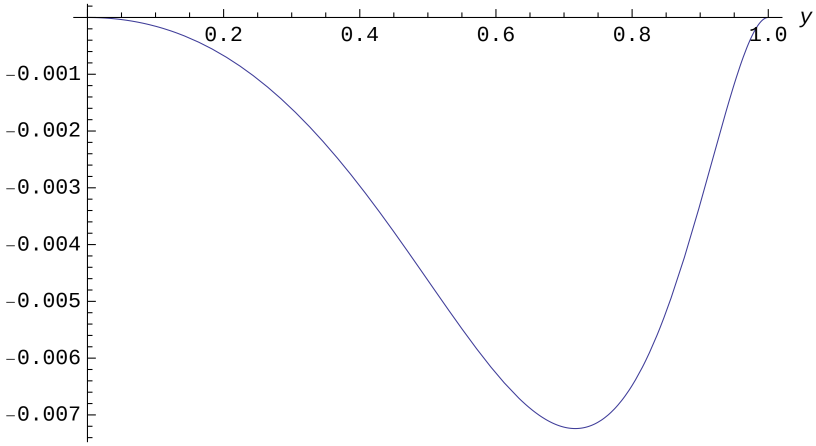

Figure 1(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\sigma = 0.15$, $\beta = 0.05$)

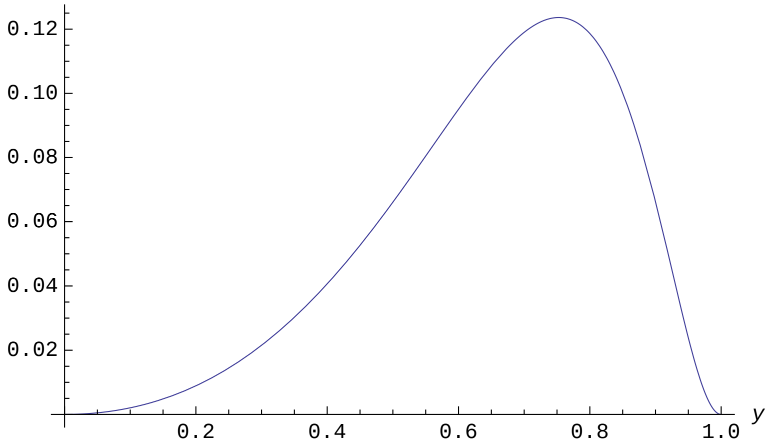

Figure 1(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\sigma = 0.15$, $\beta = 0.05$)

