Strategic Asset Allocation with Predictable Returns and Transaction Costs*

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Abstract

We propose a factor-based model that incorporates common factor shocks for the security returns. Under these realistic factor dynamics, we solve for the dynamic trading policy in the class of linear policies analytically. Our model can accommodate stochastic volatility and liquidity as a function of same factor exposures. Calibrating our model with empirical data, we show that our trading policy achieves superior performance particularly in the presence of common factor shocks.

1. Introduction

Strategic asset allocation is a central objective for institutional investors. In response to time-varying expected future returns and volatility asset managers need to continuously update the constituents of the portfolio and their corresponding weights. While determining the new portfolio weights, investors also consider how much transaction costs will be paid during the transition to the new portfolio. In a nutshell, this trade-off is what complicates the portfolio decision process: the investor is forward-looking while trying to predict future returns and volatility but is backward-looking in order to keep transaction costs to minimum. Furthermore, the drivers of the portfolio

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decision have complex interdependence: expected future returns are often negatively correlated with volatility and transaction costs increase with higher volatility. Characterizing an optimal trading rule under these interactions is a difficult task.

Many dynamic portfolio choice models need to impose restrictive assumptions, yet often unrealistic, about return generating model in order to achieve a tractable solution. There is little academic work on combining dynamic asset allocation with return predictability and transaction cost models. In this paper, we aim to incorporate many of the empirical facts we know related to the equity price dynamics. We propose a tractable approach that utilizes linear policies in strategic asset allocation with a model including predictability, transaction costs, stochastic volatility and general interdependence between these factors. We obtain a closed-form solution for our policy parameters that allows us interpret the impact of various characteristics on our trading policy.

In our model, the dynamics of the return predicting factors is assumed to be arbitrary. We allow for factor dependent covariance structure in returns driven by common factor shocks i.e., stochastic volatility. Furthermore, we can also have time-varying liquidity costs which are correlated with the expected returns of the factors. Our model involves the standard wealth equation in dollars and is built with nonlinear dynamics in our portfolio holdings.

We provide a well-calibrated simulation study to analyze the performance metrics of our approach. Our simulation study shows that best linear policy provides significant benefits compared to other frequently used policies in the literature, especially when returns evolve according to factor dependent covariance structure. Unlike other parametric approaches studied so far, our approach provides a closed form solution and the driver of the policy dynamics can be analyzed in full detail.

1.1. Related literature

The vast literature on dynamic portfolio choice starts with the seminal paper by Merton (1971) which studies the optimal dynamic allocation of one risky asset and one bond in the portfolio in a continuous-time setting. Following this seminal paper, there has been a significant literature aiming to incorporate the impact of various frictions on the optimal portfolio choice. For a survey on this literature, see Cvitanic (2001). Constantinides (1986) studies the impact of proportional transaction costs on the optimal investment decision and observes path dependence in the optimal
policy. Similarly, [Davis and Norman (1990)] and [Dumas and Luciano (1991a)] study the impact of transaction costs on the optimal investment and consumption decision by formally characterizing the trade and no-trade regions. One drawback of all these papers is that the optimal solution is only computed in the case of a single stock and bond. [Liu (2004)] extends this result to multiple assets but assumes that asset returns are not correlated.

There is a growing literature on portfolio selection that incorporates return predictability with transaction costs. [Balduzzi and Lynch (1999)] and [Lynch and Balduzzi (2000)] illustrate the impact of return predictability and transaction costs on the utility costs and the optimal rebalancing rule by discretizing the state space of the dynamic program. Recently, [Brown and Smith (2010)] provides heuristic trading strategies and dual bounds for a general dynamic portfolio optimization problem with transaction costs and return predictability. [Brandt et al. (2009a)] parameterizes the rebalancing rule as a function of security characteristics and estimates the parameters of the rule from empirical data without modeling the distribution of the returns and the return predicting factors. Our approach is also a linear parametrization of return predicting factors, but at the micro-level, we seek to obtain a policy that is coherent with the update of the position holdings in a nonlinear fashion. Thus, our linear policy uses the convolution of the factors with their corresponding returns in order to correctly satisfy the wealth equation at all times. On a separate note, we solve for the optimal policy in closed-form using a deterministic linear quadratic control and can achieve greater flexibility in parameterizing the trading rule.

[Garleanu and Pedersen (2012)] achieve a closed-form solution for a model with linear dynamics in return predictors and quadratic function for transaction costs and quadratic penalty term for risk. In this paper, the model for the security returns is given in price changes which may suffer highly from non-stationarity. [Moallemi and Saglam (2012)] proposes an approximate trading rule which are linear in return predicting factors and is again based on a model with price changes. This paper emphasizes the computational tractability of linear rebalancing rules. In our methodology, we work with the standard wealth equation and incorporate quadratic transaction costs that enable us to solve for our policy parameters analytically.
2. Model

We lay out the return generating process for the set of securities, the portfolio dynamics, the agent’s optimization problem, and our solution technique.

There are $T$ periods. Each period, the $N \times 1$ vector of risky asset returns $R_{t+1}$ are realized. The returns are governed by a standard factor model, and in our basic setting are equal to the sum of a risk-free return plus the systematic returns on the $K$ factors weighted by the factor exposures, plus an idiosyncratic risk term. Also, per unit of factor exposure, the assets earn a premium of $\lambda_t$.

At time 0 the agent is endowed with a portfolio of dollar holdings of the $N$ risky assets, $x_0$. The set of risky assets earns a gross return $R_1$ over the first period. At $t = 1$, the end of the first period, the agent trades a dollar quantity $u_1$. Based on the stochastic evolution of prices, factor-exposures and risk for each asset, the agent continues to trade each period up through period $T$. The agent’s dollar holdings of the $N$ risky assets at the end of period $t+1$, $x_{t+1}$, are just equal to the holdings at the end of period $t$, multiplied by the gross returns on the assets over period $t+1$, plus the dollar trades the agent makes at the end of period $t+1$.

For reasons that will become clear, we restrict the agent’s trade of asset $i$ at time $t$ to be a linear function of: (1) the current set of factor exposures $B_{i,t}$; (2) all past factor exposures weighted by the gross return on asset $i$ since that time $B_{i,s}R_{i,s\rightarrow t}, \forall s < t$.

In Section 2.1 we describe in detail the return generating process for the set of securities. In Section 2.2 we derive the portfolio dynamics. Section 2.3 lays out the agent’s optimization problem, and sections 2.4 and 2.5 describe our solution technique and derive the solution.

