Dynamic Asset Allocation with Predictable Returns and Transaction Costs

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Abstract

We propose a simple approach to dynamic multi-period portfolio choice with quadratic transaction costs. The approach is tractable in settings with a large number of securities, realistic return dynamics with multiple risk factors, many predictor variables, and stochastic volatility. We obtain a closed-form solution for a trading rule that is optimal if the problem is restricted to a broad class of strategies we define as 'linearity generating strategies' (LGS). When restricted to this parametric class the highly non-linear dynamic optimization problem reduces to a deterministic linear-quadratic optimization problem in the parameters of the trading strategies. We show that the LGS approach dominates several alternative approaches in realistic settings. In particular, we demonstrate large performance differences when there is a dynamic factor structure in returns or stochastic volatility (*i.e.*, when the covariance matrix is stochastic), and when transaction costs covary with return volatility.

1 Introduction

The seminal contribution of Markowitz (1952) has spawned a large academic literature on portfolio choice. The literature has extended Markowitz's one period mean-variance setting to dynamic multiperiod setting with a time-varying investment opportunity set and more general objective functions.[1] Yet there seems to be a wide disconnect between this academic literature and the practice of asset allocation, which still relies mostly on the original one-period mean-variance framework. Indeed, most MBA textbooks tend to ignore the insights of this literature, and even the more advanced approaches often used in practice, such as that of Grinold and Kahn (1999), propose modifications of the single period approach with *ad-hoc* adjustments designed to give solutions which are more palatable in a dynamic, multiperiod setting.

Yet the empirical evidence on time-varying expected returns suggests that the use of a dynamic approach should be highly beneficial to asset managers seeking to exploit these different sources of predictability.[2]

One reason for this disconnect is that the academic literature has largely ignored realistic frictions such as trading costs, which are paramount to the performance of investment strategies in practice. This is because introducing transaction costs and price impact in the standard dynamic portfolio choice problem tends to make it intractable. Indeed, most academic papers studying transaction costs focus on a very small number of assets (typically two) and limited predictability (typically none).[3] Extending their approach to a large number of securities and several sources of predictability quickly runs into the curse of dimensionality.

In this paper we propose an approach to dynamic portfolio choice in the presence of transaction costs that can deal with a large number of securities and realistic return generating processes.

[1] Merton (1969), Merton (1971), Brennan, Schwartz, and Lagnado (1997), Kim and ohmberg (1991), Campbell (1999), Campbell, Chan, and Viceira (2003), Liu (2007), Detemple and Rindisbacher (2010) and many more. See Cochrane (2012) for a survey.

[2] The academic literature has documented numerous variables which forecast the cross-section of equity returns. Stambaugh, Yu, and Yuan (2011) provides a list of many of these variables, and also argue that the structure and magnitudes of this forecastability exhibits considerable time variation.

[3] Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Shreve and Soner (1994) study the two-asset (one risky-one risk-free) case with iid returns. Cvitanić (2001) surveys this literature. Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000) add some predictability in the risky asset. Lynch and Tan (2011) extend this to two risky assets at considerable computational cost. Liu studies the multiasset case under CARA preferences and for *i.i.d.* returns.

For example, our approach can handle a large number of predictors, a general factor structure for returns, and stochastic volatility. The approach relies on three features. First, we assume investors maximize the expected terminal wealth net of a risk-penalty that is linear in the variance of their portfolio return. Second, we assume that the total transaction cost for a given trade is quadratic in the dollar trade size. Third, we assume that the conditional mean vector and covariance matrix of returns are known functions of an observable state vector, and the dynamics of this state vector can be simulated. Thus, this framework nests most factor based models that have been proposed in the literature.

For a standard set of return generating processes, the portfolio optimization problem does not admit a simple solution because the wealth equation and return generating process introduce non-linearities in the state dynamics. Thus, the problem falls outside the linear-quadratic class which is known to be tractable (Litterman (2005), Gârleanu and Pedersen (2012)) even though we use the same objective function as they do. However, we identify a particular set of strategies, which we call "linearity generating strategies" (LGS), for which the problem admits a closed-form solution. An LGS is defined as a strategy for which the dollar position in each security is a weighted average of current and lagged stock "exposures" interacted with its own past returns (*i.e.*, effectively a linear combination of managed portfolios).

The exposures are selected *ex-ante* for each stock, and should include all stock specific state variables on which the optimal dollar position in each security depends: variables summarizing the conditional expected return and variance for each security, and variables summarizing the cost of trading this security. Note that the exposures can also include variables such as the vector of optimal security weights when transaction costs are zero, or the solution to a related optimization problem, such as that proposed by Litterman (2005) and Gârleanu and Pedersen (2012) or various rules of thumb (*e.g.*, Brown and Smith (2010)).

The optimal trade and position for each security will be a linear function of that security's exposures, interacted with its past-returns, for a set of lags. This implies a very high dimensional optimization problem. While one would anticipate that this high-dimensional problem is difficult to solve, we show that for strategies in the LGS class this optimization problem reduces to a deterministic linear-quadratic problem that can be solved very efficiently.

Another key question is whether the set of LGS's is sufficiently rich that the optimal LGS approximates the unconstrained optimum. This is an empirical question. However, assuming the specifications of the return generating process and transaction cost function are correct, the LGS can always be designed to perform as well as any alternative approach: the reason is that the solution of any other approach can be used as an input to the LGS approach. The magnitude of the improvement of the LGS will depend on the value of the additional exposures in getting closer to the unconstrained optimum.

We solve several realistic examples which allow us to study the magnitude of this improvement in different settings. First, we compare the performance of our approach to that of several alternatives in two benchmark simulated economies: one we call the *characteristics model* and the other the *factor model.* In both cases expected returns are driven by three characteristics which mimic the well-known reversal (Jegadeesh 1990), momentum (Jegadeesh and Titman 1993) and long-term-reversal/value (DeBondt and Thaler 1985, Fama and French 1993) effects. However, the economies differ in their covariance matrix of returns. The characteristics model assumes that the covariance matrix is constant (implying a failure of the APT in a large economy). In contrast, the factor model assumes that the three characteristics reflect loadings on common factors. Thus, they are reflected in the covariance matrix of returns. Since factor exposures are time-varying and drive both expected returns and covariances, in this model the covariance matrix is stochastic.

The characteristics model is similar to the return model used in the recent works of Litterman (2005) and Gârleanu and Pedersen (2012) (henceforth L-GP) Their linear-quadratic programming approaches provides a useful benchmark since they solve for the exact closed-form solution for strategies with a similar objective function.[4] Indeed, we find that the LGS and the L-GP closed-form of solution perform almost equally well in the characteristics based economy we simulate, as the covariance matrix is close to time-invariant.[5]

However, in the factor model economy, where the covariance matrix changes as the factor

[4]One important difference is that to obtain a closed-form solution Litterman (2005) and Gârleanu and Pedersen (2012) specify their model for price changes and not returns and the objective function of the investor in terms of number of shares. They further assume the covariance matrix of price changes is constant. This allows them to retain a linear objective function avoiding the non-linearity in the wealth equation due to the compounding of returns over time.

[5]More precisely, the GP solution is optimal if the covariance matrix of changes in the dollar price per share is time invariant.

loadings of individual securites change, the L-GP solution is further from optimal, since their approach relies on a constant covariance matrix, and their trading rule significantly underperforms our approach based on LGS. This is because the latter explicitly takes into account the dual effect of higher factor exposures in both raising expected returns and covariances. The LGS also outperforms a myopic mean-variance approach optimized for the presence of transaction-costs – as suggested by Grinold and Kahn (1999) – which is often used by practitioners. This alternative approach consists in using the one-period mean-variance solution with transaction costs, but recognizing that this approach ignores the dynamic objective function, it adds a multiplier to the transaction costs incurred when trading. This t-cost multiplier is chosen so as to maximize the actual performance of the strategy across many simulations.

We also perform an experiment with real return data. We analyze the performance of a trading strategy involving the 100 largest stocks traded on the NYSE over the time period from 1974 to 2012. We trade these stocks exclusively based on the short-term reversal factor, which is a well-known predictor of stock returns. Because the half-life of reversal is several days, portfolio turnover is high and the performance of a strategy based on this factor is highly dependent on transaction costs. Also, the literature suggests that strategy performance is dependent on volatility (Khandani and Lo (2007), Nagel (2012)). We therefore use a realistic return process that features GARCH in the common market factor as well as in the cross-sectional idiosyncratic variance. This captures salient empirical features of the reversal factor as documented in Collin-Dufresne and Daniel (2013). In our experiment the costs of trading shares of an individual firm depend on that firm's return volatility, consistent with the findings in the transaction cost literature. Thus, transaction costs are stochastic. We solve for the optimal trading strategy using our LGS and backtest our strategy in comparison with a myopic t-cost optimized strategy. We find that our approach outperforms this benchmark significantly.

There is a growing literature on portfolio selection that incorporates return predictability with transaction costs. Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000) illustrate the impact of return predictability and transaction costs on the utility costs and the optimal rebalancing rule by discretizing the state space of the dynamic program. Their approach runs into the curse of dimensionality and only applies to very few stocks and predictors. Brown and Smith (2010) discuss

this issue and instead provide heuristic trading strategies and dual bounds for a general dynamic portfolio optimization problem with transaction costs and return predictability that can be applied to larger number of stocks.

Our approach is closest related to two strands of literature: First, Brandt, Santa-Clara, and Valkanov (2009, BSV) model the portfolio weight on each asset directly as linear functions of a set of asset "characteristics" that are determined *ex-ante* to be useful for portfolio selection.[6] The vector of characteristic weights are optimized by maximizing the average utility the investor would have obtained by implementing the policy over the historical sample period. The BSV approach explicitly avoids modeling the asset return distribution, and therefore avoids the problems associated with the multi-step procedure of first explicitly modeling the asset return distribution as a function of observable variables, and then performing portfolio optimization as a function of the moments of this estimated distribution.[7] However, the BSV approach is limited in that the optimization is performed via numerical simulation, and therefore is limited to a relatively small number of predictive variables. Further, since the performance of the objective function is optimized in sample, restricting to a small number of parameters and predictors is desirable to avoid over-fitting. Our contribution is that we identify a set of trading strategies for which the optimization can be performed in closed-form using deterministic linear quadratic control for very general return processes in a dynamic setting with transaction costs. We can thus achieve a greater flexibility in parameterizing the trading rule.

As noted earlier, our approach is also closely related to the L-GP approach – as proposed by Litterman (2005) and Gârleanu and Pedersen (2012). L-GP obtain a closed-form solution for the optimal portfolio choice in a model where: (1) expected price change per share for each security is a linear, time-invariant function of a set of predictor variables; (2) the covariance matrix of price changes per share is time-invariant; and (3) trading costs are a quadratic function of the number of shares traded, and investors have a linear-quadratic objective function expressed in terms of number of shares. Their approach relies heavily on linear-quadratic stochastic programming (*e.g.*, Ljungqvist and Sargent (2004)). Our approach considers a problem that is more general, in that

[6] See also Aït-Sahalia and Brandt (2001), Brandt and Santa-Clara (2006) and Moallemi and Saglam (2012).

[7] See Black and Litterman (1991), Chan, Karceski, and Lakonishok (1999), as well as references given in footnote 2 of BSV (p. 3412).

our return generating process can allow for a general factor structure in the covariance matrix with stochastic volatility, the transaction costs can be stochastic, and our objective function is written in terms of dollar holdings. In general, such a problem does not belong to the linear-quadratic class and thus does not admit a simple closed-form along the lines of the L-GP solution. Our contribution is to find a special parametric class of portfolio policies, such that when the portfolio choice problem is considered in that class it reduces to a deterministic linear-quadratic program in the policy parameters.

2 Model

In this section we lay out the return generating process for the set of securities our agent can trade. Then we describe the portfolio dynamics in the presence of transaction costs. Finally, we present the agent's objective function and our solution technique.

2.1 Security and factor dynamics

We consider a dynamic portfolio optimization problem where an agent can invest in N risky securities with price $S_{i,t}$ $i = 1, \ldots, N$ and a risk-free cash money market with value $S_{0,t}$. We assume that security i pays a dividend $D_{i,t}$ at time t. The gross return to our securities is thus defined by $R_{i,t+1} = \frac{S_{i,t+1}+D_{i,t+1}}{S_{i,t}}$. We assume that the conditional mean return vector and covariance matrix of security returns are both known functions of an observable vector of state variables X_t:

$$\mathsf{E}_t[R_{t+1}] = 1 + m(X_t, t) \tag{1}$$

$$\mathsf{E}_t[(R_{t+1} - \mathsf{E}_t[R_{t+1}])(R_{t+1} - \mathsf{E}_t[R_{t+1}])'] = \Sigma_{t \to t+1}(X_t, t) \tag{2}$$

The vector of observable state variable X_t may include both individual security characteristics (such as individual firms' book to market ratios, past returns or idiosyncratic volatilities) as well as common drivers of security returns (such as market volatility, and market or industry factors).

It is important for our approach that the dynamics of X_t are known so as to us to simulate the behavior of the conditional moments of security returns. An example that nests many return

generating processes used in the literature is:

$$R_{i,t+1} = g(t, \beta_{i,t}^{\top}(F_{t+1} + \lambda_t) + \epsilon_{i,t+1}) \qquad i = 1, \ldots, N \tag{3}$$

for some function $g(t, \cdot) : \mathbb{R} \to \mathbb{R}$, increasing in the second argument, and where:

- $\beta_{i,t}$ is the $(K, 1)$ vector of firm i's factor exposures at time t.
- F_{t+1} is the $(K, 1)$ vector of random (as of time t) factor realizations over period $t + 1$. F_{t+1} is mean 0, and follows a multivariate GARCH process with conditional covariance matrix $\Omega_{t,t+1}$.
- $\epsilon_{i,t+1}$ is security i's idiosyncratic return over period $t + 1$.
  We assume that $\epsilon_{\cdot,t+1}$ are mean zero, have a time-invariant covariance matrix Σ_ϵ, and are uncorrelated with the contemporaneous factor realizations.
- λ_t is the $(K, 1)$ vector of conditional expected factor returns at time t.

In this case the vector of state variables $X_t = [\beta_{1,t}; \beta_{2,t}; \ldots \beta_{N,t}; \lambda_t; \Omega_{t,t+1}]$ has $NK+K+K\cdot(K+1)/2$ elements. We further assume that $\beta_{i,t}$ and λ_t are observable and follow some known dynamics. In the empirical applications below, we assume that both λ_t and the $\beta_{i,t}$ follow Gaussian AR(1) processes.

Note that this setting captures two standard return generating processes from the literature:

1. The **"discrete exponential affine"** model for security returns in which log-returns are affine in factor realizations:[8]

$$\log R_{i,t+1} = \alpha_i + \beta_{i,t}^{\top}(F_{t+1} + \lambda) + \epsilon_{i,t+1} - \frac{1}{2}\left(\sigma_i^2 + \beta_{i,t}^{\top}\Omega\beta_{i,t}\right)$$

2. The **"linear affine factor model"** where returns (and therefore also excess returns) are

[8] The continuous time version of this model is due to Vasicek (1977), Cox, Ingersoll, and Ross (1985), and generalized in Duffie and Kan (1996). The discrete time version is due to Gourieroux, Monfort, and Renault (1993) and Le, Singleton, and Dai (2010).

affine in factor exposures:

$$R_{i,t+1} = \alpha_i + \beta_{i,t}^\top (F_{t+1} + \lambda_t) + \epsilon_{i,t+1}$$

As we show below, our portfolio optimization approach is equally tractable for both of these return generating processes. We emphasize that the approach does not rely on this factor structure assumption. All that is required is that there be some known relation between the conditional first and second moments of security returns and the known state vector X_t so that conditional means and variances of security returns can be simulated along with the state vector.

2.2 Cash and security position dynamics

We assume discrete time dynamics. At the end of each period t the agent buys $u_{i,t}$ dollars of security i at price $S_{i,t}$. All trades in risky securities incur transaction costs which are quadratic in the dollar trade size. Trades in risky securities are financed using the cash money market position, which we assume incurs no trading costs. The cash position (w_t) and dollar holdings ($x_{i,t}$) in each security $i = 1, \ldots, N$ held at the end of each period t are thus given by:

$$\begin{aligned} x_{i,t} &= x_{i,t-1} R_{i,t} + u_{i,t} \qquad i = 1, \ldots, N \\ w_t &= w_{t-1} R_{0,t} - \sum_{i=1}^{N} u_{i,t} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,t} \Lambda_t(i,j) u_{j,t}, \end{aligned}$$

or, in vector notation,

$$x_t = x_{t-1} \circ R_t + u_t \tag{4}$$

$$w_t = w_{t-1} R_{0,t} - \mathbf{1}^\top u_t - \frac{1}{2} u_t^\top \Lambda_t u_t \tag{5}$$

where the operator $\circ$ denotes element by element multiplication if the matrices are of same size or if the operation involves a scalar and a matrix, then that scalar multiplies every entry of the

matrix.[9]

The matrix Λ_t captures (possibly time-varying) quadratic transaction/price-impact costs, so that $\frac{1}{2}u_t^\top \Lambda_t u_t$ is the dollar cost paid given a vector of trades at time t of (dollar) size u_t. Without loss of generality, we assume this matrix is symmetric. Gârleanu and Pedersen (2012) discuss some micro-economic foundations for such quadratic costs. It is also very convenient analytically.

