

Optimal consumption and savings with stochastic income and recursive utility

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Abstract

We develop a tractable continuous-time consumption-savings model for a liquidity-constrained agent who faces both permanent and transitory income shocks under incomplete markets. We derive an explicitly-solved consumption function and show that the marginal (certainty equivalent) value of liquidity measures the effects of financial frictions on welfare. We further analytically characterize steady-state target savings and demonstrate that risk aversion and inter-temporal substitution have very different effects on savings and the dispersions of wealth and consumption.

Keywords: buffer stock; precautionary savings; incomplete markets; borrowing constraints; permanent income; non-expected utility

JEL Classification: G11, G31, E2

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1 Introduction

Income shocks and borrowing constraints can have significant effects on households' consumption and savings decisions especially when markets offer limited opportunities for households to manage these uninsurable income shocks. The literature by now has a large class of income-fluctuations and self-insurance models where the household maximizes intertemporal utility by optimally saving in a risk-free asset paying a constant rate of interest r over time to buffer against idiosyncratic income shocks subject to a no-borrowing limit, as analyzed in Deaton (1991) and Carroll (1997).

In this paper, we revisit the classic self-insurance problem by developing an analytically tractable continuous-time incomplete-markets model with the following important new features. First, we choose the non-expected recursive utility developed by Epstein and Zin (1989) and Weil (1990), as we will show that risk aversion and intertemporal substitution have very different qualitative and quantitative effects on optimal consumption and savings. Recent asset pricing research provides arguments in support of separating risk aversion from intertemporal substitution.¹ Second, our model allows us to incorporate a range of income processes with (a) permanent and transitory shocks, (b) diffusive/continuous and discrete/jump shocks, and (c) mean-reverting income growth shocks. Our model's flexibility and tractability in dealing with income processes offer us advantages when analyzing dynamics and (long-run) steady-state consumption-income and wealth-income ratios. Third, despite capturing a richer preference (than standard expected iso-elastic utility) and income shocks, our model remains highly tractable and the model's economic intuition is transparent. Additionally, our model can be readily used for quantitative exercises with plausible parameter values in general equilibrium Aiyagari-Bewley settings.

By exploiting the homogeneity property (implied by preferences and the income process), we characterize the optimal consumption-income ratio $c(w)$ and the value function via the effective state variable, denoted by w , the ratio between financial wealth and contemporane-

¹See Bansal and Yaron (2004) and the long-run risk literature. In a representative-agent equilibrium framework with recursive utility, Lustig and Van Nieuwerburgh (2008) show that the restrictions on the joint distribution of financial wealth returns, human wealth returns, and consumption will depend only on the elasticity of intertemporal substitution, not on the coefficient of risk aversion. Tallarini (2000) is an early important application of Epstein-Zin utility to asset pricing with production. Also for related applications of risk-sensitive preferences to asset pricing, see Hansen, Sargent, and Tallarini (1999).

ous labor income.² Intuitively speaking, the larger this liquidity ratio w , the less financially constrained the agent. Our model captures this insight by showing that the household's welfare, measured by the certainty equivalent wealth per unit of income $p(w)$, is increasing and concave in w , which further implies that the marginal value of wealth $p'(w)$ is higher for a lower value of w . Importantly, we show that this marginal value of liquidity $p'(w)$ plays a critical role in determining consumption and savings under incomplete markets.

Our continuous-time model offers several advantages. First, it gives an economically intuitive closed-form formula for the optimal consumption-income ratio $c(w)$ that depends on both the level of the certainty equivalent wealth $p(w)$ and also the slope of $p(w)$, which we refer to as the marginal value of liquidity $p'(w)$. While it is straightforward to see the intuitive result that consumption increases with certainty equivalent wealth $p(w)$, it is less obvious to see that the higher the marginal value of liquidity $p'(w)$, the lower the agent's consumption $c(w)$. The intuition for the latter result is that current consumption is more costly in terms of forgone future consumption when the household's marginal value of liquidity, $p'(w)$, is higher, *ceteris paribus*. Our paper is the first to explicitly show that the marginal value of liquidity $p'(w)$ plays a critical role determining optimal consumption.

Naturally, the next question is how we solve the certainty equivalent wealth $p(w)$ and its slope, the marginal value of liquidity $p'(w)$. This leads to the second advantage of our formulation. We show that the certainty equivalent wealth $p(w)$ satisfies a simple nonlinear ordinary differential equation (ODE), which is easy to solve with high numerical precision.

Third, whether the liquidity constraint $w \geq 0$ binds or not only matters at the left boundary $w = 0$ via a simple check on the optimal consumption at $w = 0$. If $c(0) < 1$, then the liquidity constraint $w > 0$ never binds as the agent's voluntary saving's motive is sufficiently strong at $w = 0$ inducing the household to stay away from the no-borrowing constraint $w \geq 0$ with probability one by choosing $C < Y$ when $W = 0$. In this case, lifting the borrowing constraint has no impact on consumption and $p(w)$. However, it is possible for a rational household with no liquid wealth to voluntarily choose to live from paycheck to paycheck or hand-to-mouth with no savings, i.e., $c(0) = 1$. This is consistent with the empirical findings in Campbell and Mankiw (1989). For these hand-to-mouth households, $w = 0$ is an absorbing state, as predicted by our model. That is, whether the left boundary

²This homogeneity property is also exploited in the discrete-time expected isoelastic utility formulation by Carroll (1997).

of the ODE for $p(w)$ binds or not informs us whether the household is hand-to-mouth (the binding case) or not (the non-binding case.)

Finally, the boundary condition (as w approaches ∞) for the ODE is a natural complete-markets (CM) perfect risk-sharing case: When the agent's liquidity w reaches infinity, self insurance via savings achieves the first-best CM result, in that $\lim_{w \rightarrow \infty} p'(w) = 1$.

In summary, we provide an explicit characterization of $c(w)$ through a simple nonlinear optimal consumption function and a tractable nonlinear ODE for the scaled certainty equivalent wealth $p(w)$ with intuitive boundary conditions reflecting the economics of the borrowing constraint and CM perfect risk sharing limiting-case solution.³

Additionally, we show that intertemporal substitution and risk aversion have very different effects on consumption and buffer-stock savings behavior, both qualitatively and quantitatively. For example, changing the coefficient of relative risk aversion (e.g. from two to four) while holding EIS fixed leads to quantitatively very large increases in (1) the level of buffer-stock savings and (2) the dispersions of both wealth and consumption (scaled by income) distributions. In our baseline calculation, as we increase risk aversion from two to four, the steady-state target for the wealth-income ratio increases significantly from 2.6 to 16.1.

Related Literature. Hall (1978) formalizes Friedman's seminal permanent-income hypothesis via the martingale (random-walk) consumption in a dynamic programming framework.⁴ Zeldes (1989) and Deaton (1991, 1992) numerically solve the optimal consumption rule for expected constant relative risk averse (CRRA) utility. Carroll (1997) notes the importance of impatience and develops buffer-stock savings models for expected CRRA utility.⁵

While the expected CRRA utility is the most commonly used preferences in the literature, for tractability reasons, constant absolute risk averse (CARA) utility has also been used largely for its tractability. Caballero (1990, 1991) solves the optimal consumption and saving

³Nonlinearity is important in both single-agent models such as ours and general equilibrium models. For example, see Brunnermeier and Sannikov (2011) for an illustration on the importance of nonlinearity in a macro equilibrium model with financial frictions.

⁴For early contributions on consumption and savings in stochastic settings, see Leland (1968), Levhari and Srinivasan (1969), Sandmo (1970), Dreze and Modigliani (1972), and Schechtman (1976), among others.

⁵For other related work, see He and Pages (1993), Duffie et al. (1997), Koo (1998), Chamberlain and Wilson (2000), Gourinchas and Parker (2002), Storesletten, Telmer, and Yaron (2004), Aguiar and Hurst (2005), Guvenen (2006, 2007), Fernández-Villaverde and Krueger (2007), and Kaplan and Violante (2010), among others.

rules in closed form by assuming CARA utility and conditionally homoskedastic (autoregressive and moving average) labor-income processes.⁶ Wang (2006) generalizes Caballero (1990, 1991) to allow for income to be conditionally heteroskedastic and obtains a closed-form optimal consumption rule.⁷ Unlike Caballero (1991) where precautionary savings demand is constant, Wang (2006) finds precautionary savings to be stochastic and the MPC out of human wealth to be lower than that out of financial wealth. Weil (1993) solves a discrete-time precautionary-saving model in closed form under the assumption of a non-expected utility with CARA and a constant elasticity of intertemporal substitution (EIS) and finds a constant precautionary savings demand as Caballero (1990, 1991) due to the assumption of atemporal CARA utility. Naturally, it is hard to calibrate the CARA coefficient in Weil (1993). Unlike these CARA-based expected or non-expected utility models, our model captures wealth effects, includes both uninsurable income shocks and liquidity constraints, and can be readily used for quantitative exercises.

2 Model

We consider a continuous-time environment where an infinitely-lived agent receives an exogenously given perpetual stream of stochastic labor income. The agent saves via a risk-free asset that pays interest at a constant rate $r > 0$. There exist no other financial assets.⁸ Hence, markets are incomplete with respect to labor-income shocks.

A typical specification of a labor-income process in the literature involves both permanent and transitory shocks.⁹ For expositional convenience, we first specify the labor income process Y as one with permanent diffusion shocks only and leave the generalization to allow for large jump shocks to Section 6 and the generalization to allow for both permanent and transitory shocks to Section 7.

⁶Merton (1971) also solves an incomplete-markets consumption/portfolio choice model with CARA utility and labor income process. Kimball and Mankiw (1989) use CARA utility and obtain an effectively explicit consumption rule with a two-state Markov chain for the income in an incomplete-markets setting with taxes.

⁷See Cox, Ingersoll, and Ross (1985) for a seminal contribution and Duffie (2001) for a textbook treatment on “affine” models in finance.

⁸Duffie, Fleming, Soner, and Zariphopoulou (1997) discuss the existence of the solution for an expected utility model with a risky financial asset but do not analyze the impact on consumption.

⁹See MaCurdy (1982), Abowd and Card (1989), and Meghir and Pistaferri (2004), for example. Given our focus, we also ignore life-cycle variations and various fixed effects including education and gender.

First, we consider the following widely used labor-income process specification:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t, \quad Y_0 > 0, \quad (1)$$

where B is a standard Brownian motion, μ is the expected income growth rate, and σ measures income growth volatility. The labor-income process (1) implies that the growth rate of income, dY_t/Y_t , is independently, and identically distributed (i.i.d.).

We may also write the dynamics for logarithmic income, $\ln Y$, by using Ito's formula as,

$$d \ln Y_t = (\mu - \sigma^2/2) dt + \sigma dB_t. \quad (2)$$

Equation (2) is an arithmetic Brownian motion, which implies that $\ln Y$ is a unit-root process. The change of income $\Delta \ln Y$ has mean $(\mu - \sigma^2/2)$ and volatility σ per unit of time. While income shocks are i.i.d. for the income growth rate, they are permanent in *levels* of Y .

The widely-used standard preference in the consumption/savings literature is the expected utility with constant relative risk aversion, which ties the elasticity of intertemporal substitution (EIS) to the inverse of the coefficient of relative risk aversion. Conceptually, risk aversion and the EIS are fundamentally different and have different effects on consumption-savings decisions. We thus use the non-expected recursive utility developed by Epstein and Zin (1989) and Weil (1990), who build on Kreps and Porteous (1978), which allows economically meaningful separation between risk aversion and the EIS. Additionally, we show that the EIS and risk aversion have very different effects on optimal consumption. Specifically, we use the continuous-time formulation of this non-expected utility developed by Duffie and Epstein (1992a), and write the recursive utility process V as follows,

$$V_t = \mathbb{E}_t \left[\int_t^\infty f(C_u, V_u) du \right], \quad (3)$$

where $f(C, V)$ is known as the normalized aggregator for consumption C and utility V . Duffie and Epstein (1992a) show that $f(C, V)$ for this recursive utility is given by

$$f(C, V) = \frac{\rho}{1 - \psi^{-1}} \frac{C^{1-\psi^{-1}} - ((1 - \gamma)V)^\theta}{((1 - \gamma)V)^{\theta-1}}, \quad (4)$$

and

$$\theta = \frac{1 - \psi^{-1}}{1 - \gamma}. \quad (5)$$

Here, ψ is the EIS, γ is the coefficient of relative risk aversion, and ρ is the subjective discount rate. The widely used time-additive separable CRRA utility is a special case of the recursive utility where the coefficient of relative risk aversion γ equals the inverse of the EIS, $\gamma = \psi^{-1}$ implying $\theta = 1$. For the expected utility special case, we thus have $f(C, V) = U(C) - \rho V$, which is additively separable in C and V , with $U(C) = \rho C^{1-\gamma}/(1-\gamma)$. For the general specification of the recursive utility, $\theta \neq 1$ and $f(C, V)$ is non-separable in C and V .

