Contractible Signals and Security Design*

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Abstract

An entrepreneur has ongoing project whose random payoff will be realized at some future, terminal date. At an interim date the entrepreneur will learn about the profitability of the project. This information is known only to him. At some earlier initial date the entrepreneur decides to design and issue a security to hedge the risks his project is exposed to. Should the security payoff correlate with his own private information if this is contractible? We prove that in general it should and that equity can never replicate the allocation of the optimal linear security. These results stand in contrast with recent results by Rahi (1996) and DeMarzo and Duffie (1999).

I. INTRODUCTION

An entrepreneur has ongoing project whose random payoff will be realized at some future, terminal date. At an interim date the entrepreneur will learn about the profitability of the project. This information is known only to him. At some earlier initial date the entrepreneur decides to design and issue a security to hedge the risks his project is exposed to. Should the security payoff correlate with his own private information if this is contractible? This paper is concerned with this classic question in the literature on security design.

The choice of the particular security issued has important consequences, and the problem is by now well known. The entrepreneur is faced with uninformed outside investors who are sensitive to the informational content of the security, and he is faced with the standard "lemons problem." This lemons problem interferes with his own hedging needs. The entrepreneur will naturally weight the adverse selection costs that an informationally sensitive security entails against the exposure he will have to bear if he chooses a design with no private information whatsoever.

This set up has been applied in the literature to the problem faced by a corporation that chooses the design of the security to spread the risk of his ongoing projects. The corporation learns about several aspects of its profitability before outside investors, and he is concerned about how the design of the security affects the depth of the secondary market where the security is to be traded. The entrepreneur prefers deeper markets to improve hedging possibilities, but deeper markets will leave him exposed to considerable amounts of risk. Our goal is to characterize in a standard set up the securities that are likely to arise, and to ask ourselves whether they bear any resemblance to the securities we actually see. We consider both the case where the entrepreneur is a monopolist, and takes into account the impact his trades have on the price of the traded security, and a second case where he behaves competitively and ignores the price impact of his trade.

This work is closely related to advances on the literature on security design, but in particular to recent work by Demange and Laroque (1995), Rahi (1996) and DeMarzo and Duffie (1999). Those acquainted with the second paper will immediately notice the heavy debt the present paper owes to Rahi's work. We share with all of them the timing of security design, information learning, and trading, but also have substantial differences on the nature of the private information at the entrepreneur's disposal. Like Rahi (1996) we concentrate on both the problem of the monopolistic entrepreneur and that of the price taker. Instead, Demange and Laroque (1995) investigate only the problem of a competitive entrepreneur, whereas DeMarzo and Duffie (1999) focus on the monopolistic entrepreneur.

In the presence of noise traders, Demange and Laroque (1995) show it is in the best interest of the informed entrepreneur to make the payoff of security sensitive to private information. Rahi (1996) rightly points out that the conclusion reached by the work of Demange and Laroque (1995) is indeed sensitive to the assumption on the existence of noise traders. He writes a formal model where all outside investors are rational. He then goes to prove that the optimal security design is independent of any private information the entrepreneur may have whether he behaves monopolistically or competitively. Rahi's result is indeed more surprising that it may seem at first. The entrepreneur's private information may be an important component of the risks he is exposed to. Still, he is left with no choice but to completely exclude private information from the optimal design. The optimal security design in the context of the model presented by Demange and Laroque (1995) yields a partially revealing rational expectations equilibrium supported by the presence of noise traders. Rahi's entrepreneur prefers a fully revealing security under any circumstance. This result is confirmed in DeMarzo and Duffie (1999), albeit in a rather different model. DeMarzo and Duffie (1999) though note that the contractibility of the private signal is, in general, too strong an assumption, and that typically it is advantageous to add a marginal amount of information sensitivity to the design. These findings beg the question of whether noise trading is necessary to support informationally sensitive securities given the contractibility of the private signal.

To answer this question we write a slightly richer model than the one considered by Rahi

(1996) but that includes it as a special case. More precisely we assume that the entrepreneur has private information about *several* aspects of his project, and, to make our results stronger, we assume that the private information is indeed contractible. We prove that in this case the entrepreneur chooses an informationally sensitive security that supports a partially revealing equilibrium independently of whether he behaves monopolistically or competitively. We further identify the sources of private information the entrepreneur includes in the security as those that are correlated with the project's payoff realization. Any private information that is uncorrelated with the project's payoff realization is excluded from the design of the security. Unlike Rahi (1996), equity does not support the allocation of the optimal linear security this paper concentrates on.

The sharp difference between our results and those of Rahi (1996), hinges on the ability of our entrepreneur to achieve a non revealing rational expectations equilibrium through the introduction of *several* sources of private information in the security design that are correlated with the project's payoff. This commits to the entrepreneur to trade private information about his security's payoff in fixed proportion limiting the amount of informational trade he can undertake. Judiciously mixing private information about the security's payoff with private information uncorrelated with the security's payoff can indeed commit the entrepreneur not to trade "too strongly" on his privileged information. But this restrain comes at the cost of considerable exposure to the risks associated with the private information. Private extraneous information does not compensate the agent for this commitment, as it is uncorrelated with the project's payoff realization. As a result it is not beneficial to include private information about the project's payoff together with extraneous private information. When both sources of private information are correlated with the project's information this commitment not to trade to strongly on private information is compensated by the hedging it brings along making the introduction of private information worthwhile.

We show then that the optimal security design would support a non revealing rational expectations equilibrium and include contractible private information, even in the absence of irrational noise traders. This result is independent of whether the entrepreneur behaves monopolistically or competitively.

The paper proceeds as follows. In the second section we present the model. The third section is concerned with the case of the monopolistic entrepreneur, whereas the fourth deals with the case of a price taker. The last section concludes. All proofs are to be found in the appendix. Throughout, and for ease of comparison, we have tried to stay as close as possible to the notation of Rahi's paper.

II. THE MODEL

All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. Throughout we denote by $V[\chi]$, the unconditional variance of random variable χ, and by $V[\chi|\eta]$, the conditional variance of χ given η. Similarly $C[\chi, \xi]$ denotes the unconditional covariance between χ and ξ, and $C[\chi, \xi|\eta]$ the conditional covariance.

There are two agents, with von Neumann-Morgensten utility functions displaying constant absolute risk aversion. We refer to the first agent as the entrepreneur and we assume that he has an exogenously determined project which yields an stochastic payoff e at some terminal date. The second agent is simply an outside investor. There are three dates in the economy. At the ex-ante stage, date 0, the entrepreneur designs an security with payoff f, and for this reason we sometimes refer to the entrepreneur as the issuer. At the interim stage, date 1, the entrepreneur receives some private signal about the payoff of the project. Right after the signal is received trading between the entrepreneur and the uninformed outside investor takes place. At the final stage, date 2, payoffs are realized and all signals become public.