2.3 Objective function

We assume that the agent is endowed with a portfolio of dollar holdings in securities x_0 and an initial amount of cash w_0. We assume that the investor's objective function is to maximize his expected terminal wealth net of a risk penalty which, following L-GP, we take to be linear in the sum of per-period variances. For simplicity, we also assume that the risk-free rate is zero, i.e., $R_{0,t} = 1$.[10] Thus the objective is:

$$\max_{u_1,\ldots,u_T} \mathsf{E}\left[w_T + x_T^\top \mathbf{1} - \sum_{t=0}^{T-1} \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t\right] \tag{8}$$

Recall that $\Sigma_{t\to t+1} = \mathsf{E}_t\left[(R_{t+1} - \mathsf{E}_t[R_{t+1}])(R_{t+1} - \mathsf{E}_t[R_{t+1}])^\top\right]$ is the conditional one-period variance-covariance matrix of returns and γ can be interpreted as the coefficient of risk aversion.

[9]The timing convention could be changed so that the agent buy $u_{i,t}$ dollars of security i at price $S_{i,t}$ *at the beginning of period* t. In that case the dynamics would be:

$$x_{t+1} = (x_t + u_t) \circ R_{t+1} \tag{6}$$

$$w_{t+1} = (w_t - \mathbf{1}^\top u_t - \frac{1}{2} u_t^\top \Lambda_t u_t) R_{0,t+1} \tag{7}$$

All our results go through for this alternative timing convention. We make the choice in the text because, for one parameterization of our objective function identified below, it allows us to closely approximate the objective function of Litterman (2005) and Gârleanu and Pedersen (2012) and thus makes the link between the two frameworks more transparent.

[10]It is straightforward to extend our approach to a non-zero risk-free rate and to an objective function that is linear-quadratic in the position vector (i.e., $F(x_t, w_T) = w_T + a_1^\top x_T - \frac{1}{2} x_T^\top a_2\, x_T$) rather than linear in total wealth. See Appendix A.

Note that by recursion we can write:[11]

$$x_T = x_0 + \sum_{t=0}^{T-1} x_t \circ r_{t+1} + \sum_{t=1}^{T} u_t \tag{9}$$

$$w_T = w_0 - \sum_{t=1}^{T} (u_t^\top \mathbf{1} + \frac{1}{2} u_t^\top \Lambda_t u_t) \tag{10}$$

where we have defined the *net return* $r_{t+1} = R_{t+1} - 1$ with corresponding expected net return $m_t = \mathsf{E}_t[R_{t+1}] - 1$. Inserting in the objective function and simplifying we find the optimization reduces to:

$$\max_{u_1,\ldots,u_T} \mathsf{E}\left[\sum_{t=0}^{T-1} \left\{ x_t^\top m_t - \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t - \frac{1}{2} u_{t+1}^\top \Lambda_{t+1} u_{t+1} \right\}\right] \quad s.t. \quad \text{eq (4)} \tag{11}$$

We see that this objective function is very similar to that used in L-GP (see, *e.g.*, equation (4) of GP): we maximize the expected sum of local-mean-variance objectives, net of transaction costs paid.[12] However, there are several notable and important differences. First, our objective function is in terms of dollar holdings (x_t, w_t) and trades (u_t). In contrast, the L-GP objective function is terms of number of shares held and traded (their x_t and Δx_t). For the price processes, our expected returns (m's) and covariance matrix ($\Sigma_{t-1\to t}$) are in terms of returns, while in the L-GP framework r_{t+1} and Σ necessarily denote the expected price change and the price-change variance, both on a per share basis.

At first glance this may appear to be an innocuous change of units. However, to obtain an analytical solution, the L-GP framework requires a constant covariance matrix of price changes. This implies that the *return* variance will be inversely related to the security price squared: if a security's price falls from \$100/share to \$50/share, the return variance must quadruple. It also requires that the transaction cost function – as measured in the transaction costs per share traded – must be independent of the share price. This is generally inconsistent with empirical evidence on

[11] Indeed, $x_T = x_{T-1} \circ (R_T - 1) + x_{T-1} + u_T = x_{T-1} \circ (R_T - 1) + x_{T-2} \circ (R_{T-1} - 1) + x_{T-2} + u_{T-1} + u_T = \ldots$.

[12] While, to our knowledge, there is no utility based axiomatic foundation for this objective function, it is useful to point out that for the case where $\gamma = 1$ this objective function is essentially logarithmic. Indeed, assuming the terminal wealth follows a continuous time diffusion process we can write $\mathsf{E}[\log W_T] = \log W_0 + \mathsf{E}[\int_0^T dW_t - \frac{1}{2} dW_t^2]$, which is a continuous time version of our objective function. As is clear from this example, in the absence of transaction costs, the objective function is myopic.

security return dynamics. As a result some of the implications of the L-GP framework seem to go against the intuition developed in previous literature.

To better illustrate this, we first focus on the special case where expected return and variances are constant, which can be solved for in closed-form before turning to the more general case with predictability.

2.4 Constant expectation and variance of price changes or of returns?

If $m_t, \Sigma_t,$ and Λ_t are constant, then the optimal portfolio choice problem in equation (11) admits a closed-form solution. In Appendix (A.1) we derive this solution for comparison with the GP framework in an infinite horizon stationary model, i.e., we consider the problem:

$$\max_{u_1,\ldots,} \mathsf{E}\left[\sum_{t=0}^{\infty} \rho^t \left\{ x_t^\top m - \frac{\gamma}{2} x_t^\top \Sigma x_t - \frac{1}{2} u_{t+1}^\top \Lambda u_{t+1} \right\}\right] \quad s.t. \quad \text{eq (4)} \tag{12}$$

We show that the optimal dollar trade u_t is linear-affine in the current position, i.e.,

$$u_t = a_0 + a_1 x_t \tag{13}$$

where the coefficients are given explicitly in equation (84) in the Appendix A.1. Instead, if one assumes that the expected price change and the variance of price changes are constant, then the optimal policy would imply an optimal trade such that the number of shares traded h_t^* is linear affine in the number of shares held, n_t:

$$h_t = b_0 + b_1 n_t \tag{14}$$

where the coefficients b_0, b_1 are given in equation (89) in the appendix. Clearly, these two trading rules are inconsistent (since by definition $u_t = h_t S_t$ and $x_t = n_t S_t$ both equations (13) and (14) cannot both hold at the same time). As expected, the optimal trading strategy obtained for constant covariance of returns differs from that obtained for a constant covariance of price changes.

One important difference between the two solutions is that if the covariance of price changes is constant, then if at some point we hold the mean-variance optimal portfolio (*i.e.*, if $x_t = (\gamma\Sigma)^{-1} m$ or

equivalently $n_t = (\gamma\Sigma_s)^{-1}\mu_s$ where $\Sigma_s = \Sigma * S_t^2$ and $\mu_s = m * S_t$ are defined as the (constant) variance and expectation of price changes respectively) then it is optimal to never trade hence-forth (see Appendix B.6). This implies that if we held the mean-variance optimal portfolio, and the price of a security were to fall by a factor of two, the optimal solution would be not to trade. Intuitively, there is no trade to rebalance the portfolio because, given the assumed dynamics (constant expectation and variance of price changes), when the price halves, the security's expected return and return volatility both double, meaning the optimal dollar holdings also halve, so there is no motive for rebalancing.

If instead we were to assume that the expectation and variance of returns (rather than price changes) were constant, then there would be no position such that it is never optimal to trade at all future dates. Indeed, this is is because random shocks to return induce random changes in future dollar positions via equation (4), which in turn would lead to deviations in dollar portfolio holdings from the first best, and thus to a rebalancing motive for trading even in the *i.i.d* case. This rebalancing motive for trading is the one investigated in the traditional transaction cost literature (such as Constantinides (1986)). In addition, we point out in the appendix that in the *i.i.d.* case, there exists a position x_{no} given in equation (94) such that it is optimal not to trade for one period (i.e., if $x_t = x_{no}$ then $u_t = 0$). However, interestingly this no-trade position is not equal to the mean-variance efficient portfolio. The intuition is that the current position does not reflect where it is expected to be in one period, since it will experience random return shocks. So in effect, even in the *i.i.d.* case, current optimal trades reflect a trade-off between where we are today and where we expect to be in the future given the return shocks we will experience.

While we can obtain a closed-form solution in the *i.i.d.* case, the general framework we lay out in the previous section allows for security price processes to have more general dynamics, with time-varying expected returns, variances and trading costs. So in general, we cannot obtain a closed-form solution. However, just as in the *i.i.d.* case the model will typically capture this rebalancing motive for trading (which is, for example, at the heart of the classic Merton (1969) dynamic portfolio optimization with constant investment opportunity set). The *i.i.d.* solution is also interesting as it motivates our choice of focusing on 'linearity generating strategies.' Indeed, combining the linearity of the trading rule in (13) and the dynamics of the state in (4) and iterating

backwards we see that both the optimal state and the optimal trade are of the form

$$u_t = \sum_{s \leq t} \pi_{s,t} R_{s \to t} \tag{15}$$

$$x_t = \sum_{s \leq t} \theta_{s,t} R_{s \to t} \tag{16}$$

where we define the holding period returns $R_{s \to t} = R_{s \to s+1} R_{s+1 \to s+2} \dots R_{t-1 \to t}$. The optimal loadings $\pi_{s,t}, \theta_{s,t}$ are constant and obtained from the optimal solution. They can be shown to be related (by equation (4)) such that:

$$\begin{cases} \theta_{s,t} = \theta_{s,t-1} + \pi_{s,t} & \text{for } s < t \\ \theta_{t,t} = \pi_{t,t} & \text{for } s = t \end{cases}$$

For the general case, where the investment opportunity set is time-varying, we will seek a solution within a set of LGS that has the same structure, but where the loadings on past holding period returns can be increased or decreased depending on a set of instruments that can be stochastic. We now turn to the general case and introduce the set of 'linearity generating strategies' that we consider.

2.5 Linearity generating strategies

Even though the objective function is similar to that of a linear-quadratic problem which are known to be very tractable (*e.g.*, Litterman (2005), Gârleanu and Pedersen (2012)) our problem is not in that class because of the non-linearity introduced by the state equation, and because of the general return process, which may display stochastic volatility (and thus make the matrix Q_t stochastic). Thus the problem appears difficult to solve in full generality, even numerically. Instead, we introduce a specific set of 'linearity generating trading strategies' (LGS) for which the problem remains tractable. The idea of restricting the set of strategies to make the problem tractable is not new. For example, this is the idea underlying Brandt, Santa-Clara, and Valkanov (2009), who consider strategies which are restricted to be linear in security characteristics and numerically optimize directly the empirical objective function on a sample of data over the parameters of the

trading strategy. Because their approach relies on a numerical in-sample optimization, they have to specify fairly simple strategies so as to not over-fit the data. In contrast, with our approach the optimization is done in closed-form so, assuming our specification is correct, we can consider a rich class of path-dependent strategies. This is particularly useful in optimization problems with transaction costs.[13]

The remarkable result we demonstrate below is that, for linearity generating strategies, the problem reduces to a deterministic linear-quadratic optimization problem in the parameters of the policy. The only other approaches in the literature that yield a closed form solution – the L-GP approach – makes some strong assumptions about the return generating process and the objective function to obtain a closed-form solution. Specifically, these approaches require that the covariance matrix of price changes per share and the per share transaction cost function be time-invariant, and require that the agents' objective function be expressed in number of shares rather than dollars. With these assumptions, the L-GP solution is the exact optimal solution. However, it is the solution to a problem which will be an accurate representation of reality in only a very limited number of situations.

The advantage to our approach is that we can determine the optimal solution given a wide range of security price dynamics. The drawback to our approach is that the solution we derive is only optimal among the set of all solutions that are linear functions of the exposures we select. So the key to getting a good solution with the LGS methodology is selecting a set of exposures that come close to spanning the globally optimal solution. One advantage that our method has on this front is that virtually any variable in the information set can be used as an exposure. So, for example, the solution to the simple myopic or the more complex L-GP problem, or both can be chosen as exposures. In this case, our methodology will assign weights to additional exposures – including scaled-lagged exposures — if and only if they provide an improvement over and above what can be obtained with the myopic or L-GP solution. For example, in a setting where the L-GP solution was optimal, these additional exposures would add nothing, consequently they would get

[13]One advantage of the Brandt, Santa-Clara, and Valkanov (2009) approach is that they dispense with specifying the return generating process altogether, instead relying on the empirical performance of there propose strategies. Instead, for our approach we need to specify the return generating process, and in particular, the way in which expected returns and variances depend on the characteristics used for the trading rule.

no weight and our solution would be identical to the L-GP solution.

The magnitude of the improvement of LGS over alternative solutions depends on how much improvement these additional exposures provide. In Section 3, we investigate this via simulations. First though, we describe the strategy set we consider and explain how the portfolio optimization can be done in closed-form, within that restricted set.

2.5.1 Derivation of the LGS Solution

At this stage it is convenient to introduce the following notation (inspired from *Matlab*): We write $[A; B]$ (respectively $[A\,B]$) to denote the vertical (respectively horizontal) concatenation of two matrices.

To define our set of LGS we first specify, for each security, a K-vector $B_{i,t}$ of "security exposures." The exposures are typically non-linear transformations of the general state vector X_t (*i.e.*, $B_{i,t} = h_i(X_t)$). For example, $B_{i,t}$ may include the individual security's conditional expected return divided by its conditional variance (see, *e.g.*, Aït-Sahalia and Brandt (2001)), the optimal dollar position in the security in the absence of transaction costs given by the myopic solution, or a t-cost aware solution from another method. More generally, it would include security specific factor exposures, conditional variances and other relevant information for portfolio formation.

The restriction for our set of strategies is that the dollar holdings and dollar trades of security i must be specified as linear functions of current and lagged exposures via sets of K-dimensional vectors of parameters, $\pi_{i,s,t}$ and $\theta_{i,s,t}$, defined for all $i = 1, \ldots, N$ and for all $s \leq t$. These parameters fully determine the dollar trades ($u_{i,t}$) and the corresponding positions ($x_{i,t}$) for asset i via the parametric relations:

$$x_{i,t} = \sum_{s=0}^{t} \theta_{i,s,t}^{\top} \mathcal{B}_{i,s\to t} \qquad \text{for } t = 0, ..., T \tag{17}$$

$$u_{i,t} = \sum_{s=0}^{t} \pi_{i,s,t}^{\top} \mathcal{B}_{i,s\to t} \qquad \text{for } t = 1, ..., T \tag{18}$$

where $\mathcal{B}_{i,s\to t}$ is defined as the vector of time s exposures $B_{i,s}$, scaled by the gross-return on security

i between s and t:

$$\mathcal{B}_{i,s\to t} = B_{i,s} R_{i,s\to t}. \tag{19}$$

In effect, the dollar trades and dollar positions in security i at time t in asset i ($x_{i,t}$) can be thought of as a weighted sum of simple buy and hold trading strategies that went long the security at past dates ($s < t$) proportionally to time s exposures and held the security until date t.

However in the LGS framework, this time-s scaled exposure can be built up gradually after time s, and then sold gradually. Scaled exposure, because it is scaled by the firm's cumulative gross return, is time invariant: if you bought one unit of scaled-exposure at time s and didn't trade further, you would still hold one unit at all future times. The value of a unit of scaled time-s exposure at time t is given by $\mathcal{B}_{i,s\to t}$. The number of units of time-s exposure purchased at time $t \geq s$ is given by $\pi_{i,s,t}$, and the number of units held at time t ($\theta_{i,s,t}$) is just the sum of the number of units purchased between s and t.

Perhaps the easiest way to illustrate this is to examine the equations for the dollar positions and trades of firm i at $t = 0, 1, 2$, as given below:

$x_{i,0} = \theta_{i,0,0}^\top B_{i,0}$			
$u_{i,1} = \pi_{i,0,1}^\top \mathcal{B}_{i,0\to 1}$	$+\pi_{i,1,1}^\top B_{i,1}$		
$x_{i,1} = (\underbrace{\theta_{i,0,0} + \pi_{i,0,1}}_{=\theta_{i,0,1}})^\top \mathcal{B}_{i,0\to 1}$	$+ \underbrace{\pi_{i,1,1}^\top}_{=\theta_{i,1,1}^\top} B_{i,1}$		
$u_{i,2} = \pi_{i,0,2}^\top \mathcal{B}_{i,0\to 2}$	$+\pi_{i,1,2}^\top \mathcal{B}_{i,1\to 2}$	$+\pi_{i,2,2}^\top B_{i,2}$	
$x_{i,2} = (\underbrace{\theta_{i,0,0} + \pi_{i,0,1} + \pi_{i,0,2}}_{=\theta_{i,0,2}})^\top \mathcal{B}_{i,0\to 2}$	$+(\underbrace{\pi_{i,1,1} + \pi_{i,1,2}}_{=\theta_{i,1,2}})^\top \mathcal{B}_{i,1\to 2}$	$+ \underbrace{\pi_{i,2,2}^\top}_{=\theta_{i,2,2}^\top} B_{i,2}$	

The first equation gives the initial position as a function of the time 0 exposures. Since the initial position is generally not a choice variable, the vector $\theta_{i,0,0}$ must be constrained so that the first equation holds. [14]

The second equation gives the first trade, $u_{i,1}$. Note that this trade is a function of both the lagged exposures for time 0, scaled by $R_{i,0\to 1}$, and the current ($t = 1$) exposures. The dependence

[14] In general, one of the elements of the vector $B_{i,0}$ will be a one, so a straightforward way to impose this constraint is to require that the corresponding elements of $\theta_{i,0,0}$ be equal to the initial dollar position $x_{i,0}$.

on the time zero exposure is important here, because the optimal trade at $t = 1$ and later are dependent on the initial position. Intuitively, if we are given a large initial position in a security, the strategy will start trading out of that position with the first trade at time 1 – how quickly it trades out will be determined by $\pi_{i,0,1}$.