The agent's liquid financial wealth W accumulates as follows

$$dW_t = (rW_t + Y_t - C_t)dt, \quad t \geq 0. \quad (6)$$

We assume that the agent cannot borrow against future incomes, i.e. liquid financial wealth is non-negative at all times,

$$W_t \geq 0, \quad \text{for all } t \geq 0. \quad (7)$$

Despite being unable to borrow, the agent optimally adjusts the rate of wealth accumulation/de-accumulation to partially smooth consumption over time.

In summary, the agent maximizes the non-expected recursive utility given in (3)-(4) subject to the labor-income process (1), the wealth accumulation process (6), and the non-negative wealth constraint (7). Before analyzing the incomplete-markets model, we first present the complete-markets (CM) solution, which we will use as a natural benchmark for comparisons with our incomplete-markets model.

3 The Complete-Markets (CM) PIH Benchmark

Friedman (1957) and Hall (1978) define *human wealth* H as the expected present value of future labor income, discounted at the risk-free interest rate r , in that

$$H_t = \mathbb{E}_t \left(\int_t^\infty e^{-r(u-t)} Y_u du \right). \quad (8)$$

To ensure that human wealth is finite, which is necessary for convergence under CM, we assume that the income growth rate μ is lower than the interest rate r , in that

$$\mathbf{Condition 1} : r > \mu. \quad (9)$$

Under **Condition 1** given in (9), human wealth H is finite and is given by

$$H_t = hY_t = \frac{Y_t}{r - \mu}. \quad (10)$$

Next, we define

$$m^* = r + \psi(\rho - r). \quad (11)$$

As we will show, m^* equals to the marginal propensity to consume for CM or Ramsey models. To ensure that the problem is well posed under CM, we require the following:

$$\textbf{Condition 2} : m^* > 0 \text{ or equivalently, } \rho > (1 - \psi^{-1})r. \quad (12)$$

Intuitively, the agent's discount rate ρ needs to be sufficiently large for convergence.

Under CM, we use the Arrow-Debreu result to decompose the optimization problem into two: "total" wealth maximization and utility maximization, and the "total" wealth maximization is independent of the agent's preference.¹⁰ We next summarize the CM results.

Proposition 1 *Under CM with **Condition 1** and **Condition 2** given in (9) and (12), the agent's value $V^*(W, Y)$ is given by*

$$V^*(W, Y) = \frac{(bP^*(W, Y))^{1-\gamma}}{1-\gamma}, \quad (13)$$

where $w = W/Y$ is the wealth-income ratio, the "total" wealth $P^*(W, Y) = p^*(w)Y$ with

$$p^*(w) = w + \frac{1}{r - \mu}, \quad (14)$$

and

$$b = \rho \left[\frac{r + \psi(\rho - r)}{\rho} \right]^{\frac{1}{1-\psi}}. \quad (15)$$

The optimal consumption-income ratio $c = C/Y$ is given by

$$c^*(w) = m^*p^*(w), \quad (16)$$

where m^* is the MPC under CM and is given by (11).

Intuitively, consumption is described by the PIH rule (16). Next, we characterize the solution for the case where the agent faces both the borrowing constraint and uninsurable shocks.

¹⁰Technically, we introduce a tradable risky asset, which is subject to the same shock as the labor-income process to complete markets. The Arrow-Debreu theorem implies that the agent maximizes the preference-free market value of total wealth. See the Appendix for details.

4 The Incomplete-Markets Setting

We first analyze the agent's consumption policy rule in the interior region with positive wealth, i.e. $W_t > 0$, and then discuss the boundary conditions. In the interior region, the agent chooses consumption to satisfy the Hamilton-Jacobi-Bellman (HJB) equation,¹¹

$$0 = \max_{C>0} f(C, V) + (rW + Y - C)V_W(W, Y) + \mu Y V_Y(W, Y) + \frac{\sigma^2 Y^2}{2} V_{YY}(W, Y). \quad (17)$$

The first-order condition (FOC) for consumption is given by

$$f_C(C, V) = V_W(W, Y), \quad (18)$$

which equates the marginal benefit of consumption $f_C(C, V)$ with the marginal value of wealth $V_W(W, Y)$. For the special CRRA utility case, $f_C(C, V)$ equals the marginal utility of consumption $U'(C)$, which is independent of the agent's continuation value V . More generally, for a non-expected utility, $f_C(C, V)$ depends on both current consumption C and the agent's continuation value V , as $f(C, V)$ is not additively separable in C and V .

We show that the agent's value function is given by

$$V(W, Y) = \frac{(bP(W, Y))^{1-\gamma}}{1-\gamma}, \quad (19)$$

where b is given by (15). By comparing $V(W, Y)$ given in (19) with $V^*(W, 0)$ given by (13) for the CM benchmark with no labor income, we may interpret $P(W, Y)$ as the certainty equivalent wealth, the minimal wealth level for which the agent is willing to permanently give up the labor income process Y and wealth W , $V(W, Y) = V^*(P(W, Y), 0)$. Certainty equivalent wealth is a monotonic transformation of the agent's value function expressed in units of consumption goods rather than utils. It is more intuitive and also analytically more convenient to express the agent's welfare in $P(W, Y)$. As we will show, $P(W, Y)$ and its sensitivity $P_W(W, Y)$ naturally appear in the agent's optimal consumption function with intuitive interpretations.

With the Epstein-Zin utility and a geometric labor-income process, our model has the homogeneity property as the expected CRRA utility does.¹² We use the lower case to denote

¹¹Duffie and Epstein (1992b) generalize the standard HJB equation for the expected-utility case, to allow for non-expected recursive utility such as the Epstein-Weil-Zin utility used here.

¹²Carroll (1997) shows the homogeneity property for the CRRA utility case and numerically solves for the optimal consumption rule in the discrete-time setting.

the corresponding variable in the upper case scaled by contemporaneous labor income Y . For example, $w_t = W_t/Y_t$ denotes the wealth-income ratio, $c_t = C_t/Y_t$ is the consumption-income ratio, and $p(w) = P(W, Y)/Y$ is the scaled certainty equivalent wealth. The following theorem summarizes the main results. See the Appendix for details.

Theorem 1 *The optimal consumption-income ratio $c(w)$ is given by*

$$c(w) = m^* p(w) (p'(w))^{-\psi}, \quad (20)$$

where m^* as given in (11) is the agent's MPC under CM and $p(w)$ solves the following ODE:

$$\begin{aligned} 0 = & \left(\frac{m^* (p'(w))^{1-\psi} - \psi \rho}{\psi - 1} + \mu - \frac{\gamma \sigma^2}{2} \right) p(w) + p'(w) + (r - \mu + \gamma \sigma^2) w p'(w) \\ & + \frac{\sigma^2 w^2}{2} \left(p''(w) - \gamma \frac{(p'(w))^2}{p(w)} \right). \end{aligned} \quad (21)$$

The above ODE for $p(w)$ is solved subject to the following condition:

$$\lim_{w \rightarrow \infty} p(w) = p^*(w) = w + h = w + \frac{1}{r - \mu}, \quad (22)$$

Additionally, the ODE (21) for $p(w)$ satisfies the following constraint for $c(\cdot)$ at the origin,

$$0 < c(0) \leq 1. \quad (23)$$

Consumption $c(w)$ depends on both the scaled certainty equivalent wealth $p(w)$ and its slope $p'(w)$, the marginal (certainty equivalent) value of wealth. Uninsurable labor income shocks and the borrowing constraint cause $p(w)$ to be highly non-linear unlike in the CM case. First, $p(w)$ is lower than its first-best CM value $p^*(w) = w + h$. Second, the marginal value of liquidity $p'(w)$ is greater than unity, $p'(w) > 1$. Therefore, consumption $c(w)$ given by (20) is lower than the CM benchmark level, $c^*(w)$, as $p(w) < p^*(w) = w + h$ and $p'(w) \geq 1$.

The ODE (21) describes the nonlinear certainty equivalent valuation $p(w)$ in the interior region $w > 0$. In the limit as $w \rightarrow \infty$, wealth completely buffers labor income shocks. Therefore, $\lim_{w \rightarrow \infty} p(w) = w + h$, $\lim_{w \rightarrow \infty} p'(w) = 1$, and $\lim_{w \rightarrow \infty} c(w) = m^*(w + h)$. This CM result in the limit serves as one natural boundary condition for the ODE (21).

For the borrowing constraint $W_t \geq 0$, we only need to check whether it binds or not at the boundary $W = 0$ as consumption is a “flow” and wealth is a “stock” variable. Therefore,

the borrowing constraint $W_t \geq 0$ is equivalent to $C_t \leq Y_t$ when $W_t = 0$, which implies the boundary condition (23) for the ODE (21).¹³

There are two sub-cases: $c(0) = 1$ and $c(0) < 1$.

- If $c(0) = 1$, the constraint $W \geq 0$ binds permanently (once wealth reaches zero) and the agent permanently saves nothing becoming “hand-to-mouth.” Campbell and Mankiw (1989) find that about 50% of households in their sample do not save. We show that these consumers’ behavior can be optimal in our model. For “hand-to-mouth” consumers, relaxing the borrowing constraint can cause the consumption profile to change and hence can be welfare-enhancing.
- If $c(0) < 1$, the borrowing constraint $W \geq 0$ never binds. In this case, the agent’s voluntary savings demand is sufficiently high so that wealth always remains positive with probability one. Therefore, relaxing the borrowing constraint (e.g. by offering a credit line) has no effect on optimal consumption and savings.

5 Quantitative Analysis

We now analyze our model’s predictions on consumption, savings, and the stationary distributions for scaled wealth and consumption. Parameter values are annualized and continuously compounded when applicable. We set the subjective discount rate $\rho = 5.5\%$ and the risk-free rate $r = 5\%$, which imply that the agent is relatively impatient (compared with the market), with a wedge $\rho - r = 0.5\%$. We choose the expected income growth rate $\mu = 1\%$, the volatility of income growth $\sigma = 10\%$, and the EIS parameter $\psi = 0.5$. We consider three values for the coefficient of relative risk aversion, $\gamma = 0, 2, 4$ and we will show that the quantitative effects of risk aversion on consumption and welfare are large. The case with $\gamma = 2$ corresponds to the expected CRRA utility with $\psi = 1/\gamma = 0.5$. The case with risk neutrality ($\gamma = 0$) and a positive EIS is proposed by Farmer (1990) and used by Gertler (1999) in his study of social security in a life-cycle economy.

¹³Also technically speaking, there is an additional condition at $w = 0$:

$$0 = \left(\frac{m^*(p'(0))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p(0) + p'(0).$$

But this condition simply follows from the continuity of the ODE (21) for $p(w)$ at $w = 0$. For expositional simplicity, we do not include in the main text.

