LARRY SELDEN*

A New Approach to the Joint Consumption-Portfolio Problem

1. INTRODUCTION AND SUMMARY

OVER THE PAST TWO DECADES, extensive research has been directed at solving, for an individual economic agent, the problems of (1) selecting an optimal portfolio of financial securities in a "risky" one-period setting (e.g., the pioneering contributions of Markowitz and Tobin) and (2) determining an optimal multiperiod consumption plan in a world of perfect certainty (e.g., the early work of Brumberg, Modigliani, and Friedman). These two very important kernels of modern financial economics developed in virtual isolation from one another until the recent contributions of Drèze and Modigliani [6], Sandmo [29], Samuelson [28], Merton [19], Hakansson [12], and Fama [7]. It is now well accepted that one cannot ordinarily determine his optimal financial asset holdings independently of his optimal multiperiod consumption plan, and vice versa. Focusing on the two-period case, one will discover that virtually all efforts at simultaneously solving these two decision problems employ the TPC (two-period cardinal) expected utility hypothesis.

We present in this article a new formulation of the joint consumption-portfolio problem which enables us to (1) distinguish between the roles played by "risk" preferences and "time" preferences in determining optimal consumption and asset demand; (2) generalize to an uncertain setting the classic Fisherian two-period diagrammatics and notion of the marginal rate of time preference; and (3) expand substantially the set of consumption-portfolio optima consistent with utility maximi-

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LARRY SELDEN *is associate professor, Graduate School of Business, Columbia University.*

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zation. A key element in our analysis is the OCE (ordinal certainty equivalent) representation of preferences over certain-uncertain consumption pairs developed in Selden [30]. This proposed alternative to the standard *two-period* expected utility model is based on a set of conditional *second-period* expected utility functions and a *two-period ordinal* time preference index. The resulting OCE representation includes the TPC paradigm as a limited special case. The latter requires additional axiomatic structure which results in a specific, strong interdependence between risk and time preferences (cf., Rossman and Selden [26]). Consistent with much of the literature on the joint consumption-portfolio problem, we shall assume in this article that period-two risk preferences are independent of first-period consumption.

In the next section, we discuss notation and formally define the joint consumption-portfolio problem together with the standard TPC formulation. The OCE representation hypothesis is reviewed in section 3. Then in section 4 we formulate the OCE approach to the joint decision problem and provide a graphical analysis. The notions of financial and consumption opportunity sets are introduced in section 5 together with several results on their shape. The final section examines the consumer-investor's personal equilibrium under our new formulation and also compares it with the standard characterization. We conclude the article with a numerical example based on an OCE representation defined by logarithmic risk preferences and CES (constant elasticity of substitution) time preferences. A set of unique consumption and portfolio optima, depending continuously on the value of the elasticity of substitution, is obtained. Only one element in this set of optima corresponds to a TPC representation.

2. PRELIMINARIES

Let us begin by agreeing on the following notation:

c_t: value of real, generalized consumption flow in time-period t $(= 1, 2)$

y_1: (positive) certain income to be received in c_1-units at the beginning of time-period one

$(n, n_F) \equiv (n_1, \ldots, n_{m-1}, n_F)$: vector of security holdings for $m - 1$ risky assets and the single risk-free asset (denoted F)

$(p, p_F) \equiv (p_1, \ldots, p_{m-1}, p_F)$: vector of first-period security prices. Each component is stated in terms of the numeraire commodity c_1, the price of which has, by convention, been set equal to unity

$\tilde{e}_j$: random variable mapping states of nature into real (gross) returns for security j $(= 1, \ldots, m - 1)$, payable in c_2-units at the beginning of time-period two

$(\tilde{e}, e_F) \equiv (\tilde{e}_1, \ldots, \tilde{e}_{m-1}, e_F)$: vector of $m - 1$ random, (gross) real returns on the risky securities and the certain return on the risk-free asset

R_F: real, *gross* rate of return on the riskless security F (denominated c_2-units of return per c_1-unit of investment)[1]

$\tilde{c}_2$: random variable which denotes the consumer's real, generalized consumption flow for time-period two.

Consider the case of an economic agent endowed with y_1 units of first-period consumption and possessing intertemporal consumption preferences which are "two-period myopic." He only cares about consumption in the first and second time periods and has no bequest motive. This individual confronts the fundamental intertemporal allocation problem of determining both an optimal plan for consumption in time periods one and two and an optimal program for investing his unconsumed period-one endowment in the set of one risk-free[2] and m-1 risky securities. The realized returns on these assets provide him with second-period consumption (for simplicity, we assume no second-period endowment).

Before formally characterizing this joint decision problem, let us first set forth the following standard market structure or institutional assumption:

Assumption 1. (1) There exist current markets for both the current consumption commodity and also all investment securities; (2) there exist riskless bonds, denoted asset F, that one can borrow or lend in unlimited quantities at the market risk-free rate of interest—there are also no restrictions on short sales of the risky financial assets; and (3) all economic agents act as price takers in both the consumption and financial securities markets, which are costless barter markets (i.e., "transactions perfect").

Definition 1. The consumption-portfolio decision problem for an individual is: Find that $(c_1{}^*, n^*, n_F{}^*)$ which for him "produces" a consumption plan $(c_1{}^*, \tilde{c}_2{}^*)$ that is *"preferred"* to all other feasible consumption plans.

All starred variables will be understood to be *optimal* in the sense of this definition.