More precisely then the entrepreneur has the utility function $E[-\exp(-r_i W)]$ over the consumption W at date 2, is endowed with an asset that has a payoff e at date 2, and receives a vector S of information signals at date 1. The outside investor has the utility function $E[-\exp(-r_u W)]$ over the consumption W at date 2, but has neither endowment of an asset nor an information signal. Let θ_i and θ_u be the positions of informed and uninformed agents respectively. It follows then that the terminal wealth of each agent is given by:

$$W_i = e + \theta_i (f - p) \text{ and } W_u = \theta_u (f - p),$$

where p is the asset price.[1]

For tractability we assume that e is given by a product of two random variables x and z, namely $e = xz$, and without loss of generality, we assume that

$$z = \zeta + s. \tag{1}$$

We are ready now to describe the nature of the private information. The vector of information signals S that the entrepreneur receives at date 1 is given by $S = (s, x, y)$, where y is any other private information the entrepreneur may have at date 1 that is uncorrelated to the project's payoff realization. Because all signals are made public at the terminal date the security payoff may be made contingent in all of them if it is in the interest of the entrepreneur. Then any security f can be written in the form

$$\begin{aligned} f &= \overline{f} + a\zeta + a_s s + bx + cy + d\epsilon \\ &= \overline{f} + \mathbf{v} \cdot \phi \end{aligned} \tag{2}$$

where ϵ is any other random variable whose realization occurs at the second date, and $\overline{f}$, a, a_s, b, c, and d are constants. Clearly $\mathbf{v} = \begin{bmatrix} a & a_s & b & c & d \end{bmatrix}^\top$ and $\phi = \begin{bmatrix} \zeta & s & x & y & \epsilon \end{bmatrix}^\top$.

All underlying random variables are independently and are normally distributed i.e.,

$$\phi \sim N \left[\mathbf{0}, diag(V[\zeta], V[s], V[x], V[y], V[\epsilon]) \right], \tag{3}$$

where $\mathbf{0}$ is a vector of zeroes and $diag(V[\zeta], V[s], V[x], V[y], V[\epsilon])$ is a matrix whose diagonal elements are $V[\zeta]$, $V[s]$, $V[x]$, $V[y]$, and $V[\epsilon]$.

[1] It is assumed that the asset is either a futures contract or that there exists a riskless asset whose rate of return is normalized to one.

The uninformed outside investor is assumed to represent a large number of identical investors and hence behaves competitively. He has rational expectations and uses the observed market price p to update beliefs about the payoff of the traded security. The informed entrepreneur understands this and designs the security accordingly. We label a security design optimal if there is no other that would yield a higher ex-ante (at date 0) utility level to its designer. We make the following standard assumption to guarantee a well defined problem:

Assumption 1: $r_i^2 V[z]V[x] < 1$.

The main difference between the set up we propose here and the one advanced by Rahi (1996) is the fact that the issuer receives also a signal about the random variable z. As shown in equation (1) the entrepreneur receives a signal s, one component of variable z. Our model nests Rahi's by simply making $V[s] = 0$, and throughout we discuss the implications of such an assumption.

III. THE MONOPOLISTIC ENTREPRENEUR

We first consider the case where the entrepreneur, after she has created a security with payoff f, trades the security with the outside investor monopolistically, that is taking into account the impact of his asset position on the price. Attention will be restricted to equilibrium with a linear price function of the form:[2]

$$p(\theta_i) = \overline{p} + \delta\theta_i \qquad (\overline{p}, \delta) \in R^2, \tag{4}$$

where $\frac{1}{\delta}$ is a measure of market depth. The entrepreneur will take into account how does his choice of design affect market depth.

At date 1, the informed entrepreneur solves the following optimization problem:

$$\max_{\theta_i} E[-\exp(-r_i W_i)|S],$$

[2] See Bhattacharya et al. (1995) and Rahi (1996) for justification of this assumption.

where

$$W_i = e + \theta_i[f - p(\theta_i)]. \tag{5}$$

Since conditional on $S = (s, x, y)$, W_i is normally distributed, by the usual argument, this optimization problem is equivalent to the following one:

$$\max_{\theta_i} E[W_i|S] - \frac{r_i}{2} V[W_i|S].$$

subject to (5) .

The first-order condition is

$$E[f|S] - \overline{p} - 2\delta\theta_i - r_i \{C[f, z|S]x + V[f|S]\theta\} = 0,$$

and the second-order condition is

$$2\delta + r_i V[f|S] > 0. \tag{6}$$

If $2\delta + r_i V[f|S] < 0$, this optimization problem has no solution, and there exists no equilibrium. If $2\delta + r_i V[f|S] = 0$, it has a solution if and only if $E[f|S] - \overline{p} - r_i C[f, z|S]x = 0$ for all S, and $\theta_i = 0$ is an optimal solution. If $2\delta + r_i V[f|S] > 0$, the issuer's demand of the security is

$$\theta_i = \frac{E[f|S] - \overline{p} - r_i C[f, z|S]x}{2\delta + r_i V[f|S]}. \tag{7}$$

On the other side, the uninformed outside investor solves

$$\max_{\theta_u} E[-\exp(-r_u W_u)|p(\theta_i)],$$

which is equivalent to

$$\max_{\theta_u} E[W_u|p(\theta_i)] - \frac{r_u}{2} V[W_u|p(\theta_i)].$$

Thus, as long as $V[f|p(\theta_i)] > 0$, the outside investor's demand is given by

$$\theta_u = \frac{E[f|p(\theta_i)] - p}{r_u V[f|p(\theta_i)]}. \tag{8}$$

A *linear rational expectation equilibrium* is a set $(p(\cdot), \theta_i, \theta_u)$ such that (a) $p(\cdot)$ is given by (4), (b) agents maximize, and (c) markets clear, that is $\theta_i + \theta_u = 0$.

We call an equilibrium *trivial* if it entails zero amount of trade, and concentrate our attention to the non-trivial equilibrium. Our first result, Lemma 3.1, provides a condition under which a non trivial equilibrium exists.

Lemma 3.1: *With a monopolist issuer a non-trivial linear equilibrium exists if and only if*

$$\begin{aligned} J &\equiv a^2 r_i^2 V^2[\zeta] V[x] \quad (9) \\ &\quad - \left[b^2 V[x] + a_s^2 V[s] + c^2 V[y]\right] \quad (10) \\ &> 0 \end{aligned}$$

Lemma 3.1 shows that if $J \leq 0$ the economy fails to have an equilibrium or it is trivial in the sense that optimal positions are zero. The first term of J, expression (9), is related to the hedging demand of the informed entrepreneur as indicated by its dependence on the degree of risk aversion. The second term, expression (10), is related to the informational motive for trading. A non-trivial equilibrium exists if the former dominates the latter. If this is not the case, the adverse selection problem is so severe that no equilibrium exists.

The next lemma provides a closed form solution for the ex-ante utility of the entrepreneur.