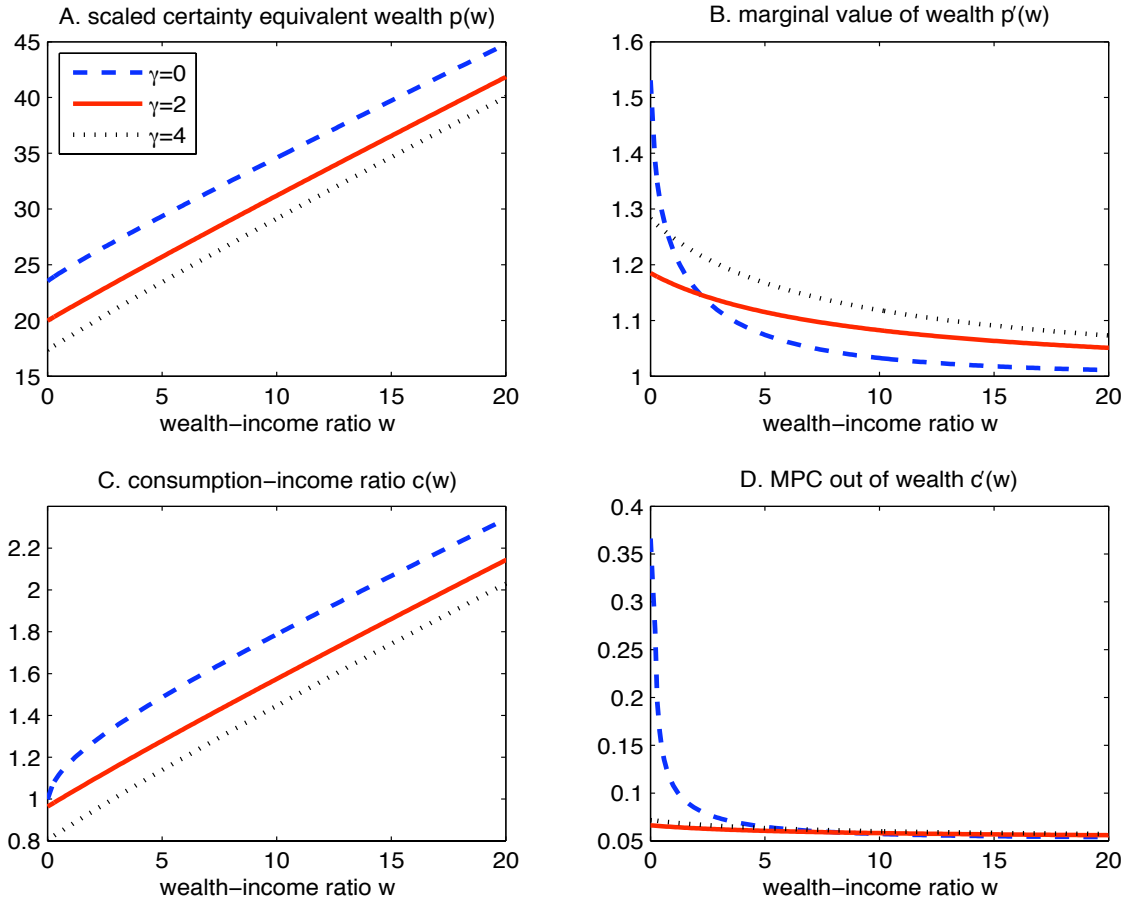


Figure 1: Scaled certainty equivalent wealth $p(w)$, marginal (certainty equivalent) value of wealth $p'(w)$, consumption-income ratio $c(w)$, and the MPC out of wealth $c'(w)$. Parameter values are: $r = 5\%$, $\rho = 5.5\%$, $\sigma = 10\%$, $\mu = 1\%$, and $\psi = 0.5$.

5.1 Marginal Value of Liquidity and Optimal Consumption

Panels A and B of Figure 1 plot $p(w)$ and $p'(w)$, respectively. Uninsurable labor income shocks and the borrowing constraint cause $p(w)$ to be concave in w . Because wealth buffers labor-income shocks and mitigates the impact of the borrowing constraint on consumption, the marginal (certainty equivalent) value of liquid wealth $p'(w)$ is greater than one. Intuitively speaking, $p'(w)$ decreases with w , as a wealthier agent is less concerned about uninsurable labor-income shocks and the borrowing constraint, *ceteris paribus*. As $w \rightarrow \infty$, self insurance achieves the first-best outcome, and hence $p'(w)$ approaches one. The risk-neutral agent ($\gamma = 0$ but with $\psi = 0.5$) with $W = 0$ values a unit of windfall wealth at $p'(0) = 1.53$,

which is 53% higher than its face value reflecting the premium for liquid financial wealth over illiquid human wealth.

Panels C and D of Figure 1 plot $c(w)$ and the MPC out of wealth, $c'(w)$, respectively, and show that consumption is increasing and concave in wealth.¹⁴ The risk-neutral agent with no wealth consumes all labor income, $c(0) = 1$, and wealth is permanently absorbed at $W = 0$. The MPC out of wealth at $W = 0$ is $c'(0) = 36.6\%$, which is much higher than the CM benchmark value $m^* = 0.0525$ and reflects the significant cost of the borrowing constraint. With risk neutrality, $\gamma = 0$, the marginal value of wealth $p'(0)$ is greater than one, if and only if the agent is constrained.

For $\gamma = 2$, $p'(0) = 1.19$, and $c(0) = 0.96$. With no wealth, the agent consumes 96% of labor income, saves the remaining 4%, and values wealth marginally at $p'(0) = 1.19$. The MPC at the origin, $c'(0)$, is 6.6%, which is slightly higher than $m^* = 0.0525$, the MPC for the first-best CM benchmark. Similarly, for $\gamma = 4$ (but again with $\psi = 0.5$), $p'(0) = 1.28$, $c(0) = 0.80$, and the MPC $c'(0) = 0.072$. For both $\gamma = 2$ and $\gamma = 4$, the borrowing constraint does not bind, $c(0) < 1$. Thus, relaxing the borrowing constraint and allowing the agent to be in debt ($W < 0$) will not change the agent's behavior at all. Our continuous-time formulation clearly demonstrates the effect of the borrowing constraint as the corresponding boundary behavior is explicitly characterized in our model.

Figure 1 shows that consumption decreases with risk aversion γ for a given value of w . While seemingly intuitive, this result is in sharp contrast with the first-best CM result, where risk aversion γ has no effect on consumption. In Figure 1, $c^*(w) = m^*(w + h) = 5.25\% \times (w + 25)$ for $\gamma = 0, 2, 4$. With uninsurable income shocks and the borrowing constraint, the quantitative effect of risk aversion on consumption is large. Panel C illustrates significant variations of $c(w)$ with respect to risk aversion γ . Even when wealth is 20 times current income ($w = 20$), consumption $c(w)$ is still 8.4% and 13.2% lower than the first-best CM level for $\gamma = 2$ and $\gamma = 4$, respectively.

5.2 Buffer-Stock Savings

We next turn to the model's implications on buffer stock savings. While the endogenous wealth process is not stationary, the wealth-income ratio $w = W/Y$ can be stationary and

¹⁴Carroll and Kimball (1996) show that the consumption function is concave under certain conditions for the expected-utility case.

may have a target as emphasized in the buffer-stock savings literature (Carroll, 1997).

Using Ito's formula, we obtain the following dynamics for $\ln(w)$:

$$d\ln(w_t) = \left[r - \mu + \frac{\sigma^2}{2} + \frac{1 - c_t}{w_t} \right] dt - \sigma dB_t = \mu_w(w_t)dt - \sigma dB_t, \quad (24)$$

where $\mu_w(w)$ denotes the expected change (drift) of w and is given by

$$\mu_w(w) = r - \mu + \frac{\sigma^2}{2} + \frac{1 - c_t}{w_t}. \quad (25)$$

Because income Y is stochastic and wealth W (earning a constant rate of return r) is locally deterministic, $\ln(w)$ has constant volatility σ . The negative sign for the diffusion term in (24) indicates that w decreases as income Y receives a positive shock.

How much wealth should the agent accumulate? Since W is non-stationary, we measure the target level of savings per unit of labor-income Y in the long-run sense. Let w^{ss} denote the steady-state wealth-income ratio. Intuitively, around the steady-state savings target for w^{ss} , we expect that w mean reverts; the agent increases w in expectation if w lies below the target w^{ss} , and decreases w on average if otherwise. At the steady state, the expected change of w , $\mu_w(w)$, equals zero, i.e. $\mu_w(w^{ss}) = 0$, which in turn implies that the steady-state consumption-income ratio, $c^{ss}(w)$, is given by

$$c^{ss}(w) = 1 + \left(r - \mu + \frac{\sigma^2}{2} \right) w^{ss}. \quad (26)$$

Intuitively, the steady-state consumption per unit of income $c^{ss}(w)$ equals one plus a term that is proportional to the steady-state savings target, w^{ss} , with coefficient $\left(r - \mu + \frac{\sigma^2}{2} \right)$. This coefficient is given by the interest rate r minus the expected income growth rate μ , plus $\sigma^2/2$, the Jensen's inequality term.

There are two cases, $w^{ss} = 0$ and $w^{ss} > 0$. If $w^{ss} = 0$, the borrowing constraint binds and $c(w^{ss}) = 1$. With no initial wealth, this agent lives from paycheck to paycheck, or "hand to mouth." Campbell and Mankiw (1989) find that many consumers behave in this way. We show that these consumers may behave optimally. Panel A of Figure 2 shows that for the risk-neutral case with $\gamma = 0$ and $\psi = 0.5$, the steady-state w^{ss} is at the corner ($w = 0$) and hence households become wealth-less (i.e. $w = 0$) and "hand-to-mouth" permanently in the limit. Note that $w^{ss} = 0$ is the intersection between the optimal consumption function $c(w)$ and the straight line $(r - \mu + \sigma^2/2)w + 1$.

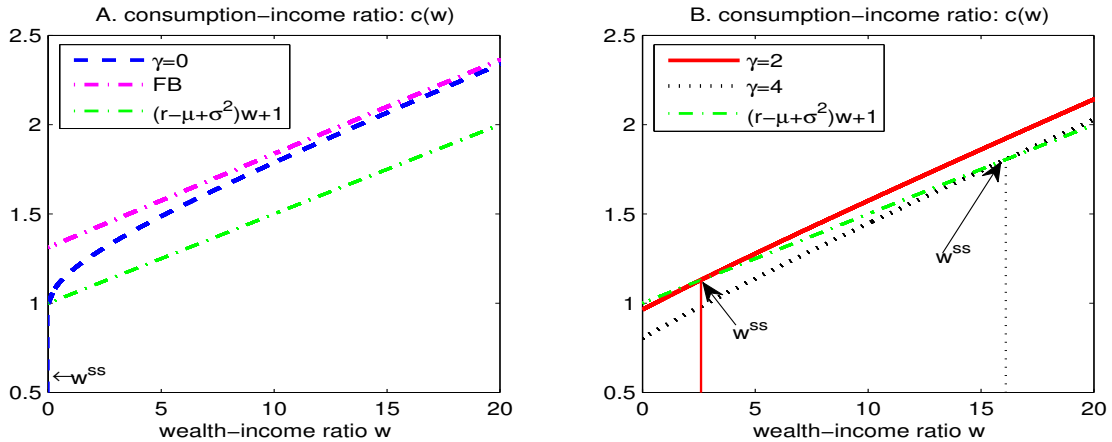


Figure 2: **The steady-state wealth-income ratio w^{ss} for $\gamma = 0, 2, 4$ and $\psi = 0.5$.** Panel A shows that for $\gamma = 0$, the steady-state wealth-income ratio $w^{ss} = 0$ corresponding to the “hand-to-mouth” case. Panel B shows that $w^{ss} = 2.6$ and $w^{ss} = 16.1$ for $\gamma = 2$ and $\gamma = 4$, respectively. Parameter values are: $r = 5\%$, $\rho = 5.5\%$, $\sigma = 10\%$, $\mu = 1\%$, and $\psi = 0.5$.

If $w^{ss} > 0$, the wealth-income ratio w mean reverts. The following condition ensures that the wealth-income ratio w is stationary:

$$\rho - r > -\psi^{-1} \left(\mu - \frac{\sigma^2}{2} \right), \quad (27)$$

because $\lim_{w \rightarrow \infty} \mu_w(w) = \lim_{w \rightarrow \infty} 1 + \left[-\mu + \frac{\sigma^2}{2} - \psi(\rho - r) \right] w < 0$ under the above condition. Intuitively speaking, (27) requires the agent to be sufficiently impatient.¹⁵

Panel B of Figure 2 graphically determines the steady-state target w^{ss} for the cases with $\gamma = 2$ and $\gamma = 4$, again as the intersection of the corresponding optimal consumption rule $c(w)$ with the straight line $(r - \mu + \sigma^2)w + 1$. Note that $w^{ss} = 16.1$ for the case with $\gamma = 4$, which is much higher than the savings target $w^{ss} = 2.6$ for the case with $\gamma = 2$. The higher the coefficient of relative risk aversion γ , the stronger the savings motive and the higher the steady-state target savings w^{ss} . This substantial increase of w^{ss} from 2.6 to 16.1

¹⁵Carroll (1997) provides a similar condition under discrete-time for the case of expected CRRA utility. Interestingly, risk aversion γ does not determine the stationarity of w as we see from (27). This is because in the limit, it is the CM PIH rule that determines the agent’s optimal consumption, and risk aversion plays no role in the CM PIH-based consumption policy. The standard impatience condition $\rho > r$ is neither necessary nor sufficient to ensure that w is stationary.

as γ increases from 2 to 4 indicates that risk aversion may have a first-order effect on the dispersion of wealth in the long run.