2.1. Security and factor dynamics

We consider a dynamic portfolio optimization problem with $K$ factors and $N$ securities. Let $S_{i,t}$ be the discrete time dynamics for the price of the security that pays a dividend $D_{i,t}$ at time $t$. We assume that the gross return to our security defined by $R_{i,t+1} = \frac{S_{i,t+1} + D_{i,t+1}}{S_{i,t}}$ have the following

\[ R_{i,t+1} = \frac{S_{i,t+1} + D_{i,t+1}}{S_{i,t}} \]

In general, the agent will probably choose to not trade at $T$, but for some objective functions he may.
for some family of functions \(g(t, \cdot) : \mathbb{R} \to \mathbb{R}\), increasing in their second argument, and where we further introduce the following notation:

- \(B_{i,t}\) is the \((K, 1)\) vector of exposures to the factors.
- \(F_{t+1}\) is the \((K, 1)\) vector of random (as of time \(t\)) factor realizations, with mean 0 and conditional covariance matrix \(\Omega_{t,t+1}\).
- \(\epsilon_{i,t+1}\) is the idiosyncratic risk of stock \(i\).

We assume that \(\epsilon_{\cdot,t+1}\) are mean zero, have a time-invariant covariance matrix \(\Sigma_{\epsilon}\), and are uncorrelated with the contemporaneous factor realizations.

- \(\lambda_{t}\) is the \((K, 1)\) vector of conditional expected factor returns.

We assume that \(B_{i,t}\) and \(\lambda_{t}\) are observable and follow some known dynamics, which for now we leave unspecified (when we solve a special example below, we assume that \(\lambda_{t}\) is constant and that the \(B_{i,t}\) follow a Gaussian AR(1) process, but our approach could apply to more complex dynamics).

As we show below, our approach can be extended to account for time varying factor expected returns (i.e., \(\lambda_{t}\) could be stochastic), and non-normal factor or idiosyncratic risk distributions (e.g., GARCH features can easily be added).

Note that this setting captures two standard return generating processes:

1. The “discrete exponential affine” model for security returns in which log-returns are affine in factor realizations:\(^2\)

\[
\log R_{i,t+1} = \alpha_{i} + B_{i,t}^\top(F_{t+1} + \lambda) + \epsilon_{i,t+1} - \frac{1}{2} \left( \sigma_{i}^2 + B_{i,t}^\top \Omega B_{i,t} \right)
\]

\(^2\)The continuous time version of this model is due to Vasicek (1977), Cox et al. (1985), and generalized in Duffie and Kan (1996). The discrete time version is due to Gourieroux et al. (1993) and Le et al. (2010).
2. The “linear affine factor model” where returns (and therefore also excess returns) are affine in factor exposures:

\[ r_{i,t+1} = \alpha_i + B_{i,t}^\top (F_{t+1} + \lambda) + \epsilon_{i,t+1} \]

As we show below, our portfolio optimization approach is equally tractable for both these return generating processes.

2.2. Cash and stock position dynamics

We will assume discrete time dynamics for our cash \((w(t))\) position and dollar holdings \((x_i(t))\) in stocks. We assume that

\[
x_{i,t+1} = x_{i,t} R_{i,t+1} + u_{i,t+1} \quad i = 1, \ldots, N
\]

\[
w_{t+1} = w_t R_{0,t+1} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,t+1} A_{t+1}(i,j) u_{j,t+1}
\]

where \(R_{i,t+1} = \frac{S_{i,t+1} + D_{i,t+1}}{S_{i,t}}\) is the total gross return (capital gains plus dividends) on the security \(i\). We are here effectively assuming that each position in security \(i = 1, \ldots, n\) is financed by a short position in a \((e.g.,\) risk-free) benchmark security \(0\), which we assume can be traded with no transaction costs. We denote by \(x_{i,t}\) the dollar investment in asset \(i\), by \(w_t\) the total cash balances (invested in the risk-free security \(S_0\)), and \(u_{i,t+1}\) is the dollar amount of security \(i\) we will trade at price \(S_{i,t+1}\). In vector notation,

\[(1)\]

\[
x_{t+1} = x_t \circ R_{t+1} + u_{t+1}
\]

\[(2)\]

\[
w_{t+1} = w_t R_{0,t+1} - \mathbf{1}^\top u_{t+1} - \frac{1}{2} u_{t+1}^\top A_{t+1} u_{t+1}
\]

where the operator \(\circ\) denotes element by element multiplication if the matrices are of same size or if the operation involves a scalar and a matrix, then that scalar multiplies every entry of the matrix.

The matrix \(A_t\) captures (possibly time-varying) quadratic transaction/price-impact costs, so
that $\frac{1}{2}u_t^\top \Lambda_t u_t$ is the dollar cost paid when realizing a trade at time $t$ of size $u_t$. For simplicity we assume this matrix is symmetric.\footnote{The symmetry assumption could easily be relaxed.} Garleanu and Pedersen (2012) present some micro-economic foundations for such quadratic costs. As they show, the quadratic form is analytically very convenient.

### 2.3. Objective function

We assume that the investor’s objective function is to maximize a linear quadratic function of his terminal cash and stock positions $F(w_T, x_T) = w_T + a^\top x_T - \frac{1}{2}x_T^\top b x_T$, net of a risk-penalty which we take to be proportional to the per-period variance of the portfolio. We assume $a$ is a $(N, 1)$ vector and $b$ a $(N, N)$ symmetric matrix.\footnote{The symmetry assumption on $b$ could easily be relaxed.} So we assume the objective function is simply:

$$
\max_{u_1, \ldots, u_T} \mathbb{E} \left[ F(w_T, x_T) - \sum_{t=0}^{T-1} \frac{\gamma}{2} x_T^\top \Sigma_{t\rightarrow t+1} x_t \right]
$$

We define $\Sigma_{t\rightarrow t+1} = \mathbb{E}_t[(R_{t+1} - \mathbb{E}_t[R_{t+1}])[R_{t+1} - \mathbb{E}_t[R_{t+1}]]']$ to be the conditional one-period variance-covariance matrix of returns and $\gamma$ can be interpreted as the coefficient of risk aversion.

The $F(\cdot, \cdot)$ function parameters can be chosen to capture different objectives, such as maximizing the terminal gross value of the position ($w_T + 1^\top x_T$) or the terminal liquidation (i.e., net of transaction costs) value of the portfolio ($w_T + 1^\top x_T - \frac{1}{2}x_T^\top \Lambda_T x_T$), or any intermediate situation.