Definition 1 implies that the criterion of optimality is some (presumably well-behaved) complete preference preordering[3] over a set of possible certain first-period and random second-period consumption pairs. However, it leaves unanswered what is an acceptable representation of preferences. One way of answering this question is to assume that there exists a TPC, von Neumann-Morgenstern utility function W. One can then formulate the consumption-portfolio decision in the following conventional fashion (see, e.g., [6, 8, 29]):

$$\max_{c_1, n} E\,[W\,(c_1, \tilde{c}_2)]$$

$$s.t.\ \tilde{c}_2 - (y_1 - c_1 - \sum_{j=1}^{m-1} n_j\, p_j)\, R_F - \sum_{j=1}^{m-1} n_j\, \tilde{e}_j = 0\,. \tag{1}$$

[1]Although the topic of market equilibrium is not broached in this paper, it is nevertheless important to understand that e_F is the basic exogenous datum. Equilibrium determines the current price and then the "equilibrium" yield may be computed $R_F = e_F/p_F$.

[2]For the purposes of this paper, we choose not to employ the generalization of Black [3] based on the assumption of no riskless asset F.

[3]A preference relation over some set Z is said to be complete if every pair of elements z' and z'' in Z satisfies $z' \preccurlyeq z''$, $z'' \preccurlyeq z'$, or $z' \sim z''$. The relation is called a preordering if it is transitive and reflexive. See Debreu [5].

3. OCE UTILITY

In this section we summarize the OCE representation results obtained in Selden [30]. Let us continue to suppose that c_1 and c_2 denote real consumption in time-periods one and two. Let F and G be c.d.f.'s (cumulative distribution functions) defined on second-period consumption. Define S to be some (suitable) space of (c_1,F)- pairs. Assume that a consumer possesses a complete preordering over S, denoted $\lesseqgtr$. Further, let this ordering be representable by a continuous utility function Ψ defined on S, i.e.,

$$(c_1', F) \lesseqgtr (c_1'', G) \Leftrightarrow \Psi(c_1', F) \leq \Psi(c_1'', G) \tag{2}$$

for any values of period-one consumption c_1' and c_1'' and c_2-c.d.f.'s F and G.

Given a complete preordering on S, each "cross-section" thereof S_{c_1} (defined by a value c_1 of first-period consumption) will possess an ordering denoted $\lesseqgtr c_1$. The set of these orderings, $\{\lesseqgtr c_1\}$, will be referred to as the consumer's conditional risk preferences. These preferences over c.d.f.'s on second-period consumption are assumed to be unaffected by the level of consumption in period one—identified as the *risk preference independence* postulate. This assumption implies that each of the conditional orderings is identical. (Note that in a large number of TPC formulations of the consumption-portfolio problem, the two-period NM—von Neumann-Morgenstern—index W is assumed to be additively separable, which clearly implies risk preference independence.) Let us further suppose that the (common) conditional ordering, $\lesseqgtr c_1$, can be represented by a (single-attribute) *second*-period expected utility function; i.e., there exists a continuous (strictly monotonically increasing) period-two NM index V such that for any pair of c_2-c.d.f.'s F and G

$$F \lesseqgtr c_1\, G \Leftrightarrow \int V(c_2)\, dF(c_2) \leq \int V(c_2) dG(c_2)\ . \tag{3}$$

Then the intertemporal choice between pairs such as (c_1', F) and (c_1'', G) can be decomposed into two steps. First, these pairs can be converted into the certain first-period, certainty equivalent second-period consumption pairs $(c_1', \hat{c}_2^F)$ and $(c_1'', \hat{c}_2^G)$ by using the consumer's second-period expected utility function, where the certainty equivalents are defined as follows:

$$\hat{c}_2^F = V^{-1}[\int V(c_2) dF(c_2)] \text{ and } \hat{c}_2^G = V^{-1}[\int V(c_2) dG(c_2)]\ . \tag{4}$$

Then the latter pairs can be ordered by a (continuous) ordinal time preference function U defined on certain consumption plans (which, as is shown in the proof of theorem 1 in [30], in essence corresponds to Ψ restricted to the set $\{(c_1, F^*)\}$, where F^* denotes a degenerate or one-point c.d.f.). Together these two steps are order-preserving in the sense that (for any c_1', c_1'', F and G)

$$\begin{aligned}(c_1', F) \lesseqgtr (c_1'', G) \Leftrightarrow \Psi(c_1', F) &= U(c_1', \hat{c}_2^F) \\ &\leq U(c_1'', \hat{c}_2^G) = \Psi(c_1'', G).\end{aligned} \tag{5}$$

This procedure will be referred to as the OCE representation. (Subject to minor changes, essentially the same argument can be used for the case when risk preference independence is not assumed—cf., [30].) Thus our proposed alternative to the *two-period* (multiattribute) expected utility model is based on a *second-period* (single-attribute) expected utility function and a *two-period* ordinal index.

Let us next consider how the OCE and two-period expected utility models are related. First of all, the preference ordering over all of S (not just each S_{c_1}) will be representable in accord with the expected utility principle if and only if there exists a (continuous) *two-period* NM index W such that (for all c_1', c_1'', F, and G)

$$(c_1', F) \overset{S}{\lesssim} (c_1'', G) \Leftrightarrow h\,[\Psi(c_1', F)] = \int W(c_1', c_2)\, dF(c_2) \leq \int W(c_1'', c_2)\, dG(c_2) = h\,[\Psi(c_1'', G)], \quad h' > 0\,. \tag{6}$$

Further assuming $\overset{S}{\lesssim}$ to exhibit risk preference independence implies that (Pollak [23] and Keeney [13])

$$W(c_1, c_2) = \alpha(c_1) + \beta(c_1)\, V(c_2), \qquad \beta(c_1) > 0\,. \tag{7}$$

Now it is shown in Selden [30] (theorem 2) that under comparable assumptions, the OCE representation hypothesis includes the two-period expected utility paradigm as a limited special case. Thus, together, the *existence of a (continuous) ordinal time preference function U and the NM representability of the consumer's conditional risk preferences are sufficient for there to exist an OCE representation of* $\overset{S}{\lesssim}$, *but are not enough for it necessarily to be linear in the probabilities as is required to have a two-period expected utility function* (cf., eq. (6)). In order to obtain the latter representation, Rossman and Selden [26] have shown that an additional axiom, referred to as "coherence," is required.