5.3 Stationary distributions

We now analyze the long-run stationary distributions of w and $c(w)$. (For space considerations, we have chosen to leave out the transitional dynamics analysis.) We start from the steady-state target w^{ss} and simulate two hundred sample paths for w by using (24). Each path is 5,000-year long with a time increment $\Delta t = 0.05$. Table 1 reports the mean, standard deviation, and various quantiles for the stationary distributions of w and $c(w)$ for $\gamma = 2, 4$.

The long-run average of w is 18.63 for $\gamma = 4$, which is about six times 3.13, the long-run average of w for $\gamma = 2$. More risk-averse agents save more on average and are more likely to be richer in the long run, consistent with our earlier findings via the steady-state savings target w^{ss} . The long-run average of $c(w)$ is 1.98 for $\gamma = 4$, which is 72% larger than 1.15, the long-run average of $c(w)$ for $\gamma = 2$. Intuitively, more risk-averse agents accumulate more wealth to dampen the impact of income shocks on consumption. As a more risk-averse agent accumulates more wealth in the long run, this agent also consumes more in the long run on average. Again, the difference in target wealth and consumption is enormous despite a moderate change of risk aversion from two to four.

Additionally, the entire distributions for both w and $c(w)$ are also much more dispersed for $\gamma = 4$ than for $\gamma = 2$. Quantitatively, the effects of changing risk aversion γ from 2 to 4 on the distributions of w and $c(w)$ are very large. Given that both $\gamma = 2$ and $\gamma = 4$ are commonly used levels of risk aversion and are sensible choices for risk aversion, our analysis suggests that quantitative results about wealth distribution are highly sensitive to the choice of risk aversion γ . Next, we analyze the impact of EIS.

5.4 Elasticity of Intertemporal Substitution (EIS)

We have shown that risk aversion has a first-order effect on consumption and welfare. We next show that the EIS also matters but in a way that is quite distinct from risk aversion.

There are significant disagreements about a reasonable value for the EIS. For example, Hall (1988) obtains an estimate of EIS near zero by using aggregate consumption data. Bansal and Yaron (2004) show that an EIS greater than one (in the range from 1.5 to

Table 1: STATIONARY DISTRIBUTIONS FOR w AND $c(w)$

This table reports mean, standard deviation, and various quantiles for the stationary distributions of the wealth-income ratio w and the consumption-income ratio $c(w)$ for $\gamma = 2, 4$.

γ	mean	std dev	1%	5%	25%	50%	75%	95%	99%
Panel A: w									
2	3.13	2.91	0.8	1.0	1.6	2.4	3.6	7.5	13.7
4	18.63	19.59	4.0	5.3	8.6	12.8	21.1	57.5	171.1
Panel B: $c(w)$									
2	1.15	0.17	1.00	1.02	1.06	1.11	1.18	1.42	1.78
4	1.98	1.42	1.06	1.15	1.35	1.61	2.08	4.09	10.2

2) is essential for consumption-based asset pricing models to fit empirical evidence. Thus, choosing an EIS parameter that is larger than one has become a common practice in the macro-finance literature. However, there is no consensus on what the sensible value of EIS should be. The Appendix in Hall (2009) provides a brief survey of estimates in the literature.

We do not take a strong view on the value of EIS. Instead, we assess the sensitivities of optimal consumption and welfare to changes in the EIS value from $\psi = 0.1$ (a low estimate as in Hall (1988)) to $\psi = 2$ (a high estimate preferred by the macro finance researchers).

Panels A and B of Figure 3 plot $p(w)$ and $p'(w)$, respectively. Perhaps surprisingly, the scaled certainty equivalent wealth $p(w)$ barely changes as we vary EIS ψ from 0.1 to 2. Even at $w = 0$ where it is most likely to find an effect of EIS ψ , $p(w)$ changes little with the EIS ψ , $p(0) = 20.00$ for $\psi = 0.1$, $p(0) = 19.98$ for $\psi = 0.5$, and $p(0) = 19.91$ for $\psi = 2$. The marginal value of wealth $p'(w)$ also varies little with the EIS ψ . At $w = 0$, $p'(0) = 1.2$ for $\psi = 0.1$, $p'(0) = 1.19$ for $\psi = 0.5$, and $p'(0) = 1.16$ for $\psi = 2$. Compared to risk aversion, the EIS ψ has much less significant effects on $p(w)$ and $p'(w)$.

Panels C and D of Figure 3 plot consumption $c(w)$ and the MPC out of wealth $c'(w)$, respectively. As $w \rightarrow \infty$, consumption $c(w)$ approaches the PIH benchmark where $c^*(w) = m^*(w + h)$ and the MPC $c'(w) = m^* = r + \psi(\rho - r)$. By continuity, for sufficiently high

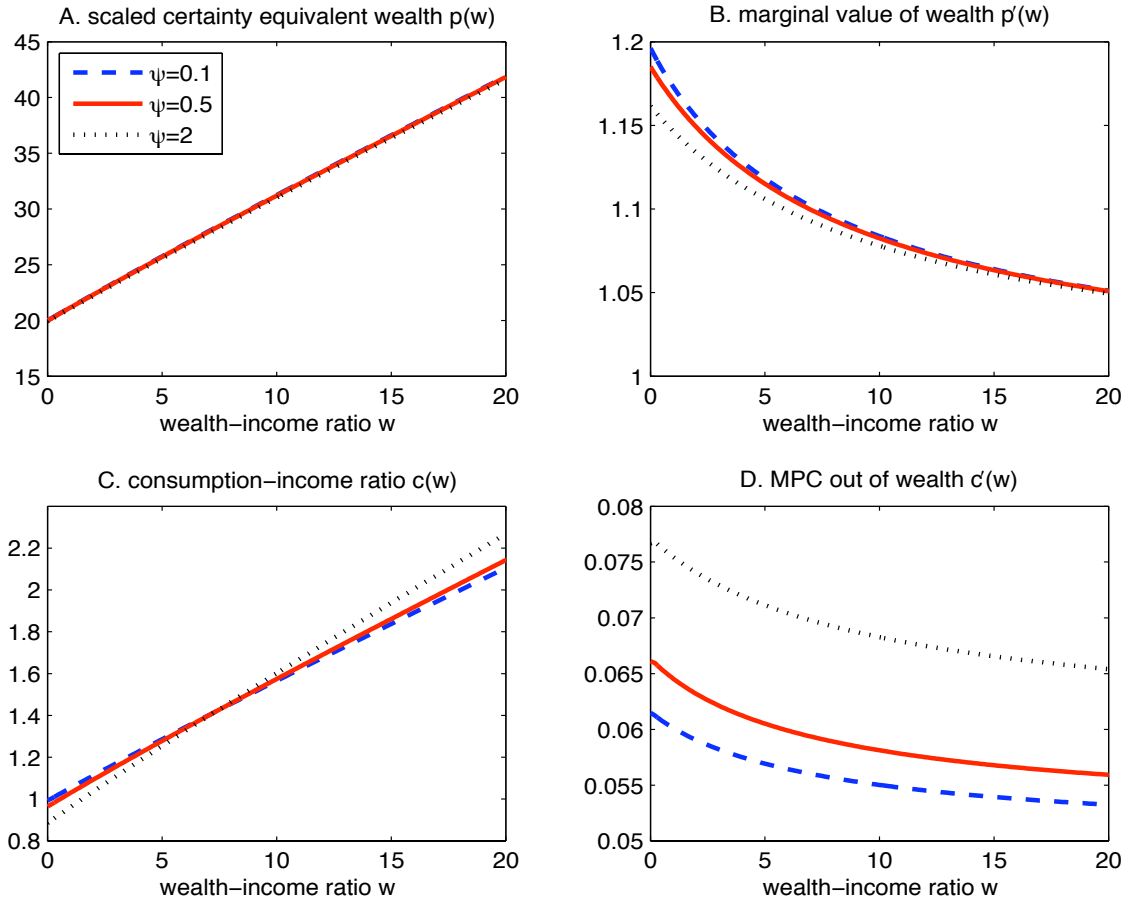


Figure 3: **The effects of EIS ψ on $p(w)$, $p'(w)$, $c(w)$, and $c'(w)$.** Parameter values: $r = 5\%$, $\rho = 5.5\%$, $\sigma = 10\%$, $\mu = 1\%$, and $\gamma = 2$.

w , $c(w)$ and $c'(w)$ must increase with the EIS ψ , as we see from Figure 3. This is in line with standard intuition. The higher the EIS, the more sensitively consumption responds to wealth and the higher the agent's consumption.

In contrast, for low values of w , $c(w)$ *decreases* with the EIS. The agent with a higher EIS ψ is more willing to substitute consumption intertemporally and thus lowers consumption $c(w)$ more near $w = 0$. Reducing consumption is especially valuable in mitigating the impact of the borrowing constraint for low values of w . At the high and low ends of w , EIS ψ has opposite effects on $c(w)$, which implies that $c(w)$ rotates somewhere around a mid-ranged value of w in a counter-clock-wise direction as we increase the EIS ψ .

We provide additional insights by observing the analytical formula for $c(w) = m^* p(w) p'(w)^{-\psi}$.

Table 2: STATIONARY DISTRIBUTIONS FOR w AND $c(w)$

This table reports mean, standard deviation, and various quantiles for the stationary distributions of scaled wealth w and scaled consumption $c(w)$ for $\psi = 0.1$, $\psi = 0.5$ and $\psi = 2$.

ψ	mean	std dev	1%	5%	25%	50%	75%	95%	99%
Panel A: w									
0.1	1.35	2.42	0.2	0.3	0.6	0.9	1.5	3.4	7.4
0.5	3.13	2.91	0.8	1.0	1.6	2.4	3.6	7.5	13.7
2	5.17	3.02	1.8	2.3	3.3	4.4	6.1	10.4	16.2
Panel B: $c(w)$									
0.1	1.06	0.13	1.004	1.01	1.03	1.05	1.08	1.19	1.42
0.5	1.15	0.17	1.00	1.02	1.06	1.11	1.18	1.42	1.78
2	1.25	0.21	1.02	1.06	1.13	1.21	1.33	1.63	2.01

Given the values of $p(w)$ and $p'(w)$, the EIS has two opposing effects on the optimal consumption rule.¹⁶ On the one hand, consumption $c(w)$ increases with the EIS ψ since the MPC for the CM PIH consumption rule, m^* , increases with ψ . This CM PIH-based intuition works well when the agent is effectively unconstrained, i.e. at high values of w . On the other hand, $c(w)$ may decrease with ψ especially for a wealth-poor (low w) agent reflected by the power $-\psi$ in the term $p'(w)^{-\psi}$ in $c(w)$. These two opposing forces imply that the comparative statics of $c(w)$ with respect to the EIS ψ is non-monotonic. Indeed, $c(w)$ rotates counter-clockwise as we increase the EIS.

Stationary distributions for w and $c(w)$. We generate the stationary distributions for w and $c(w)$ using the same procedure as in Section 5. Table 2 reports the mean, standard deviation, and various quantiles for the stationary distributions of w and $c(w)$ for three values of the EIS, $\psi = 0.1$, 0.5 , and 2 . As we increase the EIS ψ , the mean, standard deviation, and various quantile statistics for the stationary distribution of w all increase. For example, the long-run average of w equals 5.17 for $\psi = 2$, 3.13 for $\psi = 0.5$, and 1.35 for $\psi = 0.1$.