Assuming the investor starts with some initial cash balances $w_0$ and an initial position in individual stocks $x_0$, note that $x_T$ and $w_T$ can be rewritten as:

$$
x_T = x_0 \circ R_{0\rightarrow T} + \sum_{t=1}^{T} u_t \circ R_{t\rightarrow T}
$$

$$
w_T = w_0 R_{0,0\rightarrow T} - \sum_{i=1}^{T} \left( u_t^\top 1 R_{0, t\rightarrow T} + \frac{1}{2} u_t^\top \Lambda_t u_t R_{0, t\rightarrow T} \right)
$$

where we have defined the cumulative return between date $t$ and $T$ on security $i$ as:

$$
R_{i, t\rightarrow T} = \prod_{s=t+1}^{T} R_{i,s}
$$
(with the convention that \( R_{i,t \to t} = 1 \)) and the corresponding \( N \)-dimensional vector \( R_{t \to T} = [R_{1,t \to T}; \ldots ; R_{N,t \to T}] \).

Now note that

\[
\begin{align*}
    a^\top x_T &= (a \circ R_{0 \to T})^\top x_0 + \sum_{i=1}^{T} (a \circ R_{t \to T})^\top u_t \\
    x_T^\top b x_T &= x_0^\top R b R_0 x_0 + \sum_{t=1}^{T} u_t^\top R b R_t u_t + 2 \sum_{t=1}^{T} x_0 \circ R_{0,t \to T} b R_t u_t 
\end{align*}
\]

where we define the \((N,N)\)-matrix \( R b R_t \) and \( b R_t \) with respective element:

\[
\begin{align*}
    \{ R b R_t \}_{ij} &= R_{i,t \to T} b_{ij} R_{j,t \to T} \\
    \{ b R_t \}_{ij} &= b_{ij} R_{j,t \to T} 
\end{align*}
\]

Substituting we obtain the following:

\[
\begin{align*}
    F(w_T,x_T) &= F_0 + \sum_{i=1}^{T} \left\{ G_t^\top u_t - \frac{1}{2} u_t^\top P_t u_t \right\} \\
    F_0 &= w_0 R_{0,0 \to T} + (a \circ R_{0 \to T})^\top x_0 - \frac{1}{2} x_0^\top R b R_0 x_0 \\
    G_t &= a \circ R_{t \to T} + 1 \circ R_{0,t \to T} - x_0 \circ R_{0,t \to T} b R_t \\
    P_t &= (R b R_t + \Lambda_t \circ R_{0,t \to T}) 
\end{align*}
\]

Substituting into the objective function given in equation (3) it can be rewritten as:

\[
\begin{align*}
    F_0 + \max_{u_1,\ldots,u_T} \sum_{t=0}^{T-1} \mathbb{E} \left[ G_{t+1}^\top u_{t+1} - \frac{1}{2} u_{t+1}^\top P_{t+1} u_{t+1} - \frac{\gamma}{2} x_{t+1}^\top \Sigma_{t+1} x_{t+1} \right] 
\end{align*}
\]

subject to the non-linear dynamics given in equations (11) and (12). 

We next describe our set of \textit{linear policies}, which make this problem tractable. At this stage it is convenient to introduce the following notation (inspired from matlab): We write \([A;B]\) (respectively \([A,B]\)) to denote the vertical (respectively horizontal) concatenation of two matrices.
2.4. Linear policies

We consider a class of parametric linear policies that is richer than the one previously considered in the literature (see, e.g., Brandt et al. (2009b)), but nevertheless has the advantage of leading to an explicit solution for the portfolio choice problem with transaction costs. Thus, in contrast to the approach proposed in Brandt et al. (2009b), we do not need to perform a numerical optimization, and can handle transaction costs efficiently. Further, in contrast to Garleanu and Pedersen (2012), I can handle more complex asset return dynamics and explicitly formulate the problem in terms of dollar returns (as opposed to number of shares), and yet retain the analytical flexibility of the linear-quadratic framework.

These benefits come at a cost, namely that of restricting our optimization to a specific set of parametrized trading strategies. It is an empirical question whether the set we work with is sufficiently large to deliver useful results. We present some empirical tests of our approach in the next section. First, we describe the strategy set we consider. Then we explain how the portfolio optimization can be done in closed-form, within that restricted set.

We define our set of linear policies with a set of \((K + 1)\)-dimensional vectors of parameters, \(\pi_{i,s,t}\) and \(\theta_{i,s,t}\), defined for all \(i = 1, \ldots, N\) and for all \(s \leq t\). The (previously defined) time \(t\) trade of asset \(i\) \((u_{i,t})\) of and dollar investment in asset \(i\) \((x_{i,t})\) are given by:

\[
(16) \quad u_{i,t} = \sum_{u=1}^{t} \pi_{i,u,t}^\top B_{i,u,t} \\
\]

and

\[
(17) \quad x_{i,t} = \sum_{u=1}^{t} \theta_{i,u,t}^\top B_{i,u,t} \\
\]

are then vector products of \(\pi_{i,s,t}\) and \(\theta_{i,s,t}\) and a \((K + 1)\) vector

\[
(18) \quad B_{i,u,t} = [1; B_{i,t}] R_{i,u \rightarrow t}. \\
\]

\(B_{i,u,t}\) is seen to be the \((K)\) vector of time \(t\) factor exposures, augmented with a “1”, and all weighted by the cumulative return earned by security \(i\) between time \(u\) and \(t\). In other words, these policies
allow trades at time $t$ to depend on current factor exposures $B_{i,t}$, but also on all past exposures weighted by their past holding period returns.

Intuitively, the dependence on current exposures, unweighted by lagged returns, is clearly important. In fact, in a no-transaction cost affine portfolio optimization problem where the optimal solution is well-known, the optimal solution will involve only current exposures (see, e.g., Liu (2007)). Note that this is also the choice made by Brandt et al. (2009b) for their ‘parameteric portfolio policies.’ However, while Brandt et al. (2009b) specify the loadings on exposure of individual stocks to be identical, we allow two stocks with identical exposures (and with perhaps different levels of idiosyncratic variance) to have different weights and trades.\footnote{Note, for the Brandt et al. (2009b) econometric approach it is useful to have fewer parameters. This is not an issue with our approach as our solution is closed-form.}

With transaction costs, allowing portfolio weights and trades to depend on past returns interacted with past exposures seems useful. The intuition for this comes from the path-dependence we observe in known closed-form solutions (see Constantines, 1986; Davis and Norman, 1990; Dumas and Luciano, 1991b; Liu and Loewenstein, 2002, and others).