In adding that extra axiomatic structure required for $\overset{S}{\lesssim}$ to be NM representable according to (6), one however produces a *specific strong interdependence between risk and time preferences*. An individual's time preference representation U and his (two-period) NM index W, both defined on certain first- and second-period consumption pairs, are closely related; since they define the same indifference classes, each is an increasing monotonic tranformation of the other (Pollak [23]). Thus, if $\overset{S}{\lesssim}$ is NM representable and exhibits risk preference independence,

$$U(c_1, c_2) = T\,[\alpha(c_1) + \beta(c_1)\, V(c_2)], \qquad T' > 0, \tag{8}$$

(which clearly includes the two-period additively separable form as a special case). In contrast, the more general OCE representation permits one to prescribe risk preferences (V) and time preferences (U) *separately*—thereby making possible an explicit modeling of their interrelationship (including the possible cases of complete independence and the (two-period) expected utility-dependence (8)).

4. FORMULATION AND GRAPHICAL EXPOSITION

In this section, we show that under suitable conditions the *joint* consumption-portfolio decision can be *split,* both *analytically* and *graphically,* into *two distinct but generally interdependent* optimization *problems:* the conditional portfolio and consumption-savings problems. In later sections this *separation* will enable us on the one hand to utilize a number of results from both the one-period portfolio and multiperiod (certain) consumption theories, and on the other to generalize much of the classic Fisherian consumption-savings analysis to our uncertain setting. Throughout we assume the consumer's preferences over "certain-uncertain" consumption pairs to be OCE representable.

Conditional Portfolio Problem

For a given setting $(y_1, R_F, p, p_F, \tilde{e})$, an individual can be thought of as facing a *set* of one-period portfolio problems, *each* of which is *conditional upon* an assumed value of *first-period consumption* and characterized by his seeking to pick that bundle of securities producing the preferred c.d.f. on second-period consumption.

Definition 2. For a *fixed* level of first-period consumption c_1, the agent's *"conditional"* portfolio problem will be

$$\max_{n,n_F} \; EV(\tilde{c}_2) = \max_{n,n_F} \int V\left(\sum_{j=1}^{m-1} n_j e_j + n_F e_F\right) \mathrm{dK}(\cdot)$$

$$s.t.\; y_1 - c_1 - \sum_{j=1}^{m-1} n_j p_j - n_F p_F = 0, \tag{9}$$

where $K(\cdot)$ is defined to be Prob $\{\tilde{e} \leqslant e\}$.

With a view to the existence and properties of the conditional portfolio optima, we assume the following.

Assumption 2. (1) The set of feasible portfolios $\mathrm{N} \equiv \{(n, n_F)\}$ is compact, convex, and contains the zero vector.[4] (2) The *period-two* NM *index* V is third-order continuously differentiable, strictly monotone increasing, and strictly concave. (3) The *gross return variables* satisfy: (i) Prob $\{\tilde{e}_k < 0\} = 0$ and $E(\tilde{e}_k) < \infty$ (for $k = 1, \ldots, m - 1, F$) and (ii) the condition that the securities "not be perfectly correlated."

Let $K(\cdot)^{\#}$ be a particular, given joint asset return distribution. Let Γ be the space of "environments" defined as a product set of $m + 3$ compact intervals in $[0, \infty)$, where each interval corresponds respectively to one of the $m + 3$ deterministic "environmental" variables $c_1, y_1, R_F, p_1, \ldots p_{m-1}, p_F$.

[4]This condition clearly requires amendment of assumption 1 (2). Although the restriction on short-selling and borrowing implied by this boundedness assumption on N is not as general as that of Leland [16] or of Bertsekas [2], it nevertheless allows us to avoid the basic existence problem with a miminum of distraction from the main issues of the present study. Cf., Selden [31]. It should be noted that we are here ignoring the important question of default risk—see Stiglitz [33] and Smith [32]. I am indebted to the referee for his helpful comments on this point.

Assumptions 1 and 2 imply that the expected utility (indifference) hypersurfaces are strictly convex in the asset space N, and that *each* conditional portfolio problem (i.e., dependent on c_1) possesses a *unique* optimum$(n^{\#}, n_F^{\#})$ satisfying[5] the following first-order conditions:

$$E\,[V'\,(\tilde{c}_2^{\#})\,(\tilde{e}_j - p_j R_F)] = 0 \qquad j = 1, \ldots, m-1$$
$$y_1 - c_1 - \sum n_j^{\#} p_j - n_F^{\#} p_F = 0\,. \tag{10}$$

Furthermore, each conditional optimum is (by an appropriate version of the implicit function theorem) expressible as a local function of the environment and, as shown in [31] (also see [11]), these local functions can be "patched together" to obtain the *global* asset demand function $h{:}\Gamma \rightarrow N$ defined by

$$(h^1\,(c_1, y_1, R_F, p_1, \ldots, p_F; K\,(\cdot)^{\#}), \ldots, h^F\,(c_1, y_1, R_F, p_1, \ldots$$
$$\ldots, p_F; K\,(\cdot)^{\#})) = (n_1, \ldots, n_F), \tag{11}$$

(where each h^j, $j = 1, \ldots, F$, is twice continuously differentiable). Thus, for any environment $(c_1, y_1, R_F, p_1, \ldots, p_F) \in \Gamma$, the vector of asset holdings given by (11) maximizes expected utility for the corresponding "conditional" portfolio problem.