The EIS also influences the stationary distribution of $c(w)$. The mean, standard deviation,

¹⁶For simplicity and without losing key economic insights, we ignore the indirect effects of the EIS ψ on $p(w)$ and $p'(w)$, as they are not significant as we have just noted.

tion, and various quantile statistics for the stationary distribution of $c(w)$ all increase with the EIS ψ as well. For example, the long-run average of $c(w)$ equals 1.04 for $\psi = 0.1$, 1.15 for $\psi = 0.5$, and 1.25 for $\psi = 2$. Notably, the effects of the EIS on the distribution of $c(w)$ are weaker than those on the distribution of w . The optimal consumption-savings adjustments mitigate the impact of wealth dispersion on consumption dispersion in the long run. An agent with a higher EIS consumes more when w is high but consumes less when w is low as shown in Figure 3. The opposite effects of the EIS, ψ , on consumption at the two ends of w make consumption less dispersed than wealth in the long run.

We have shown that the quantitative results on the steady-state savings target w^{ss} and the stationary distributions for w and $c(w)$ critically depend on the choice of the EIS and the coefficient of relative risk aversion. Moreover, the effects of risk aversion and the EIS are quite different, both conceptually and quantitatively. It is thus desirable to separate these two important parameter values when analyzing optimal consumption and buffer stock savings.

5.5 The Case with Long-Run Risk Parameters

Explaining asset pricing facts (e.g., high equilibrium risk premium and low risk-free rate) is very challenging. One widely-used approach to match asset pricing facts (via various moments) is to incorporate long-run risk (LRR) into a representative-agent model with Epstein-Zin utility following Bansal and Yaron (2004). LRR models require both high risk aversion and high EIS, e.g., $\gamma = 10$ and $\psi = 2$. Note that a combination of a high risk aversion and a high EIS is feasible only in Epstein-Zin utility (not in expected utility.)

In this subsection, we analyze the implications of these parameter values on consumption and savings in our incomplete-markets environment.

Figure 4 reports the results for the case with LRR parameter values (e.g., $\gamma = 10$ and $\psi = 2$) and compares with the CM PIH case with $\psi = 2$. Panel A and B of Figure 4 plot the scaled certainty equivalent wealth $p(w)$ and the marginal value of liquidity $p'(w)$, respectively. Compared with the CM PIH rule (dashed lines), for our case with $\gamma = 10$ and $\psi = 2$, the certainty equivalent wealth $p(w)$ is substantially lower and the marginal value of liquidity $p'(w)$ is substantially higher than unity (for the CM benchmark.) For example, at $w = 0$, the certainty equivalent wealth is $p(0) = 13.1$, which is 47% lower than the CM benchmark certainty equivalent wealth $p(0) = h = 25$, and the marginal value of liquidity is

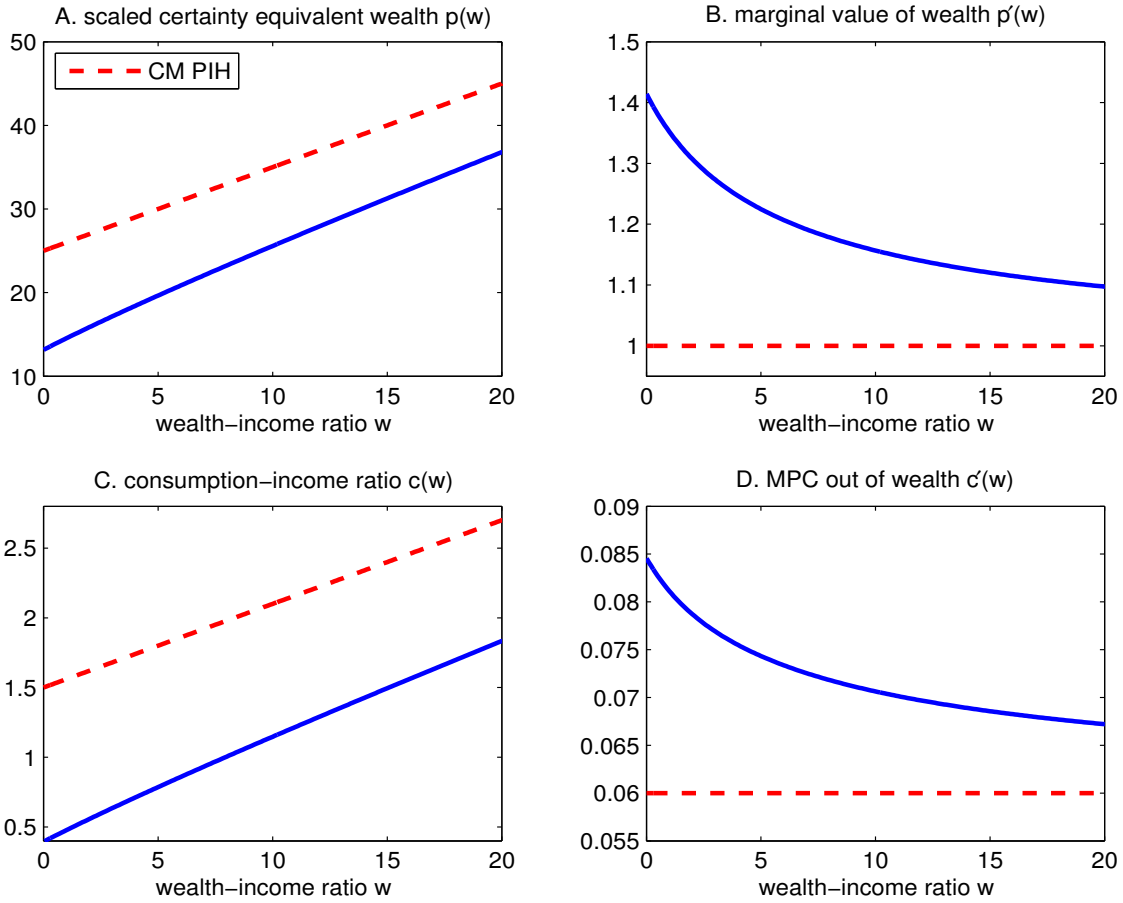


Figure 4: **The case with long-run risk parameter values: $\gamma = 10$ and $\psi = 2$.** Other parameter values: $r = 5\%$, $\rho = 5.5\%$, $\sigma = 10\%$, and $\mu = 1\%$.

$p'(0) = 1.41$, which is 41% higher than the CM benchmark result $p'(0) = 1$. Quantitatively, uninsurable income shocks clearly have a first-order effect on certainty equivalent valuation $p(w)$ and the marginal value of liquidity $p'(w)$.

Panel C and D of Figure 4 plot scaled consumption $c(w)$ and the MPC $c'(w)$, respectively. For example, with $w = 0$, the agent consumes 39% of disposable income ($c(0) = 0.39$) and saves the remaining 61% for the future. In contrast, the CM PIH agent consumes 50% more than his disposable income by borrowing 50% against the future incomes. Also as expected, the MPC $c'(0) = 0.085$ is higher than the MPC for the CM benchmark value $m^* = 5\% + 0.5\% \times 2 = 6\%$. Moreover, $c'(w)$ decreases with w .

Table 3 reports the steady-state values and also stationary distributions of w and $c(w)$.

Table 3: STATIONARY DISTRIBUTIONS FOR w AND $c(w)$

This table reports mean, standard deviation, and various quantiles for the stationary distributions of scaled wealth w and scaled consumption $c(w)$ for the case with $\psi = 2$ and $\gamma = 10$, commonly used parameters in Bansal and Yaron (2004) and the long-run risk literature.

variable	steady-state	mean	std dev	1%	5%	25%	50%	75%	95%	99%
w	30.3	32.9	30.1	9.7	12.5	18.6	25.7	37.3	73.4	147.2
$c(w)$	2.52	2.64	1.88	1.10	1.30	1.72	2.20	2.94	5.24	9.80

With $\gamma = 10$ and $\psi = 2$, the long-run average of w is 32.9 and the steady-state savings target is $w^{ss} = 30.3$. That is, the agent’s target saving in the long run is very large, about 30-33 times of contemporaneous labor income. Recall that for our baseline case with $\gamma = 2$ and $\psi = 0.5$, the long-run mean of w is about 3.13 and the steady-state savings w^{ss} is about 2.6. That is, approximately speaking, the savings target is 10 times larger under the LRR parameter values than under the baseline case (expected utility with $\gamma = 2$). In addition, the long-run average of scaled consumption $c(w)$ is 2.64, which is more than twice the long-run average of $c(w)$, 1.15 for the baseline expected utility case with $\gamma = 2$. Our table also reports more skewed and fat-tailed distributions of scaled wealth and consumption under the LRR parameter values than under the baseline expected utility case with $\gamma = 2$. In summary, risk aversion and the EIS clearly have very strong effects on consumption and savings.

6 Large Income Shocks

We have so far specified the income process with diffusive permanent shocks. It has been well documented that wages fall dramatically at job displacement, generating so-called “scarring” effects.¹⁷ Wages may be low after unemployment due to fast depreciation of skills as in Ljungqvist and Sargent (1998). Alternatively, specific human capital can be lost and it may be hard to replace upon re-entry as in Low, Meghir, and Pistaferri (2010). To capture large infrequent movements of income within a very short period, diffusive shocks may not be sufficient and jumps may be necessary. We next incorporate large income movements.

¹⁷See Jacobson, LaLonde, and Sullivan (1993) and von Wachter, Song, and Manchester (2007), for example.

6.1 Model Setup and Solution

We model large income shocks as stochastic jumps, which arrives with a constant probability λ per unit of time (i.e. Poisson arrivals). When a jump occurs, the income is changed by a stochastic fraction Z of the contemporaneous income Y , where Z follows a well-behaved probability density function (pdf) $q_Z(z)$ with $Z \geq 0$. There is no limit to the number of jump shocks and a jump does not change the likelihood of another. We write the income process, which is now subject to both diffusion and jump risks, as follows

$$dY_t = \mu Y_{t-} dt + \sigma Y_{t-} dB_t - (1 - Z) Y_{t-} dJ_t, \quad Y_0 > 0, \quad (28)$$

where J is a pure jump process. To ensure that human wealth for the case with jumps to be finite, we impose the following condition:

$$\textbf{Condition 3} : r > \mu - \lambda(1 - \mathbb{E}(Z)). \quad (29)$$

This condition states that the expected income growth rate (with jumps) $\mu - \lambda(1 - \mathbb{E}(Z))$ needs to be lower than the rate of interest r for convergence.

Under **Condition 3**, the corresponding human wealth H , as defined by (8), is proportional to the contemporaneous income Y_t , $H_t = h_J Y_t$, where h_J is given by

$$h_J = \frac{1}{r - [\mu - \lambda(1 - \mathbb{E}(Z))]} . \quad (30)$$

For each realized jump, the expected percentage loss of income is $(1 - \mathbb{E}(Z))$. Since jumps occur with probability λ per unit of time, the expected income growth is thus lowered to $\lambda(1 - \mathbb{E}(Z))$ and hence the value of labor income scaled by current Y (using the risk-free rate to discount), h_J , is given by (30). Moreover, jumps induce precautionary savings demand as the agent is prudent, financially constrained, and faces uninsurable jump shocks under incomplete markets, which we can quantify using the certainty equivalent wealth.

Similar to our analysis of the baseline model, we proceed in two steps. First, we analyze the agent's consumption policy rule in the interior region with positive wealth, i.e. $W_t > 0$, and then discuss the boundary conditions. In the interior region, the agent chooses consumption C to maximize value function $V(W, Y)$ by solving the following HJB equation:

$$0 = \max_{C > 0} f(C, V) + (rW + Y - C)V_W + \mu Y V_Y + \frac{\sigma^2 Y^2}{2} V_{YY} + \lambda \mathbb{E} [V(W, ZY) - V(W, Y)] . \quad (31)$$

The expectation $\mathbb{E}(\cdot)$ is with respect to $Q_Z(z)$, the cumulative distribution function (cdf) for Z . Other terms are essentially the same as those in the baseline model.

Proposition 2 *The optimal consumption-income ratio $c(w)$ is given by (20), the same as in the baseline model. The scaled certainty equivalent wealth $p(w)$ solves the following ODE:*

$$0 = \left(\frac{m^*(p'(w))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} \right) p(w) + p'(w) + (r - \mu + \gamma\sigma^2)wp'(w) + \frac{\sigma^2w^2}{2} \left(p''(w) - \gamma \frac{(p'(w))^2}{p(w)} \right) + \frac{\lambda}{1-\gamma} \mathbb{E} \left[\left(\frac{Zp(w/Z)}{p(w)} \right)^{1-\gamma} - 1 \right] p(w). \quad (32)$$

The above ODE for $p(w)$ is solved subject to the following conditions:

$$\lim_{w \rightarrow \infty} p(w) = w + h_J, \quad (33)$$

$$0 = \left[\frac{m^*(p'(0))^{1-\psi} - \psi\rho}{\psi - 1} + \mu - \frac{\gamma\sigma^2}{2} + \frac{\lambda}{1-\gamma} \mathbb{E}(Z^{1-\gamma} - 1) \right] p(0) + p'(0). \quad (34)$$

Additionally, we require $0 < c(0) \leq 1$, the same condition as (23).