Lemma 3.2: *The ex-ante utility of the monopolistic issuer is given by*

$$E[u_i(W_i)] = -\left[(1 - r_i^2 V[x] V[z]) + \mathcal{U}_m(\mathbf{v})\right]^{-\frac{1}{2}},$$

where

$$(11)\ \mathcal{U}_m(\mathbf{v}) = r_i J \frac{\begin{array}{c}\left(1 - r_i^2 V[x]V[z]\right)\left(a_s^2 V[s] + c^2 V[y]\right) \\ + V[x]\left[b - r_i\left(aV[\zeta] + a_s V[s]\right)\right]^2\end{array}}{\begin{array}{c}(r_i + 2r_u)\left[a_s^2 V[s] + (b - ar_i V[\zeta])^2 V[x] + c^2 V[y]\right]\left(a^2 V[\zeta] + d^2 V[\epsilon]\right) \\ + 2r_u\left(a_s^2 V[s] + c^2 V[y]\right) a^2 r_i^2 V^2[\zeta] V[x]\end{array}},$$

unless $J \leq 0$*, in which case* $\mathcal{U}_m(\mathbf{v}) = 0$.

Here, $1 - r_i^2 V[x]V[z]$ corresponds to the ex-ante utility from the initial endowment of the asset, and $\mathcal{U}_m(\mathbf{v})$ corresponds to the benefit from trading a particular security f. The optimally created security maximizes $\mathcal{U}_m(\mathbf{v})$. But note that for $\mathcal{U}_m(\mathbf{v}) > 0$, it is necessary that $J > 0$, and this, in turn, requires that $a^2 > 0$. Hence, without loss of generality, we assume that $a = 1$. Furthermore, by a judicious choice of b, a_s, and c, the issuer can always guarantee that $J > 0$. Lemma 3.2 shows that it is always optimal to do so.

Next we start by characterizing the optimal security. We first prove two preliminary propositions to then present the main result of this section.

Proposition 3.3: *The monopolistic entrepreneur's equilibrium utility is monotonically decreasing in the weight that the asset's payoff assigns to extraneous noise at the terminal date, that is,*

$$\frac{\partial \mathcal{U}_m}{\partial d^2} < 0.$$

Clearly extraneous noise at the terminal date simply increases the riskiness of the security and does not provide any informational advantage to the issuer, nor does it improve his hedging possibilities. The next proposition states that for any design that makes the payoff sensitive to s and/or y there exists another design that yields a higher utility to the entrepreneur and that is not sensitive to the extraneous private information, y.

Proposition 3.4: *For all* $\mathbf{v} = (1, a_s, b, c, 0)$ *there exists* $\mathbf{v}' = (1, a_s', b, c', 0)$ *such that* $c' = 0$ *and* $\mathcal{U}_m(\mathbf{v}) \leq \mathcal{U}_m(\mathbf{v}')$, *where the inequality is strict whenever* $c \neq 0$.

To summarize our findings so far we have that, by Lemma 3.2, the entrepreneur will always design a security for which a non trivial equilibrium exist by setting $a \neq 0$, and as a normalization we chose $a = 1$. Propositions 3.3 and 3.4 further show that any security whose payoff is sensitive to extraneous private information, whether at the interim stage or the terminal one, cannot be optimal, that is $c = d = 0$. Given these results we can simplify the expression of $\mathcal{U}_m(\mathbf{v})$ in (11) to[3]

$$\mathcal{U}_m(\mathbf{v}) = r_i J \frac{L}{N}, \tag{12}$$

where:

$$\begin{aligned}
N &\equiv (r_i + 2r_u)\left[a_s^2 V[s] + (b - r_i V[\zeta])^2 V[x]\right] V[\zeta] + 2r_u a_s^2 V[s] r_i^2 V^2[\zeta] V[x], \\
L &\equiv \left(1 - r_i^2 V[x] V[z]\right) a_s^2 V[s] + V[x]\,(b - r_i V[\zeta] - a_s r_i V[s])^2, \text{ and} \\
J &\equiv \left(r_i^2 V^2[\zeta] - b^2\right) V[x] - a_s^2 V[s]
\end{aligned}$$

It is useful to note at this point that expression (12) collapses to the one in Rahi (1996, page 293) if $V[s] = 0$. We make use of (12) to obtain the main result of this section:

Theorem 3.5: *The optimal security design for a monopolistic issuer has a payoff of the form* $f^* = \overline{f} + \zeta + a_s^* s + b^* x$ *for some* $a_s^* > 0$ *and* $b^* > 0$.

Theorem 3.5 stands in sharp contrast to Theorem 2.4 in Rahi (1996) that establishes that the optimal design yields a security payoff that is uncorrelated to any private information the issuer may posses.[4] Recall that the issuer in this model has the possibility of designing

[3]With a certain abuse of notation we keep using the symbol $\mathbf{v}$ to denote the vector defining the particular security, but once $a = 1$, and $c = d = 0$ are imposed.

[4]It is also worth comparing our result with Proposition 9 of DeMarzo and Duffie (1999) and their illuminating discussion thereafter.

a security that would make the resulting rational expectations non revealing by including extraneous private information y in the design of the asset. Because y is uncorrelated with the entrepreneur'sss payoff e, there is no hedging motive when including it the design, but purely an informational one. This greatly affects market depth that is not compensated by the gain that may result from the possibility of hedging that part of the project's realization that is correlated with the issuer's private information. Instead, in our model, when including both sources of private information, s and x, the informed entrepreneur benefits from the additional hedging possibilities of that the security brings which, more than compensates the loss of market depth that the adverse selection entails. The result is a security that is indeed sensitive to the issuer's private information.

The crucial difference then comes from the fact that equilibrium here can be noisy i.e. not fully revealing, while it is always fully revealing in Rahi (1996). The issuer can be better off, ex-ante, by creating the security whose equilibrium is not fully revealing.[5] This is because in the noisy rational expectation equilibrium, the issuer can enjoy the benefits not only from the private information, but also from hedging the risk whose realization is not fully reflected in the equilibrium price. Consequently, the issuer designs the payoff of the security to be correlated with her private signals (s, x) in order to make the equilibrium noisy.

More precisely, we can show that

$$\frac{J}{N} = \frac{1}{2\delta + r_i V[f|S]}.$$

Since $p(\theta_i) = \overline{p} + \delta\theta_i$, $\frac{J}{N}$ is a proxy for the level of market depth $\frac{1}{\delta}$ of security f: other things being equal, the deeper the market is, the larger $\frac{J}{N}$ is. We can also show that

$$L = \left(1 - r_i^2 V[x]V[z]\right) V[E(f - \overline{p}|S)|x] + \tag{13}$$

$$+V[E(f - \overline{p}|x) - r_i C(f, z|S)x - x r_i C[E(f|S), E(z|S)|x]] \tag{14}$$

(13) measures the degree of potential benefits of the trade driven by the information on s, whereas (14) measures the degree of net potential benefits of the trade driven by the information

[5]The issuer can do so by setting $a_s \neq 0$ and $r_i V[\zeta] - b \neq 0$.

on x and the hedging with security f. Thus, the benefit $\mathcal{U}_m$ from trading f is roughly given by the product of the level of market depth $\frac{J}{N}$ and the sum of the potential benefits of the trades driven by informational and hedging motives.