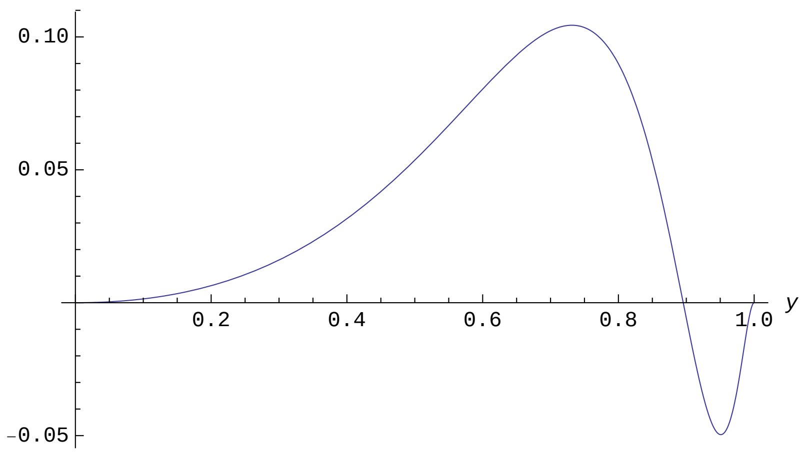

Figure 2: $s - s^{(0)}$, for comparison with $\frac{1}{\beta} s^{(1)}$

Figure 2(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

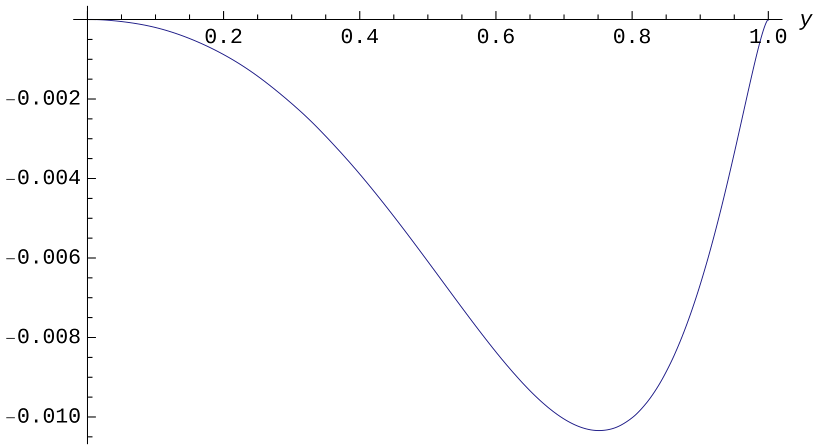

Figure 2(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

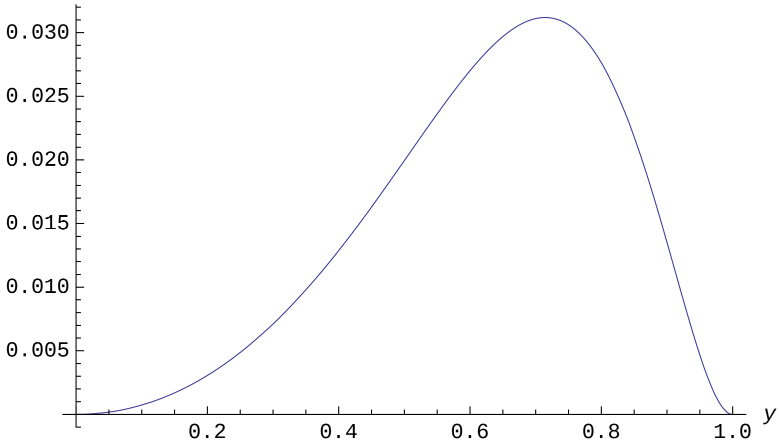

Figure 2(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)

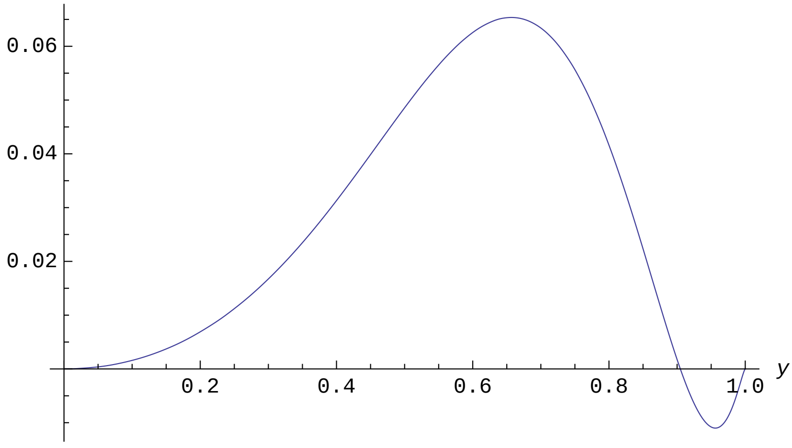

Figure 3: *B*

Figure 3(a): the Risk–Tolerant Case (with $R = 1$, $r = 0.5$)

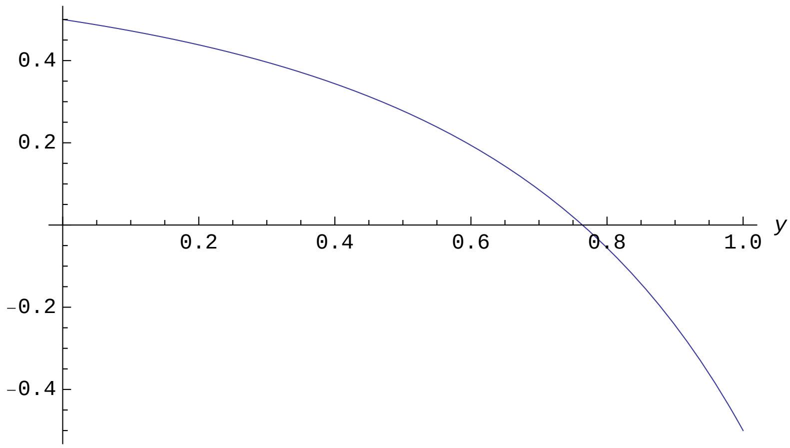

Figure 3(b): the Risk–Averse Case (with $R = 10$, $r = 2.5$)