The third equation gives the total dollar holdings of security i at $t = 1$. $x_{i,1}$ is equal to initial position, grossed up by the realized return on firm i from 0 to 1, plus $u_{i,1}$. However, note that this equation decomposes these holdings into the number of units of scaled time zero exposure $\theta_{i,0,1}$, and time 1 exposure $\theta_{i,1,1}$. Since the first time we purchase time 1 exposure is at time 1, $\theta_{i,1,1} = \pi_{i,1,1}$.

The fourth and fifth equations give, respectively, the time 2 trade and position. The trade is decomposed into the number of units of time 0, 1, and 2 scaled exposure we buy. The vector of costs of the exposures are given by the $\mathcal{B}$s. $\theta_{i,0,2}$ – the total number of units of time 0 scaled exposure held at time 2 – is the sum of the initial endowment ($\theta_{i,0,0}$) plus the number of units purchased at time 1 and at time 2. The number of units of time 1 exposure held at time 2 ($\theta_{i,1,2}$) is the sum of the number of units purchased at time 1 and 2.

In an environment with transaction costs, the position in the lagged return-scaled time s exposure will generally be accumulated gradually over time. That is, following a shock at time s to exposures that raises a security's expected return (holding constant its risk) the corresponding elements of $\pi_{i,s,t}$ will be positive for t slightly bigger than s, and then will turn negative as t increases, and then finally asymptote to zero. That is, it will be optimal to gradually trade into positions in securities, and then trade out of these positions as the expected return decays towards zero. We'll illustrate this via simulation in Section 3.5.

As is apparent in the discussion above, $\theta_{i,s,t}$ and $\pi_{i,s,t}$ must be chosen so that holdings and trades are consistent. Specifically the trades and positions in equations (17) and (18), respectively, are required to satisfy the dynamics given in equations (4) and (5). It follows that the parameter

vectors $\pi_{i,s,t}$ and $\theta_{i,s,t}$ have to satisfy the following restrictions, for all $i = 1, \ldots, N$:

$$\begin{aligned} \theta_{i,s,t} &= \theta_{i,s,t-1} + \pi_{i,s,t} && \forall t \geq 1 \text{ and } 0 \leq s < t \\ \theta_{i,t,t} &= \pi_{i,t,t} && \forall t \geq 1 \\ \theta_{i,0,0} B_{i,0} &= x_{i,0} \\ \pi_{i,0,0} &= 0 \end{aligned} \tag{20}$$

These restrictions are intuitive. The first specifies that number of units of scaled time s exposure held at time t is equal to the number of units held at time $t-1$ plus the number of units bought at time t. The second restriction specifies that the number of units of scaled time t exposure held at time t is the number bought at time t. Since $B_{i,t}$ is not in the information set until time t, you cannot buy time t exposure before time t. The last two conditions specify that the inital scaled-exposures much be chosen to match the initial holdings $x_{i,0}$, and that the time 0 trade is zero, consistent with the dynamics laid out in Section 2.2.

Intuitively, the dependence on current exposures is important. In a no-transaction cost affine portfolio optimization problem where the optimal solution is well-known, the optimal holdings will involve only today's exposures (see, *e.g.*, Liu (2007)).[15] With transaction costs, allowing today's weights and trades to also depend on lagged security exposures, scaled by each security's return up to today, is useful because these variables summarize the positions – held today – as a result of trades made in previous periods. When transactions costs are present, the optimal trades today will generaally depend on what positions were taken on in past periods. This path-dependence is observed in known closed-form solutions in environments with transaction costs. (see Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Liu and Loewenstein (2002) and others).

To proceed, we first rewrite the policies in equation (20) in a concise matrix form. First, define

[15] Note that this is also the choice made by Brandt, Santa-Clara, and Valkanov (2009) for their 'parametric portfolio policies.' However, while BSV specify the loadings on exposure of individual securities to be identical, we allow two securities with identical exposures (and with perhaps different levels of idiosyncratic variance) to have different weights and trades.

the $NK(t+1)$-dimensional vectors π_t and θ_t as

$$\pi_t = [\pi_{1,0,t}; \ldots; \pi_{n,0,t}; \pi_{1,1,t}; \ldots; \pi_{n,1,t}; \ldots; \pi_{1,t,t}; \ldots; \pi_{n,t,t}] \tag{21}$$

$$\theta_t = [\theta_{1,0,t}; \ldots; \theta_{n,0,t}; \theta_{1,1,t}; \ldots; \theta_{n,1,t}; \ldots; \theta_{1,t,t}; \ldots; \theta_{n,t,t}] \tag{22}$$

Also, we define the following (NK, N) matrices (defined for all $0 \leq s \leq t \leq T$) as the diagonal concatenations of the N vectors $\mathcal{B}_{i,s\to t}$ $\forall i = 1, \ldots, N$:

$$\mathcal{B}_{s,t} = \begin{Bmatrix} \mathcal{B}_{1,s\to t} & 0 & \cdots & 0 \\ 0 & \mathcal{B}_{2,s\to t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{B}_{n,s\to t} \end{Bmatrix}$$

Finally, we define the $(NK(t+1), N)$ matrix $\mathcal{B}_t$ by stacking the $t+1$ matrices $\mathcal{B}_{s,t}$ $\forall s = 0, \ldots, t$:

$$\mathcal{B}_t = [\mathcal{B}_{0,t}; \mathcal{B}_{1,t}; \ldots; \mathcal{B}_{t,t}] \tag{23}$$

With these definitions, it is straightforward to verify that:

$$u_t = \mathcal{B}_t^\top \pi_t \tag{24}$$

$$x_t = \mathcal{B}_t^\top \theta_t \tag{25}$$

Further, in terms of these definitions the constraints on the parameter vectors in (20) can be rewritten concisely as:

$$\theta_t = \theta_{t-1}^0 + \pi_t \tag{26}$$

where we define $x^0 = [x; \mathbf{0}_{NK}]$ to be the vector x stacked on top of an NK-dimensional vector of zeros $\mathbf{0}_{NK}$.

The usefulness of restricting ourselves to this set of 'linearity generating trading strategies' is that optimizing over this set amounts to optimizing over the parameter vectors π_t and θ_t, and that, as we show next, that problem reduces to a **deterministic** linear-quadratic control problem, which

can be solved in closed form.

Indeed, substituting the definition of our linear trading strategies from equations (24) and (25) into our objective function in equation (11) and then taking expectations gives:

$$\max_{\pi_1,\ldots,\pi_T} \sum_{t=0}^{T-1} \theta_t^\top \overline{m}_t - \frac{1}{2}\pi_{t+1}^\top \overline{\Lambda}_{t+1}\pi_{t+1} - \frac{\gamma}{2}\theta_t^\top \overline{\Sigma}_t \theta_t \tag{27}$$

$$\text{subject to} \quad \theta_t = \theta_{t-1}^0 + \pi_t \tag{28}$$

and where we define the vector $\overline{m}_t$ and the square matrices $\overline{\Sigma}_t$ and $\overline{\Lambda}_t$ for $t = 0, \ldots, T$ by

$$\overline{m}_t = \mathsf{E}_0[\mathcal{B}_t m_t] \tag{29}$$

$$\overline{\Sigma}_t = \mathsf{E}_0[\mathcal{B}_t \Sigma_{t\to t+1} \mathcal{B}_t^\top] \tag{30}$$

$$\overline{\Lambda}_t = \mathsf{E}_0[\mathcal{B}_t \Lambda_t \mathcal{B}_t^\top] \tag{31}$$

Note that the time indices also capture their size: $\overline{m}_t$ is a vector of length $NK(t+1)$, and $\overline{\Sigma}_t$ and $\overline{\Lambda}_t$ are square matrices of the same dimensionality.[16] Equation (27) is just the objective function (equation (11)) with the u_t's and x_t's rewritten as linear functions of the elements in $\mathcal{B}_t$, with coefficients π_t and θ_t, respectively. Since the policy parameters π_t and θ_t are set at time 0, they can be pulled outside of the expectation operator.

Intuitively equation (27) is a linear-quadratic function of the policy parameters π_t and θ_t, with $\overline{m}_t$, $\overline{\Sigma}_t$, $\overline{\Lambda}_t$ as the coefficients in this equation. These three components give, respectively, the effect on the objective function of: the expected portfolio returns resulting from trades at time t; the transaction costs paid as a result of trades at time t; and finally the effect of the holdings at time t on the risk-penalty component of the objective function.

Since $\overline{m}_t$, $\overline{\Sigma}_t$, $\overline{\Lambda}_t$ are not functions of the policy parameters, they can be solved for explicitly or by simulation, and this only needs to be done once. Their values will depend on the initial conditions, and on the assumptions made about the state vector X_t driving the return generating process R_t and the corresponding security-specific exposure dynamics $B_{i,t}$. But, since equation

[16] It is important to note that these matrices $\overline{m}_t, \overline{\Sigma}_t, \overline{\Lambda}_t$ will depend on the initial conditions (in particular on the initial exposures $\mathcal{B}_0$, which typically will depend on the initial positions in each stock).

(27) is a linear-quadratic equation, albeit a high-dimensional one, it can be solved using standard methods. We next calculate the closed form solution.

2.6 Closed form solution

We begin with the linear-quadratic problem defined by equations (27) and (28). Define recursively the value function starting from $V(T) = 0$ for all $t \leq T$ by:

$$V(t-1) = \max_{\pi_t} \left\{ \theta_t^\top \overline{m}_t - \frac{\gamma}{2} \theta_t^\top \overline{\Sigma}_t \theta_t - \frac{1}{2} \pi_t^\top \overline{\Lambda}_t \pi_t + V(t) \right\}$$
$$\text{subject to } \theta_t = \theta_{t-1}^0 + \pi_t$$

Then it is clear that $V(0)$ gives the solution to the problem we are seeking. To solve the problem explicitly, we guess that the value function is of the form:

$$V(t) = -\frac{\gamma}{2} \theta_t^\top M_t \theta_t + L_t^\top \theta_t + H_t \tag{32}$$

with M_t a symmetric matrix. Since $V(T) = 0$, it follows that $M_T = 0$, $L_T = 0$ and $H_T = 0$. To find the recursion plug the guess in the Bellman equation:

$$V(t-1) = \max_{\pi_t} \left\{ \theta_t^\top \overline{m}_t - \frac{1}{2} \pi_t^\top \overline{\Lambda}_t \pi_t - \frac{\gamma}{2} \theta_t^\top (\overline{\Sigma}_t + M_t) \theta_t + L_t^\top \theta_t + H_t \right\} \tag{33}$$
$$\text{subject to } \theta_t = \theta_{t-1}^0 + \pi_t \tag{34}$$

Now plugging in the constraint, it is simpler to optimize over the state θ_t rather than π_t, so we obtain:

$$V(t-1) = \max_{\theta_t} \left\{ \theta_t^\top \overline{m}_t - \frac{1}{2} (\theta_t - \theta_{t-1}^0)^\top \overline{\Lambda}_t (\theta_t - \theta_{t-1}^0) - \frac{\gamma}{2} \theta_t^\top (\overline{\Sigma}_t + M_t) \theta_t + L_t^\top \theta_t + H_t \right\} \tag{35}$$

The first order condition gives the optimal position vector:

$$\theta_t = \left[\overline{\Lambda}_t + \gamma (\overline{\Sigma}_t + M_t) \right]^{-1} \left(\overline{m}_t + L_t + \overline{\Lambda}_t \theta_{t-1}^0 \right),$$

and plugging into the state equation (equation (28)) we find the optimal trade vector:

$$\pi_t = [\overline{\Lambda}_t + \gamma(\overline{\Sigma}_t + M_t)]^{-1} \left(\overline{m}_t + L_t - \gamma(\overline{\Sigma}_t + M_t)\theta^0_{t-1}\right).$$

Next, substitute these optimal policies into the Bellman equation in (33), giving:

$$V(t-1) = \frac{1}{2}\left(\overline{m}_t + L_t + \overline{\Lambda}_t\theta^0_{t-1}\right)^\top [\overline{\Lambda}_t + \gamma(\overline{\Sigma}_t + M_t)]^{-1}\left(\overline{m}_t + L_t + \overline{\Lambda}_t\theta^0_{t-1}\right) + H_t - \frac{1}{2}\left(\theta^0_{t-1}\right)^\top \overline{\Lambda}_t\theta^0_{t-1} \tag{36}$$

Comparing this equation and the conjectured specification for $V(t-1)$ in equation (32) shows that this specification will be correct if H_t, L_t, and M_t are chosen to satisfy the recursions:

$$\begin{aligned} H_{t-1} &= H_t + \frac{1}{2}\left(\overline{m}_t + L_t\right)^\top [\overline{\Lambda}_t + \gamma(\overline{\Sigma}_t + M_t)]^{-1}\left(\overline{m}_t + L_t\right) \\ L^\top_{t-1} &= \underline{\left(\overline{m}_t + L_t\right)^\top [\overline{\Lambda}_t + \gamma(\overline{\Sigma}_t + M_t)]^{-1}\overline{\Lambda}_t} \\ \gamma M_{t-1} &= \underline{\overline{\Lambda}_t - \overline{\Lambda}_t[\overline{\Lambda}_t + \gamma(\overline{\Sigma}_t + M_t)]^{-1}\overline{\Lambda}_t} \end{aligned}$$

with initial conditions $H_T = 0$, $L_T = 0$ and $M_T = 0$ and where $\underline{Y}$ denotes the vector (or matrix) obtained from Y by deleting the last NK rows (or rows and columns).

We have thus derived the optimal value function and the optimal trading strategy in the LGS class.

Before discussing some specific examples it is useful to introduce a set of LGS strategies which uses the exposures lagged at most ℓ periods. This set of "restricted lag" LGS is useful in applications when the time horizon is fairly long, and for signals that have a relatively fast decay rate, so that the dependence on lagged exposures can be restricted without a significant cost. We next show that the same tractability obtains for the restricted lag setting.

2.7 LGS with finite number of lags

In the baseline LGS, trades and positions are a linear function of return-scaled-exposures (*i.e.*, $\mathcal{B}_{i,s,t}$ for $0 \leq s < t$). In most settings, the coefficients in both the position and the trade equations ($\theta_{i,s,t}$ and $\pi_{i,s,t}$) should converge to zero for $s << t$. Indeed, we shall show via impulse response functions

in Section 3.5 that this is exactly the behavior that is observed.[17] Thus, to reduce complexity it can be advantageous to use strategies for which the trades are dependent on scaled exposures lagged at most ℓ periods.

We first specify that the trading rule will only trade based on at most ℓ lags, *i.e.* such that:

$$u_{i,t} = \sum_{s=t-\ell\vee 0}^{t} \pi_{i,s,t}^{\top} \mathcal{B}_{i,s\to t} \tag{37}$$

where $t-\ell\vee 0$ denotes the maximum of $t-\ell$ and 0. If we want the holdings to remain linear and of the form:

$$x_{i,t} = \sum_{s=0}^{t} \theta_{i,s,t}^{\top} \mathcal{B}_{i,s\to t} \tag{38}$$

Then we see that the linear constraints in equations (20) have to be modified so as to still satisfy the wealth dynamics in equations (4) and (5). Specifically, we require that :

$$\begin{aligned} &\theta_{i,t,t} = \pi_{i,t,t} && \forall \quad t \geq 1 \\ &\theta_{i,s,t} = \theta_{i,s,t-1} + \pi_{i,s,t} && \text{for } t-\ell\vee 0 \leq s < t \\ &\theta_{i,s,t} = \theta_{i,s,t-1} && \text{for } 0 < s < t-\ell \end{aligned} \tag{39}$$

Since this is still a set of linear constraints we can straightforwardly extend the previous method to derive the optimal LGS strategy with trades that only look back ℓ periods.

However, it is also generally the case that it will not be optimal to have any weight on scaled-exposures that are sufficiently old. Inspecting these constraints, we see that if we impose the additional constraint that $(\pi_{i,t-\ell,t} = -\theta_{i,t-\ell,t-1})\ \forall\, t > \ell$ (*i.e.*, that we completely trade out of any remaining time-$(t-\ell)$ scaled-exposure at time t), then it follows that $\theta_{i,s,t} = 0\ \forall\, 0 < s \leq t-\ell$. In other words, by imposing one additional linear constraint on the trading strategy one can find a set of LGS where the trading strategy u_t look-backs at most ℓ periods and the dollar position x_t look-backs at most $\ell-1$ periods. Formally, we have

$$u_{i,t} = \sum_{s=t-\ell\vee 0}^{t} \pi_{i,s,t}^{\top} \mathcal{B}_{i,s\to t}$$

[17] See, in particular Figures 2 and 4, and the related discussion.

and

$$x_{i,t} = \sum_{s=t-\ell+1\vee 0}^{t} \theta_{i,s,t}^{\top} \mathcal{B}_{i,s\to t}$$

We summarize this second set of linear constraints as:

$$\begin{aligned}
\theta_{i,t,t} &= \pi_{i,t,t} && \forall t \geq 1 \\
\theta_{i,s,t} &= \theta_{i,s,t-1} + \pi_{i,s,t} && \forall \text{ and } t-\ell \vee 1 \leq s < t \\
\pi_{i,s,t} &= -\theta_{i,s,t-1} && \text{for } 0 < s = t-\ell \\
\theta_{i,s,t} &= 0 && \text{for } 0 < s \leq t-\ell
\end{aligned}$$

Because these constraints are linear, we can follow the approach above and derive the optimal trading strategy coefficients by solving a deterministic dynamic programming problem.