Now, consider the case where $a_s \geq 0$ and $b \geq 0$.[6] Then, it is easy to see that as a_s increases, L increases but market depth, as proxied by $\frac{J}{N}$, decreases. This is for the following reason: as a_s becomes larger, the potential benefits from trading f for private information and hedging on s increases, which increases L. However, this in turn exacerbates the adverse selection against the uninformed trader in trading f, which decreases the market depth $\frac{J}{N}$. To attain the optimal security design, the issuer should balance these two effects, and the net benefit of making f correlated with s is larger than keeping f uncorrelated with s. This explains $a_s > 0$. Similar argument applies to the case of b and the trade-off between the effects on market depth and potential benefits from the trades leads to $b > 0$.

Rahi (1996) further shows that, in the economy he proposes, straight equity is equivalent to the optimal linear security. We investigate next whether this is the case in the more general economy presented in this paper. The answer to this question turns to be negative as the following theorem shows.

Theorem 3.6: *If $V[s] > 0$, the equilibrium allocation that the optimal security in Theorem 3.5 attains differs from the one that would result from issuing straight equity, $e = xz$.*

This result stands again in contrast to Rahi's results where the optimal security design yields an allocation equivalent to that of issuing straight equity on the project. Theorem 3.6 shows that this is only the case when $V[s] = 0$.

IV. THE COMPETITIVE ENTREPRENEUR

In this section we assume that the entrepreneur is a price taker, by which we mean that he ignores the effect that his asset position has in the price of the traded asset. That is,

[6] It is immediate from the proof of Theorem 2.5 that these conditions are necessary for the optimal design.

following the notation introduced in the previous section, he assumes "$\delta = 0$". In Rahi (1996) the main result (Theorem 3.4, page 295) establishes that now it is optimal to include private information in the optimal (linear) security payoff. The equilibrium is fully revealing though[7] and the situation is identical to one where there is no private information. Instead in our economy the equilibrium retains the non revealing aspect and it is still optimal to include *both* sources of private information in the security design.

Following steps similar to the ones of the previous section we can prove that the demand of the informed entrepreneur is given by:

$$\theta_i = \frac{E[f|S] - \hat{p}(\theta_i) - r_i C[f,z|S]x}{r_i V[f|S]} \tag{15}$$

as long as $V[f|S] > 0$, whereas that of the outside investor is

$$\theta_u = \frac{E[f|\hat{p}(\theta_i)] - \hat{p}(\theta_i)}{r_u V[f|\hat{p}(\theta_i)]}$$

as long as $V[f|\hat{p}(\theta_i)] > 0$.

The first two lemmas present conditions under which a non trivial equilibrium exists and a closed form representation of the equilibrium utility of the informed issuer for an arbitrary security.

Lemma 4.1: *With a competitive issuer a non-trivial linear equilibrium exists if and only if* $b \neq r_i a V[\zeta]$ $(a \neq 0)$.

Lemma 4.2: *The equilibrium utility of the competitive entrepreneur is given by:*

$$E[u_i(W_i)] = -\left[(1 - r_i^2 V[x]V[z]) + \mathcal{U}_c(\mathbf{v})\right]^{-\frac{1}{2}}$$

$$\mathcal{U}_c(\mathbf{v}) = r_i^2 \hat{J}^2 \frac{\hat{L}}{\hat{N}^2}, \tag{16}$$

[7] As it was in the case of the monopolistic issuer when asset shows no sensitivity to the extraneous information y. Recall that in both the model proposed by Rahi (1996) and the one presented in this paper c is optimally set equal to zero.

where

$$\begin{aligned}
\hat{J} &= ar_iV[\zeta](b - ar_iV[\zeta])V[x], \\
\hat{L} &= \left(a^2V[\zeta] + d^2V[\epsilon]\right)\{(1 - r_i^2V[x]V[z])(a_s^2V[s] + c^2V[y]) \\
&\quad + V[x]\,(b - ar_iV[\zeta] - a_sr_iV[s])^2\}, \textit{ and} \\
\hat{N} &= (r_i + r_u)\left(a_s^2V[s] + (b - ar_iV[\zeta])^2V[x] + c^2V[y]\right)\left(a^2V[\zeta] + d^2V[\epsilon]\right) \\
&\quad + r_u\left(a_s^2V[s] + c^2V[y]\right)a^2r_i^2V^2[\zeta]V[x],
\end{aligned}$$

unless there is a market breakdown in which case $\mathcal{U}_c(\mathbf{v}) = 0$.

Without loss of generality, assume $a = 1$. In the same way as the monopolistic case, we can show the following:

Proposition 4.3: *The competitive entrepreneur's equilibrium utility is monotonically decreasing in the weight that the asset's payoff assigns to extraneous noise at the terminal date, that is,*

$$\frac{\partial \mathcal{U}_c}{\partial d^2} < 0.$$

Proposition 4.4: *For all* $\mathbf{v} = (1, a_s, b, c, 0)$ *there exists* $\mathbf{v}' = (1, a_s', b, c', 0)$ *such that* $c' = 0$ *and* $\mathcal{U}_c(\mathbf{v}) \leq \mathcal{U}_c(\mathbf{v}')$*, where the inequality is strict whenever* $c \neq 0$.

As in the previous section then, we are entitled to set $a = 1$ and $b = c = 0$, and (16) collapses to:

$$\mathcal{U}_c(\mathbf{v}) = r_i^2\widehat{J}^2\frac{\widehat{L}}{\widehat{N}^2},$$

where

$$\begin{aligned}
\widehat{J} &= r_iV[\zeta](b - r_iV[\zeta])V[x], \\
\widehat{L} &= V[\zeta]\left\{\left(1 - r_i^2V[x]V[z]\right)a_s^2V[s] + V[x]\,(b - r_iV[\zeta] - a_sr_iV[s])^2\right\}, \text{ and} \\
\widehat{N} &= (r_i + r_u)\left\{a_s^2V[s] + (b - r_iV[\zeta])^2V[x]\right\}V[\zeta] + r_ua_s^2V[s]r_i^2V^2[\zeta]V[x]
\end{aligned}$$

Armed with these results we are ready to characterize the optimal linear security, which we can identify as the choice of a_s and b. Some preliminaries are useful in providing an intuitive description of the optimal security. Inspection of these equations proves that for the optimal (a_s, b), the following conditions hold:[8] (1) $b - r_i V[\zeta] \neq 0$ for any a_s, (2) $a_s > 0 \Rightarrow b - r_i V[\zeta] < 0$, and (3) $a_s < 0 \Rightarrow b - r_i V[\zeta] > 0$.

Next define,

$$\xi = \frac{a_s}{b - r_i V[\zeta]}$$

Clearly then $\xi \leq 0$ with $\xi = 0$ if and only if $a_s = 0$. As is usual in the literature on noisy rational expectation equilibrium, we interpret x as the endowment of the units of the asset, whose final payoff per unit is z, where the endowment $e = xz$, then, ξ can be interpreted as the ratio of the sensitivities of the security payoff to the information on the per unit final payoff of the asset and to that on the units of the asset endowed.