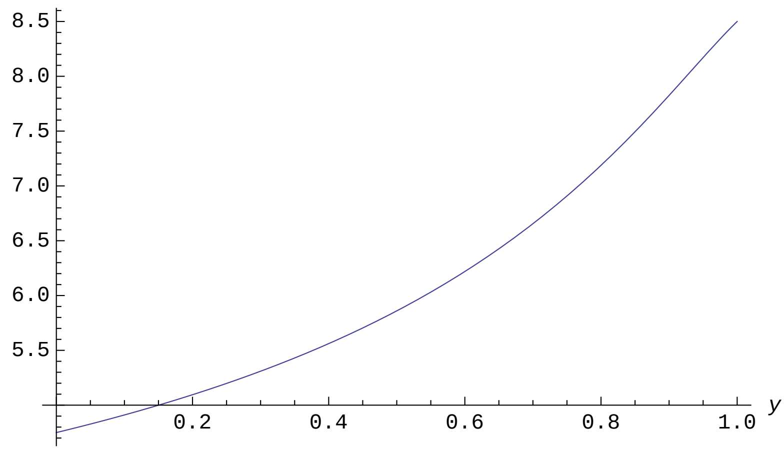

Figure 3(c): the Mixed Case (with $R = 8$, $r = 1.3$)