3 Simulation Experiment

How much our proposed method improved on approaches proposed in the literature remains an empirical question. In this section we present several experiments to illustrate the usefulness of our portfolio selection approach. We compare portfolio selection in a characteristics-based versus factors-based return generating environment.

As we show below the standard linear-quadratic portfolio approach proposed in Litterman (2005) and Gârleanu and Pedersen (2012) is well-suited to the characteristics-based environment, but in a factor-based environment, since it cannot adequately capture the systematic variation in the covariance matrix due to variations in the exposures it is less successful. Instead, our approach can handle this feature and thus performs better.

3.1 Characteristics versus Factor-based return generating model

We wish to compare the following two environments:

- The factor-based return generating process with excess return and exposure dynamics

$$\begin{aligned} r_{i,t+1} &= \beta_{i,t}^{\top}(F_{t+1} + \lambda) + \epsilon_{i,t+1}, \\ \beta_{i,t+1}^{k} &= (1 - \phi_k)\beta_{i,t}^{k} + \epsilon_{i,t+1}. \end{aligned} \tag{40}$$

- The characteristics based return generating process with excess return and exposure dynamics

$$\begin{aligned} r_{i,t+1} &= \beta_{i,t}^{\top}\lambda + \epsilon_{i,t+1} \\ \beta_{i,t+1}^{k} &= (1 - \phi_k)\beta_{i,t}^{k} + \nu\epsilon_{i,t+1}. \end{aligned} \tag{41}$$

where in both cases we assume that $\beta_{i,t}$ is a $(3, 1)$ vector with elements corresponding to firm is exposure to (1) short term reversal (Jegadeesh 1990, Lehmann 1990), (2) medium term momentum (Jegadeesh and Titman 1993), and (3) long-term reversal (DeBondt and Thaler 1985), which we henceforth label *str*, *mom* and *ltr*. We set the half-life of the *str* factor is 5 days, that of the *mom* factor is 150 days, and that of the *ltr* factor is 700 days. These half lives are designed to roughly match the documented horizons at which short-term reversal, momentum, and long-term reversal are typically found.

In both frameworks, expected returns are the product of the *ex-ante* observable factor exposures and the factor premia, $\beta_{i,t}^{\top}\lambda$. However, consistent with the specification in equation (41), in the characteristics based framework, we assume that the conditional covariance matrix of security returns is constant, *i.e.* $\Sigma_{t\to t+1} = \mathsf{E}_t[\epsilon_{t+1}\epsilon_{t+1}^{\top}] = \Sigma$. In contrast, in the factor-based framework, the residual covariance matrix is constant, $\mathsf{E}_t[\epsilon_{t+1}\epsilon_{t+1}^{\top}] = \Sigma$, but the conditional covariance matrix of returns is time varying: $\Sigma_{t\to t+1} = \beta_t\Omega\beta_t^{\top} + \Sigma$ where $\beta_t = [\beta_{1,t}^{\top}; \beta_{2,t}^{\top}; \ldots; \beta_{n,t}^{\top}]$ is the (N, K) matrix of factor exposures, and Ω is the (time-invariant) (K, K) factor covariance matrix. Finally, ν is a free parameter used to match the sharpe ratios generated in both environments for a myopic investor trading costlessly.

Note that the innovations in the factor exposure are driven entirely by idiosyncratic return shocks consistent with their interpretation as 'technical' return based factors. The AR(1) representation has the convenient representation as a weighted average of past shocks where the weights

Table 1: **Parameters for Simulation Experiment**

The table presents the parameters estimated using the proceedure described in Appendix C, and used in the simulation exercise. The three factors are designed to capture the short-term reversal, momentum, and long-term reversal effects. $\hat{h}_k$ is the factor half-life in days, ϕ_k is the factor decay rate, λ_k the factor premium, and $\sigma_{f,k}$ the factor volatility. The final three columns give the estimated factor correlations. The factor covariance matrix Ω is equal to $\text{diag}(\sigma_f)\rho\,\text{diag}(\sigma_f)$
All of the numbers given in this table are daily.

						$\hat{\rho}$ (correlations)		
k	Factor	$\hat{h}_k$	ϕ_k	$\hat{\lambda}_k$	$\hat{\sigma}_{f,k}$	1	2	3
1	str	3	0.206299	-0.093482	0.406887	1	-0.366	0.167
2	mom	150	0.004610	0.001484	0.006999	-0.366	1	-0.576
3	ltr	700	0.000990	-0.000400	0.001764	0.167	-0.576	1

depend on the ϕ_k. This makes the interpretation as short, medium and long-term return based factors transparent.

The value of ϕ_k is tied to its half-life (expressed in number of days) $\hat{h_k}$ by the simple relation $\phi_k = 1 - (\frac{1}{2})^{1/\hat{h}_k}$.

3.2 Calibration of main parameters

The number of assets in our experiment is 15. Our trading horizon is 26 weeks with weekly rebalancing. Our objective is to maximize the net terminal wealth minus penalty terms for excessive risk (see Section 2.3).

We calibrate the factor mean, λ, and covariance matrix, Ω, using the Fama-French decile portfolios sorted on short-term reversal, momentum, and long term reversal. The calibration is described in Appendix C. The parameters obtained from this calibration and used in the simulation are given in Table 1.

For our simulations, we assume that both F and ϵ vectors are serially independent and normally distributed with zero mean and covariance matrix Ω and Σ, respectively. We calibrate Σ using our empirical return data on 100 largest firms by market capitalization[18]. We randomly choose 15 stocks, estimate the daily variance-covariance matrix from their returns, and calibrate Σ by converting it to its weekly counterpart. We set initial exposures to zero, i.e., $\beta^k_{i,0} = 0\ \forall i, k$. Finally, ν is computed to be 0.2498 so that the Sharpe ratios generated in both models in the absence of

[18]See Section 4.1 for details of the data construction

transaction costs are equal.

The transaction cost matrix, Λ is assumed to be a constant multiple of the conditional covariance matrix, Σ or $\beta_t \Omega \beta_t^\top + \Sigma$, with proportionality constant η in characteristics or factor-based return generating model respectively. Consequently, in the factor-based framework, trading a constant dollar amount will be time-varying.

We use a rough estimate of η according to widely used transaction cost estimates reported in the algorithmic trading community [19]. We provide three regimes: low, medium and high transaction cost environment. The slippage values for these three regimes are assumed to be around 2.5 bps, 5 bps and 10 bps respectively. Therefore, we expect that a trade with a notional value of \$1 million results in \$250, \$500 and \$1,000 of transaction costs in these regimes. In our model, $\eta\sigma_\epsilon^2 u^2$ measures the corresponding transaction cost of trading u dollars. Using weekly volatility of $\sigma_\epsilon = 0.05$, this yields an η roughly around 1×10^{-7}, 2×10^{-7} and 4×10^{-7} for the low, medium and high transaction cost regimes respectively.

Finally, we assume that the coefficient of risk aversion γ equals 10^{-8}, which can be thought of as corresponding to a relative risk aversion of 1 for an agent with \$100 million under management and trades 1% of the fund every week on average.

3.3 Approximate policies

Due to the nonlinear dynamics in our wealth function, solving for the globally optimal policy even in the case of a concave objective function is intractable due to the curse of dimensionality. Thus to assess the performance of the LGS, we compare it to alternative policies suggested in the literature and used in practice. In this section, we lay out how we implement these policies and discuss the implementation of the optimal LGS, which we label the Best Linear or BL strategy.

[19] See Moallemi, Saglam, and Sotiropoulos (2014) for a recent empirical study.

3.3.1 Myopic Policy (MP):

We can solve for the myopic policy using only one-period data. We solve the myopic problem given by

$$\max_{x_t} \mathsf{E}\left[\left(x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t - \frac{\eta}{2} u_t^\top \Sigma_{t\to t+1} u_t\right)\right]. \tag{42}$$

Using the dynamics for r_{t+1}, the optimal myopic policy is given (in the factor based framework) by

$$x_t^{\mathsf{MP}} = \left((\eta+\gamma)\,\Sigma_{t\to t+1}\right)^{-1}\left(\beta_t\lambda + \eta\Sigma_{t\to t+1}\left(x_{t-1}\circ R_t\right)\right) \tag{43}$$

3.3.2 Myopic Policy with Transaction Cost Multiplier (MP-TC):

A problem with the myopic optimization problem in equation (42) is that R_{t+1} and Σ have units of time^{-1} (*i.e.*, return or return variance per unit time), but the transaction costs are unitless. Thus, the myopic policy may give nonsensical solutions, particularly if the period length does not line up with the units in which expected returns and variances are measured. For this reason, it is common among practitioners to modify the myopic policy by multiplying the final terms in (42) by an amortization factor τ (with units of time^{-1}).[20] In our implementation, we choose this multiplier so as to maximize the unconditional performance (i.e., across all simulations) of the trading strategy. This modified problem gives a solution of:

$$x_t^{\mathsf{MP\text{–}TC}} = \left((\tau^*\eta+\gamma)\,\Sigma_{t\to t+1}\right)^{-1}\left(\beta_t\lambda + \tau^*\eta\Sigma_{t\to t+1}\left(x_{t-1}^{\mathsf{MP\text{–}TC}}\circ R_t\right)\right)$$

where τ^* is given by

$$\begin{aligned}\tau^* = &\operatorname*{argmax}_{\tau}\ \mathsf{E}\left[\left(x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t - \frac{\tau\eta}{2} u_t^\top \Sigma_{t\to t+1} u_t\right)\right],\\ &\text{subject to } x_t = \left((\tau\eta+\gamma)\,\Sigma_{t\to t+1}\right)^{-1}\left(\beta_t\lambda + \tau\eta\Sigma_{t\to t+1}\left(x_{t-1}\circ R_t\right)\right).\end{aligned}$$

[20] see, e.g., Grinold and Kahn (1999).

3.3.3 Gârleanu & Pedersen Policy (GP):

Using the estimation methodology in Gârleanu and Pedersen (2012), we can construct an approximate trading policy in closed-form calibrated from a model using price changes instead of percentage returns. Using a simulated data from our characteristics and factor-based framework, we can estimate the model parameters. Assuming an initial stock price of \$1 for each security and using percentage returns from the simulated data, we can obtain the price change vector $\Delta S_{t+1} = S_{t+1} - S_t$ for our two regimes. We can estimate the predictive ability of the each characteristic, ℓ^k from the following regression:

$$\Delta S_{i,t+1} = \ell^1 \beta^1_{i,t} + \ell^2 \beta^2_{i,t} + \ell^3 \beta^3_{i,t} + \varepsilon_{i,t+1}. \tag{44}$$

We estimate the constant covariance matrix, $\bar{\Sigma}^{pc}$, taking the unconditional covariance of price changes, that is to say, $\bar{\Sigma}^{pc} = \mathrm{Var}(\Delta S_t)$. Since Gârleanu and Pedersen (2012) also uses AR(1) representation for exposure dynamics, decay rate parameters, ϕ, do not need to be estimated. Transaction cost matrix, $\bar{\Lambda}^{pc}$, is a constant multiple of the covariance matrix which is given by $\eta\bar{\Sigma}^{pc}$.

Using these estimated parameters, a suboptimal trading policy can be obtained based on Gârleanu and Pedersen (2012) seeking optimal number of shares, h_t, to hold maximizing the following objective function:

$$\max_{h_1,\ldots,h_T} \mathsf{E}\left[\sum_{t=1}^{T}\left(h_t^\top \Delta S_{t+1} - \frac{\gamma}{2} h_t^\top \bar{\Sigma}^{pc} h_t - \frac{1}{2} n_t^\top \bar{\Lambda}^{pc} n_t\right)\right] \tag{45}$$

$$\text{subject to } h_t = h_{t-1} + n_t \tag{46}$$

The optimal solution to this problem is given by

$$h_t = \left(\bar{\Lambda}^{pc} + \gamma\bar{\Sigma}^{pc} + A^t_{xx}\right)^{-1}\left(\bar{\Lambda}^{pc} h_{t-1} + \left(C + A^t_{xf}\left(I - \Phi\right)\right)\beta^{\mathsf{st}}_t\right)$$

where $\beta^{\mathsf{st}}_t = [\beta^1_{:,t}; \ldots; \beta^3_{:,t}]$ is the stacked vector of factor exposures, $C = \ell^\top \otimes I_{N\times N}$ and $\Phi =$

$\text{diag}(\phi \otimes I_{N\times 1})$ and A_{xx}^{t-1} and A_{xf}^{t-1} satisfy the following recursions,

$$A_{xx}^{t-1} = -\bar{\Lambda}^{pc}\left(\bar{\Lambda}^{pc} + \gamma\bar{\Sigma}^{pc} + A_{xx}^{t}\right)^{-1}\bar{\Lambda}^{pc} + \bar{\Lambda}^{pc},$$
$$A_{xf}^{t-1} = \bar{\Lambda}^{pc}\left(\bar{\Lambda}^{pc} + \gamma\bar{\Sigma}^{pc} + A_{xx}^{t}\right)^{-1}\left(A_{xf}^{t}\left(I-\Phi\right) + C\right),$$

with $A_{xx}^{T} = \mathbf{0}$ and $A_{xf}^{T} = \mathbf{0}$.

3.3.4 Gârleanu & Pedersen Policy with Resolve (GP-R):

GP policy as calibrated from data will not work very well especially in the factor-based model as the price-change implementation with constant covariance matrix and factor returns will not capture the true dynamics of the return generating process. We can improve the performance by using a similar strategy but now resolving the set of Riccati recursions at every trading period. For this implementation, we need to compute conditional factor returns and covariance matrix and solve for optimal number of shares to hold as if they will stay constant in the future. However, in the next period these quantities will change and we will resolve the recursions using the realizations in the next period. Thus, clearly this approach will be computationally demanding especially for large number of periods.

We will input the conditional covariance matrix of price changes by letting $\bar{\Sigma}_t^{pc} = \text{diag}(S_t)\Sigma\,\text{diag}(S_t)$ in the characteristics based model and $\bar{\Sigma}_t^{pc} = \text{diag}(S_t)\left(\beta_t\Omega\beta_t^{\top} + \Sigma\right)\text{diag}(S_t)$ in the factor-based model. Transaction cost matrix, $\bar{\Lambda}_t^{pc}$, will be also time-varying with $\eta\bar{\Sigma}_t^{pc}$. With this parametrization, GP-R will choose the following suboptimal solution:

$$h_t = \left(\bar{\Lambda}_t^{pc} + \gamma\bar{\Sigma}_t^{pc} + A_{xx}^{t,t}\right)^{-1}\left(\bar{\Lambda}_t^{pc}h_{t-1} + \left(C_t + A_{xf}^{t,t}\left(I-\Phi\right)\right)\beta_t^{\mathsf{st}}\right)$$

where $\beta_t^{\mathsf{st}} = [\beta_{:,t}^1; \ldots; \beta_{:,t}^3]$ is the stacked vector of factor exposures, $C_t = \lambda^{\top} \otimes \text{diag}(S_t)$ and $\Phi = \text{diag}(\phi \otimes I_{N\times 1})$ and $A_{xx}^{t,t-1}$ and $A_{xf}^{t,t-1}$ satisfy the following recursions,

$$A_{xx}^{t,t-1} = -\bar{\Lambda}_t^{pc}\left(\bar{\Lambda}_t^{pc} + \gamma\bar{\Sigma}_t^{pc} + A_{xx}^{t,t}\right)^{-1}\bar{\Lambda}_t^{pc} + \bar{\Lambda}_t^{pc},$$
$$A_{xf}^{t,t-1} = \bar{\Lambda}_t^{pc}\left(\bar{\Lambda}_t^{pc} + \gamma\bar{\Sigma}_t^{pc} + A_{xx}^{t,t}\right)^{-1}\left(A_{xf}^{t,t}\left(I-\Phi\right) + C_t\right),$$

with $A_{xx}^{t,T} = \mathbf{0}$ and $A_{xf}^{t,T} = \mathbf{0}$. Here, we have double time superscripts in $A_{xx}^{t,t}$ and $A_{xf}^{t,t}$ in order to illustrate the necessity to resolve the Riccati recursion at each time period.

3.3.5 Best Linear Policy (BL):

We define the relevant stock exposure variables for each security to be the stock specific myopic portfolio holdings and a constant term, i.e., $B_{i,t} = [x_{i,t}^{\mathsf{MP}}; 1]$. We then follow the methodology developed in Section 2 to determine the optimal LGS satisfying our nonlinear state evolution:

$$\begin{aligned} u_t^{\mathsf{BL}} &= \mathcal{B}_t^\top \pi_t^* \\ x_t^{\mathsf{BL}} &= \mathcal{B}_t^\top \theta_t^* \end{aligned}$$

where as before $\mathcal{B}_t$ is constructed from the return-scaled exposures $\mathcal{B}_{i,s\to t} = B_{i,s} R_{s\to t}$, where π_t^* and θ_t^* solve:

$$\begin{aligned} &\max_{\pi_1,\ldots,\pi_T} \sum_{t=1}^{T} \theta_t^\top \overline{m}_t - \frac{1}{2}\pi_t^\top \overline{\Lambda}_t \pi_t - \frac{\gamma}{2}\theta_t^\top \overline{\Sigma}_t \theta_t \\ &\text{subject to } \ \theta_t = \theta_{t-1}^0 + \pi_t \end{aligned}$$

3.4 Simulation Results

We now discuss the performance of the approximate policies and the best linear (LGS) policies in the simulation for both the factor- and the characteristics-models, for low, medium and high transaction costs. We also provide performance statistics for a zero transaction-cost setting as a benchmark case.