With a certain abuse of notation, simple manipulations allows us to write $\mathcal{U}_c$ as a function of ξ as follows:

$$\mathcal{U}_c(\xi) = r_i^2 (r_i V[\zeta] V[x])^2 \frac{\hat{l}}{\hat{n}^2}, \tag{17}$$

where,

$$\begin{aligned} \hat{l} &= V[\zeta] \left\{ (1 - r_i^2 V[x] V[z]) \xi^2 V[s] + V[x] (1 - \xi r_i V[s])^2 \right\} \\ \hat{n} &= (r_i + r_u) \left(\xi^2 V[s] + V[x] \right) V[\zeta] + r_u V[s] r_i^2 V^2[\zeta] V[x] \xi^2 \end{aligned}$$

Expression (17) means that the optimal choice of security reduces to the choice of ξ. The next proposition proves that it is never optimal to set $\xi = 0$.

Proposition 4.5: *Define $\mathcal{U}_c(\xi)$ as in equation (17) then*

$$\frac{\partial \mathcal{U}_c}{\partial \xi}|_{\xi=0} < 0.$$

[8] Observe that $\forall b$, $\exists b'$, $(b - r_i V[\zeta])^2 = (b' - r_i V[\zeta])^2$ and $-(b - r_i V[\zeta]) = b' - r_i V[\zeta]$.

Let ξ^* be the solution of the equation $G(\xi^*) = 0$, where the function $G(\cdot)$ is defined in equation (22) in appendix III. It can be easily proved that $\frac{dG}{d\xi}|_{\xi=\xi^*} < 0$.[9] We have then the main result of this section:

Theorem 4.6: *The optimal security design by a competitive issuer exists, and the class of assets that are optimal is given by:*

$$\{f^* = \overline{f} + \zeta + a_s^* s + b^* x \mid a_s^* \neq 0, b^* - r_i V[\zeta] \neq 0, \frac{a_s^*}{b^* - r_i V[\zeta]} = \xi^* < 0\}.$$

Thus, for the competitive case, there is a whole collection of securities that are optimal, and

that are characterized by the set of (a_s^*, b^*) that attains the optimal ratio ξ^*. Nonetheless, since $a_s^* \neq 0$ is necessary for optimality, the security whose payoff is uncorrelated with the issuer's private information cannot be an element of such optimal class.

Furthermore, we can show that:

$$f^* - \widehat{p}(\theta_i) = \zeta + As + Bx \text{ with } A, B > 0, \tag{18}$$

where A and B are given by equations (19) and (20) in appendix II. That is, the optimal equilibrium net payoff $f^* - \widehat{p}(\theta_i)$ depends on the choice of (a_s, b). This is because the equilibrium is noisy so that the price does not fully reflect $a_s s + bx$, the part of payoff of $f = \overline{f} + \zeta + a_s s + bx$ about which the issuer has private information. On the other hand, if $V[s] = 0$, which corresponds to the case that Rahi (1996) investigated, we can show that the net payoff is equal to $\zeta + \frac{r_i r_u}{r_i + r_u} V[\zeta] x$ and is independent of the choice of (a_s, b). This is because in this case, the equilibrium is fully revealing and the price fully reflects the issuer's information signals (s, x). To build some intuition about the proposed solution assume that bot s and x are both large an positive. The entrepreneur knows then that the security's payoff is likely to be high. He

[9] Since $G(\cdot)$ is cubic, such ξ^* exists.

may be tempted to trade heavily on this information, but basic manipulations of equation (15) yield:

$$\theta_i = \frac{a_s s + (b - r_i V[\zeta])\, x}{r_i V[\zeta]}.$$

Because $(b - r_i V[\zeta]) < 0$, his trade is much smaller than otherwise it would be for purely informational reasons. His hedging needs are also low as s and x are high. As a consequence the outside investor is given some protection against informational trade while keeping the hedging possibilities of the entrepreneur open. If, on the other hand, s is, say positive, and x is negative, then the position of the investor will tend to be high. But observing large positions in the market the outside investor correctly infers that the hedging needs of the entrepreneur must be large and not based on any informational advantage. In this case the outside investor is willing to provide the required liquidity, even when failing to back out the private signals.

It is important to stress that this commitment to trade in fixed proportions could also be achieved through the inclusion of extraneous private information. Indeed, at the interim stage, by simply substituting $a_s s$ by cy, we would obtain the same formulas if the entrepreneur were to include extraneous private information together with information on x. It is at the ex-ante date where this substitution makes the difference as the inclusion of y in the security design provides no additional insurance whereas the inclusion of s does.

The intuition behind this theorem is similar to that for the monopolistic case. The issuer can be better off, ex-ante, by creating a security whose equilibrium price does not fully reveal his private information on (s, x). This is because by doing so, he can enjoy the benefits both from the private information and from the hedging of the risks whose realization is not fully reflected in the equilibrium price.

Differently from the monopolistic case where the correlation between the payoff of the security and the information signal affects both the degree of information transmitted through the market and the price impact of the informed issuer's trade, in the competitive case, the correlation between the payoff of the security and the information signals affects only the degree of information transmitted through the security price. For this reason, the optimal security design here is determined solely by choosing the optimal degree of the information transmitted in equilibrium.

As in the monopolistic case, we can show that $(\frac{\widehat{J}}{\widehat{N}})^2 = (\frac{1}{\delta + r_i V[f|S]})^2$. Hence, $(\frac{\widehat{J}}{\widehat{N}})^2$ indicates the level of market depth $\frac{1}{\delta}$ of security f: other things being equal, the deeper the market is, the larger $\frac{\widehat{J}}{\widehat{N}}$ is. Similarly, we can interpret that the first term of $\widehat{L}$, $V[\zeta](1 - r_i^2 V[x]V[z])a_s^2 V[s]$, measures the degree of potential benefits of the trade driven by the information on s and that the second term $V[\zeta]V[x](b - r_i V[\zeta] - a_s r_i V[s])^2$ measures the degree of net potential benefits of the trade driven by the information on x and the hedging with security f. Thus, again, the benefit $\mathcal{U}_c$ from trading f is roughly given by the product of the level of market depth and the sum of the potential benefits of the trades driven by informational and hedging motives. The more correlated the security payoff is with the information signals, the less deep the market is, but the larger the potential benefits are. The optimal security design maximizes the total benefit $\mathcal{U}_c$ by balancing the trade-off between these two effects.

As in the monopolistic case, we obtain the following result about the non-equivalence between an equity and the optimal security:

Theorem 3.7: *If $V[s] > 0$, the equilibrium allocation that the optimal security in Theorem 3.5 attains differs from the one that would result from issuing straight equity, $e = xz$.*

Again, as long as we consider an economy whose information structure is conventional in economics and finance literature on asymmetric information, issuing the equity cannot be equivalent to creating the optimally designed security. They can be equivalent only when $V[s] = 0$, which is the case considered by Rahi (1996).