To proceed, we note that the assumed linear position and trading strategies in equations (16) and (17) have to satisfy the dynamics given in equations (1) and (2). It follows that the parameter vectors $\pi_{i,s,t}$ and $\theta_{i,s,t}$ have to satisfy the following restrictions, for all $i = 1, \ldots, N$:

\begin{align}
\pi_{i,s,t} &= \theta_{i,s,t} - \theta_{i,s,t-1} & \text{for } s < t \\
\pi_{i,t,t} &= \theta_{i,t,t} 
\end{align}

We can rewrite these policies in a concise matrix form. First, define the ($N(K + 1)t, 1$) vectors $\pi_t$ and $\theta_t$ as

\begin{align}
\pi_t &= \begin{bmatrix} 
\pi_{1,1,t}; \ldots; \pi_{n,1,t}; \pi_{1,2,t}; \ldots; \pi_{n,2,t}; \ldots; \pi_{1,t,t}; \ldots; \pi_{n,t,t} 
\end{bmatrix} \\
\theta_t &= \begin{bmatrix} 
\theta_{1,1,t}; \ldots; \theta_{n,1,t}; \theta_{1,2,t}; \ldots; \theta_{n,2,t}; \ldots; \theta_{1,t,t}; \ldots; \theta_{n,t,t} 
\end{bmatrix}
\end{align}

Further, let’s define the following ($N(K + 1), N$) matrices (defined for all $1 \leq s \leq t \leq T$) as the
diagonal concatenations of the $N$ vectors $B_{i,s,t} \forall i = 1, \ldots, N$:

$$
B_{s,t} = \begin{pmatrix}
B_{1,s,t} & 0 & 0 & \ldots & 0 \\
0 & B_{2,s,t} & 0 & \ldots & 0 \\
& & & \ldots & \\
0 & \ldots & 0 & B_{n,s,t}
\end{pmatrix}
$$

Then we can define the $(N(K+1)t, N)$ matrix $B_t$ by stacking the $t$ matrices $B_{s,t} \forall s = 1, \ldots, t$:

$$
B_t = [B_{1,t}; B_{2,t}, \ldots, B_{t,t}]
$$

It is then straightforward to check that:

\begin{align}
(23) & \quad u_t = B_t^T \pi_t \\
(24) & \quad x_t = B_t^T \theta_t
\end{align}

Further, in terms of these definitions the constraints on the parameter vector in (19) can be rewritten concisely as:

\begin{align}
(25) & \quad \theta_{t+1} - \theta_0^t = \pi_{t+1}
\end{align}

where we define $\theta_0^t = [\theta_t; 0_{K+1}]$ to be the vector $\theta_t$ stacked on top of a $(K + 1, 1)$ vector of zeros $0_{K+1}$.

The usefulness of restricting ourselves to this set of ‘linear trading strategies’ is that optimizing over this set amounts to optimizing over the parameter vectors $\pi_t$ and $\theta_t$, and that, as I show next, that problem reduces to a deterministic linear-quadratic control problem, which can be solved in closed form.

Indeed, substituting the definition of our linear trading strategies from equation (23) into our objective function we may rewrite the original problem given in equation (15) as follows.
\[
F_0 + \max_{\pi_1, \ldots, \pi_T} \sum_{t=0}^{T-1} g_{t+1}^\top \pi_{t+1} - \frac{1}{2} \pi_{t+1}^\top P_{t+1} \pi_{t+1} - \frac{\gamma}{2} \theta_{t+1}^\top Q_{t+1} \theta_{t+1}
\]

(27) \hspace{1cm} s.t. \theta_{t+1} - \theta_0 = \pi_{t+1}

and where we define the vectors \(G_t\) and the matrices \(P_t\) and \(Q_t\) defined for all \(t = 0, \ldots, T\) by

\[
G_t = \mathbb{E}[B_t G_t]
\]

(28)
\[
P_t = \mathbb{E}[B_t P_t B_t^\top]
\]

(29)
\[
Q_t = \mathbb{E}[B_t \Sigma_t B_t^\top]
\]

(30)

Note that we choose the time indices for the matrices \(G_t, P_t, Q_t\) to reflect their (identical) size (index \(t\) denotes a square-matrix or vector of row-length \(N(K+1)t\). The matrices \(G_t, P_t, Q_t\) can be solved for explicitly or by simulation depending on the assumptions made about the return generating process \(R_t\) and the factor dynamics \(B_{i,t}\). But once these expressions have been computed or simulated (and this only needs to be done once), then the explicit solution for the optimal strategy can be derived using standard deterministic linear-quadratic dynamic programming. We derive the solution next.

### 2.5. Closed form solution

Define the value function

\[
V(n) = \max_{\pi_{n+1}, \ldots, \pi_T} \sum_{t=n}^{T-1} g_{t+1}^\top \pi_{t+1} - \frac{1}{2} \pi_{t+1}^\top P_{t+1} \pi_{t+1} - \frac{\gamma}{2} \theta_t^\top Q_t \theta_t
\]

Now at \(n = T - 1\) we have

\[
V(T - 1) = \max_{\pi_T} g_T^\top \pi_T - \frac{1}{2} \pi_T^\top P_T \pi_T - \frac{\gamma}{2} \theta_{T-1}^\top Q_{T-1} \theta_{T-1},
\]
which yields the solution $\pi_T^* = \mathcal{P}_T^{-1} \mathcal{G}_T$ and the value function $V(T-1) = \frac{1}{2} \mathcal{G}_T^\top \mathcal{P}_T^{-1} \mathcal{G}_T - \frac{\gamma}{2} \theta_{T-1}^\top \mathcal{Q}_{T-1} \theta_{T-1}$.

We therefore guess that the value function is of the form:

\begin{equation}
V(n) = -\frac{1}{2} \theta_n^\top \mathcal{M}_n \theta_n + L_n^\top \theta_n + H_n
\end{equation}

The Hamilton-Jacobi-Bellman equation is

\begin{equation}
V(t) = \max_{\pi_{t+1}} \left\{ \mathcal{G}_{t+1}^\top \pi_{t+1} - \frac{1}{2} \pi_{t+1}^\top \mathcal{P}_{t+1} \pi_{t+1} - \frac{\gamma}{2} \theta_t^\top \mathcal{Q}_t \theta_t + V(t+1) \right\}
\end{equation}

\begin{equation}
s.t. \quad \theta_{t+1} - \theta_t^0 = \pi_{t+1}
\end{equation}

The first order condition is:

\begin{equation}
\mathcal{G}_{t+1} + \mathcal{L}_{t+1} - (\mathcal{P}_{t+1} + \mathcal{M}_{t+1}) \pi_{t+1} = \mathcal{M}_{t+1} \theta_t^0
\end{equation}

which gives the optimal trade (and corresponding) state equation:

\begin{equation}
\pi_{t+1} = (\mathcal{P}_{t+1} + \mathcal{M}_{t+1})^{-1} (\mathcal{G}_{t+1} + \mathcal{L}_{t+1} - \mathcal{M}_{t+1} \theta_t^0)
\end{equation}

\begin{equation}
\theta_{t+1} = (\mathcal{P}_{t+1} + \mathcal{M}_{t+1})^{-1} (\mathcal{G}_{t+1} + \mathcal{L}_{t+1} + \mathcal{P}_{t+1} \theta_t^0)
\end{equation}