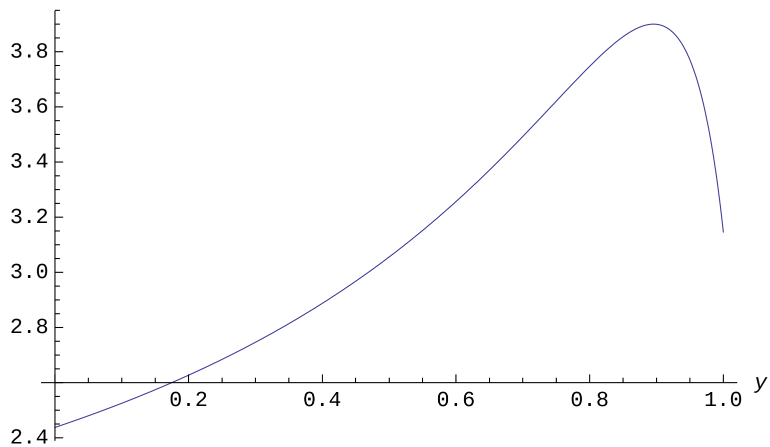

Figure 4: $\frac{s^{(0)}}{1-y}$ and $\frac{s}{1-y}$

Figure 4(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

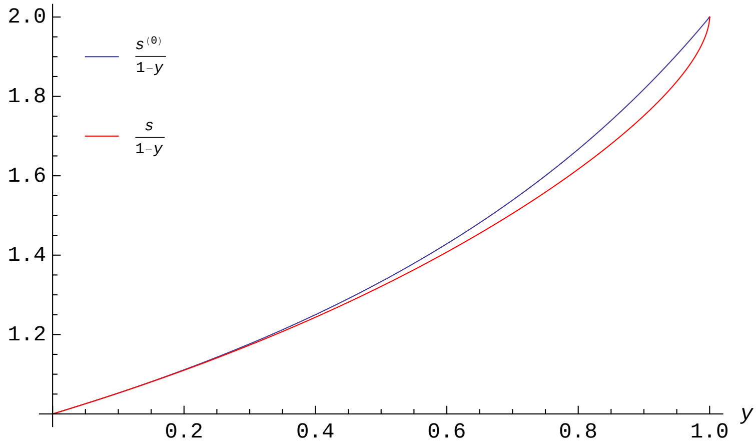

Figure 4(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

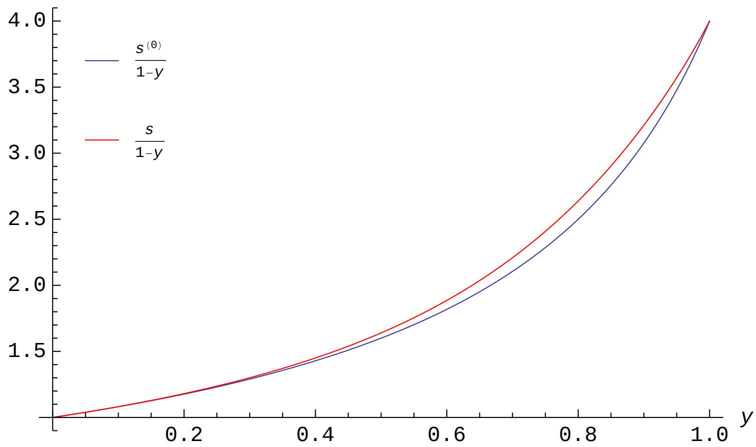

Figure 4(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)

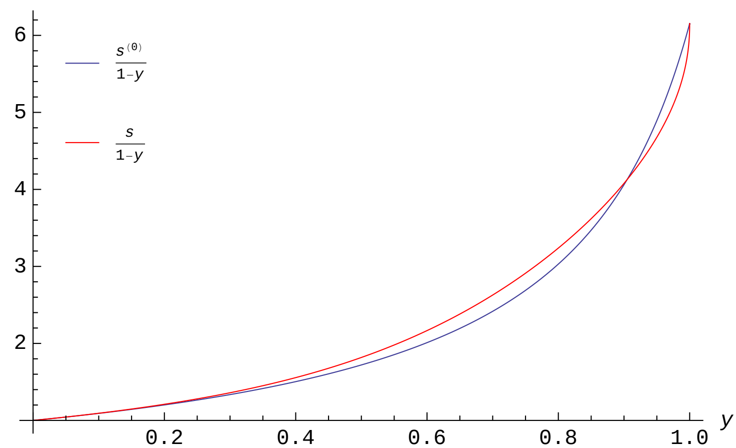

Figure 5: $\frac{s}{1-y} - \frac{s^{(0)}}{1-y}$

Figure 5(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

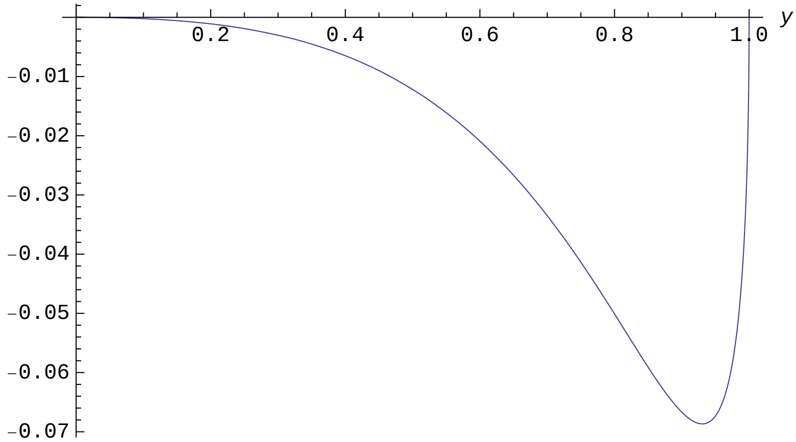

Figure 5(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

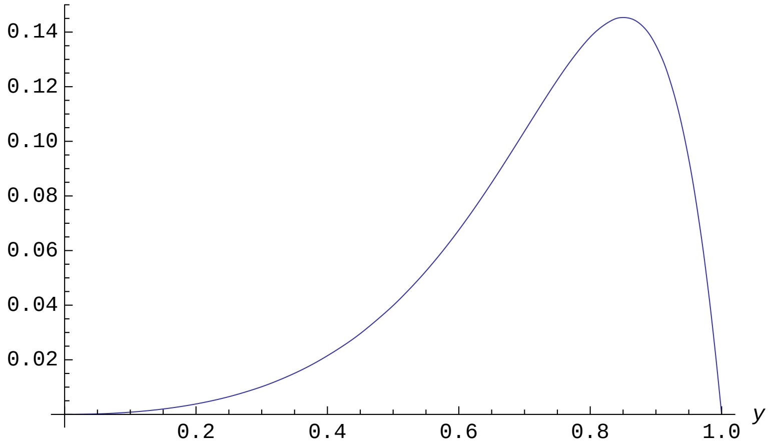

Figure 5(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)