V. CONCLUSIONS

We have studied security design when the entrepreneur has several sources of private information about the profitability of his project. In this case we saw that the entrepreneur prefers a non revealing rational expectations equilibrium to a fully revealing one. He achieves it by including several private signals as long as they are correlated with the project's payoff. This result holds under the strong assumption that the private signals are contractible and in sharp contrast with results by Rahi (1996) and DeMarzo and Duffie (1999).

That entrepreneurs are better informed about the project at any point in time than outside investors is indeed a very realistic assumption. They are indeed informed about several aspects of the production process, whether there were any problems in the production line or whether the product is selling well or not. Indeed they continuously receive data about sales and earnings, and these are communicated to outside investors only so often. Once communicated, these early signals about the overall profitability of the project become known and, if we abstract from truth telling considerations, they could potentially be contracted on. Our current research is directed towards understanding whether making periodic announcements can serve the purpose of relieving some of the adverse selection costs suffered by the entrepreneur and hence improve his hedging possibilities. For instance, in the current set up the terminal date can be taken to be the one where an announcement is made. Indeed this seems the interpretation one should give to the model advanced by Rahi (1996) for it is at this terminal date when private signal become public. Would it be in the best interest of the firm to announce private signals at the interim stage? How would the optimal security design be affected in that case? These questions have received no attention in the literature so far, but firms do make announcement reporting profits, intermediate sales, and so on. The next step in this research agenda will be to understand how announcements can affect the liquidity of the secondary market and how do they affect the optimal security design in the presence of private information.

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APPENDIX I: JUSTIFICATION OF $z = \zeta + s$

Suppose that the information signal that is correlated with z is given by s_0 and that $(z, s_0)^\top$ are joint normally distributed with mean $(0,0)^\top$ and variance

$$\begin{bmatrix} V[z] & C[z,s_0] \\ C[z,s_0] & V[s_0] \end{bmatrix}$$

where $C[z, s_0] \neq 0$.

Now, define $\zeta \equiv z - E[z|s_0]$ and $s \equiv E[z|s_0]$. Then, $z = \zeta + s$ and $C[\zeta, s] = 0$. Furthermore,

$$\begin{aligned} f &= \overline{f} + az + a_0 s_0 + bx + cy + d\epsilon \\ &= \overline{f} + a\zeta + (a + a_0 \frac{V[s_0]}{C[z,s_0]})s + bx + cy + d\epsilon. \end{aligned}$$

By defining $a_s \equiv a + a_0 \frac{V[s_0]}{C[z,s_0]}$, we obtain the formulation in this paper.

APPENDIX II: COEFFICIENTS IN FORMULA (18)

$$A \equiv \frac{-r_i^2 V^2[\zeta]V[x]\xi^*}{(r_i + r_u)(V[s]\xi^{*2} + V[x])V[\zeta] + r_u r_i^2 V^2[x]V[s]\xi^{*2}} \tag{19}$$

$$B \equiv \frac{\{r_i V[s]V[\zeta]\xi^{*2} + r_u(V[s]\xi^{*2} + V[x] + r_i^2 V[\zeta]V[s](r_i + r_u)[x])V[\zeta]\}r_i V[\zeta]}{(r_i + r_u)(V[s]\xi^{*2} + V[x])V[\zeta] + r_u r_i^2 V^2[x]V[s]\xi^{*2}}. \tag{20}$$

APPENDIX III: PROOFS

Throughout and for ease of exposition we refer to the informed agent as simply "i", and to the uninformed agent as "u". Also we omit the superscript in $\mathcal{U}$ as well as a the argument in the function as it should be transparent from the discussion in the text.

Proof of lemma 3.1

Since $p(\theta_i) = \overline{p} + \delta\theta_i$, information that the investor u can infer from observing $p(\theta_i)$ is equivalent to that from observing θ_i. Define

$$q = E[f|S] - r_i C[f, z|S]x - \overline{f}.$$

By equation (5), observing θ_i is equivalent to observing q. Thus, for u, observing $p(\theta_i)$ is equivalent to observing q.

Now, imposing the market clearing condition, we obtain $\overline{p} = \overline{f}$ and

$$\delta = \frac{\frac{C[f,q]}{V[q]} r_i V[f|S] + r_u V[f|q]}{1 - 2\frac{C[f,q]}{V[q]}}$$

Equilibrium exists if and only if i's second-order condition is satisfied i.e., $2\delta + r_i V[f|S] > 0$. This is equivalent to

$$\left(a^2 r_i^2 V^2[\zeta] - b^2\right) V[x] - a_s^2 V[s] - c^2 V[y] > 0$$

□

Proof of lemma 3.2

Recall that i's information at date 1 is given by $S = (s, x, y)$. i's date 1 interim utility level $E[-\exp(-r_i W_i)|S]$ in equilibrium is given, in terms of the certainty equivalent, by

(21) $$-\exp\left(-r_i\left[xE[z|S] + \theta_i(E[f|S] - p(\theta_i)) - \frac{r_i}{2}\left(x^2 V[z|S] + 2\theta_i C[f,z|S]x + \theta_i^2 V[f|S]\right)\right]\right).$$

Write expression (21) by $-\exp(u - v^2)$, where

$$\begin{aligned} u &\equiv -r_i\left(xE[z|S] - \frac{r_i}{2}x^2 V[z|S]\right) \\ v &\equiv \sqrt{\frac{r_i}{2}}\frac{E[f|S] - r_i C[f,z|S]x - \overline{p}}{\sqrt{2\delta + r_i V[f|S]}}. \end{aligned}$$

Since (u, v) are joint normally distributed,

$$E[-\exp(u - v^2)|x] = \frac{-1}{\sqrt{1 + 2V[v|x]}}\exp\left(E[u|x] + \frac{1}{2}V[u|x] - \frac{(E[v|x] + C[u,v|x])^2}{1 + 2V[v|x]}\right)$$

where

$$\begin{aligned} E[u|x] &= \frac{r_i^2}{2}x^2 V[z|S] = \frac{r_i^2}{2}x^2 V[s], \\ V[u|x] &= r_i^2 x^2 V[E[z|S]|x] = r_i^2 x^2 V[s] \\ E[v|x] &= \sqrt{\frac{r_i}{2}}\frac{E[f - \overline{p}|x] - r_i C[f,z|S]x}{\sqrt{2\delta + r_i V[f|S]}} = \sqrt{\frac{r_i}{2}}\frac{(b - ar_i V[\zeta])x}{\sqrt{2\delta + r_i V[f|S]}}, \\ V[v|x] &= \frac{r_i}{2}\frac{V[E[f - \overline{p}|S]|x]}{2\delta + r_i V[f|S]} = \frac{r_i}{2}\frac{a_s^2 V[s] + c^2 V[y]}{2\delta + r_i V[f|S]} \\ C[u,v|x] &= C\left[-r_i x E[z|S], \sqrt{\frac{r_i}{2}}\frac{E[f|S]}{\sqrt{2\delta + r_i V[f|S]}}|x\right] = r_i\sqrt{\frac{r_i}{2}}x\frac{a_s V[s]}{\sqrt{2\delta + r_i V[f|S]}} \end{aligned}$$