The HJB equation can be rewritten with our guess as

\begin{equation}
V(t) = \pi_{t+1}^\top \left( \mathcal{G}_{t+1} + \mathcal{L}_{t+1} - \frac{1}{2} (\mathcal{P}_{t+1} + \mathcal{M}_{t+1}) \pi_{t+1} \right) - \frac{1}{2} \theta_t^0 \mathcal{M}_{t+1} (\theta_t^0)^\top - \pi_{t+1}^\top \mathcal{M}_{t+1} \theta_t^0 - \frac{\gamma}{2} \theta_t^\top \mathcal{Q}_t \theta_t + H_{t+1} + L_{t+1}^\top \theta_t^0
\end{equation}

Now, for a $[N(K+1) t, N(K+1) t]$ dimensional square matrix $X_t$ we define $\overline{X}_t$ to be the upper left-hand corner square submatrix with dimensions $[N(K+1)(t-1), N(K+1)(t-1)]$. Using this definition and substituting the FOC we get:

\begin{equation}
V(t) = \frac{1}{2} (\mathcal{G}_{t+1} + \mathcal{L}_{t+1} - \mathcal{M}_{t+1} \theta_t^0)^\top (\mathcal{P}_{t+1} + \mathcal{M}_{t+1})^{-1} (\mathcal{G}_{t+1} + \mathcal{L}_{t+1} - \mathcal{M}_{t+1} \theta_t^0) - \frac{1}{2} \theta_t^\top (\overline{\mathcal{M}}_{t+1} + \gamma \mathcal{Q}_t) \theta_t
\end{equation}

\begin{equation}
+ H_{t+1} + L_{t+1}^\top \theta_t^0
\end{equation}
which we can simplify further:

\[
V(t) = \frac{1}{2}(G_{t+1} + L_{t+1})^\top [P_{t+1} + M_{t+1}]^{-1}(G_{t+1} + L_{t+1}) - \frac{1}{2} \theta_t^\top \left( M_{t+1} + \gamma Q_t \right) - M_{t+1}[P_{t+1} + M_{t+1}]^{-1}M_{t+1} \theta_t + H_{t+1} + \left( L_{t+1} + M_{t+1}[P_{t+1} + M_{t+1}]^{-1}(G_{t+1} + L_{t+1}) \right)^\top \theta_t^0
\]

Thus we confirm our guess for the value function and find the system of recursive equations:

\[
M_t = \overline{M}_{t+1} + \gamma Q_t - M_{t+1}[P_{t+1} + M_{t+1}]^{-1}M_{t+1}
\]

(36)

\[
L_t = L_{t+1} + M_{t+1}[P_{t+1} + M_{t+1}]^{-1}(G_{t+1} + L_{t+1})
\]

(37)

\[
H_t = H_{t+1} + \frac{1}{2}(G_{t+1} + L_{t+1})^\top [P_{t+1} + M_{t+1}]^{-1}(G_{t+1} + L_{t+1})
\]

(38)

3. Experiment

In this section we present several experiments to illustrate the usefulness of our portfolio selection approach. We compare portfolio selection in a characteristics-based versus factors-based return generating environment. As we show below the standard linear-quadratic portfolio approach is well-suited to the characteristics-based environment, but in a factor-based environment, since it cannot adequately capture the systematic variation in the covariance matrix due to variations in the exposures it is less successful. Instead, our approach can handle this feature.

3.1. Characteristics versus Factor-based return generating model

We wish to compare the following two environments:

- The factor-based return generating process

\[
R_{i,t+1} = \alpha_i + B_{i,t}^\top \left( F_{t+1} + \lambda \right) + \epsilon_{i,t+1}
\]

(39)

- The characteristics based return generating process:

\[
R_{i,t+1} = \alpha_i + B_{i,t}^\top \lambda + \omega_{i,t+1}
\]

(40)
where in both cases we assume that there are three return generating factors corresponding to (1) short term (5-day) reversal, (2) medium term (1 year) momentum, (3) long-term (5 year) reversal (and potentially a common market factor).

Note the difference between the two frameworks. In the characteristics based framework, the conditional covariance of returns is constant $\Sigma_{t \rightarrow t+1} = \Sigma_\omega$ and is therefore not affected by the factor exposures. Instead, in the factor-based framework, the conditional covariance matrix of returns is time varying: $\Sigma_{t \rightarrow t+1} = B_t \Omega B_t^\top + \Sigma_\epsilon$ where $B_t = [B_{1,t}^\top; B_{2,t}^\top; \ldots; B_{n,t}^\top]$ is the $(N, K)$ matrix of factor exposures.

We assume that the half-life of the 5-day factor is 3 days, that of the one-year factor is 150 days, that of the 5-year factor is 700 days. We define the exposure dynamics using the simple auto-regressive process:

$$B_{i,t+1}^k = (1 - \phi_k) B_{i,t}^k + \epsilon_{i,t+1}.$$ 

The value of $\phi_k$ is tied to its half-life (expressed in number of days) $\hat{h}_k$ by the simple relation $\phi_k = (\frac{1}{2})^\hat{h}_k$.

For the case, where we investigate the ‘Characteristics based’ model we set the constant covariance matrix $\Sigma_\omega$ so that it matches the unconditional covariance matrix of the factor based return generating process, i.e., we set

$$\Sigma_\omega = E[B_t \Omega B_t^\top + \Sigma_\epsilon]$$

Note that $B_t \Omega B_t^\top = \sum_{l,m=1}^K \Omega_{l,m} B_{l,t}^\top (B_{m,t}^\top)$

where $B_{k,t}^k$ is the factor values of each asset corresponding to the $k$th factor at time $t$.

### 3.2. Calibration of main parameters

The number of assets in our experiment is 15. One can think of these as a collection of portfolios instead of individual stocks, e.g., stock or commodity indices. Our trading horizon is 26 weeks with weekly rebalancing. Our objective is to maximize net terminal wealth minus penalty terms for excessive risk. This requires us to set $a = 1$ and $b = 0$ in our objective function.
We calibrate the factor mean, $\lambda$, and covariance matrix, $\Omega$, using Fama-French 10 portfolios sorted on short-term reversal, momentum, and long term reversal. Using monthly returns, we compute the performance of the long-short portfolio for the highest and lowest decile in each factor data. Obtaining 3 long-short portfolios, we set $\lambda$ to be its mean and $\Omega$ to be its covariance matrix. Table 1 illustrates the estimated values for $\lambda$ and $\Omega$.