3.4.1 Characteristics Model Simulation Results

The upper panel of Table 2 shows the results when the simulated returns are generated according to the characteristics model in equation (41), (*i.e.*, when there are no common factors, and thus all return variance is idiosyncratic), and when transaction costs are zero. Because there are no transaction costs, the myopic policy is optimal. Since all other policies except GP (MP-TC, GP-

Table 2: **Policy performance: no common factor, zero t-cost environment.**

This table summarizes the performance of each policy in the characteristics environment (no common factors) for four different levels of transaction costs. For each policy, we report the average across the 10,000 runs of the: the objective value, standard error of the objective value, terminal wealth, the standard error of the terminal wealth, and the information ratio using myopic policy as a benchmark. (Dollar values are in hundred thousands of dollars.)

	MP	**MP-TC**	**GP**	**GP-R**	**BL**	**BL-GPR**
Zero Transaction Costs						
Avg Objective	2739.75	2739.75	1704.41	2739.75	2739.65	-0.10
Std Err	22.59	22.59	8.98	22.59	22.59	0.34
Avg Wealth	5487.37	5487.37	2159.97	5487.37	5488.35	0.98
Std Err	22.01	22.01	8.87	22.01	22.01	0.34
TC	0.00	0.00	0.00	0.00	0.00	0.00
IR	NA	NA	-3.47	0.05	0.04	NA
SR	3.53	3.53	3.44	3.53	3.53	0.04
Low Transaction Costs ($\eta = 1 \times 10^{-7}$)						
Avg Objective	254.34	255.08	207.59	327.39	329.55	2.16
Std Err	5.37	5.11	1.46	3.46	3.47	0.56
Avg Wealth	427.69	412.19	221.89	404.08	407.30	3.21
Std Err	5.06	4.82	1.44	3.38	3.39	0.53
TC	226.71	194.49	40.65	252.25	249.86	-2.39
IR	NA	-0.79	-0.74	-0.12	-0.10	NA
SR	1.20	1.21	2.19	1.69	1.70	0.09
Medium Transaction Costs ($\eta = 2 \times 10^{-7}$)						
Avg Objective	147.83	147.92	119.73	187.74	190.04	2.30
Std Err	3.39	3.25	0.95	2.20	2.24	0.55
Avg Wealth	219.03	213.83	125.82	218.98	222.82	3.84
Std Err	3.24	3.12	0.94	2.16	2.19	0.53
TC	123.79	111.54	25.55	157.38	155.94	-1.44
IR	NA	-0.57	-0.52	0.00	0.03	NA
SR	0.96	0.97	1.90	1.43	1.44	0.10
High Transaction Costs ($\eta = 4 \times 10^{-7}$)						
Avg Objective	85.28	85.33	66.26	103.22	105.46	2.24
Std Err	1.96	1.99	0.60	1.34	1.42	0.52
Avg Wealth	110.00	110.70	68.67	114.92	118.77	3.85
Std Err	1.91	1.93	0.59	1.32	1.39	0.51
TC	64.93	66.96	15.07	92.02	91.34	-0.69
IR	NA	0.40	-0.40	0.06	0.10	NA
SR	0.82	0.81	1.65	1.23	1.20	0.11

R, and BL) nest the unconstrained myopic one, we see that all policies all achieve the same high objective functions, and corresponding Sharpe ratios of 3.53. We note that BL nests the myopic strategy because we use as stock exposures the myopic strategy holdings. This illustrates the necessity of choosing a large enough set of exposures for the LGS to span a large enough set of strategies.

The second panel of Table 2 illustrates that even when transaction costs are relatively low, the dynamic trading strategies can improve on the myopic policy. However, GP does underperform as the model is estimated as if the data is generated from an arithmetic process.

We see that for non zero t-costs BL and the GP-R perform very closely, with BL outperforming slightly. What makes GP-R slightly non-optimal here however is that, in our simulations, returns have constant variance rather than price changes. Thus we suspect that the small performance difference between GP-R and BL is due to the fact that the log-normal return dynamics we simulate do not exactly conform to the assumed (normal) return dynamics assumed in the GP-R solution.

3.4.2 Factor Model Simulation Results

Now we turn to the results for the "factor model" simulations, where cross-sectional variation in expected returns are linked to common factor loadings as given in equation (40). The upper panel of table 3 below shows the results with zero t-costs. As before the myopic strategy is optimal, and since it is nested by all strategies considered, all achieve the same objective value to within experimental error.

First, we note that BL significantly outperforms all other strategies in terms of objective value (up to 14% higher than the second-best)[21]. Even GP-R is substantially suboptimal compared to BL in this factor-based model as the return dynamics now differ substantially from the model that underlies GP-R. The underperformance is very large for the case of data-driven GP. GP-R has the second-best performance in all cases except in the low transaction cost regime in which MP-TC achieves a higher objective value. Also, recall that the t-cost multiplier for the MP-TC strategy is

[21]Note that the Sharpe and information ratios for the strategies do not always line up with the average objective functions. The reason is that these ratios are not the objective function that is optimized and hence can be a misleading performance criterion. For example, in the medium transaction cost case MP achieves the second-lowest objective but has the highest Sharpe ratio.

chosen via simulation so as to maximize the objective function.

We observe that BL and MP-TC seem to be much more aggressive in trading compared to MP and GP-R as measured by the size of average transaction costs. In the next section, we illustrate how all strategies differ in trading in various regimes in a simplified framework.

3.5 Discussion of the Trading Rules

In this section, we construct impulse response functions for the BL, GP-R, MP and MP-TC policies described in Section 3.3. We do this for the two sets of return dynamics we considered in Section 3.4: *i.e.*, Equation (40) for the economy with both factor and residual risk, and Equation (41) for the economy with residual risk only. This analysis provides some insights into the differences in performance uncovered in our analysis in Section 3.4.

The basic environment is the same as in the preceding section. We begin by setting the time 1 positions and exposures for each security equal to the long run mean of zero: $x_{j,1} = \beta_{j,rev,1} = \beta_{j,mom,1} = \beta_{j,value,1} = 0 \; \forall j$. We further constrain the residuals for all securities over week 1 to be zero. Over week 2, we "shock" the idiosyncratic return of security i with a positive 2-standard-deviation shock, *i.e.,* $\epsilon_{i,2} = 2\sigma_i$, but set the idiosyncratic shocks for all other assets to zero ($\epsilon_{j,2} = 4 \forall i \neq j$). From week 3 to week 26, all future shocks are set to zero so that the path of realized return is equal to the path of expected returns.

3.5.1 Characteristics Model Results

The upper panel of Figure 1 plots the path of realized returns of security i for this experiment. The positive return at time 2 is the shock itself. The subsequent effects are due to the interplay between reversal, momentum and value which affect the future expected return path of the security subsequent to the time 2 shock.

Indeed, the lower panel of Figure 1 shows the path of the factor exposures. All three factor exposures at the end of week 2 are equal to approximately one-fourth of idiosyncratic shock (recall that $\nu = 0.2498$ per equation (41)), but then decay at very different rates. In the determination of the expected return, the reversal effect dominates for the first few weeks, which leads to a negative expected return. After week 11, the positive (but much smaller) premium for momentum a positive

Table 3: **Policy Performance: Common factor noise**

This table summarizes the performance of each policy in the factor model environment for four different levels of transaction costs. For each policy, we report the average across the 10,000 runs of the: the objective value, standard error of the objective value, terminal wealth, the standard error of the terminal wealth, and the information ratio using myopic policy as a benchmark. (Dollar values are in hundred thousands of dollars.)

	MP	**MP-TC**	**GP**	**GP-R**	**BL**	**BL-GPR**
Zero Transaction Costs						
Avg Objective	3001.31	3001.31	-98689.21	3001.31	3000.60	-0.71
Std Err	24.39	24.39	391.02	24.39	24.40	0.32
Avg Wealth	6007.09	6007.09	37659.88	6007.09	6006.76	-0.34
Std Err	24.37	24.37	163.16	24.37	24.38	0.34
TC	0.00	0.00	0.00	0.00	0.00	0.00
IR	NA	NA	3.17	0.59	-0.01	NA
SR	3.49	3.49	3.26	3.49	3.49	-0.01
Low Transaction Costs ($\eta = 1 \times 10^{-7}$)						
Avg Objective	429.61	452.37	-8036.32	447.94	490.57	42.63
Std Err	3.18	4.02	37.02	3.21	4.19	1.78
Avg Wealth	489.25	546.45	-5756.90	509.32	593.44	84.12
Std Err	3.16	4.00	31.14	3.18	4.14	1.72
TC	234.41	385.42	10297.69	236.30	385.78	149.48
IR	NA	0.94	-3.01	0.17	0.72	NA
SR	2.19	1.93	-2.61	2.26	2.03	0.69
Medium Transaction Costs ($\eta = 2 \times 10^{-7}$)						
Avg Objective	229.21	242.62	-5002.27	251.86	279.06	27.20
Std Err	1.74	2.23	22.25	2.09	2.62	1.56
Avg Wealth	247.82	272.87	-4200.74	280.71	323.65	42.94
Std Err	1.74	2.23	20.05	2.07	2.59	1.53
TC	132.27	223.98	6812.61	158.05	232.64	74.59
IR	NA	0.71	-3.28	0.31	0.59	NA
SR	2.02	1.73	-2.96	1.91	1.77	0.40
High Transaction Costs ($\eta = 4 \times 10^{-7}$)						
Avg Objective	118.48	125.57	-2990.79	136.65	155.83	19.17
Std Err	0.92	1.19	12.92	1.44	1.79	1.42
Avg Wealth	123.88	134.51	-2727.93	152.10	179.26	27.16
Std Err	0.92	1.19	12.15	1.44	1.77	1.41
TC	70.44	120.45	4169.41	98.60	130.81	32.21
IR	NA	0.55	-3.44	0.31	0.50	NA
SR	1.90	1.60	-3.18	1.49	1.44	0.27

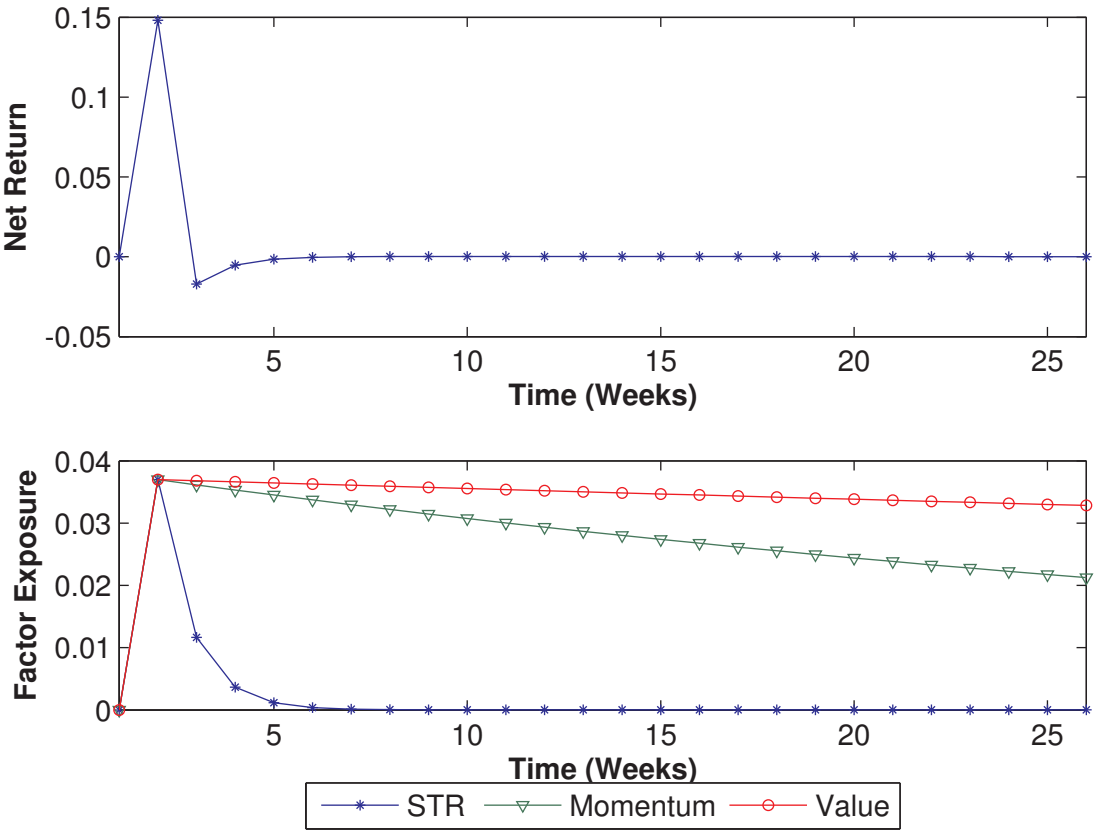

Figure 1: **Security i returns and factor exposures following a 2σ idiosynctratic shock in characteristics based model**

The top panel plots the realized return across one sample path, and the bottom panel plots the factor exposures for reversal, momentum and value in this sample path for security i. In this experiment, security i experiences a two standard deviation idiosyncratic volatility shock in week 2.

expected return, but because the premium for momentum is about two orders of magnitude smaller than that for reversal – as seen in Table 1 – the momentum effect is difficult to see in the plot. Of course, because momentum is much longer-lived than reversal, the cumulative effects are more comparable.

Figure 2 plot the dollar trades and corresponding positions in security i for the four policies, for the characteristics-based setting when the return generating process is given by equation (41) and the transactions costs are "medium."

It is clear from these plots that GP-R and BL trades and positions are nearly identical in this characteristics environment. One would expect that, since apart from modeling price changes versus dollar returns, the setting is very similar to that used in GP-R and for that setting the GP-R policy is close to optimal. The comparison of the optimal strategy with that used by the Myopic strategies

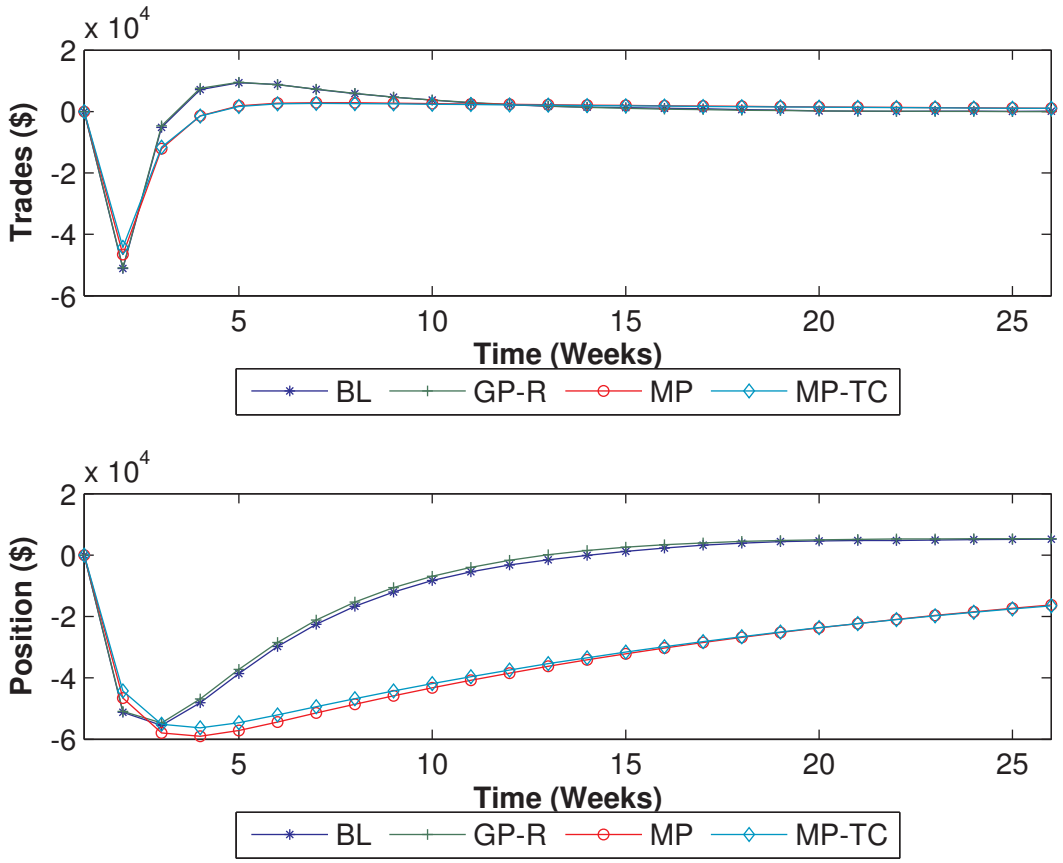

Figure 2: **Security i Trades and Positions – Characteristics Environment with Medium Transaction Costs**

The upper panel plots the dollar size of trades, and the lower panel the dollar size of positions in security i for various trading policies, following a two standard deviation idiosyncratic volatility shock in week 2. The characteristic-based return generating process is used (equation (41)), and the transaction costs parameter corresponds to "medium" (*i.e.*, $\eta = 2 \times 10^{-7}$)

is instructive. While the myopic strategy trades into the position very similarly to GP-R and BL it does not trade out of the position fast enough. The reason for this is that the myopic policy trade is based only on the expected returns and covariance at any point in time, and not on how quickly the expected return and covariance will change. In contrast, both GP-R and BL optimally incorporate the expected return dynamics of the security.