Hence, we obtain

$$E[-\exp(-r_iW_i)|x] = E[-\exp(u-v^2)|x] = -\left(\frac{2\delta + r_iV[f|x]}{2\delta + r_iV[f|S]}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^2M\right)$$

where

$$M \equiv -r_i^2V[z] + r_i\frac{\left(\frac{C[f,x]}{V[x]} - r_iC[f,z]\right)^2}{2\delta + r_iV[f|S]}.$$

Finally, taking the unconditional expectation, we obtain

$$E[-\exp(-r_iW_i)] = \left(\frac{2\delta + r_iV[f|x]}{2\delta + r_iV[f|S]}\left[(1 - r_i^2V[z]V[x]) + V[x]r_i\frac{(\frac{C[f,x]}{V[x]} - r_iC[f,z])^2}{2\delta + r_iV[f|S]}\right]\right)^{-\frac{1}{2}}$$

Substitution and slight manipulation yields the desired result. □

Proof of proposition 3.3

Immediate from the definition of $\mathcal{U}$. □

Proof of proposition 3.4

Note that for $J > 0$, it is necessary that

$$a^2r_i^2V^2[\zeta] - b^2 = (ar_iV[\zeta] - b)(ar_iV[\zeta] + b) > 0.$$

Thus, $b - ar_iV[\zeta] < 0$, and for given (a_s, c), we can choose (a_s', c') to satisfy $c' = 0$, $(a_s')^2V[s] = a_s^2V[s] + c^2V[y]$ and

$$(b - ar_iV[\zeta] - a_s'r_iV[s])^2 > (b - ar_iV[\zeta] - a_sr_iV[s])^2.$$

And the result follows. □

Proof of theorem 3.5

Note first that for $J > 0$, it is necessary that

$$r_i^2V^2[\zeta] - b^2 = (r_iV[\zeta] - b)(r_iV[\zeta] + b) > 0,$$

and hence $b - r_iV[\zeta] < 0$.

Second, by substitution, we can show that

$$\begin{aligned}\mathcal{U}|_{a_s=0,b=0} - \mathcal{U} &= r_i[r_i^2V^2[\zeta]V[x]\{2r_u a_s^2 V[s]r_i^2V^2[\zeta]V[x] + (r_i+2r_u)V[\zeta]r_i^2V[x]V[s]a_i^2V[\zeta] \\ &+2(r_i+2r_u)V[\zeta](b-r_iV[\zeta])a_s r_i V[s]V[x]\} \\ &+(r_i+2r_u)V[\zeta](b^2V[x]+a_s^2V[s])(1-r_i^2V[x]V[z])a_s^2V[s] \\ &+(r_i+2r_u)V[\zeta](b^2V[x]+a_s^2V[s])(b-r_iV[\zeta]-a_s r_i V[s])^2V[x]]\end{aligned}$$

is strictly greater than 0 if $a_s < 0$ since $b - r_iV[\zeta] < 0$. Thus, $\mathcal{U}|_{a_s=0,b=0} \geq \mathcal{U}|_{a_s<0}$ for any b, and for the optimal innovation, $a_s \geq 0$.

Third, direct calculation shows that $\frac{\partial \mathcal{U}}{\partial a_s}|_{a_s=0} > 0$ for any b. Therefore, for the optimal innovation, $a_s > 0$ should hold.

Fourth, setting $a = 1$ and $c = d = 0$, $\mathcal{U}$ is a continuous function of (a_s, b) on the compact set

$$\left\{(a_s, b) \mid a_s \geq 0,\ r_i^2V^2[\zeta]V[x] - b^2V[x] - a_s^2V[s] \geq 0\right\}$$

Thus, there exists (a_s^*, b^*) that maximizes $\mathcal{U}$ with $a = 1$ and $c = d = 0$.

Finally, define

$$\eta = \frac{r_iV[\zeta] - b}{a_s} > 0.$$

Then, we have

$$\mathcal{U} = r_i J \frac{L'}{N'}$$

$$\begin{aligned} J &\equiv r_i^2V^2[\zeta]V[x] - b^2V[x] - a_s^2V[s] \\ L' &\equiv (1 - r_i^2V[x]V[z])V[s] + V[x]\{\eta + r_iV[s]\}^2 \\ N' &\equiv (r_i + 2r_u)\{V[s] + \eta^2V[x]\}V[\zeta] + 2r_uV[s]r_i^2V^2[\zeta]V[x]. \end{aligned}$$

Now, let $\eta^* = \frac{r_iV[\zeta]-b^*}{a_s^*}$. Then, maximizing $\mathcal{U}$ s.t.

$$\left(\frac{r_iV[\zeta] - b}{a_s}\right) = \eta^*$$

is equivalent to

$$\min_{a_s, b}\ \left\{b^2V[x] + a_s^2V[s]\right\} \quad s.t. \quad \left(\frac{r_iV[\zeta] - b}{a_s}\right) = \eta^*$$

for which (a_s^*, b^*) must be the solution. Indeed, we can solve this minimization problem to obtain

$$\begin{aligned} a_s^* &= \frac{\eta^* r_i V[\zeta] V[x]}{(\eta^*)^2 V[x] + V[s]} > 0 \text{ and} \\ b^* &= \frac{r_i V[\zeta] V[s]}{(\eta^*)^2 V[x] + V[s]} > 0. \end{aligned}$$

□

Proof of theorem 3.6

Suppose that the optimally designed security $f^* = \overline{f} + \zeta + a_s^* s + b^* x$ is issued. Denote by θ_i^* i's demand of f^* in equilibrium, and by $P^*(\theta_i^*) = \overline{p}^* + \delta^* \theta_i^*$ the rational expectation equilibrium price function. Then, issuer i's equilibrium allocation is given by

$$\begin{aligned} xz + \theta_i^* \left[f^* - P^*(\theta_i^*) \right] &= xz + \frac{E[f^*|S] - r_i C[f^*, z|S]x - \overline{f}}{2\delta^* + r_i V[f^*|S]} (\zeta + a_s^* s + b^* x) \\ &\quad + \left[\frac{E[f^*|S] - r_i C[f^*, z|S]x - \overline{f}}{2\delta^* + r_i V[f^*|S]} \right]^2 \delta^* \\ &= xz + \frac{b^* - r_i V[\zeta] - a_s^* r_i V[s]}{2\delta^* + r_i V[f^*|S]} xz \\ &\quad + \frac{b^* - r_i V[\zeta] - a_s^* r_i V[s]}{2\delta^* + r_i V[f^*|S]} x \left[(a_s^* - 1)s + b^* x \right] \\ &\quad + \frac{a_s^*}{2\delta^* + r_i V[f^*|S]} s \left[z + (a_s^* - 1)s + b^* x \right] \\ &\quad + \left[\frac{a_s^* s + (b^* - r_i V[\zeta] - a_s^* r_i V[s])x}{2\delta^* + r_i V[f^*|S]} \right]^2 \delta^*. \end{aligned}$$