Note that corresponding to the conditional asset demand relations (11) will be, if one defines investible wealth by $l \equiv l\,(c_1, y_1) = y_1 - c_1$, the alternative demand functions

$$(H^1\,(l, R_F, p_1, \ldots, p_F; \quad K\,(\cdot)^{\#}), \ldots, H^F\,(l, R_F, p_1,$$
$$\ldots, p_F; K\,(\cdot)^{\#})) = (n_1, \ldots, n_F), \tag{12}$$

where $h^k\,(c_1, y_1, R_F, \ldots) = H^k\,(l\,(c_1, y_1), R_F, \ldots)$, $k = 1, \ldots, m-1, F$.

The set of conditional portfolio problems analyzed above can be represented graphically in terms of the conventional static demand-theoretic apparatus of convex indifference curves (hypersurfaces) and linear budget lines (hyperplanes). This is illustrated in Figure 1 for the two asset (one risky and one risk-free) case. Corresponding to period-one consumption values of ${}_ac_1$, ${}_bc_1$, ${}_dc_1$, and ${}_0c_1$, four separate conditional portfolio problems are portrayed together with their respective unique optima, denoted ${}_aQ$, ${}_bQ$, ${}_dQ$, ${}_0Q$. Increases in first-period consumption correspond to decreases in investable wealth (l) and hence to parallel shifts of the budget line back toward the origin. As this is done continuously for all $c_1 \in [0, y_1]$, the "expansion path" 0_oQ is generated.

[5] We are here assuming $(n^{\#}, n_F^{\#}) \in$ Interior N.

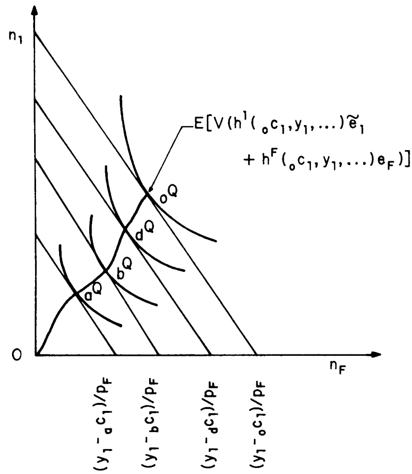

Fig. 1. Set of Conditional Portfolio Problems

Consumption-Savings Problem

Having solved the conditional portfolio problem given any level of first-period consumption, the consumer-investor then addresses the question of his optimal division of initial endowment between first-period consumption and *total* investment. Before stating formally the *OCE formulation* of this problem, let us make the following (simplifying) assumption.

Assumption 3. U is twice continuously differentiable, strictly quasi-concave, and satisfies, for all (nonnegative) pairs (c_1, c_2), $U_1(c_1, c_2)$, $U_2(c_1, c_2) > 0$.

Definition 3. Given definition 1 and assumptions 1–3, we define $(c_1{}^*, n^*, n_F{}^*)$ to be optimal if it solves

$$\max_{c_1,\, \hat{c}_2} U(c_1, \hat{c}_2)$$

$$s.t.\ \hat{c}_2 - t\,(c_1, y_1, R_F, p_1, \ldots, p_F; K(\cdot)^{\#}) = 0 \tag{13}$$

where (1) the consumption transformation function t is defined as follows:[6]

$$t\,(c_1, y_1, R_F, \ldots) = V^{-1}\,(EV\,\{[y_1 - c_1 - \textstyle\sum p_j h^j(c_1, y_1, \ldots)]\,R_F$$
$$+ \textstyle\sum \tilde{e}_j h^j\,(c_1, y_1, \ldots)\}); \tag{14}$$

(2) the set of feasible $(c_1, \hat{c}_2)$-pairs is restricted to the nonnegative orthant; (3) $h^1, \ldots, h^F$ are the *global, conditional* asset demand functions obtained from the

[6]We are ignoring the possible complication that for some "environments" and V's, $EV(\tilde{c}_2)$ and hence t may not be defined.

set of feasible conditional portfolio problems; and (4) the intertemporally optimal $(n_1{}^*, \ldots, n_F{}^*)$ is obtained from

$$n_k{}^* = h^k(c_1{}^*, \ldots) \qquad k = 1, \ldots, F .$$

One of the merits of the OCE separation of the joint consumption-portfolio problem is that the consumption-savings portion can be depicted in terms of a two-dimensional diagram paralleling the classic Fisherian (certainty) portrayal. This is illustrated in Figure 2. Building on the two-asset example of Figure 1, we know that for a given level of period-one consumption, say ${}_ac_1$, there exists a unique portfolio optimum $({}_aQ)$. Corresponding thereto is the maximal certainty equivalent[7]

$${}_a\hat{c}_2{}^{\#} = V^{-1}\{EV[h^1({}_ac_1, y_1, \ldots)\tilde{e}_1 + h^F({}_ac_1, y_1, \ldots)e_F]\}, \quad (15)$$

where the supersharp indicates that it is a conditional maximum. Repeating this process for other $c_1 \epsilon [0, y_1]$ produces the (dashed) "efficient frontier" in Figure 2. For now let us just assume that the consumption opportunity set is convex—sufficient conditions will be given in the next section.