3.5.2 Factor Model Results

The upper panel of Figure 3 plots the path of realized returns of security i and the lower panel of Figure 3 shows the path of the factor exposures. The main difference here is that all three factor exposures at the end of week 2 are now equal to the value of idiosyncratic shock (per equation (40))

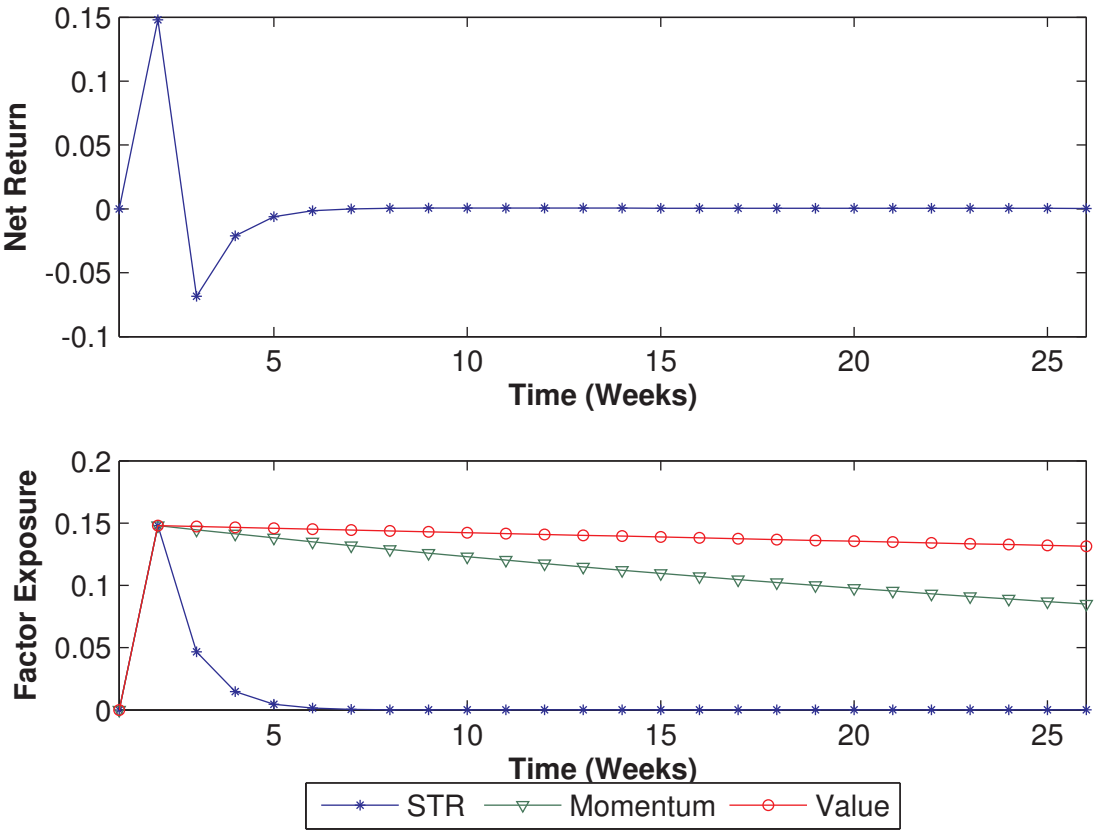

Figure 3: **Security i returns and factor exposures following a 2σ idiosynctratic shock in factor-based model**

The top panel plots the realized return across one sample path, and the bottom panel plots the factor exposures for reversal, momentum and value in this sample path for security i. In this experiment, security i experiences a two standard deviation idiosyncratic volatility shock in week 2.

and lead to four-times the magnitude of the expected returns when compared to characteristics-based model. Consequently, the sign of the realized return stay the same as in the previous model.

Figure 4 plots security i trades and positions for the four policies for the factor-based model setting (*i.e.*, when the return generating process is given by equation (40). Comparing this to Figure 2, we see that there is now substantial differences between the BL and GP-R trades immediately following the shock. It is clear that BL trades more aggressively: builds larger short position in the first few weeks (due to short-term reversal) and over time builds up a larger positive position in security i (due to momentum). The end result, as discussed previously, is that BL outperforms GP-R.

In the characteristics model of the previous section we found that BL and GP-R both outperformed the myopic strategies. The reason is that both BL and GP-R properly account for the

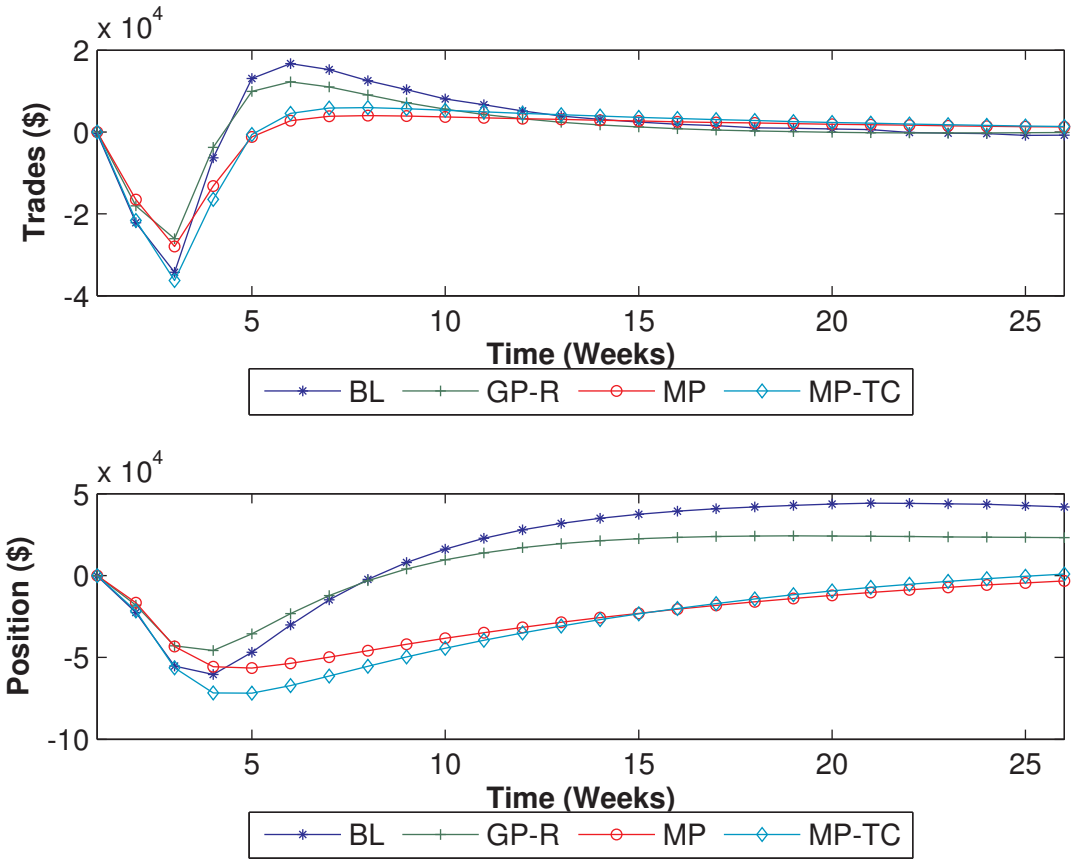

Figure 4: **Security i Trades and Positions – Factor Model with Medium Transaction Costs**

The upper panel plots the dollar size of trades, and the lower panel the dollar size of positions in security i for various trading policies, following a two standard deviation idiosyncratic volatility shock in week 2. The factor-model-based return generating process is used (equation (40)), and the transaction costs parameter corresponds to "medium" (*i.e.*, $\eta = 2 \times 10^{-7}$)

dynamics of the expected return. However, in the factor-model setting, the covariance matrix is changing with the shocks to expected return. The BL method takes into account the future dynamics associated with the changing covariance matrix. Recall that we optimize the GP-R method with respect to the instantaneous covariance matrix so in that sense it is optimal but it cannot take into account in the trading decision the future dynamics associated with this covariance matrix. For example, this may explain why the week 2 trade that is done in response to the shock is smaller for GP-R then for BL: the GP-R methodology implicitly assumes that the high risk for security i at time 2 will continue indefinitely. The BL trade incorporates the notion that, as the factor loading decays over time the risk will fall and therefore trades more aggressively.

4 Trading the Short-Term Reversal Effect

The short-term reversal anomaly was noted in Fama (1965), and was later explored more fully in Jegadeesh (1990) and Lehmann (1990). This negative serial correlation is generally interpreted as evidence consistent with incomplete liquidity provision, and much empirical evidence is consistent with this.[22] Collin-Dufresne and Daniel (2013, CD) develop a detailed dynamic model for the short-term reversal effect, and use this to examine the time variation in the reversal effect.

In this section we apply the LGS methodology we develop in Section 2 to a trading strategy based on the CD model and a simple model for transaction costs. We then apply this model to estimate the effectiveness of an LGS-based strategy using historical equity returns and simulated transaction costs, and compare the performance of the LGS strategy with other approaches suggested in the literature.

We include this for several reasons: first, we demonstrate the steps necessary for implementing the LGS methodology in a setting like that would be faced by an investor. Second, this gives us an opportunity to compare the performance of the LGS methodology to other candidate methodology using actual rather than simulated data.

The short-term-reversal strategy is a good test-bed for dynamic-portfolio-choice methods. The strategy returns, gross of transaction costs, are large. But the returns associated with short-term-reversal die out quickly: CD show that the half life associated with this decay averages 2.4 days over this sample period.

4.1 Sample and Data

We implement our LGS-based trading strategy on the 100 largest firms by market capitalization in the CRSP universe, over the period from January 1974-March 2013. At close on the last trading day of each calendar year we select the 100 firms with the largest market capitalization. We then

[22] There is now a voluminous academic literature on this topic: Chan (2003) and Tetlock (2011) show that large security price moves exhibit more reversal if they are not associated with news and Da, Liu, and Schaumburg (2013) show that within industry/residual based reversals have a far higher Sharpe ratio. In addition, Avramov, Chordia, and Goyal (2006) argue that the reversal effect is present only in small, illiquid securities with high turnover, and Khandani and Lo (2007) document a dramatic decline over time in the efficacy of the strategy. Finally, the strength of the reversal strategy appears to depend on arbitrageurs ability to access capital: Nagel (2012) documents that the return of a \$1-long/\$1-short short-term-reversal portfolio is far higher when market volatility is higher. He attributes this time variation to limited investor capital in times of high volatility.

trade those firms over the next calendar year. What this means is that our cross-section is the very largest firms in the CRSP universe. We do this to ensure that there is high liquidity in each of the securities in our sample on each trading day.

Because we trade only the largest firms in the CRSP universe, every one of the firms in our sample has a fairly liquid market. Almost every one of the firms in our sample trades every day. There are a relatively small number of delistings, and for those firms that are delisted, a reliable delisting return is available.[23]

Note the firms in our sample often change at close on the last trading day of each calendar year. If a firm leaves our sample (because it is no longer one of the 100 largest firms) we close out our position at the closing price on that date. Similarly, if a firm enters the sample of the 100 largest firms, we allow our simulation to trade into that firm starting at close on the last trading day of the year.

4.2 Model Specification

Our baseline specification for the individual firm excess returns ($\tilde{r}_{i,t+1}$) is:

$$\tilde{r}_{i,t+1} = \beta_{i,m}\tilde{r}_{m,t+1} + \underbrace{B_{i,t}\lambda_t + \sigma_i h_{\epsilon,t}\tilde{\epsilon}_{i,t+1}}_{\equiv\tilde{u}_{i,t+1}} \qquad \tilde{\epsilon}_{i,t+1} \sim \mathcal{N}(0,1) \tag{47}$$

where $\mathsf{E}_t[\tilde{\epsilon}_{i,t+1}\tilde{\epsilon}_{j,t+1}] = 0 \;\forall\; i \neq j$, and where $\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2 = 1$.

Firm i's excess return $\tilde{r}_{i,t+1}$ has a loading of $\beta_{i,m}$ a single common factor (the market). $u_{i,t+1}$ denotes firm i's residual return at time $t+1$.

4.2.1 The market return

The process for the excess market return ($\tilde{r}_{m,t+1}$) is:

$$\tilde{r}_{m,t+1} = \mu + \underbrace{h_{m,t}\tilde{v}_{t+1}}_{\equiv\tilde{u}_{m,t+1}}, \qquad \tilde{v}_{t+1} \sim \mathcal{N}(0,1) \tag{48}$$

[23] In tests of our trading strategy, if a firm is delisted, up until the delisting takes place, we trade as if we are unaware that the delisting will take place. On the delist date, we assume that any holdings, long or short, of that firm's shares earn the return on that date plus the CRSP delisting return.

where $\mathsf{E}_t[\tilde{v}_{t+1}\tilde{\epsilon}_{i,t+1}] = 0 \ \forall \ i$. The market return $\tilde{r}_{m,t+1}$ has a constant mean return, and variance that follows a Glosten, Jagannathan, and Runkle (1993) GARCH process:

$$h^2_{m,t} = \omega_m + \beta_m h^2_{m,t-1} + (\alpha_m + \gamma_m \mathrm{I}[(\tilde{r}_{m,t} - \mu) < 0]) \, u^2_{m,t}. \tag{49}$$

Maximum likelihood estimates of the parameters in equations (48) and (49) are given in Table 4.

4.2.2 Residual return specification

Expected return: Consistent with the literature showing that residual returns exhibit reversal, we specify that a firm's residual return $u_{i,t+1}$ is not mean zero, but rather has a conditional expected return that is negatively correlated with it's lagged residual returns for lags of several weeks. Specifically, equation (47) specifies that firm i's conditional expected residual return $\mathsf{E}_t[\tilde{u}_{i,t+1}] = B_{i,t}\lambda_t$ is the product of a firm specific exposure $B_{i,t}$ and a common premium λ_t. The exposure $(B_{i,t})$ is governed by an auto-regressive process:

$$\begin{aligned} B_{i,t} &= \beta_r B_{i,t-1} + (1-\beta_r)\tilde{u}_{i,t-1} \\ &= (1-\beta_r)\sum_{l=1}^{\infty} \beta_r^l u_{i,t-l} \end{aligned} \tag{50}$$

Firm i's time t exposure at time t is thus an exponentially weighted sum of its lagged daily residuals, starting at time $t-1$. Note that our specification does include the day t return in the expected return for day $t+1$ – we skip a day to avoid various econometric problems. [24] Collin-Dufresne and Daniel (2013, CD) show that the specification in equation (50) does a good job of capturing the short-term reversal process. For this sample of firms, over the 1974-2013 time period, the average λ_t is about -0.12. This means that, following a residual shock of 1%, the price of a firm i will fall (starting in one day) by about 12 basis points over the next few weeks. CD's estimation of equation (50) for daily returns gives a $\hat{\beta}_r = 0.720$, corresponding to a half-life of this price decay of 2.4 days.

[24] With this specification, the expected residual return at time $t+1$ is a function of residuals at times $t-1$ and before. This is, at least in part, to avoid potential bid-asked bounce effects. Note that every firm in our sample trades (almost) every day. See Collin-Dufresne and Daniel (2013) for details.

Variance Process Specification: Equation (47) specifies that the volatility of firm i's residual return is $\sigma_i h_{\epsilon,t}$ – the product of a time-invariant but firm-specific term σ_i, and the (common) level of cross-sectional volatility, $h_{\epsilon,t}$. Our specification is consistent with Kelly, Lustig, and Van Nieuwerburgh (2012), who argue that time variation in individual firm idiosyncratic volatilities are largely captured by a single factor structure. Here, all changes in idiosyncratic volatility a result of changes in the common level $h_{\epsilon,t}$.

Equation (47) specifies that the shock to firm i's residual returns is

$$(\tilde{u}_{i,t+1} - B_{i,t}\lambda_t) = \sigma_i h_{\epsilon,t} \tilde{\epsilon}_{i,t+1}$$

This means that:

$$\begin{aligned} \mathsf{E}_t \left[\frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_{i,t+1} - B_{i,t}\lambda_t)^2 \right] &= h_{\epsilon,t}^2 \mathsf{E}_t \left[\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \tilde{\epsilon}_{i,t+1}^2 \right] \\ &= h_{\epsilon,t}^2 \left(\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \right) \\ &= h_{\epsilon,t}^2 \end{aligned} \tag{51}$$

That is, given our restrictions that $\frac{1}{n}\sum_{i=1}^{n} \sigma_i^2 = 1$ and that the ϵ's are *i.i.d.*, unit-variance, $h_{\epsilon,t}^2$ is the conditional cross-sectional variance (as defined by the LHS of equation (51).)

The dynamics of $h_{\epsilon,t}^2$ are captured by an GARCH(1,1) process (Bollerslev 1986):

$$h_{\epsilon,t}^2 = \kappa_\epsilon + \alpha_\epsilon h_{\epsilon,t-1}^2 + \mu_\epsilon \sigma_{xs,t}^2, \tag{52}$$

where $\sigma_{xs,t}^2$ denotes the realized cross-sectional variance in period t:

$$\sigma_{xs,t}^2 \equiv \frac{1}{n} \sum_{i=1}^{n} \left(\tilde{u}_{i,t} - B_{i,t-1}\lambda_{t-1} \right)^2 .$$

For an individual firm, equation (47) specifies that $\mathsf{E}_t \left[\left(\tilde{u}_{i,t+1} - B_{i,t}\lambda_t \right)^2 \right] = \sigma_i^2 h_{\epsilon,t}^2$. That is, σ_i^2 is the ratio of an individual firm's residual variance to the average residual variance of the n firms in our sample. Following CD, in our empirical implementation, we estimate σ_i at the beginning of

each year, and then hold it fixed from the first trading day of the year through the last, consistent with the specification in equation (47). Empirically, CD find considerable cross-sectional variation in σ_i.