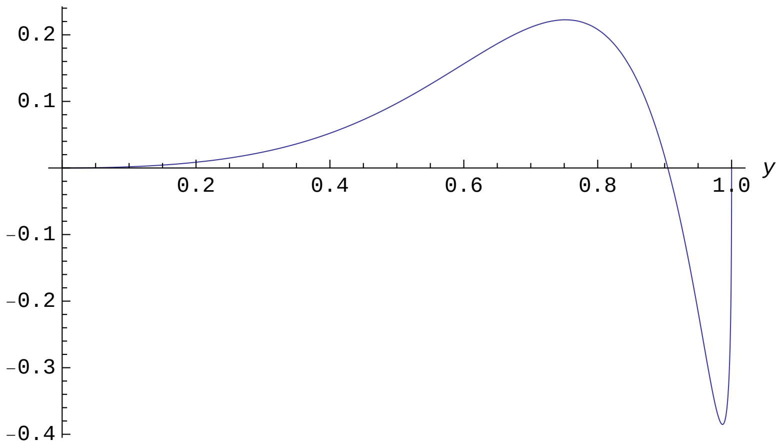

Figure 6: $\frac{b}{1-y}$

Figure 6(a): the Risk-Tolerant Case (with $R = 1$, $r = 0.5$, $\mu = 0.025$, $\sigma = 0.15$, $\beta = 0.05$)

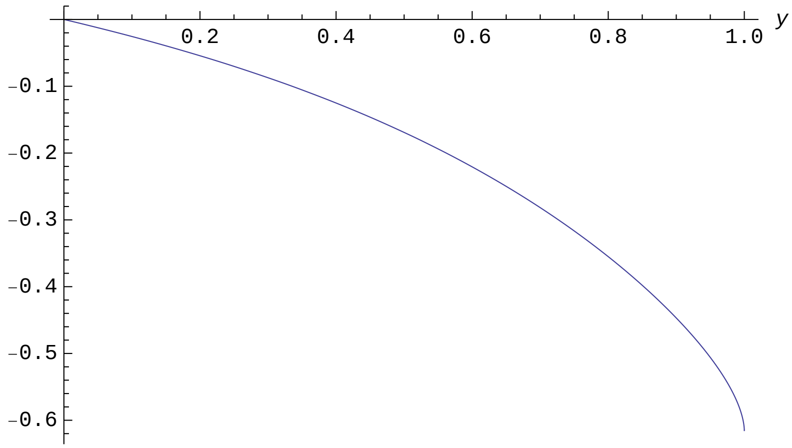

Figure 6(b): the Risk-Averse Case (with $R = 10$, $r = 2.5$, $\mu = 0.12$, $\sigma = 0.15$, $\beta = 0.05$)

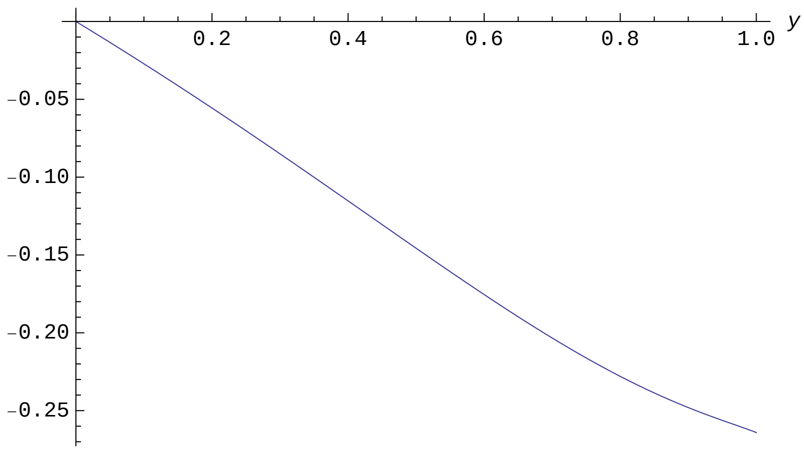

Figure 6(c): the Mixed Case (with $R = 8$, $r = 1.3$, $\mu = 0.10$, $\sigma = 0.15$, $\beta = 0.05$)