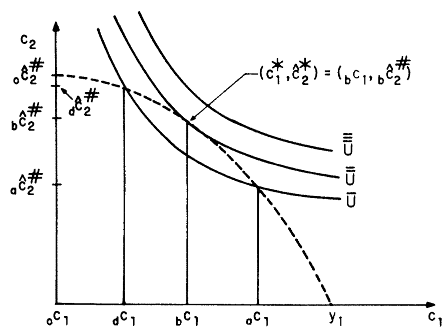

Fig. 2. Consumption-Savings Portion of the Consumption-Portfolio Problem

So far, in generating the "efficient set," only the agent's risk preferences (as represented by $EV(\tilde{c}_2)$) have been employed. However, to determine the overall optimum from this set of $(c_1, \hat{c}_2{}^{\#})$-pairs, it is necessary to employ his (ordinal) representation of preferences over certain consumption plans. As indicated in Figure 2, this results in $({}_bc_1, {}_b\hat{c}_2{}^{\#})$ being optimal.[8] The intertemporally optimal savings (or

[7]To see that *this* $\hat{c}_2$-value is a maximum, note that when $EV(\tilde{c}_2)$ is evaluated using the asset demand functions h^1 and h^F, the largest value of expected utility holds. Since $\hat{c}_2$ differs from $E[V(\tilde{c}_2)]$ only by the increasing monotonic transformation V^{-1}, $\hat{c}_2$ and $EV(\tilde{c}_2)$ must be "equivalent" representations. Thus $\hat{c}_2$ achieves a maximum when $EV(\tilde{c}_2)$ does.

[8]It is not difficult to show that the joint consumption-portfolio problem (13) possesses a solution, $(c_1{}^*, n^*, n_F{}^*)$.

total investment) and specific asset holdings, respectively, are then given by $y_1 - {}_bc_1$ and $(h^1({}_bc_1, \ldots), h^F({}_bc_1, \ldots))$ (the latter corresponding to ${}_bQ$ in Figure 1).

Remarks

1. Since (under the assumptions of section 3) every TPC representation can be expressed as an OCE representation, it follows that the TPC version of the joint consumption-portfolio problem, equation (1), can be reformulated in the above OCE framework: i.e., it can be decomposed both analytically and graphically into a set of conditional portfolio problems and a (Fisherian) consumption-savings decision problem.

2. It would be possible using the more general OCE utility hypothesis to solve the consumption-savings and portfolio problems simultaneously paralleling the conventional TPC approach. However, one of the primary objectives of this paper is to show that it is more illuminating economically to separate the problems.

3. Finally, we note that it is possible to drop the risk preference independence assumption introduced in section 3 and still *separate* the conditional portfolio and consumptions-savings problems. Let conditional risk preferences $[\lesseqgtr c_1]$ be NM representable with the period-two (conditional) NM index $V_{c_1}(c_2)$ depending differentiably on first-period consumption. Then an OCE representation will exist, as noted in section 3, and it is straightforward to show that the conditional portfolio problem can be solved using just the period-two NM index V_{c_1}, and further, that in the consumption-savings problem, just the consumption transformation function (and not the time preference indifference map) is affected by the risk preference dependence. We leave this generalization for later consideration.

5. ON THE SHAPE OF THE FINANCIAL AND CONSUMPTION OPPORTUNITY SETS: EXTENSION OF FISHERIAN TWO-PERIOD DIAGRAMMATICS

Definition 3 can be thought of as establishing a "consumer technology" in the Muth-Lancaster tradition. The consumer-investor can be interpreted as a "household" or "quasi" producer. Given the unconsumed portion of his y_1-endowment, he acquires a portfolio of financial securities, the aggregate random return on which he converts into a certainty equivalent value for second-period consumption. Thus $\hat{c}_2$ may be viewed as a final consumption good or as a "characteristic" not acquired through exchange, but rather "produced" via the portfolio optimization and certainty equivalent process from the traded inputs (n, n_F).

Key elements of this interpretation are noted in Figure 3. Define the financial transformation function T as follows

$$T(l, R_F, p_1, \ldots, p_F; K(\cdot)^{\#}) = V^{-1}\left(EV\left\{\left[l - \sum p_j H^j(l, \ldots)\right] R_F + \sum \tilde{e}_j H^j(l, \ldots)\right\}\right) = \hat{c}_2 . \tag{16}$$

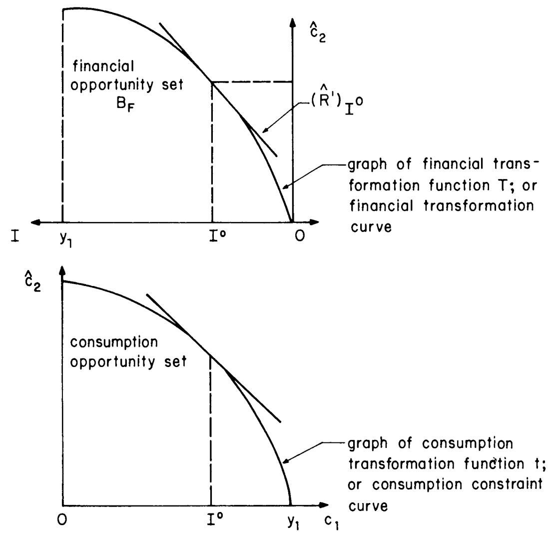

Fig. 3. Financial and Consumption Opportunity Sets

The consumption transformation function t was defined by equation (14) in definition 3. It is then perfectly consistent with standard "production" theory to refer to

$$\hat{R}' = \partial \hat{c}_2 / \partial l = \partial T(l, \ldots) / \partial l \tag{17}$$

as the "household" producer's portfolio (or overall) *marginal certainty equivalent rate of return.*

As suggested in the preceding section, the *OCE formulation of the joint consumption-portfolio problem represents a generalization of the classic Fisherian certainty consumption-savings analysis.* As a consequence, many similarities, especially in interpretation, can be noted. First, like Fisher, we view "time preferences" as being concerned essentially with one's trade-off between *certain* first- and second-period consumption even if he also confronts risky consumption possibilities.[9] Perhaps the most obvious similarity relates to our two-period graphical exposition presented in Figure 2, and the concomitant separation of the agent's "consumption possibilities" and his certain time preferences.