<table>
<thead>
<tr>
<th>Fama-French Moments</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-0.00726</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.00182</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>-0.00323</td>
</tr>
<tr>
<td>$\Omega_{11}$</td>
<td>0.00103</td>
</tr>
<tr>
<td>$\Omega_{12}$</td>
<td>0.00051</td>
</tr>
<tr>
<td>$\Omega_{13}$</td>
<td>0.00154</td>
</tr>
<tr>
<td>$\Omega_{22}$</td>
<td>0.00050</td>
</tr>
<tr>
<td>$\Omega_{23}$</td>
<td>0.00081</td>
</tr>
<tr>
<td>$\Omega_{33}$</td>
<td>0.00162</td>
</tr>
</tbody>
</table>

Table 1: Calibration results for $\lambda$ and $\Omega$.

For our simulations, we assume that both $F$ and $\epsilon$ vectors are serially independent and normally distributed with zero mean and covariance matrix $\Omega$ and $\Sigma_\epsilon$, respectively. We assume that $\Sigma_\epsilon$ is a diagonal matrix e.g., diag($\sigma_\epsilon$). Each entry in $\sigma_\epsilon$ is set randomly at the beginning of the simulation according to a normal distribution with mean 0.20 and standard deviation 0.05.

Initial distribution for $B_{t,0}^k$ is given by the unconditional stationary distribution of $B_{t,t}^k$ which is given by a normal distribution with mean zero and variance $\frac{\sigma_{\epsilon,t}^2}{2\phi - \phi^2}$.

Transaction cost matrix, $\Lambda$ is assumed to be a constant multiple of $\Sigma_\omega$ or $\Sigma_\epsilon$ with proportionality constant $\eta$ in characteristics or factor-based return generating model respectively. We use a rough estimate of $\eta$ according to widely used transaction cost estimates reported in the algorithmic trading community. We provide two regimes: low and high transaction cost environment. The slippage values for these two regimes are assumed to be around 4bps and 400bps respectively. Therefore, we expect that a trade with a notional value of $\$100,000$ results in $\$40$ and $\$4000$ of transaction costs in these regimes. In our model, $\eta_\sigma^2 u^2$ measures the corresponding transaction cost of trading $u$ dollars. Using $u = \$100,000$ and $\sigma_\epsilon = 0.20$, this yields that $\eta$ is roughly around $5 \times 10^{-6}$ and $5 \times 10^{-4}$ for the low and high transaction cost regimes respectively.
Finally, we assume that the coefficient of risk aversion, $\gamma$ equals $10^{-6}$, which we can think of as corresponding to a relative risk aversion of 1 for an agent with 1 million dollars under management.

3.3. Approximate policies

Due to the nonlinear dynamics in our wealth function, solving for the optimal policy even in the case of concave objective function is intractable due to the curse of dimensionality. In this section, we will provide various policies that will help us compare the performance of the best linear policy to the existing approaches in the literature.

**Garleanu & Pedersen Policy (GP):** Using the methodology in Garleanu and Pedersen (2012), we can construct an approximate trading policy that will work in our current set-up. A closed-form solution can be obtained if one works with linear dynamics in state and control variables:

$$
\bar{r}_{t+1} = C_t B^x_t + \bar{\epsilon}_{t+1}
$$

$$
B^x_t = (I - \Phi) B^x_t + \epsilon_{t+1}
$$

where $\bar{r}_{t+1} = S_{t+1} - S_t$ stores dollar price changes. Then, our objective function can be written as

$$
\max \mathbb{E} \left[ \sum_{t=1}^{T} \left( x^\top_{t-1} \bar{r}_t - \frac{\gamma}{2} x^\top_t \bar{\Sigma}_t x_t - \frac{1}{2} u^\top_t \bar{\Lambda} u_t \right) \right]
$$

where $\bar{\Lambda}$ and $\bar{\Sigma}_t$ are deterministic and measured in dollars and given by

$$
C_t = \mathbb{E}[\text{diag}(S_t)] \left( \lambda^\top \otimes I_{N \times N} \right)
$$

$$
\bar{\Lambda} = \mathbb{E}[S_t S_t^\top] \Lambda
$$

$$
\bar{\Sigma}_t = \text{Var}(\bar{r}_{t+1}).
$$

The optimal solution to this problem is given by

$$
x_t = \left( \bar{\Lambda} + \gamma \bar{\Sigma}_t + A^x_{t+1} \right)^{-1} \left( \bar{\Lambda} x_{t-1} + \left( A^x_t (I - \Phi) B^x_t \right) \right)
$$
with the following recursions:

\[
A_t^{-1} = -\bar{\Lambda} \left( \bar{\Lambda} + \gamma \bar{\Sigma}_t + A_{xx}^t \right)^{-1} \bar{\Lambda} + \bar{\Lambda} \\
A_{xf}^{-1} = \bar{\Lambda} \left( \bar{\Lambda} + \gamma \bar{\Sigma}_t + A_{xx}^t \right)^{-1} \left( A_{xf}^t (I - \Phi) \right) + C_t 
\]

Note that this policy uses unconditional expected returns for the stock. In order to account for the significant departures from the expected returns in some sample paths, we will scale our policy with the ratio,

\[
D_t = \text{diag} \left( \frac{S_{1,t}}{E[S_{1,t}]}, \ldots, \frac{S_{n,t}}{E[S_{n,t}]} \right).
\]

Then our scaled policy uses

\[
x_t = \left( \bar{\Lambda} + \gamma \bar{\Sigma}_t + A_{xx}^t \right)^{-1} \left( \bar{\Lambda}D_t x_{t-1} + \left( D_t A_{xf}^t (I - \Phi) \right) B_r^* \right).
\]

**Myopic Policy (MP):** We can solve for the myopic policy using only one-period data. We solve the myopic problem given by

\[
\max E \left[ \left( x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma_t x_t - \frac{1}{2} u_t^\top \Lambda u_t \right) \right].
\]

Using the dynamics for \( r_{t+1} \), the optimal myopic policy is given by

\[
x_t = \left( \Lambda + \gamma \left( B_t \Omega B_t^\top + \Sigma_e \right) \right)^{-1} \left( B_t \lambda + \Lambda \left( x_{t-1} \circ R_t \right) \right)
\]