Premium Specification: From equation (47), λ_t is the time-t expectation of the premium per unit of exposure that will be earned over period $t+1$ (*i.e.*, between t and $t+1$). The updating rule for λ_t is:

$$\lambda_{t+1} = \alpha_\lambda \lambda_t + (1-\alpha_\lambda)\tilde{u}_{B_1,t+1} \tag{53}$$

where $\tilde{u}_{B_1,t+1}$ denotes the period $t+1$ residual return of a portfolio with unit exposure to the str factor (that is, $B_{p,t} = 1$), and specifically the portfolio with time-t weights:[25]

$$w_{i,t}^{B_1} = \frac{B_{i,t}}{\sum_i B_{i,t}^2}$$

Here, λ_t should be interpreted as the econometrician's expectation of the latent period-$t+1$ premium λ_t^* – *i.e.*, $\lambda_t = E_t[\lambda_t^*]$. Similarly, equation (53) should be interpreted as describing the evolution of this expectation. It is a reduced form for the Kalman filter solution to the system of equations:

$$\begin{aligned} \lambda_{t+1}^* &= \rho\lambda_t^* + (1-\rho)\bar{\lambda}^* + \tilde{v}_{t+1} \\ \tilde{u}_{B_1,t+1} &= \lambda_t^* + \tilde{\epsilon}_{B_1,t+1} \end{aligned}$$

where λ_{t+1}^* is the latent/unobserved premium.

In summary, according to this specification, a large positive residual-return is associated with future price declines. However, the half-life associated with this decline is short, with a point estimate of 2.4 days. In contrast, the *magnitude* of the price decline varies quite slowly, following an $AR(1)$ process with a half-life of about 1 year.

Table 4: **Maximum Likelihood Estimates of Market Process Parameters**
This table presents the estimates, standard errors and t-statistics from the joint estimation GJR-GARCH process (equations (48) and (49)). All parameter estimates are obtained from an iterative ML procedure run on daily market returns, where the market is defined as the equal-weighted average of the returns of the 100 largest firms, measured at the beginning of each year. The sample period is January 2, 1974 through March 28, 2013.

Joint Estimation of Equations (48) and (49)

Variable	ML. Est.	std. err.	t-stat
$\mu^{\dagger}$	2.6119	1.0658	2.4505
$\omega_m^{\dagger}$	1.5646	0.3555	4.4008
α_m	0.0152	0.0039	3.9286
γ_m	0.1006	0.0171	5.8844
β_m	0.9207	0.0102	90.3474

†The coefficients and std. errors for μ are $\times 10^4$, and for ω are $\times 10^6$.

4.3 Model Estimation

In this section we present the results of the estimation of the reversal model described in Section 4.2 for our sample of the 100 largest CRSP firms, and over our sample period of 1974:01-2013:03. The details of the model estimation are reported in Collin-Dufresne and Daniel (2013).

We begin with the estimation of the market process (equations (48) and (49)), which we estimate jointly using a numerical maximum likelihood procedure. The results of this estimation are reported in Table 4

We also estimate the parameters of the individual firm process (equations (50), (53) and (52)). Consistent with this specification, CD find that this the auto-regressive process in equations (47) and (50) do a reasonable job of capturing the short-term reversal phenomemon. CD estimate $\hat{\beta}_r = 0.7195$ using a Fama and MacBeth (1973) regression. Based on this value of β_r, other parameters are estimated using an iterative MLE procedure. The results of this estimation are reported in Table 5.[26]

In the next section, we develop a LGS based on: (1) the process specification laid out in Section 4.2, (2) the estimated parameters, (3) a simple model of transaction costs.

[25]Note that $\sum_i w_{i,t}^{B_1} B_{i,t} = 1$. If the residuals were uncorrelated and with uniform variance, this would be the portfolio with minimum variance portfolio subject to the constraint that $B_{p,t} = 1$.

[26]The value of $\hat{\alpha}$ in this table is s.t. $\alpha = 1$ can't be rejected. Should probably instead use unit-root estimation techniques to see whether we can reject $\alpha = 1$.

Table 5: **Maximum Likelihood Estimates of Individual Firm Model Parameters**
This table presents the estimates, standard errors and t-statistics for the model parameters in equations (50), (52) and (53). All estimates are obtained from an iterative ML procedure run on daily returns from the 100 largest market capitalization at the start of each year. The sample period is January 2, 1974 through March 28, 2013. Note that $\hat{\lambda}_0$ is the starting value of λ_t that maximizes the the likelihood function.

param.	ML-est.	std. err	t-stat
$\hat{\beta}_r$	0.751470		
$\hat{\lambda}_0$	0.084259	0.046678	1.805094
α_λ	0.997765	0.001298	768.672674
$\kappa_\epsilon^\dagger$	4.741034	1.109324	4.273805
α_ϵ	0.723503	0.021953	32.956604
μ_ϵ	0.260749	0.019914	13.093699

†The coefficient and std. error for κ_ϵ are $\times 10^6$.

4.4 An LGS-based methodology applied to short-term reversal

In this section we document the construction of the objective function for the short term reversal trading strategy based on the methodology developed in Section 2, and the estimated model of short-term-reversal described in Sections 4.2 and 4.3.

4.4.1 Objective function

We use an objective function similar to that used in recent literature as discussed in Section 2.3.

$$\max \sum_{t=1}^{T} \mathsf{E}\left[x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t - \frac{\eta}{2} u_t^\top I_N \Sigma_{t\to t+1} u_t\right],$$

where $\Sigma_{t\to t+1} = h_{m,t}^2 \beta\beta^\top + h_{e,t}^2 \operatorname{diag}(\sigma_1, \ldots, \sigma_N)$ from the dynamics of the security returns. With this objective function we assume that transaction cost of trading a particular asset varies linearly with the variance of only this asset's return but does not depend on the variance of the remaining security returns (i.e., no cross-impact). We multiply $\Sigma_{t\to t+1}$ in the transaction cost term by I_N in order to ensure this, i.e., diagonal transaction cost matrix. Note that this is chosen for brevity as our methodology is general enough to incorporate any quadratic function, deterministic or stochastic.

We calibrate η to be 2×10^{-7}, as in the medium transaction cost environment in the simulation experiment. Note that as η approaches to zero, the complexity of the problem decreases drastically

and myopic policy becomes optimal. Finally, we assume that the coefficient of risk aversion, γ, equals 1×10^{-8} which we can think of as corresponding to a relative risk aversion of 1 for an agent with \$100 Million under management.

4.4.2 Policies

We compare the gains from trading according to a Myopic Policy (MP) and the Restricted Best Linear (RBL) policy.

We use a similar approach undertaken in Section 3.3 to compute both trading policies. Let x_t^{MP} be the vector of dollar positions that the myopic policy chooses in each asset. Then,

$$x_t^{\mathsf{MP}} = \left(I_N \Sigma_{t\to t+1} + \gamma \Sigma_{t\to t+1}\right)^{-1} \left(B_t \lambda_t + \Lambda \left(x_{t-1}^{\mathsf{MP}} \circ R_t\right)\right),$$

where $\Sigma_{t\to t+1} = h_{m,t}^2 \beta\beta^\top + h_{e,t}^2 \operatorname{diag}(\sigma_1, \ldots, \sigma_N)$. Note that in the absence of transaction costs, x_t^{MP} simplifies to $\frac{1}{\gamma}\Sigma_{t\to t+1}^{-1} B_t \lambda_t$.

We will compare the myopic policy with restricted best linear policy that uses two most recent myopic positions. Formally, we solve for the optimization

$$\begin{aligned}
&\max_{\theta,\pi} \sum_{t=1}^{T} \mathsf{E}\left[x_t^\top r_{t+1} - \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t - \frac{\eta}{2} u_t^\top I_N \Sigma_{t\to t+1} u_t\right], \\
\text{subject to } & x_{i,t} = \theta_{i,t} x_{i,t}^{\mathsf{MP}}, \\
& u_{i,t} = \pi_{i,1,t} x_{i,t}^{\mathsf{MP}} + \pi_{i,2,t} x_{i,t-1}^{\mathsf{MP}}, \\
& \pi_{i,1,t} = \theta_{i,t} \text{ for } t > 1, \\
& \pi_{i,2,t} = -\theta_{i,t-1} \text{ for } t > 1, \\
& \pi_{i,1,1} = 0.
\end{aligned}$$

4.4.3 Results

We run the performance statistics of our myopic and restricted best linear policy for the short-term reversal experiment. Table 6 illustrates that the restricted best linear policy outperforms the

Table 6: **Short-Term Reversal Experiment: Policy Performance.**

For the myopic policy (MP) and the restricted best-linear policy (RBL), we report average terminal wealth, average objective value, Sharpe ratio for terminal wealth at the end of each trading year and Sharpe ratio for daily trading gains. (Dollar values are in thousands of dollars.)

	MP	**RBL**
Avg Objective	718.6	724.4
Avg Wealth	1313	1047
Std Dev	1966	1446
TC	178.1	166.1
Sharpe Ratio of Terminal Wealth	0.9108	0.962
Sharpe Ratio of Daily Profit	0.7164	0.7883

myopic policy in terms of average objective value and Sharpe ratios for the one-year and daily trading profits. The improvement is significant, but clearly both appear to do very well in this medium t-cost environment (which is our calibration assumption for this experiment). It would be interesting to investigate higher t-cost levels and perhaps a greater dependence of t-costs on market volatility to see how they affect the relative performance of the dynamic versus myopic trading strategy and the overall profitability of the reversal strategy.

Figure 5 illustrates the back-tested cumulative profits of trading the reversal strategy using the BL trading strategy. We do see a marked tapering during the recent years of the average profitability of the trading strategy as well as significant drawdowns.

5 Conclusion and Future Directions

The LGS framework we propose here accommodates complex return predictability models studied in the literature in multiperiod models with transaction costs. Our return predicting factors do not need to follow any pre-specified model but instead can have arbitrary dynamics. We allow for factor dependent covariance structure in returns driven by common factor shocks, as well as time varying transaction costs.

The main insight is that for the class of LGS the optimal policy can be computed in closed-form by solving a deterministic linear quadratic problem, which is computationally very efficient.

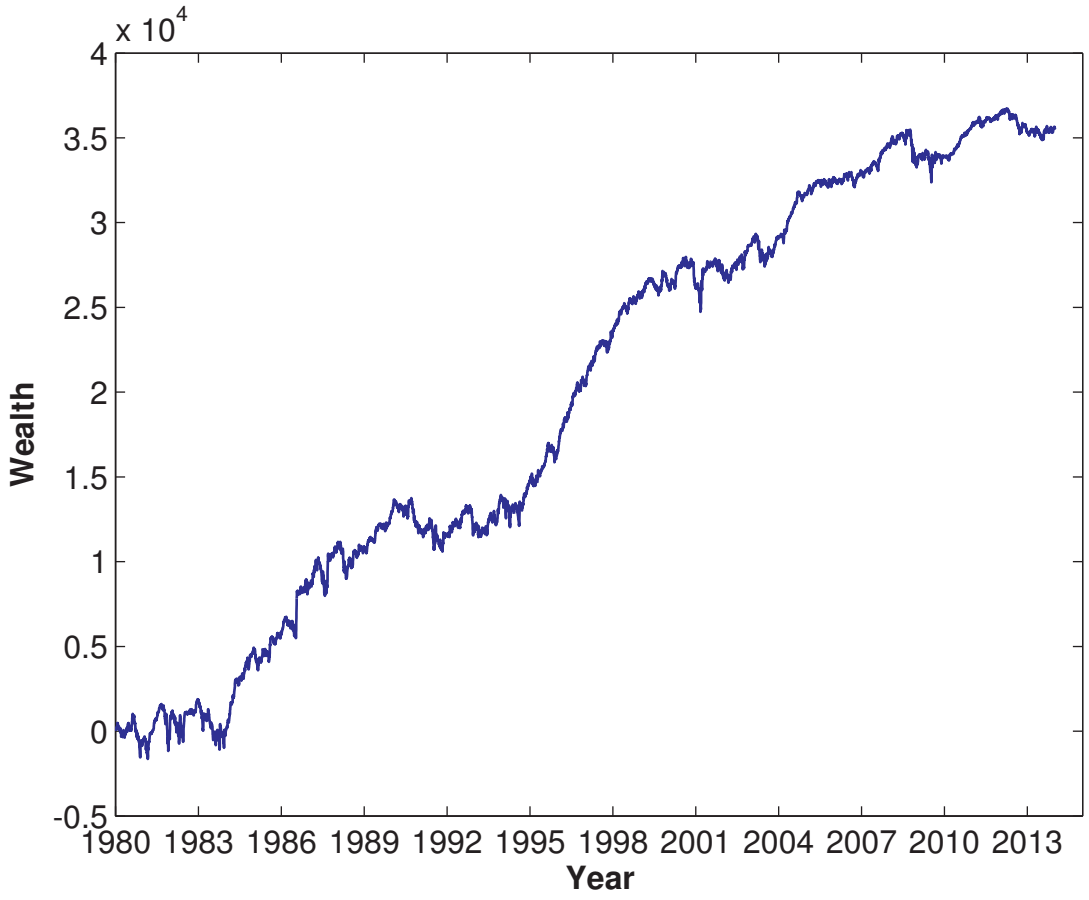

Figure 5: **Cumulative profits net of t-costs of trading short-term-reversal using the BL strategy**

Numerical experiments show that the LGS performs similarly to the linear-quadratic solutions of Litterman (2005) and Gârleanu and Pedersen (2012) when the return generating process is close to having a constant covariance matrix of price changes (where L-GP provide the optimal solution). Instead, the LGS framework performs better in other situations, such as when returns display stochastic volatility, e.g., when they are driven by a factor model. We also investigate the performance of the LGS framework when trading a high turnover strategy based on return reversal, for which transaction costs are a first order concern. The benefits to using a dynamic framework appear significant compared to a widely used approach that relies on a myopic objective function with a transaction cost multiplier that is chosen to maximize the in-sample performance.

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A General quadratic objective function

It is straight-forward to extend our approach to a non-zero risk-free rate $R_{0,t}$ and an objective function that is linear-quadratic in the position vector (i.e., $F(x_t, w_T) = w_T + a_1^\top x_T - \frac{1}{2} x_T^\top a_2 \, x_T$) rather than linear in total wealth. The $F(\cdot,\cdot)$ function parameters could be chosen to capture different objectives, such as maximizing the terminal gross value of the position ($w_T + \mathbf{1}^\top x_T$) or the terminal liquidation (i.e., net of transaction costs) value of the portfolio ($w_T + \mathbf{1}^\top x_T - \frac{1}{2} x_T^\top \Lambda_T x_T$), or the terminal wealth penalized for the riskiness of the position ($w_T + \mathbf{1}^\top x_T - \frac{\gamma}{2} x_T^\top \Sigma_T x_T$), or some intermediate situation.

Suppose the objective function is:

$$\max_{u_1,\ldots,u_T} \mathsf{E}\left[F(w_T, x_T) - \sum_{t=0}^{T-1} \frac{\gamma}{2} x_t^\top \Sigma_{t\to t+1} x_t \right] \tag{54}$$

By recursive substitution x_T and w_T can be rewritten as:

$$x_T = x_0 \circ R_{0\to T} + \sum_{t=1}^{T} u_t \circ R_{t\to T} \tag{55}$$

$$w_T = w_0 R_{0,0\to T} - \sum_{t=1}^{T} \left(u_t^\top \mathbf{1} R_{0,t\to T} + \frac{1}{2} u_t^\top \Lambda_t u_t R_{0,t\to T} \right) \tag{56}$$

where we have defined security i's cumulative return between date t and T as:

$$R_{i,t\to T} = \prod_{s=t+1}^{T} R_{i,s} \tag{57}$$

(with the convention that $R_{i,t\to t} = 1$) and the corresponding N-dimensional vector $R_{t\to T} = [R_{1,t\to T}; \ldots; R_{N,t\to T}]$.