Under this OCE separation what can be said about the shape of the financial transformation and consumption constraint curves? First of all, in addition to being continuous, the financial (consumption) transformation function is strictly monotone increasing in l (decreasing in c_1). Second, the following establishes a relatively simple

[9]This is reflected in Figure 2 by the fact that the consumer's indifference map, which is employed in making choices between different $(c_1, \hat{c}_2)$-pairs, is *based* on his preferences over *certain* consumption pairs. Also see the rather extensive, related discussion in section 6—especially note the quoted passage from [9] on the "marginal rate of time preference" in the presence of uncertainty.

sufficient condition for the consumer-investor's "financial opportunity set" B_F of nonnegative $(l, \hat{c}_2)$-pairs satisfying $\hat{c}_2 - T(l, R_F, \ldots) \leq 0$ to exhibit decreasing, constant, or increasing returns to scale.[10]

THEOREM 1. *Given definition 3 and interpreting the consumer-investor as a "household" producer, B_F exhibits {decreasing, constant, increasing} returns to scale at $l^o \in (0, y_1)$ if*

$$\frac{\partial}{\partial l} \log V'(\hat{c}_2) \Bigg|_{l^0} \{>, =, <\} \frac{\partial}{\partial l} \log E[V'(\tilde{c}_2)] \Bigg|_{l^0} . \tag{18}$$

(The proof of this result as well as the proofs of the corollary below and theorem 2 have been deleted due to space limitations; however, they are available upon request from the author.)

It follows from theorem 1 that the consumption constraint curve (cf., Fig. 3) will be {concave, linear, convex} if[11] the rate of proportional change in the marginal utility of the ("conditional" portfolio optimum) second-period certainty equivalent is {>, =, <} the rate of proportional change in the expected marginal utility of ("conditional" portfolio optimum) second-period consumption for an increase in investable (endowed) wealth.

The condition in theorem 1 depends on both risk preferences and probabilities. However, there exists a simple sufficient condition, involving *just* V, for B_F to exhibit constant returns to scale for all $l \in (0, y_1)$. Before giving this result, let us define the standard measure of absolute risk aversion [1, 24]:

$$\rho_A(c_2) = -V''(c_2)/V'(c_2) . \tag{19}$$

COROLLARY. *If the consumer-investor's period-two NM index satisfies*

$$[-V''(c_2)/V'(c_2)] = (a + bc_2)^{-1}, \tag{20}$$

then his "financial opportunity set" exhibits constant *returns to scale for all $l \in (0, y_1)$ and the consumption constraint curve is linear for all $c_1 \in (0, y_1)$.*[12]

The family of NM indices satisfying (20) is frequently referred to as the HARA (hyperbolic absolute risk aversion) class (e.g., Merton [20] and Rubinstein [27]). Also, this condition has been shown by Cass and Stiglitz [4] and Leland [15] to be necessary and sufficient for portfolio separation (assuming arbitrary asset return distributions and the presence of a risk-free asset).

Given the Fisherian interpretation proposed earlier, the corollary returns one essentially to a consumption-savings setting identical in *form* to that of the classical

[10] B_F will be said to exhibit decreasing, constant, or increasing returns to scale at l^0 if and only if $T(\lambda l^0; R_F, p_1, \ldots, p_F; K(\cdot)^*)$ [<, =, >] $\lambda T(l^0; R_F, p_1, \ldots, p_F; K(\cdot)^*)$, $\lambda > 1$ and $l^0 \in (0, y_1)$. See Malinvaud [17, pp. 43ff].

[11] We are here assuming no "isolated singularities."

[12] I am grateful to Jim Scott for suggesting a generalization of an earlier version of this result.

perfect capital markets two-period paradigm (but where the OCE consumption constraint curve slope need not equal the Fisherian value of $-R_F$).

6. PERSONAL EQUILIBRIUM

An interior consumption optimum $(c_1^*, \hat{c}_2^*)$ for the OCE version of the joint consumption-portfolio problem will satisfy the following first-order condition:

$$U_1(c_1^*, \hat{c}_2^*)/U_2(c_1^*, \hat{c}_2^*) = R_F E\,[V'(\tilde{c}_2^*)]/[V'(\hat{c}_2^*)] = (\hat{R}')^*, \tag{21}$$

where the far RHS was referred to [cf., eq. (17)] as the consumer-investor's (gross) portfolio (or overall) marginal certainty equivalent rate of return. If the only investment vehicle available is asset F, this expression collapses to the standard certainty result

$$U_1(c_1^*, c_2^*)/U_2(c_1^*, c_2^*) = R_F\,. \tag{22}$$

In seeking to interpret (22), economists invariably introduce the Fisherian notion of the "marginal rate of time preference" defined by

$$MTP(c_1, c_2) = (-dc_2/dc_1) - 1 = [U_1(c_1, c_2)/U_2(c_1, c_2)] - 1\,. \tag{23}$$

Combining (22) and (23) yields the famous Fisherian personal equilibrium condition

$$MTP(c_1^*, c_2^*) = R_F - 1, \tag{24}$$

where the *MTP function* is evaluated at the optimum consumption plan (c_1^*, c_2^*).

Given the OCE representation of preferences reviewed in section 3, equation (23) is highly suggestive of how to extend the notion of the "marginal rate of time preference" to the "certain-uncertain" intertemporal setting:

$$MTP(c_1, \hat{c}_2) = (-d\hat{c}_2/dc_1) - 1 = [U_1(c_1, \hat{c}_2)/U_2(c_1, \hat{c}_2)] - 1\,. \tag{25}$$

That this is quite consistent with the views of Irving Fisher can be seen from the following quote: "uncertainty must naturally have an influence on the rate of time preference . . . of its possessor; it is to be remembered that the [rate of time preference] is the percentage preference for \$1 *certain* of immediate [consumption], over \$1, *also certain* of [consumption] of one year hence, *even if all the [consumption] except that dollar be uncertain*" [9, pp. 76–77].[13]

[13] The bracketed words have been substituted for the original. It seems clear from this quote that for Irving Fisher the required increment in period-two consumption must be *certain*, and that is exactly what $U_1(c_1,c_2)/U_2(c_1,c_2)-1$ measures. But according to the OCE representation hypothesis so does $U_1(c_1,\hat{c}_2)/U_2(c_1,\hat{c}_2)-1$, since under the specified set of axioms the decision maker does not distinguish between the required premium in second-period consumption being a "strictly certain" quantity (as, for instance, obtained from $n_F e_F$) or a certainty equivalent (as, for instance, obtained from $V^{-1}EV(n_1\tilde{e}_1)$)—they are for him perfect substitutes.