Jumps have two effects. First, the expected value of future labor incomes, h_J , depends on jumps and hence influences the CM PIH solution, as we see from the boundary condition (33). Second, the agent is prudent and financially constrained, jumps thus induce additional precautionary savings demand since jump risk is not spanned (incomplete markets).

6.2 An Example

The solution presented above applies to any well-behaved distribution for Z . For the numerical example, we consider the case where jumps lead to income losses, i.e. $Z < 1$. We assume that Z follows a power distribution over $[0, 1]$ with parameter $\alpha > 0$. Thus, the cdf is $Q_Z(z) = z^\alpha$ and the corresponding pdf is

$$q_Z(z) = \alpha z^{\alpha-1}, \quad 0 \leq z \leq 1. \quad (35)$$

A large value of α implies a small expected income loss of $\mathbb{E}(1-Z) = 1/(\alpha+1)$ in percentages. For $\alpha = 1$, Z follows a uniform distribution. For any $\alpha > 0$, (35) implies that $-\ln Z$ is exponentially distributed with mean $\mathbb{E}(-\ln Z) = 1/\alpha$.¹⁸

¹⁸We have also considered other distributions, for example, a log-normal distribution for Z . Due to space constraints, we will leave out the details that are available upon requests.

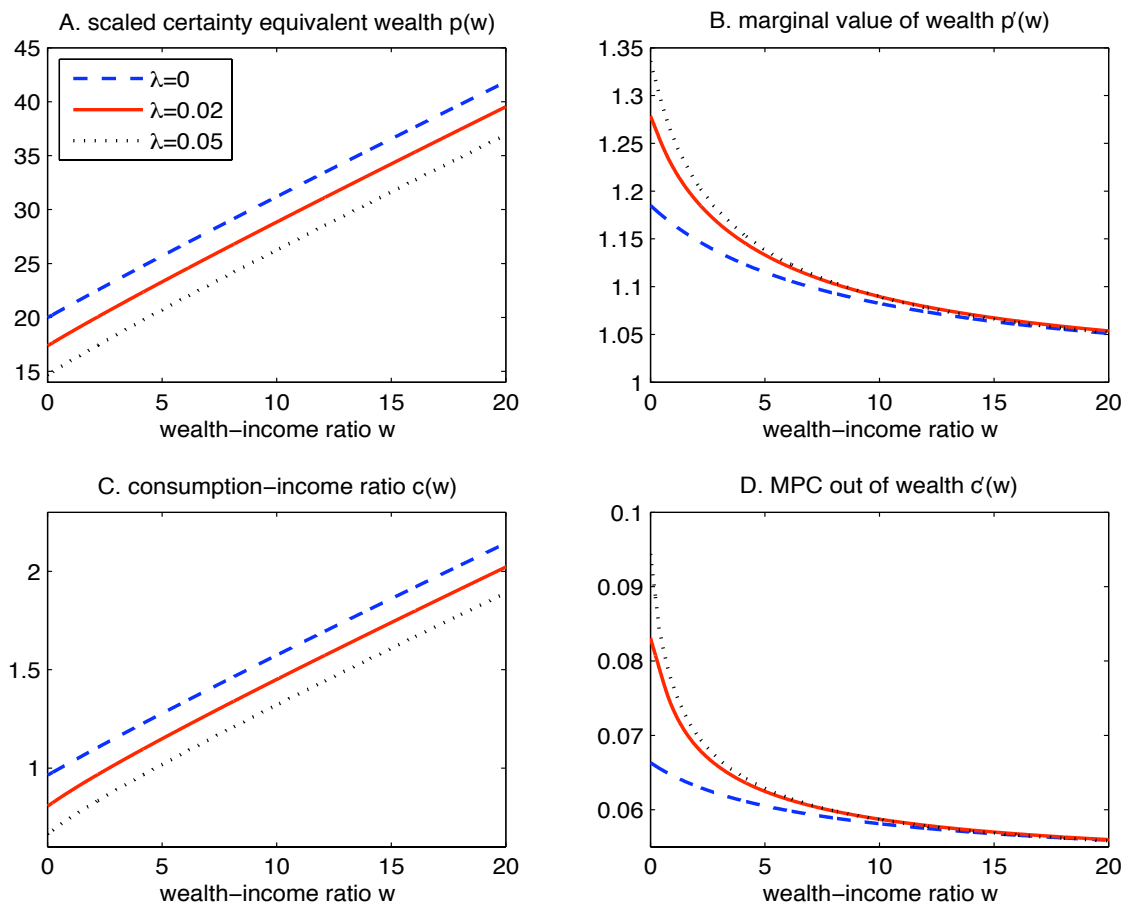


Figure 5: **Large income shocks: The effects of the mean arrival rate λ .** Parameter value: $r = 5\%$, $\rho = 5.5\%$, $\sigma = 10\%$, $\mu = 1\%$, $\gamma = 2$, $\psi = 0.5$, and $\alpha = 3$. The expected income loss in percentage upon jumps is $\mathbb{E}(1 - Z) = 1/(1 + \alpha) = 25\%$.

Figure 5 demonstrates the effects of the jump's mean arrival rate λ on $p(w)$, the marginal value of wealth, $p'(w)$, consumption, $c(w)$, and the MPC, $c'(w)$. The no-jumps case ($\lambda = 0$) corresponds to our baseline case (see the dashed line). For the case with jumps, we set $\alpha = 3$ so that the implied average loss $\mathbb{E}(1 - Z)$ is 25% when a jump occurs.

With $\lambda = 0.02$, large income shocks occur on average once every fifty years. The expected loss that is purely due to large income shocks is $\lambda\mathbb{E}(1 - Z) = 0.5\%$ per year, which implies that the human capital multiple h_J given in (30) decreases by about 11.1% from $h = 25$ with no jumps to $h_J = 22.22$ with jumps. The decrease is significant as jump shocks are permanent. At $w = 0$, the inclusion of the jump risk causes consumption to drop by about 16% from $c(0) = 0.96$ for the baseline case to $c(0) = 0.81$. The strong consumption response

for low values of w indicates the strong precautionary savings demand for a risk-averse agent even when the jump shock only occurs on average once every fifty years.

With $\lambda = 0.05$, large income shocks occur on average once every twenty years. The human wealth multiple, h_J , equals 19.04, which is about 23.8% lower than the multiple $h = 25$ with no jumps. At $w = 0$, we have $p(0) = 14.67$, which is 26.6% lower than $p(0) = 19.98$ under no jumps. Moreover, consumption $c(0) = 0.67$, which is about 30% lower than consumption, $c(0) = 0.96$, under no jumps. In summary, large income shocks even occurring with low frequency are very costly in terms of consumption as Figure 5 shows.

7 Transitory and Permanent Income Shocks

Empirical specifications of the income process often feature both permanent and transitory shocks. Meghir and Pistaferri (2011) provide a comprehensive survey. We next generalize the income process to have both permanent and transitory components. We show that transitory income shocks also have an important effect on consumption, especially for the wealth-poor.

We continue to use Y given in (1) to denote the permanent component of income. Let x denote the transitory component of income. The total income (in levels), denoted by X , is given by the product of Y and x , $X_t = x_t Y_t$. Empirical researchers often express the income process in logs, $\ln X_t = \ln Y_t + \ln x_t$. In our model, the logarithmic permanent component $\ln Y$ given by (2) follows an arithmetic Brownian motion.

Let $\{s_t : t \geq 0\}$ denote the transitory income state. For simplicity, we suppose that s_t is in one of the two states, G and B , which we refer to the good and bad state respectively. The transitory income value x equals x_G in state G and equals x_B in state B , with $x_B < x_G$. Over a small time period $(t, t + \Delta t)$, if the current state is G , the transitory state switches from x_G to x_B with probability $\phi_G \Delta t$, and stays unchanged with the remaining probability $1 - \phi_G \Delta t$. Similarly, the transition probability from B to G over a small time period Δt is $\phi_B \Delta t$. Technically, we model the transitory income state via a two-state Markov chain.¹⁹

Before analyzing the general incomplete-markets case, we first summarize the first-best CM setting where the modified PIH consumption rule is optimal.

¹⁹Markov chain specifications of the income process are often used in macro consumption-savings literature. Our model can be generalized to allow for multiple discrete states for the transitory income component.

Human wealth and the PIH consumption rule. As before, we define human wealth H_t under state s_t as the present discount value of future labor incomes, discounted at the risk-free rate,

$$H_t = \mathbb{E}_t \left(\int_t^\infty e^{-r(u-t)} x_u Y_u du \right). \quad (36)$$

Note that transitory income $\{x_u : u \geq 0\}$ follows a stochastic process. Transitory income shocks affect human wealth in an economically relevant and interesting way. We denote by h_t the agent's human wealth scaled by the permanent component Y , i.e. $h_t = H_t/Y_t$.

In the Appendix, we show that h_B and h_G have explicit forms given by

$$h_G = \frac{x_G}{r - \mu} \left(1 + \frac{\phi_G}{r - \mu + \phi_G + \phi_B} \frac{x_B - x_G}{x_G} \right), \quad (37)$$

$$h_B = \frac{x_B}{r - \mu} \left(1 + \frac{\phi_B}{r - \mu + \phi_G + \phi_B} \frac{x_G - x_B}{x_B} \right). \quad (38)$$

As the formula for h_G is symmetric to that for h_B , we only discuss h_G . First consider the special case where state G is absorbing, i.e. the probability of leaving state G is zero, $\phi_G = 0$. Transitory shocks become permanent and $h_G = x_G/(r - \mu)$. More generally, transitory shocks ($\phi_G > 0$) induce mean reversion between G and B , which we see from the second term in (37) for h_G . The higher the mean arrival rate ϕ_G from state G to B , the lower the value of h_G . Also, the larger the gap $(x_G - x_B)$, the lower the value of h_G .

With CM, consumption is given by the PIH rule, $c_s^*(w) = m^*(w + h_s)$, which implies that consumption is proportional to total wealth $x + h_s$. As expected, the MPC is the same as (11) for the baseline case. This is due to the Arrow-Debreu result that utility maximization and total wealth maximization are separate under CM. We next solve for the general incomplete-markets case with both permanent and transitory income shocks.

Incomplete-markets solution. Let $V(W, Y; s)$ denote the agent's value function with liquid wealth W , the permanent component of income Y , and the transitory income state s . Using the principle of optimality for recursive utility, in the interior region with positive wealth, i.e. $W_t > 0$, we have the following HJB equation,

$$0 = \max_{C > 0} f(C, V) + (rW + x_s Y - C)V_W(W, Y; s) + \mu Y V_Y(W, Y; s) + \frac{\sigma^2 Y^2}{2} V_{YY}(W, Y; s) + \phi_s [V(W, Y; s') - V(W, Y; s)], \quad (39)$$

where s' denotes the other discrete state. Note that the total income flow is $X = x_s Y$. The last term in (39) gives the conditional expected change of $V(W, Y; s)$ due to transitory income shocks. Consumption satisfies the FOC, $f_C(C, V) = V_W(W, Y; s)$. Using the homogeneity property, we write the certainty equivalent wealth $P(W, Y; s) = p(w; s)Y$. The following proposition summarizes the main results for $p(w; s) \equiv p_s(w)$ and the consumption rule $c(w; s) \equiv c_s(w)$.