**Myopic Policy with Transaction Cost Aversion (MP-TC):** Since myopic policy only considers the current state of the return predicting factors, it realizes substantial transaction costs. This policy can be significantly improved by considering another optimization problem on the transaction cost matrix which ultimately tries to control the amount of transaction costs incurred by the policy. Thus, this policy uses

\[
x_t = \left( \tau^* \Lambda + \gamma \left( B_t \Omega B_t^\top + \Sigma_e \right) \right)^{-1} \left( B_t \lambda + \tau^* \Lambda \left( x_{t-1} \circ R_t \right) \right)
\]
where $\tau^*$ is given by

$$
\arg\max_{\tau} \mathbb{E} \left[ \left( x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma_t x_t - \frac{1}{2} u_t^\top \Lambda u_t \right) \right]
$$

with

$$
x_t = \left( \tau \Lambda + \gamma \left( B_t \Omega B_t^\top + \Sigma_t \right) \right)^{-1} \left( B_t \lambda + \tau \Lambda \left( x_{t-1} \circ R_t \right) \right)
$$

**Best Linear Policy (BL):** Using the methodology in Section ??, we can find the optimal linear policy that satisfies our nonlinear state evolution:

$$
u_t = B_t^\top \pi_t^*$$

$$x_t = B_t^\top \theta_t^*
$$

where $\pi_t^*$ and $\theta_t^*$ solve the following program

$$
\max_{\pi_1, \ldots, \pi_T} \sum_{t=0}^{T-1} G_{t+1}^\top \pi_{t+1} - \frac{1}{2} \pi_{t+1}^\top P_{t+1} \pi_{t+1} - \frac{\gamma}{2} \theta_t^\top Q_t \theta_t
$$

s.t. $\theta_{t+1} - \theta_t^0 = \pi_{t+1}$

**Restricted Best Linear Policy (RBL):** Instead of using the whole history of stochastic factors in our policy, we can restrict the best linear policy to use only a fixed number of periods. In this experiment, we will use only the last observed exposures in our position vector, $x_t$, and the last two period’s exposures and the last period’s return in our trade vector, $u_t$. Formally, we will let

$$
x_{i,t} = \theta_{i,t}^\top B_{i,t,t},
$$

$$u_{i,t} = \pi_{i,1,t}^\top B_{i,t-1,t} + \pi_{i,2,t}^\top B_{i,t,t},$$
where we need

\[ \pi_{i,2,t} = \theta_{i,t}, \]
\[ \pi_{i,1,t} = -\theta_{i,t-1}, \]
\[ \pi_{i,1,1} = 0, \]

in order to satisfy the nonlinear state dynamics in (1) and (2).

**Myopic Policy without Transaction Costs (NTC):** Without transaction costs, our trading problem is easy to solve, namely, the myopic policy will be optimal. Thus, using the myopic policy in the absence of transaction costs, i.e.,

\[ x_t = \left( \gamma \left( B_t \Omega B_t^\top + \Sigma_e \right) \right)^{-1} (B_t \lambda) \]

and applying it to the objective function without the transaction cost terms will provide us an upper bound for the optimal objective value of the original dynamic program. This policy will help us to evaluate how suboptimal the approximate policies are in the worst case.

### 3.4. Simulation Results

We run the performance statistics of our approximate policies in the presence and lack of factor noise and low and high transaction costs. We observe that in all of these cases, best linear policy performs very well compared to the other approximate policies and when compared to the upper bound it achieves near-optimal performance.

Table 2 illustrates that when transaction costs are relatively small, myopic policies are also near-optimal but even in this case best linear policy dominates in terms of performance. Garleanu & Pedersen policy does not perform very well mainly due to the return dynamics expressed in percentage terms versus dollar units. Table 3 underlines the amount of improvement introduced with the best linear policy. In this case, myopic policies perform significantly worse than the best linear policy.

Table 4 and Table 5 depict the impact of common factor shocks in the terminal wealth statistics.
Table 2: Summary of the performance statistics of each policy in the case of no common factor noise and low transaction cost environment. For each policy, we report average terminal wealth, average objective value, variance of the terminal wealth, average terminal sharpe ratio in the presence and lack of transaction costs and average weekly sharpe ratio in the presence of transaction costs. (Dollar values are in thousands of dollars.)

<table>
<thead>
<tr>
<th></th>
<th>GP</th>
<th>MP</th>
<th>MP-TC</th>
<th>RBL</th>
<th>BL</th>
<th>NTC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg Wealth</td>
<td>263.9</td>
<td>573</td>
<td>574.6</td>
<td>547.5</td>
<td>568.5</td>
<td>594.3</td>
</tr>
<tr>
<td>Avg Objective</td>
<td>175.8</td>
<td>281.9</td>
<td>282.4</td>
<td>281.1</td>
<td>291.0</td>
<td>297.0</td>
</tr>
<tr>
<td>Variance</td>
<td>3.11e+04</td>
<td>1.37e+05</td>
<td>1.37e+05</td>
<td>1.23e+05</td>
<td>1.30e+05</td>
<td>1.44e+05</td>
</tr>
<tr>
<td>TC</td>
<td>5.65</td>
<td>9.967</td>
<td>11.71</td>
<td>14.66</td>
<td>13.45</td>
<td>0</td>
</tr>
<tr>
<td>Sharpe with TC</td>
<td>2.13</td>
<td>2.188</td>
<td>2.196</td>
<td>2.207</td>
<td>2.231</td>
<td>2.215</td>
</tr>
<tr>
<td>Sharpe w/o TC</td>
<td>2.15</td>
<td>2.194</td>
<td>2.204</td>
<td>2.22</td>
<td>2.244</td>
<td>2.215</td>
</tr>
<tr>
<td>Weekly Sharpe with TC</td>
<td>2.73</td>
<td>3.39</td>
<td>3.393</td>
<td>3.383</td>
<td>3.443</td>
<td>3.453</td>
</tr>
</tbody>
</table>

Table 3: Summary of the performance statistics of each policy in the case of no common factor noise and high transaction cost environment. For each policy, we report average terminal wealth, average objective value, variance of the terminal wealth, average terminal sharpe ratio in the presence and lack of transaction costs and average weekly sharpe ratio in the presence of transaction costs. (Dollar values are in thousands of dollars.)