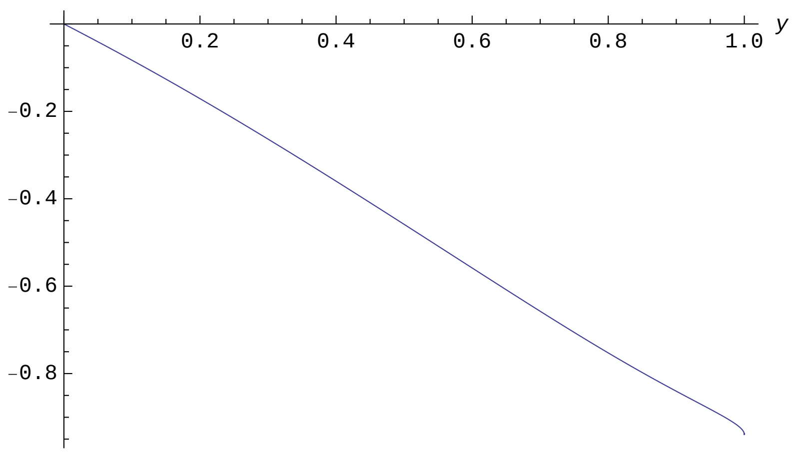

Figure 7: $\frac{b}{1-y}$

Figure 7(a): the Risk-Tolerant Case with $R = 1$, $r = 0.5$, $\sigma = 0.15$,

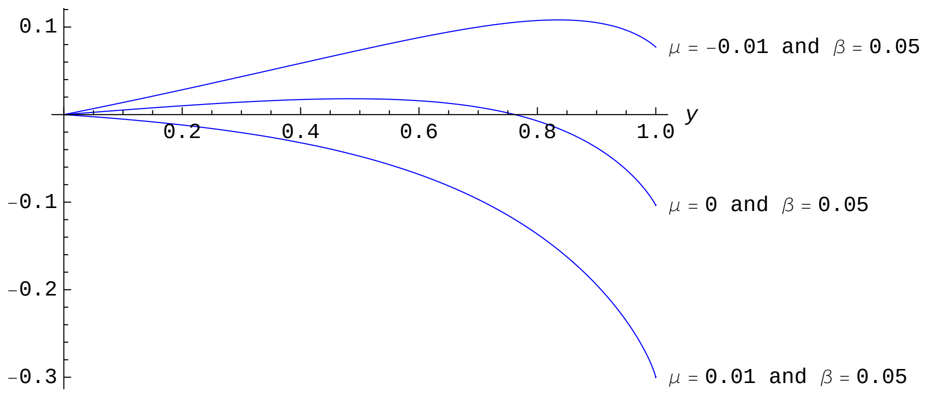

Figure 7(b): the Risk-Averse Case with $R = 10$, $r = 2.5$, $\sigma = 0.15$,

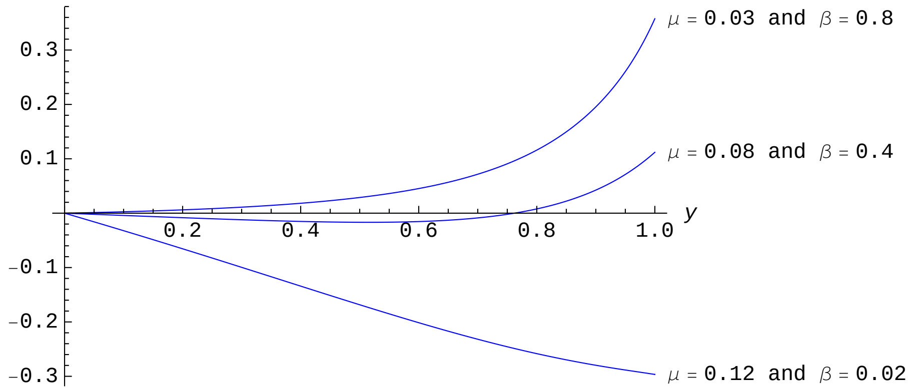

Figure 7(c): the Mixed Case with $R = 8$, $r = 1.3$, $\sigma = 0.15$,