Proposition 3 *The optimal consumption-income ratio $c_s(w)$ is given by*

$$c_s(w) = m^* p_s(w) (p'_s(w))^{-\psi}, \quad s = G, B, \quad (40)$$

where m^* is given in (11) and $p_s(w)$ solves the following system of ODEs:

$$\begin{aligned} 0 = & \left(\frac{m^* (p'_s(w))^{1-\psi} - \psi \rho}{\psi - 1} + \mu - \frac{\gamma \sigma^2}{2} \right) p_s(w) + x_s p'_s(w) + (r - \mu + \gamma \sigma^2) w p'_s(w) \\ & + \frac{\sigma^2 w^2}{2} \left(p''_s(w) - \gamma \frac{(p'_s(w))^2}{p_s(w)} \right) + \phi_s (p_{s'}(w) - p_s(w)), \quad s, s' = G, B. \end{aligned} \quad (41)$$

The above system of ODEs is solved with the following boundary conditions. First,

$$\lim_{w \rightarrow \infty} p_s(w) = w + h_s, \quad s = G, B, \quad (42)$$

where h_s is given by (37) and (38) for state G and B , respectively. Second, at $w = 0$,

$$0 = \left[\frac{m^* (p'_s(0))^{1-\psi} - \psi \rho}{\psi - 1} + \mu - \frac{\gamma \sigma^2}{2} - \phi_s \right] p_s(0) + x_s p'_s(0) + \phi_s p_{s'}(0), \quad s, s' = G, B. \quad (43)$$

Finally, consumption at the origin cannot exceed total income which implies

$$0 < c_s(0) \leq x_s, \quad s = G, B. \quad (44)$$

We now have two inter-linked ODEs that jointly characterize $p_G(w)$ and $p_B(w)$. Liquidity constraints in the two states are now different. In state G , $c_G(0)$ can possibly exceed one as the transitory income shock $x_G > 1$. In contrast, consumption at $w = 0$ in state B cannot exceed its transitory component, i.e. $c_B(0) < x_B < 1$. Therefore, the borrowing constraint is tighter in state B than in state G .

Figure 6 demonstrates the effects of transitory income shocks on optimal consumption $c_s(w)$. We choose $x_G = 1.2$ in state G and $x_B = 0.8$ in state B . The mean transition rate

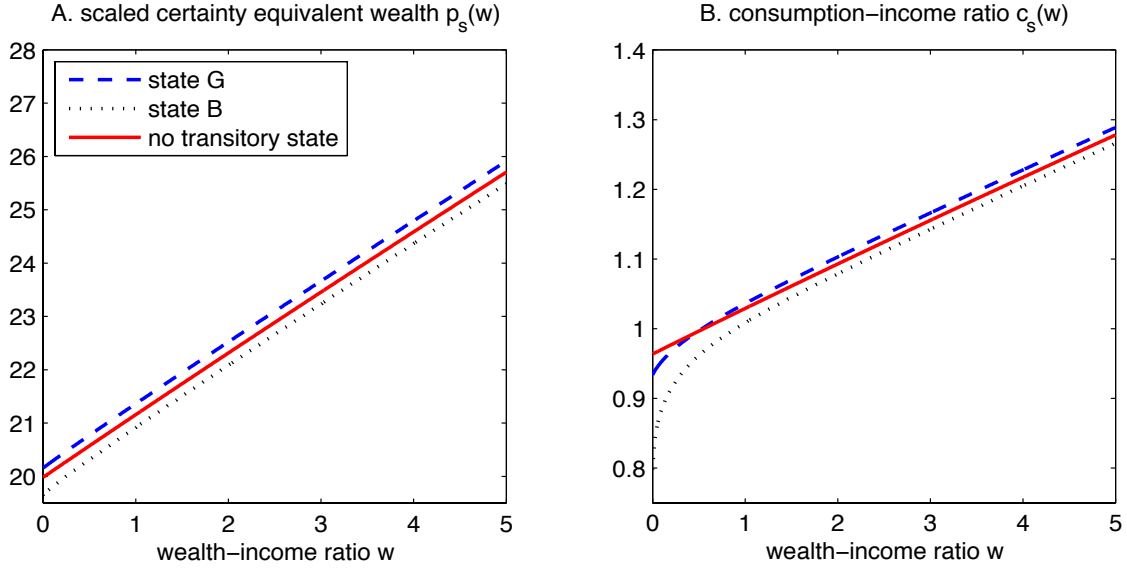


Figure 6: **The case with both permanent and transitory income shocks.** Parameter value: $r = 5\%$, $\rho = 5.5\%$, $\sigma = 10\%$, $\mu = 1\%$, $\gamma = 2$, $\psi = 0.5$, $\phi_G = 0.5$, $\phi_B = 0.5$, $x_G = 1.2$, $x_B = 0.8$, $h_G = 25.19$ and $h_B = 24.81$.

from state B to G is set at $\phi_B = 0.5$ which implies that the expected duration of state B is 2 years. Similarly, the expected duration of state G is set to 2 year as $\phi_G = 0.5$. Using the formulas for h_B and h_G , we obtain $h_B = 24.81$ and $h_G = 25.19$. Because $\phi_G = \phi_B$, the probability mass for the stationary distribution is $\pi_G = \pi_B = 1/2$. Therefore, the long-run average of human wealth is given by $\bar{h} = h_G\pi_G + h_B\pi_B = 25$.

As $w \rightarrow \infty$, self insurance is sufficient to achieve the first-best outcome and the CM PIH rule is optimal. However, this convergence is rather slow. Even for large values of w , transitory income shocks matter. For example, with $w = 5$, the certainty equivalent wealth is $p_G(5) = 25.91$, which is 14% lower than the CM “total” wealth, $w + h_G = 30.19$. Similarly, $p_B(5) = 25.49$, which is 15% lower than the “total” wealth in the limit $w + h_B = 29.81$. For consumption, $c_G(5) = 1.29$, which is 19% lower than the CM PIH level, $c_G^* = 1.58$. Even at $w = 20$, $c_G(20) = 2.15$, which is only 90.6% of the CM PIH level consumption.

Intuitively, one can view our exercise in this section as a dynamic “mean-preserving” spread of transitory income shocks around the baseline case where $x = 1$. The precautionary savings demand induced by this mean-preserving spread of transitory shocks generates large curvatures for consumption rules $c_s(w)$ in both state G and B especially for low values of w .

The agent becomes constrained at $w = 0$ in state B , $c_B(0) = x_B = 0.8$, and hence the agent is a hand-to-mouth consumer in state B . Interestingly, when the transitory income switches out of state B and transitions to state G , scaled consumption jumps from $c_B(0) = x_B = 0.8$ to $c_G(0) = 0.93$ and the agent is then no longer constrained and saves with a positive target wealth-income ratio w_G^{ss} . Specifically, in state G , the agent saves $0.27Y$, which is much larger than $0.03Y$, savings in the benchmark case with permanent income shocks only. In summary, transitory income shocks are critically important in understanding consumption and savings for the poor and can generate very large precautionary savings demand in state G for the wealth poor (low w).

8 Conclusions

We develop an analytically tractable continuous-time framework for the classic incomplete-markets income fluctuation/savings problem. Key features include (1) recursive utility which separates the coefficient of relative risk aversion from the elasticity of intertemporal substitution (EIS); (2) the borrowing constraint; (3) permanent and transitory income shocks; and (4) both diffusive/continuous and discrete/jump components for permanent shocks.

Our continuous-time model offers several advantages. First, it provides a simple characterization of the optimal consumption rule through a closed-form formula and a numerically easily solvable ordinary differential equation (ODE) with high precision. Second, the model allows us to clearly identify the impact of liquidity constraints and uninsurable income shocks by analyzing the appropriate boundary conditions.

Additionally, our model allows us to easily separate the effect of elasticity of intertemporal substitution and risk aversion on consumption. We find that the separation of risk aversion from the EIS is important to understand the economics of savings, both quantitatively and conceptually. For example, changing the coefficient of relative risk aversion (e.g. from two to four) leads to a quantitatively enormous increase of buffer-stock savings target w^{ss} (e.g. from 2.6 to 16.1 in our baseline calculation) and quantitatively much more dispersed wealth and consumption distributions.

Our tractable quantitative model can be used to build equilibrium models to analyze wealth distributions in classic incomplete-markets Bewley economies.²⁰ Permanent income

²⁰See Aiyagari (1994), Huggett (1993), and Krusell and Smith (1998) for important contributions. This

shocks (as opposed to transitory income shocks as in Aiyagari (1994) and Huggett (1993)) can lead to much larger precautionary savings demand and can potentially generate a skewed and a more empirically plausible wealth distribution. Due to space considerations, we leave the general equilibrium analysis of wealth distribution for future work.

incomplete-markets equilibrium framework is widely used in the literature. For example, De Nardi (2004) evaluates the importance of bequest motives and intergenerational transmission of ability to explain wealth distribution. Quadrini (2000), Cagetti and De Nardi (2006), and Buera and Shin (2011) show that entrepreneurship is critical in explaining the cross-sectional wealth distribution.

References

- Abowd, J. M., and D. Card, 1989, "On the covariance structure of earnings and hours changes," *Econometrica*, 57(2), 411-445.
- Aguiar, M., and E. Hurst, 2005, "Consumption vs. expenditure," *Journal of Political Economy*, 113(5), 919-948.
- Aiyagari, S. R., 1994, "Uninsured idiosyncratic risk and aggregate saving," *Quarterly Journal of Economics*, 99(3), 659-684.
- Bansal, R., and A. Yaron, 2004, "Risks for the long run: A potential resolution of asset pricing puzzles," *Journal of Finance*, 59(4), 1481-1509.
- Buera, F. J., and Y. Shin, 2011, "Self-insurance vs. self-financing: A welfare analysis of the persistence of shocks," *Journal of Economic Theory*, 146(3), 845-862.
- Brunnermeier, M. K., and Y. Sannikov, 2011, "A macroeconomic model with a financial sector," working paper, Princeton University.
- Caballero, R. J., 1990, "Consumption puzzles and precautionary savings," *Journal of Monetary Economics*, 25(1), 113-136.
- Caballero, R. J., 1991, "Earnings uncertainty and aggregate wealth accumulation," *American Economic Review*, 81(4), 859-871.
- Cagetti, M., and M. De Nardi, 2006, "Entrepreneurship, Frictions, and Wealth," *Journal of Political Economy*, 114(5), 835-869.
- Carroll, C. D., 1997, "Buffer-stock saving and the life cycle/permanent income hypothesis," *Quarterly Journal of Economics*, 112(1), 1-55.
- Carroll, C. D., and M. S. Kimball, 1996, "On the concavity of the consumption function," *Econometrica*, 64(4), 981-992.
- Campbell, J. Y., and N. G. Mankiw, 1989, "Consumption, income, and interest rates: Reinterpreting the time series evidence" in O. J. Blanchard and S. Fischer eds. *NBER Macroeconomics Annual*, MIT Press: Cambridge, MA, 185-216.

- Chamberlain, G., and C. A. Wilson, 2000, "Optimal intertemporal consumption under uncertainty," *Review of Economic Dynamics*, 3, 365-395.
- Cox, J. C., Ingersoll Jr., J. E., and S. A. Ross, 1985. "A theory of the term structure of interest rates," *Econometrica*, 53, 385-407.
- Deaton, A., 1991, "Saving and liquidity constraints," *Econometrica*, 59, 122-148.
- Deaton, A., 1992, "Understanding consumption," *Oxford University Press*, Oxford.
- De Nardi, M., 2004, "Wealth inequality and intergenerational links," *The Review of Economic Studies*, 71, 743-768.
- Dreze, J. H., and F. Modigliani, 1972, "Consumption decisions under uncertainty," *Journal of Economic Theory*, 5, 308-335.
- Duffie, D., 2001, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton.
- Duffie, D., and L. G. Epstein, 1992a, "Stochastic differential utility," *Econometrica*, 60, 353-394.
- Duffie, D., and L. G. Epstein, 1992b, "Asset pricing with stochastic differential utility," *Review of Financial Studies*, 5(3), 411-436.
- Duffie, D., W. Fleming, H. M. Soner, and T. Zariphopoulou, 1997, "Hedging in incomplete markets with HARA utility," *Journal of Economic Dynamics and Control*, 21, 753-782.
- Epstein, L. G., and S. E. Zin, 1989, "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework," *Econometrica*, 57(4), 937-969.
- Farmer, R., 1990, "Rince preferences," *Quarterly Journal of Economics*, 105(1), 43-60.
- Friedman, M., 1957, "A theory of the consumption function," *Princeton University Press*, Princeton.
- Gertler, M., 1999, "Government debt and social security in a life-cycle economy," *Carnegie-Rochester Conference Series on Public Policy*, 50, 61-110.