Now, suppose that instead of f^*, an equity with the payoff $e = xz$ is issued. The equity is traded in the same timing as f^*. Suppose also that an equilibrium exists for the market of the equity. Denote by θ_{ei} i's equilibrium demand of the equity and by P_e the equilibrium price function. As natural requirements, we restrict that both θ_{ei} and P_e are measurable with respect to the information $S \equiv (s, x, y)$ available to i in trading. i's equilibrium allocation when the equity is traded is given by $xz + \theta_{ei}\{xz - P_e\}$. In order for this allocation to be equal to that attained by the optimal security f^*, it is necessary that

$$\theta_{ei} = \frac{b^* - r_i V[\zeta] - a_s^* r_i V[s]}{2\delta^* + r_i V[f^*|S]}$$

and

$$P_e = \frac{2\delta^* + r_i V[f^*|S]}{b^* - r_i V[\zeta] - a_s^* r_i V[s]} \left[\frac{b^* - r_i V[\zeta] - a_s^* r_i V[s]}{2\delta^* + r_i V[f^*|S]} x \left[(a_s^* - 1)s + b^* x \right] \right.$$

$$+\frac{a_s^* s}{2\delta^* + r_i V[f^*|S]}[z + (a_s^* - 1)s + b^* x]$$
$$+\left[\frac{a_s^* s + (b^* - r_i V[\zeta] - a_s^* r_i V[s])x}{2\delta^* + r_i V[f^*|S]}\right]^2 \delta^*].$$

However, P_e cannot be measurable with respect to $S = (s, x, y)$ as long as $V[s] > 0$. □

Proof of lemma 3.7

Immediate from the proof of theorem 2.5. □

Proof of lemma 4.1

Similarly to the proof of lemma 2.1, we can obtain the equilibrium price function $\hat{p}(\theta_i) = \overline{p} + \hat{\delta}\theta_i$ such that $\overline{p} = \overline{f}$ and

$$\hat{\delta} = \frac{1}{1 - C[f,q]/V[q]}\{\frac{C[f,q]}{V[q]} r_i V[f|S] + r_u V[f|q]\}$$

where

$$q = E[f|S] - r_i C[f, z|S]x - \overline{f}.$$

This is well defined if and only if

$$V[q] - C[f,q] = -a r_i V[\zeta](b - a r_i V[\zeta]) \neq 0.$$

□

Proof of lemma 4.2

Mimic the proof of lemma 2.2.□

Proof of proposition 4.3

Direct computation shows that

$$\begin{aligned}
\frac{\partial \mathcal{U}}{\partial d^2} &= r_i^2 \frac{\widehat{J}^2 \widehat{L} V[\epsilon]}{\widehat{N}^3 (a^2 V[\zeta] + d^2 V[\epsilon])}[-(r_i + r_u)\{a_s^2 V[s] + c^2 V[y] + \\
&\quad (b - a r_i V[\zeta])^2 V[x]\}(a^2 V[\zeta] + d^2 V[\epsilon]) \\
&\quad + r_u (a_s^2 V[s] + c^2 V[y]) a^2 V[\zeta] r_i^2 V[\zeta] V[x]] \\
&\leq 0
\end{aligned}$$

since

$$r_i^2 V[\zeta]V[x] < r_i^2 V[z]V[x] < 1.$$

□

Proof of proposition 4.4

Take a_s' and c to satisfy $c = 0$, $(a_s')^2 V[s] = a_s^2 V[s] + c^2 V[y]$, and

$$(b - a r_i V[\zeta] - a_s V[s])^2 < (b - a r_i V[\zeta] - a_s' V[s])^2.$$

□

Proof of proposition 4.5

Direct computation shows that

$$\frac{\partial \mathcal{U}}{\partial \xi} = 2r_i^2 \frac{(r_i V[\zeta]V[x])^2}{\widehat{n}^3} V^2[\zeta]V[s]G(\xi)$$

Define $G(\xi)$ as,

$$\begin{aligned} G(\xi) \quad \equiv \quad & -V[s](1 - r_i^2 V[x]V[\zeta])\{r_i + r_u + r_u r_i^2 V[x]V[\zeta]\}\xi^3 \\ & + r_i V[x]V[s]\{r_i + r_u + r_u r_i^2 V[x]V[\zeta]\}\xi^2 \\ & - V[x]\{r_i + r_u + (r_i + 3r_u) r_i^2 V[x]V[\zeta]\}\xi \\ & - V^2[x] r_i (r_i + r_u). \end{aligned} \tag{22}$$

Clearly,

$$\frac{\partial \mathcal{U}}{\partial \xi}|_{\xi=0} = 2r_i^2 \frac{(r_i V[\zeta]V[x])^2}{\{(r_i + r_u)V[\zeta]V[x]\}^3} V^2[\zeta]V[s]\{-V^2[x] r_i (r_i + r_u)\} < 0.$$

□

Proof of theorem 4.6

Since

$$\mathcal{U}|_{\xi=0} = r_i^2 (r_i V[\zeta]V[x])^2 \frac{1}{(r_i + r_u)V[\zeta]} > 0$$

and $\lim_{\xi \to -\infty} \mathcal{U} = 0$, there exists some $\overline{\xi} < 0$ such that for any $\xi' < \overline{\xi}$, $\mathcal{U}|_{\xi=\xi'} < \mathcal{U}|_{\xi=0}$. Because $\mathcal{U}$ is continuous in ξ , it attains its maximum at some ξ in a compact set $[\overline{\xi}, 0]$.

At the maximizer ξ^*, the first-order condition $\frac{\partial \mathcal{U}}{\partial \xi}|_{\xi=\xi^*} = 0$ and the second-order condition $\frac{\partial^2 \mathcal{U}}{\partial \xi^2}|_{\xi=\xi^*} < 0$ must hold.

Direct calculation shows that

$$\frac{\partial \mathcal{U}}{\partial \xi}|_{\xi=\xi^*} = 2r_i^4 \frac{V^4[\zeta]V^2[x]V[s]}{(\widehat{n})^3} G(\xi^*)$$

and

$$\frac{dG}{d\xi}|_{\xi=\xi^*} = 2r_i^4 V^4[\zeta]V^2[x]V[s]\frac{1}{(\widehat{n})^4}\left(-3\frac{\partial \widehat{n}}{\partial \xi}|_{\xi=\xi^*} G(\xi^*) + \widehat{n}\frac{dG}{d\xi}|_{\xi=\xi^*}\right).$$

Thus, $\frac{\partial \mathcal{U}}{\partial \xi}|_{\xi=\xi^*} = 0$ and $\frac{\partial^2 \mathcal{U}}{\partial \xi^2}|_{\xi=\xi^*} < 0$ is equivalent to $G(\xi^*) = 0$ and $\frac{dG}{d\xi}|_{\xi=\xi^*} < 0$. □