It is then a simple matter of substitution to obtain what we view as a natural extension of the Fisherian certainty rule (24)

$$MTP(c_1{}^*, \hat{c}_2{}^*) = (\hat{R}')^* - 1 = R_F\, E[V'(\tilde{c}_2{}^*)]/V'(\hat{c}_2{}^*) - 1 \,. \tag{26}$$

Thus an individual will balance his first-period consumption and total savings so as to equate his marginal rate of time preference (for consumption in time-period one relative to the certainty equivalent consumption in period-two) and his (net) portfolio marginal certainty equivalent rate of return.[14]

As the following theorem establishes, under the standard assumption of decreasing absolute risk aversion, the consumer will require a premium in certainty equivalent second-period consumption, in excess of the net risk-free rate of interest, to be willing to postpone one unit of first-period consumption.

THEOREM 2.

$$MTP\ (c_1{}^*, \hat{c}_2{}^*) \gtreqless R_F - 1 \text{ as } \rho_A{}'\ (c_2) \lesseqgtr 0, \tag{27}$$

where $\rho_A(c_2)$ was defined by equation (19) to be the Arrow-Pratt measure of absolute risk aversion.

(This important inequality condition between the agent's marginal rate of time preference and R_F can be explained quite simply in terms of an *indirect* (household) "*production*" *effect*.)

Let us next compare these personal equilibrium results with those traditionally obtained under the TPC formulation of the joint consumption-portfolio problem, equation (1). Performing the indicated constrained maximization yields the following analogue of (21):

$$E\ [W_1\ (c_1{}^*, \tilde{c}_2{}^*)]/\ E\ [W_2\ (c_1{}^*, \tilde{c}_2{}^*)] = R_F \,. \tag{28}$$

Note that this expression holds for any TPC representation of $\precsim$. However, if one assumes, as is frequently done in TPC analyses of the joint consumption-portfolio problem, that $\precsim$ exhibits risk preference independence, then W takes the special form of equation (7). But then the two-period expected utility function is readily expressible as an OCE representation:

$$\begin{aligned} EW\ (c_1, \tilde{c}_2) &= \alpha\ (c_1) + \beta\ (c_1)\ EV\ (\tilde{c}_2) = \alpha\ (c_1) + \beta\ (c_1)\ V\ (\hat{c}_2) \\ &\equiv U\ (c_1, \hat{c}_2) \,. \end{aligned} \tag{29}$$

[14]Condition (26) makes explicit the respective roles of "time" and "risk" preferences in the consumer's optimum balancing of consumption and savings. Some care, however, must be taken in interpreting this comment. Under our new representation of preferences, the marginal time preference *function* is determined solely by the consumer's exogenously prescribed (riskless) time preference index U, and similarly his marginal certainty equivalent rate of return *function* is determined by his conditional risk preference index V (together with other nonpreference data). However, the optimum $(c_1{}^*, \hat{c}_2{}^*)$ and hence the *values of the MTP* and $\hat{R}'$ functions depend simultaneously on *both* U and V.

Substituting into equation (21) yields

$$[\alpha'(c_1^*) + \beta'(c_1^*)\, V(\hat{c}_2^*)]/[\beta(c_1^*)\, V'(\hat{c}_2^*)] = R_F EV'(\tilde{c}_2^*)/V'(\hat{c}_2^*) \quad (30)$$

or, cancelling in the denominators and noting that $V(\hat{c}_2) = EV(\tilde{c}_2)$,

$$[\alpha'(c_1^*) + \beta'(c_1^*)\, EV(\tilde{c}_2^*)]/[\beta(c_1^*)\, EV'(\tilde{c}_2^*)] = R_F . \quad (31)$$

These two expressions illustrate several very important points. First, it is evident from (30) that as a consequence of making the seemingly innocuous assumption that the representation of $\overset{s}{\precsim}$ be "linear in the probabilities," a very strong interconnection is produced between the consumer's utility functions U and V and hence between his *MTP* and $\hat{R}'$ functions. There is no quarrel that it might be reasonable to *model* some relationship between U and V. What seems quite unsatisfactory is to produce this interconnection as a byproduct of making a *simplifying* assumption concerning the *functional form* of the representation (cf., equation (8) and attendant discussion).

A second point can be drawn from comparing (30) and (31). *Even if* one chooses to employ axioms which imply the $\overset{s}{\precsim}$ to be *representable by an expected TPC utility function, the OCE formulation of the joint consumption-portfolio decision problem* allows one to segregate the roles of "time" and "risk" preferences. Thus the LHS of the OCE-personal equilibrium condition (30) is interpretable as one plus the marginal time preference function, and the RHS as the value of the marginal certainty equivalent rate of return function. Assuming that there exists a TPC utility function does not alter the fact that the *MTP*-function is determined by the time preference index U, and the $\hat{R}'$-function is determined by the conditional risk preference index V; but rather it establishes, as we argued above, an interdependence between U and V and hence also between the marginal time preference and marginal certainty equivalent rate of return functions.