- Gourinchas, P. O., and J. A. Parker, 2002, "Consumption over the life cycle," *Econometrica*, 70(1), 47-89.
- Guvenen, F., 2006, "Reconciling conflicting evidence on the elasticity of intertemporal substitution: A macroeconomic perspective," *Journal of Monetary Economics*, 53(7), 1451-72.
- Guvenen, F., 2007, "Learning your earning: Are labor income shocks really very persistent?" *American Economic Review*, 97(3), 687-712.
- Hall, R. E., 1978, "Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence," *Journal of Political Economy*, 86, 971-987.
- Hall, R. E., 1988, "Intertemporal substitution in consumption," *Journal of Political Economy*, 96(2), 339-357.
- Hall, R. E., 2009, "Reconciling cyclical movements in the marginal value of time and the marginal product of labor," *Journal of Political Economy*, 117(2), 281-323.
- Hansen, L. P., T. Sargent, and T. Tallarini, 1999, "Robust permanent income and pricing," *The Review of Economic Studies*, 66(4), 873-907.
- He, H., and H.F. Pages, 1993, "Labor income, borrowing constraints, and equilibrium asset prices," *Economic Theory*, 3, 663-696.
- Huggett, M., 1993, "The risk free rate in heterogeneous agent incomplete insurance economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953-969.
- Jacobson, L., LaLonde, R., and D. Sullivan, 1993, "Earnings losses of displaced workers," *American Economic Review*, 83(4), 685-709.
- Fernández-Villaverde, J. and D. Krueger, 2007, "Consumption over the life cycle: Facts from consumer expenditure survey data," *The Review of Economics and Statistics*, 89(3), 552-565.
- Kaplan, G., and G. L. Violante, 2010, "How much consumption insurance beyond self-insurance?" *American Economic Journal: Macroeconomics*, 2(4), 53-87.

- Koo, H. K., 1998, "Consumption and portfolio selection with labor income: A continuous time approach," *Mathematical Finance*, 8, 49-65.
- Kreps, D. M., and E. L. Porteus, 1978, "Temporal resolution of uncertainty and dynamic choice theory," *Econometrica*, 46(1), 185-200.
- Krusell, P., and T. Smith, 1998, "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867-896.
- Leland, H. 1968, "Saving and uncertainty: The precautionary demand for saving," *Quarterly Journal of Economics*, 82(3), 465-473.
- Levhari, D., and T. N. Srinivasan, 1969, "Optimal savings under uncertainty," *Review of Economic Studies*, 36(2), 153-163.
- Low, H., C. Meghir, and L. Pistaferri, 2010, "Wage risk and employment risk over the life cycle," *American Economic Review*, 100, 1432-1467.
- Lustig, H., and S. Van Nieuwerburgh, 2008, "The returns on human capital: Good news on wall street is bad news on main street," *Review of Financial Studies*, 21(5), 2097-2137.
- MaCurdy, T. E., 1982, "The use of time series processes to model the error structure of earnings in a longitudinal data analysis," *Journal of Econometrics*, 18, 83-114.
- Kimball, M. S., and N. G. Mankiw, 1989, "Precautionary saving and the timing of taxes," *Journal of Political Economy*, 97(4), 863-879.
- Meghir, C., and L. Pistaferri, 2004, "Income variance dynamics and heterogeneity," *Econometrica*, 72(1), 1-32.
- Meghir, C., and L. Pistaferri, 2011, "Earnings, consumption and lifecycle choices," Handbook of Labor Economics, O. Ashenfelter and D. Card (eds), vol. 4B, 774-854, 2011.
- Merton, R. C., 1971, "Consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 3, 373-413.
- Quadrini, V., 2000, "Entrepreneurship, Saving, and Social Mobility," *Review of Economic Dynamics*, 3, 1-40.

- Sandmo, A., 1970, "The effect of uncertainty on saving decisions," *Review of Economic Studies*, 37(3), 353-360.
- Schechtman, J., 1976, "An income fluctuation problem," *Journal of Economic Theory*, 12(2), 218-241.
- Storesletten, K., C. I. Telmer, and A. Yaron, 2004, "Consumption and risk sharing over the life cycle," *Journal of Monetary Economics*, 51(3), 609-633.
- Tallarini, T. D., 2000 "Risk-sensitive real business cycles," *Journal of Monetary Economics*, 45(3), 507-532.
- von Watchter, T., J. Song, and J. Manchester, 2007, "Long-term earnings losses due to job separation during the 1982 recession: an analysis using longitudinal administrative data from 1974 to 2004," working paper, Columbia University.
- Wang, N., 2006, "Generalizing the permanent-income hypothesis: revisiting Friedman's conjecture on consumption," *Journal of Monetary Economics*, 53(4), 737-752.
- Weil, P., 1990, "Non-expected utility in macroeconomics," *Quarterly Journal of Economics*, 105, 29-42.
- Weil, P., 1993, "Precautionary savings and the permanent income hypothesis," *Review of Economic Studies*, 60(2), 367-384.
- Zeldes, S. P., 1989, "Consumption and liquidity constraints: An empirical investigation," *Journal of Political Economy*, 97(2), 305-346.

Appendix

A Technical details

This appendix provides technical details for the main results of the paper.

The homogeneity property of the value function holds for all the cases in our paper. Therefore, we conjecture that the value function is given by (19). Therefore, we have

$$V_W = b^{1-\gamma}(p(w)Y)^{-\gamma}p'(w), \quad (\text{A.1})$$

$$V_Y = b^{1-\gamma}(p(w)Y)^{-\gamma}(p(w) - wp'(w)), \quad (\text{A.2})$$

$$V_{WW} = b^{1-\gamma}(p(w)Y)^{-1-\gamma} (p(w)p''(w) - \gamma(p'(w))^2), \quad (\text{A.3})$$

$$V_{WY} = b^{1-\gamma}(p(w)Y)^{-1-\gamma} (-wp(w)p''(w) - \gamma p'(w)(p(w) - wp'(w))), \quad (\text{A.4})$$

$$V_{YY} = b^{1-\gamma}(p(w)Y)^{-1-\gamma} (w^2p(w)p''(w) - \gamma(p(w) - wp'(w))^2). \quad (\text{A.5})$$

A.1 Complete-Markets Solution

To complete the markets, we need to introduce an additional asset which is perfectly correlated with the labor income process. By assuming that this newly introduced asset earns no expected excess return, we construct a benchmark case where idiosyncratic labor income risk carries no risk premium under complete markets. That is, we assume the dynamics for the value of this newly introduced asset is given by

$$dS_t = S_t(rdt + \sigma_S dB_t), \quad (\text{A.6})$$

where σ_S is the volatility parameter and B is the same Brownian motion driving the labor income process (1). Let η_t denote the fraction of financial wealth allocated to this risky asset. Then, liquid wealth W accumulates as follows:

$$dW_t = (rW_t + Y_t - C_t)dt + \sigma_S \eta_t W_t dB_t. \quad (\text{A.7})$$

Using the standard principle of optimality, we may write the HJB equation as follows:

$$0 = \max_{C, \eta} f(C, V) + (rW + Y - C)V_W(W, Y) + \mu Y V_Y(W, Y) \\ + \frac{\eta^2 \sigma_S^2 W^2}{2} V_{WW}(W, Y) + \eta \sigma_S \sigma W Y V_{WY}(W, Y) + \frac{\sigma^2 Y^2}{2} V_{YY}(W, Y). \quad (\text{A.8})$$

Substituting (19) and (A.1)-(A.5) into (A.8) and using the FOCs for consumption and portfolio allocation, we obtain the following decision rules:

$$c(w) = m^* p(w) (p'(w))^{-\psi}, \quad (\text{A.9})$$

$$\eta(w) = \frac{\sigma}{\sigma_S} \left(1 - \frac{\gamma p(w) p'(w)}{w(\gamma (p'(w))^2 - p(w) p''(w))} \right). \quad (\text{A.10})$$

After simplifying, we have the following ODE for $p(w)$:

$$0 = \left(\frac{m^* (p'(w))^{1-\psi} - \psi \rho}{\psi - 1} + \mu - \frac{\gamma \sigma^2}{2} \right) p(w) + [(r - \mu)w + 1] p'(w) + \frac{\gamma^2 \sigma^2 p(w)}{2 \left(\gamma - \frac{p(w) p''(w)}{(p'(w))^2} \right)}. \quad (\text{A.11})$$

With CM, $p^*(w) = w + h$ is the solution to the ODE, which makes intuitive sense. Consumption $c(w)$ is given by (16). The optimal hedging portfolio η is given by

$$\eta(w) = \frac{\sigma}{\sigma_S} \left(1 - \frac{p^*(w)}{w} \right) = -\frac{\sigma h}{\sigma_S w}. \quad (\text{A.12})$$

The total amount of wealth in the hedging portfolio is then $\eta(w)W = -\sigma h Y / \sigma_S$, which is proportional to income Y . Using Ito's formula, we obtain

$$\begin{aligned} dw_t &= d \frac{W_t}{Y_t} = W_t \left(-\frac{dY_t}{Y_t^2} + \frac{1}{2} \frac{2(dY_t)^2}{Y_t^3} \right) + \frac{dW_t}{Y_t} - \frac{dW_t dY_t}{Y_t^2} \\ &= [(r - \mu + \sigma^2) - c(w_t)](w_t + h) dt - (w_t + h) \sigma dB_t. \end{aligned} \quad (\text{A.13})$$

We may thus write $w + h$ as a geometric Brownian motion,

$$\frac{d(w_t + h)}{w_t + h} = [\sigma^2 - \mu - \psi(\rho - r)] dt - \sigma dB_t. \quad (\text{A.14})$$

Since $w_t + h > 0$, we have $c_t > 0$ for all t .

A.2 Model Solution: Incomplete Markets

Proof of Theorem 1. We conjecture that the value function is given by (19). Using the FOC (18) for C and letting $P(W, Y) = p(w)Y$, we obtain the following:

$$c(w) = m^* p(w) (p'(w))^{-\psi}. \quad (\text{A.15})$$

Substituting the above results into the normalized aggregator (4), we have

$$f(C, V) = \frac{\rho}{1 - \psi^{-1}} \left(\frac{(bp(w)Y)^{1-\gamma} (bp'(w))^{1-\psi}}{\rho^{1-\psi}} - (bp(w)Y)^{1-\gamma} \right). \quad (\text{A.16})$$

Substituting (A.15), (A.16), the value function (19), (A.1), (A.2), and (A.5) into the HJB equation (17), and simplifying, we obtain the ODE (21) in Theorem 1.

Next, we turn to the boundary conditions. When w approaches infinity, the certainty equivalent wealth approaches the CM benchmark value, i.e. $\lim_{w \rightarrow \infty} p(w) = p^*(w) = w + h = w + 1/(r - \mu)$. And then substituting $w = 0$ into the ODE (21), we have (13). Finally, borrowing constraints imply $0 < c(0) \leq 1$.

Proof of Proposition 2. We extend the methodology for the baseline model to account for jumps. We substitute the value function $V(W, Y)$ given by (19) into the HJB equation (31) and using (A.1), (A.2), and (A.5), we obtain the ODE (32) for the scaled certainty equivalent wealth $p(w)$ and the same consumption rule (20) for $c(w)$ as in the baseline case.

Similarly, in the limit as $w \rightarrow \infty$, we conjecture $p(w)$ takes the following form

$$\lim_{w \rightarrow \infty} p(w) = w + h_J, \quad (\text{A.17})$$

and then substituting it into ODE (32), and after some algebras we obtain (30).

Proof of Proposition 3. We conjecture that the value function is given by

$$V(W, Y; s) = \frac{(bP(W, Y; s))^{1-\gamma}}{1-\gamma}, \quad (\text{A.18})$$

where b is given in (15) and $P(W, Y; s)$ is the certainty equivalent wealth. Using (A.18) and the consumption FOC, we jointly solve $p_s(w)$ and the consumption $c_s(w)$ via (40) and the interrelated ODEs (41) for G and B . As $w \rightarrow \infty$, $p_s(w) \rightarrow w + h_s$, where h_s is the corresponding human wealth under state s . Substituting $p_s(w) = w + h_s$ into (41), we obtain (37) and (38) for h_G and h_B , respectively. At $w = 0$, we have (43). Using the consumption constraint $C(0, Y; s) \leq X = x_s Y$, we have (44).