<table>
<thead>
<tr>
<th></th>
<th>GP</th>
<th>MP</th>
<th>MP-TC</th>
<th>RBL</th>
<th>BL</th>
<th>NTC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg Wealth</td>
<td>114.2</td>
<td>180.9</td>
<td>52.19</td>
<td>74.76</td>
<td>232.2</td>
<td>594.3</td>
</tr>
<tr>
<td>Avg Objective</td>
<td>85.61</td>
<td>-98.41</td>
<td>25.31</td>
<td>59.58</td>
<td>138.1</td>
<td>297.0</td>
</tr>
<tr>
<td>Variance</td>
<td>8.96e+03</td>
<td>1.72e+05</td>
<td>2.28e+04</td>
<td>3.77e+03</td>
<td>3.14e+04</td>
<td>1.44e+05</td>
</tr>
<tr>
<td>TC</td>
<td>23.59</td>
<td>7.744</td>
<td>0.2971</td>
<td>44.43</td>
<td>43.97</td>
<td>0</td>
</tr>
<tr>
<td>Sharpe with TC</td>
<td>1.706</td>
<td>0.6168</td>
<td>0.4886</td>
<td>1.722</td>
<td>1.853</td>
<td>2.215</td>
</tr>
<tr>
<td>Sharpe w/o TC</td>
<td>1.753</td>
<td>0.5421</td>
<td>0.4896</td>
<td>2.222</td>
<td>1.94</td>
<td>2.215</td>
</tr>
<tr>
<td>Weekly Sharpe with TC</td>
<td>2.30</td>
<td>2.132</td>
<td>2.098</td>
<td>2.003</td>
<td>2.517</td>
<td>3.453</td>
</tr>
</tbody>
</table>

It is important to note that in this regime, sharpe ratios are significantly lower. In both cases, best linear policy achieves the best objective value statistics.

Table 4: Summary of the performance statistics of each policy in the case of common factor noise and low transaction cost environment. For each policy, we report average terminal wealth, average objective value, variance of the terminal wealth, average terminal sharpe ratio in the presence and lack of transaction costs and average weekly sharpe ratio in the presence of transaction costs. (Dollar values are in thousands of dollars.)

<table>
<thead>
<tr>
<th></th>
<th>GP</th>
<th>MP</th>
<th>MP-TC</th>
<th>RBL</th>
<th>BL</th>
<th>NTC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg Wealth</td>
<td>53.14</td>
<td>38</td>
<td>39.44</td>
<td>19.3</td>
<td>39.23</td>
<td>41.81</td>
</tr>
<tr>
<td>Avg Objective</td>
<td>-336.9</td>
<td>15.38</td>
<td>19.28</td>
<td>9.785</td>
<td>20.51</td>
<td>20.75</td>
</tr>
<tr>
<td>Variance</td>
<td>7.93e+04</td>
<td>2.25e+04</td>
<td>4.07e+03</td>
<td>2.04e+03</td>
<td>9.19e+03</td>
<td>4.21e+03</td>
</tr>
<tr>
<td>TC</td>
<td>7.57</td>
<td>1.186</td>
<td>0.98</td>
<td>0.2683</td>
<td>1.785</td>
<td>0</td>
</tr>
<tr>
<td>Sharpe with TC</td>
<td>0.27</td>
<td>0.3586</td>
<td>0.87</td>
<td>0.604</td>
<td>0.5786</td>
<td>0.911</td>
</tr>
<tr>
<td>Sharpe w/o TC</td>
<td>0.30</td>
<td>0.8848</td>
<td>0.9</td>
<td>0.6121</td>
<td>0.586</td>
<td>0.911</td>
</tr>
<tr>
<td>Weekly Sharpe with TC</td>
<td>0.40</td>
<td>0.9058</td>
<td>0.92</td>
<td>0.7347</td>
<td>0.8756</td>
<td>0.9436</td>
</tr>
</tbody>
</table>
Table 5: Summary of the performance statistics of each policy in the case of common factor noise and high transaction cost environment. For each policy, we report average terminal wealth, average objective value, variance of the terminal wealth, average terminal sharpe ratio in the presence and lack of transaction costs and average weekly sharpe ratio in the presence of transaction costs. (Dollar values are in thousands of dollars.)

4. Interpretation of the Trading Rule

In this section, we highlight how BL and GP differ while trading with noisy factor exposures. In order to simplify the analysis, we will assume that there is a single stock driven by a single factor, $B_t$, with a corresponding positive expected factor return, $\lambda$ (e.g., momentum). The factor is driven by an AR(1) process and the initial factor realizes a shock which is normally distributed shock with positive mean and very large standard deviation, i.e., low sharpe signal. We assume that there is no idiosyncratic shock but only factor shock as in Equation (39). Figure 1 illustrates the mean return and factor realizations and the corresponding trades executed by GP and BL policies. We observe that since the GP policy cannot correctly account for the conditional variance of the returns, it trades aggressively on the signal. On the other hand, due to the factor shock, BL considers the signal weak and trades considerably less on the signal compared to the GP policy. Ultimately, BL achieves a sharpe ratio of 0.25 whereas GP achieves 0.06.

Suppose now that the factor realizes a positive shock which is deterministic, i.e., high sharpe signal. We still assume that factor shock exists. In this case, we find that both policies trade very similarly. Figure 2 illustrates the mean return and factor realizations and the corresponding trades executed by GP and BL policies. We observe that when the factor is deterministic, BL and GP trade quite similarly. We suspect that the small difference is due to the different formulations: number of shares versus dollar holdings. Trading similar amounts results in very similar sharpe ratios, as expected.
Figure 1: Differences in the trades undertaken by BL and GP policies when there is a single factor exposure that realizes a normally distributed shock with positive mean and very large standard deviation, i.e., low sharpe signal. The corresponding expected factor return is positive so after the shock the return of the security has a positive expected jump. Due to the positive mean, GP policy trades aggressively on the signal whereas BL policy trades very conservatively.

On the other hand, if we also eliminate factor shock, we again find that both policies trade very similarly. Figure 3 illustrates the mean return and factor realizations and the corresponding trades executed by GP and BL policies. We observe that in the absence of factor shock, BL and GP trade quite similarly. Only difference is due to the different formulations in number of shares and dollar holdings. Ultimately they both achieve a sharpe ratio of 2.60.

5. Conclusion and Future Directions

In this essay, we provide a methodology that accommodates complex return predictability models studied in the literature in multi-period models with transaction costs. Our return predicting factors does not need to follow any pre-specified model but instead can have arbitrary dynamics. We allow for factor dependent covariance structure in returns driven by common factor shocks. On an interesting future direction, we can also have time-varying liquidity costs which are correlated
Figure 2: Differences in the trades undertaken by BL and GP policies when there is a single factor which is completely deterministic. We still have factor risk. We observe that GP and BL policy trades very similarly.

Figure 3: Differences in the trades undertaken by BL and GP policies when there is a single factor without factor risk. GP and BL policy trades very similarly.
with the expected returns of the factors.

Our simulation study shows that best linear policy provides significant benefits compared to other frequently used policies in the literature, especially when the transaction costs are high and returns evolve according to factor dependent covariance structure. Unlike other parametric approaches studied so far, our approach provides a closed form solution and the driver of the policy dynamics can be analyzed in full detail.
References