The restrictiveness of assuming $\overset{s}{\precsim}$ to be NM representable reflects itself in still another, and perhaps more fundamental, way.

PROPOSITION. *The set of possible consumption-portfolio optima consistent with utility maximization (assuming risk preference independence) is "far larger" under the OCE representation hypothesis than under the TPC hypothesis.*

We shall illustrate this result with an example.

Example

Consider the case of a consumer-investor confronting the joint consumption-portfolio decision problem. There are only two financial assets, one risky (asset 1) and one risk-free (asset F). The problem is defined by a specific environment (i.e., values of y_1, R_F, p_1, and p_F and a particular distribution function $K(e_1)$). We assume there are only two states of nature, $\Omega = \{\omega_1, \omega_2\}$, with corresponding state probabilities of

$\pi = (1/2)$ and $(1 - \pi) = (1/2)$. The two complex securities are characterized by the following prices and return prospects:

$$\text{asset 1: } p_1 = 6\ \tilde{e}_1(\omega) = \begin{cases} 12 \text{ for } \omega_1 \\ 5 \text{ for } \omega_2 \end{cases}$$

$$\text{asset } F\text{: } p_F = 5\ e_F(\omega) = \begin{cases} 6 \text{ for } \omega_1 \\ 6 \text{ for } \omega_2 \end{cases}$$

Let $y_1 = 20{,}000$. Our individual's conditional risk preferences are NM representable with

$$V(c_2) = 1n\,(c_2)\,. \tag{32}$$

Under an *OCE FORMULATION of the joint decision problem* and the assumed "environment," it is straightforward to obtain the set of conditional asset demand functions and expansion path given in Figure 4a. It is then a simple matter to derive the linear consumption constraint curve portrayed in Figure 4b.

Up to this point the example is *exactly the same regardless of whether* we

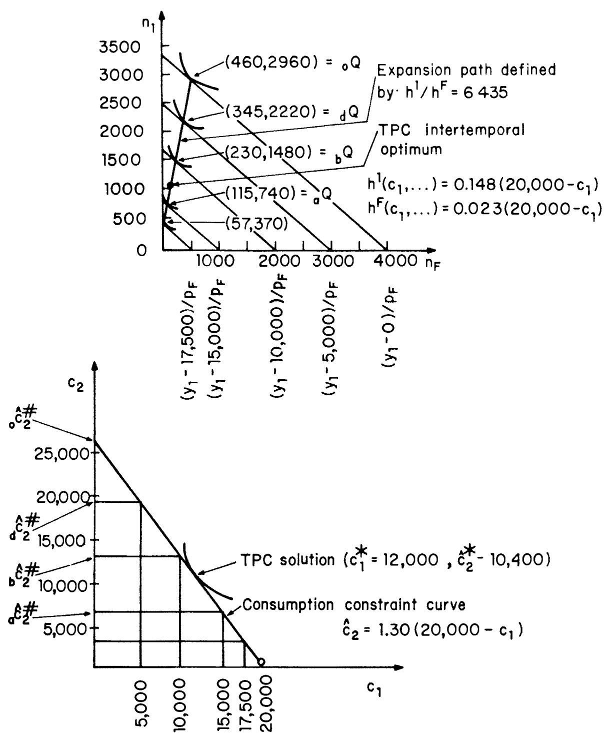

Fig. 4. Example

assume $\lesssim$ to be *TPC or OCE representable* (cf., first remark in section 4). What changes is the set of possible indifference maps over certain consumption pairs and hence the consumption optimum and intertemporally optimal asset demands. To see this, suppose that our individual's time preferences are representable by the following ordinal index

$$U(c_1, c_2) = -\alpha\, c_1{}^{-\delta}/\delta - (1-\alpha)\, c_2{}^{-\delta}/\delta, \tag{33}$$

where $0 < \alpha < 1$ and $-1 < \delta < \infty$. Now the *only* (up to a positive affine transform) TPC representation consistent with the assumption of risk preference independence and specifications (32) and (33) is the following (see [30]):

$$EW(c_1, \tilde{c}_2) \doteq \alpha \ln(c_1) + (1-\alpha)\, E \ln(\tilde{c}_2)\,. \tag{34}$$

Conversely, there will be an OCE representation corresponding to *every* value of δ in $(-1, \infty)$,

$$U(c_1, \hat{c}_2) = (-\alpha c_1{}^{-\delta}/\delta) - [(1-\alpha)/\delta]\,\{\ln^{-1}[E\, \ln(\tilde{c}_2)]\}^{-\delta} \tag{35}$$

(note that (34) follows from (35) when $\delta \to 0$).

For purposes of computation, let $\alpha = 0.6$ and suppose δ is restricted to the compact interval $[-0.9, 10]$. Then corresponding to the TPC representation (34) is the *single* joint consumption-portfolio optimum

$$(c_1{}^*, n_1{}^*, n_F{}^*) = (12000, 1184, 184)\,.$$

In striking contrast, making the same underlying assumptions produces for the OCE representation a *set* of optima (depending on δ), which corresponds to the following intervals[15]:

$$c_1{}^* \in [11375, 16894],\ n_1{}^* \in [1277, 460],\ n_F{}^* \in [198, 71]\,.$$

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[15]Two important caveats deserve mention. First, *given* a logarithmic conditional NM index V, it is not necessarily the case that all values of the time preference parameter $\delta \in [-0.9, 10]$ are equally plausible a priori. However, it is clear that producing a representation of $\lesssim$ linear in the probabilities is hardly a satisfactory (economic or decision theoretic) justification for restricting the time preference parameter δ to be equal to zero. But this is another story.

Second, there is a question of *approximation*. For some joint consumption-portfolio problems—i.e., specifications of the "environment" and choices of V and U—the set of optima produced by the richer *OCE* representation model may not differ *materially* from the *single TPC* solution.

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