Now note that:

$$a_1^\top x_T = (a_1 \circ R_{0\to T})^\top x_0 + \sum_{t=1}^{T} (a_1 \circ R_{t\to T})^\top u_t \tag{58}$$

Substituting, we obtain the following:

$$
\begin{aligned}
F(w_T, x_T) &= F_0 + \sum_{t=1}^{T} \left\{ G_t^\top u_t - \frac{1}{2} u_t^\top P_t u_t \right\} - \frac{1}{2} x_T^\top a_2 \, x_T && (59)\\
F_0 &= w_0 R_{0,0\to T} + (a_1 \circ R_{0\to T})^\top x_0 && (60)\\
G_t &= a_1 \circ R_{t\to T} - \mathbf{1} \circ R_{0,t\to T} && (61)\\
P_t &= \Lambda_t \circ R_{0,t\to T} && (62)
\end{aligned}
$$

With these definitions, the objective function in equation (54) it can be rewritten as:

$$
F_0 - \frac{\gamma}{2} x_0^\top Q_0 x_0 + \max_{u_1,\ldots,u_T} \sum_{t=1}^{T} \mathsf{E} \left[G_t^\top u_t - \frac{1}{2} u_t^\top P_t u_t - \frac{\gamma}{2} x_t^\top Q_t x_t \right] \tag{63}
$$

subject to the non-linear dynamics given in equations (4) and (5) and where we have defined

$$
Q_t = \begin{cases} \Sigma_{t\to t+1} & \text{for } t < T \\ \frac{1}{\gamma} a_2 & \text{for } t = T \end{cases} \tag{64}
$$

Indeed, substituting the definition of our linear trading strategies from equations (24) and (25) into our objective function in equation (63) and then taking expectations gives:

$$
F_0 - \frac{\gamma}{2} x_0^\top Q_0 x_0 + \max_{\pi_1,\ldots,\pi_T} \sum_{t=1}^{T} \mathcal{G}_t^\top \pi_t - \frac{1}{2} \pi_t^\top \mathcal{P}_t \pi_t - \frac{\gamma}{2} \theta_t^\top \mathcal{Q}_t \theta_t \tag{65}
$$

$$
\text{subject to } \quad \theta_t = \theta_{t-1}^0 + \pi_t \tag{66}
$$

and where we define the vector $\mathcal{G}_t$ and the square matrices $\mathcal{P}_t$ and $\mathcal{Q}_t$ for $t = 1, \ldots, T$ by

$$
\begin{aligned}
\mathcal{G}_t &= \mathsf{E}_0[\mathcal{B}_t G_t] && (67)\\
\mathcal{P}_t &= \mathsf{E}_0[\mathcal{B}_t P_t \mathcal{B}_t^\top] && (68)\\
\mathcal{Q}_t &= \mathsf{E}_0[\mathcal{B}_t Q_t \mathcal{B}_t^\top] && (69)
\end{aligned}
$$

Note that the time indices for $\mathcal{G}_t, \mathcal{P}_t, \mathcal{Q}_t$ also capture their size: $\mathcal{G}_t$ is a vector of length $NK(t+1)$,

and $\mathcal{P}_t$ and $\mathcal{Q}_t$ are square matrices of the same dimensionality.[27] Equation (65) is just the objective function (equation (63)) with the u_t's and x_t's rewritten as linear functions of the elements in $\mathcal{B}_t$, with coefficients π_t and θ_t, respectively. Since the policy parameters π_t and θ_t are set at time 0, they can be pulled outside of the expectation operator.

Intuitively equation (65) is a linear-quadratic function of the policy parameters π_t and θ_t, with $\mathcal{G}_t$, $\mathcal{P}_t$, $\mathcal{Q}_t$ as the coefficients in this equation. These three components give, respectively, the effect on the obective funtion of: the expected portfolio returns resulting from trades at time t; the transaction costs paid as a result of trades at time t; and finally the effect of the holdings at time t on the risk-penalty component of the objective function.

Since $\mathcal{G}_t$, $\mathcal{P}_t$, $\mathcal{Q}_t$ are not functions of the policy parameters, they can be solved for explicitly or by simulation, and this only needs to be done once. Their values will depend on the initial conditions, and on the assumptions made about the state vector X_t driving the return generating process R_t and the corresponding security-specific exposure dynamics $B_{i,t}$. But, since equation (27) is a linear-quadratic equation, albeit a high-dimensional one, it can be solved using standard methods. We next calculate the closed form solution.

A.1 Closed form solution

We begin with the linear-quadratic problem defined by equations (65) and (66). Define recursively the value function starting from $V(T) = 0$ for all $t \leq T$ by:

$$V(t-1) = \max_{\pi_t} \left\{ \mathcal{G}_t^\top \pi_t - \frac{1}{2}\pi_t^\top \mathcal{P}_t \pi_t - \frac{\gamma}{2}\theta_t^\top \mathcal{Q}_t \theta_t + V(t) \right\}$$
$$\text{subject to } \ \theta_t = \theta_{t-1}^0 + \pi_t$$

Then it is clear that $V(0)$ is the solution to the problem we are seeking. To solve the problem explicitly, we guess that the value function is of the form:

$$V(t) = -\frac{\gamma}{2}\theta_t^\top M_t \theta_t + L_t^\top \theta_t + H_t \tag{70}$$

[27] It is important to note that these matrices $\mathcal{G}_t, \mathcal{P}_t, \mathcal{Q}_t$ will depend on the initial conditions (in particular on the initial exposures $\mathcal{B}_0$, which typically will depend on the initial positions in each stock).

with M_t a symmetric matrix. Since $V(T) = 0$, it follows that $M_T = 0$, $L_T = 0$ and $H_T = 0$. To find the recursion plug the guess in the Bellman equation:

$$V(t-1) = \max_{\pi_t} \left\{ \mathcal{G}_t^\top \pi_t - \frac{1}{2}\pi_t^\top \mathcal{P}_t \pi_t - \frac{\gamma}{2}\theta_t^\top (\mathcal{Q}_t + M_t)\theta_t + L_t^\top \theta_t + H_t \right\}$$
$$\text{subject to } \theta_t = \theta_{t-1}^0 + \pi_t$$

Now plugging in the constraint, we can simplify the Bellman equation using the following notation $\overline{x}$ is the vector (submatrix) obtained from x by deleting the last NK rows (rows and columns). In Matlab notation $\overline{x} = x[1 : end - NK, 1 : end - NK]$.

$$V(t-1) = \max_{\pi_t} \left\{ (\mathcal{G}_t + L_t)^\top \pi_t - \frac{1}{2}\pi_t^\top [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]\pi_t - \frac{\gamma}{2}\theta_{t-1}^\top (\overline{\mathcal{Q}}_t + \overline{M}_t)\theta_{t-1} \right.$$
$$\left. -\gamma\theta_{t-1}^{0\top}[\mathcal{Q}_t + M_t]\pi_t + \overline{L}_t^\top \theta_{t-1} + H_t \right\} \tag{71}$$

The first order condition gives:

$$\pi_t = [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1} \left(\mathcal{G}_t + L_t - \gamma(\mathcal{Q}_t + M_t)^\top \theta_{t-1}^0 \right),$$

and plugging into the state equation (equation (66)) we find

$$\theta_t = [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1} \left(\mathcal{G}_t + L_t + \mathcal{P}_t^\top \theta_{t-1}^0 \right).$$

Next, substitute these optimal policies into the Bellman equation in (71), giving:

$$V(t-1) = \frac{1}{2}(\mathcal{G}_t + L_t - \gamma(\mathcal{Q}_t + M_t)^\top \theta_{t-1}^0)^\top [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1} \left(\mathcal{G}_t + L_t - \gamma(\mathcal{Q}_t + M_t)^\top \theta_{t-1}^0 \right)$$
$$- \frac{\gamma}{2}\theta_{t-1}^\top (\overline{\mathcal{Q}}_t + \overline{M}_t)\theta_{t-1} + \overline{L}_t^\top \theta_{t-1} + H_t$$

Setting $\Psi_t = [\mathcal{P}_t + \gamma(\mathcal{Q}_t + M_t)]^{-1}$ and expanding we find:

$$\begin{aligned} V(t-1) = H_t &+ \frac{1}{2}(\mathcal{G}_t + L_t)^\top \Psi_t (\mathcal{G}_t + L_t) \\ &- \gamma(\mathcal{G}_t + L_t)^\top \Psi_t (\mathcal{Q}_t + M_t)^\top \theta_{t-1}^0 + \overline{L}_t^\top \theta_{t-1} \\ &- \frac{\gamma}{2}\theta_{t-1}^\top \left[\overline{\mathcal{Q}}_t + \overline{M}_t - \gamma(\overline{\mathcal{Q}}_t + \overline{M}_t)^\top \overline{\Psi}_t (\overline{\mathcal{Q}}_t + \overline{M}_t)\right] \theta_{t-1} \end{aligned}$$

Comparing this equation and the conjectured specification for $V(t)$ in equation (70) shows that this specification will be correct if H_t, L_t, and M_t are chosen to satisfy the recursions:

$$\begin{aligned} H_{t-1} &= H_t + \frac{1}{2}(\mathcal{G}_t + L_t)^\top \Psi_t (\mathcal{G}_t + L_t) \\ L_{t-1} &= \overline{L}_t - \gamma \overline{(\mathcal{Q}_t + M_t)\Psi_t (\mathcal{G}_t + L_t)} \\ M_{t-1} &= \overline{\mathcal{Q}}_t + \overline{M}_t - \gamma(\overline{\mathcal{Q}}_t + \overline{M}_t)^\top \overline{\Psi}_t (\overline{\mathcal{Q}}_t + \overline{M}_t) \end{aligned}$$

with initial conditions $H_T = 0$, $L_T = 0$ and $M_T = 0$.

We have thus derived the optimal value function and the optimal trading strategy in the LGS class.

Before discussing some specific examples it is useful to introduce a set of LGS strategies which uses the exposures lagged at most ℓ periods. This set of 'restricted lag' LGS is useful in applications when the time horizon is fairly long, and for signals that have a relatively fast decay rate, so that the dependence on lagged exposures can be restricted without a significant cost. We next show that the same tractability obtains for the restricted lag setting.

B Constant variance of returns versus price changes

B.1 In dollars

Suppose x_t is vector of dollar holdings in risky shares and u_t is dollar trade at time t. R_f is the risk-free rate and R_t is the vector of Gross returns. The net returns are given by $r_t = R_t - \mathbf{1}$ and $r_f = R_f - 1$.

Then we have with the convention that we trade at the end of the period:

$$x_{t+1} = x_t . * R_{t+1} + u_{t+1} \tag{72}$$

$$W_{t+1} = W_t R_f + x_t'(R_{t+1} - R_f) - \frac{1}{2} u_{t+1} \Lambda_d u_{t+1} \tag{73}$$

B.2 In shares

Suppose n_t is vector of number of shares held in risky shares and h_t is number of shares traded at time t. R_f is the risk-free rate and $dS_{t+1} = S_{t+1} - S_t$ is the vector of price changes (Assume no dividends for simplicity).

Then we have with the convention that we trade at the end of the period:

$$n_{t+1} = n_t + h_{t+1} \tag{74}$$

$$W_{t+1} = W_t R_f + n_t'(dS_{t+1} - r_f S_t) - \frac{1}{2} h_{t+1} \Lambda_s h_{t+1} \tag{75}$$

B.3 The objective function

For simplicity we set $r_f = 0$ and as in GP we solve the infinite horizon problem where the investor maximizes the discounted value of mean-variance objective functions.

In dollars

$$\mathsf{E}\left[\sum_{t=0}^{\infty} \rho^t \left\{ x_t \mu_d - \frac{1}{2} u_t \Lambda_d u_t - \frac{\gamma}{2} x_t' \Sigma_d x_t \right\}\right] \tag{76}$$

or, equivalently, in shares:

$$\mathsf{E}\left[\sum_{t=0}^{\infty} \rho^t \left\{ n_t \mu_s - \frac{1}{2} h_t \Lambda_s h_t - \frac{\gamma}{2} n_t' \Sigma_s n_t \right\}\right]$$

Now, note that by definition:

$$\begin{aligned} x_t &= n_t . * S_t && (77) \\ u_t &= h_t . * S_t && (78) \\ \mu_s &= \mu_d . * S_t && (79) \\ \Sigma_s &= I_{S_t} \Sigma_d I_{S_t} && (80) \\ \Lambda_s &= I_{S_t} \Lambda_d I_{S_t} && (81) \end{aligned}$$

So clearly, assuming that the expectation and variance of dollar returns are constant is inconsistent with assuming that the expectation and variance of price changes are constant. We compare both cases next.

B.4 Constant expectation and variance of dollar returns

Let's assume that the expectation and variance of returns are constant. Then it is helpful to introduce the state variable $\overline{x}_t = x_t - u_t$, so that

$$\overline{x}_{t+1} = (\overline{x}_t + u_t) . * R_{t+1} \tag{82}$$

We can define the value function recursively by:

$$J(\overline{x}_t) = \max_{u_t} \left\{ (\overline{x}_t + u_t)\mu_d - \frac{1}{2} u_t \Lambda_d u_t - \frac{\gamma}{2} (\overline{x}_t + u_t)' \Sigma_d (\overline{x}_t + u_t) + \rho \mathsf{E}_t [J(\overline{x}_{t+1})] \right\} \tag{83}$$

Guess that the value function is quadratic.

$$J(x) = M_0 + M_1' \overline{x} + \overline{x}' M_2 \overline{x}$$

Let's first consider the one risky asset case. Then the solution is simply:

$$u_t + \overline{x}_t = \frac{\overline{x}_t \Lambda_d + \mu_d + M_1 \rho \mu_d}{\Lambda_d + \gamma \Sigma_d - 2 M_2 \rho (\mu_d^2 + \Sigma_d)} \tag{84}$$

where the coefficient of the optimal value function are given by:

$$M_2 = -\frac{\sqrt{\left(\gamma\Sigma - \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)\right)^2 + 4\gamma\Lambda\rho\Sigma\left(\mu^2+\Sigma\right)} - \gamma\Sigma + \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)}{4\rho\left(\mu^2+\Sigma\right)} \tag{85}$$

$$M_1 = \frac{2\Lambda\mu}{\sqrt{\left(\gamma\Sigma - \Lambda\left(\rho\left(\mu^2+\Sigma\right)-1\right)\right)^2 + 4\gamma\Lambda\rho\Sigma\left(\mu^2+\Sigma\right)} + \gamma\Sigma + \Lambda\mu^2\rho - 2\Lambda\mu\rho + \Lambda\rho\Sigma + \Lambda} \tag{86}$$

and

$$M_0 = \frac{-\mu^2\left(\mu^2+\Sigma\right)}{(\rho-1)\left(\left(\Sigma-\mu^2(\rho-1)\right)\sqrt{\gamma^2\Sigma^2 + 2\gamma\Lambda\Sigma\left(\rho\left(\mu^2+\Sigma\right)+1\right) + \Lambda^2\left(\rho\left(\mu^2+\Sigma\right)-1\right)^2} + \gamma\Sigma\left(\mu^2(\rho+1)+\Sigma\right) + \Lambda\left(\mu^2(\rho(\mu(\mu\rho+\mu-4)+1)+1)+\mu\rho\Sigma(\mu(\rho+2)-4)+\rho\Sigma^2+\Sigma\right)\right)} \tag{87}$$

B.5 Constant expectation and variance of price changes

For comparison purposes we make the same change of variables $\overline{n}_t = n_t - h_t$ so that

$$\overline{n}_{t+1} = \overline{n}_t + h_t$$

Then we define the value function recursively by:

$$J(\overline{n}_t) = \max_{h_t}\left\{(\overline{n}_t + h_t)\mu_s - \frac{1}{2}h_t\Lambda_s h_t - \frac{\gamma}{2}(\overline{n}_t + h_t)'\Sigma_s(\overline{n}_t + h_t) + \rho\mathsf{E}_t[J(\overline{n}_{t+1})]\right\} \tag{88}$$

Guess that the value function is quadratic.

$$J(x) = N_0 + N_1'\overline{n} + \overline{n}'N_2\overline{n}$$

Let's first consider the one risky asset case. Then we can solve everything in closed-form and we obtain:

$$h_t + \overline{n}_t = \frac{\overline{n}_t\Lambda_s + \mu_s + N_1\rho}{\Lambda_s + \gamma\Sigma_s - 2N_2\rho} \tag{89}$$

where the coefficient of the optimal value function are given by:

$$N_2 = \frac{-\sqrt{(\gamma\Sigma + \Lambda(-\rho) + \Lambda)^2 + 4\gamma\Lambda\rho\Sigma} + \gamma\Sigma + \Lambda(-\rho) + \Lambda}{4\rho} \tag{90}$$

$$N_1 = \frac{2\Lambda\mu}{\sqrt{(\gamma\Sigma + \Lambda(-\rho) + \Lambda)^2 + 4\gamma\Lambda\rho\Sigma} + \gamma\Sigma + \Lambda(-\rho) + \Lambda} \tag{91}$$

and

$$N_0 = \left\{ -\frac{\mu^2 \left((\rho - 1)\sqrt{\gamma^2\Sigma^2 + 2\gamma\Lambda(\rho + 1)\Sigma + \Lambda^2(\rho - 1)^2} + \gamma(\rho + 1)\Sigma + \Lambda(\rho - 1)^2 \right)}{4\gamma^2(\rho - 1)\rho\Sigma^2} \right\} \tag{92}$$

B.6 Comparing the two solutions

The most obvious difference between the two solutions is that in "shares" example there exists a no-trade solution.

Indeed, if at some t, we hold $\overline{n}_t$ shares such that:

$$\frac{\overline{n}_t\Lambda_{_s} + \mu_{_s} + N_1\rho}{\Lambda_{_s} + \gamma\Sigma_{_s} - 2N_2\rho} = \overline{n}_t$$

which is equivalent to

$$n_{no} = \frac{\mu}{\gamma\Sigma} \tag{93}$$

So if $\overline{n}_t = n_{no}$ at some time t, then it is optimal to **NEVER** trade from then on, since $h_t = 0$ and therefore $\overline{n}_{t+s} = \overline{n}_{t+1} = \overline{n}_t = n_{no}\ \forall s > 0$ by induction. Instead, in the dollar case, we see the the system can never settle into a no-trade equilibrium, since the dynamics of the state always lead to $\overline{x}_{t+1} \neq \overline{x}_t$ even if $u_t = 0$.

Further, it is interesting to note that the state where it is optimal not to trade **for one period** at time t in the dollar case, is actually NOT the mean-variance efficient portfolio. Indeed, the no trade position for that case corresponds to a dollar position such that:

$$\overline{x}_t = \frac{\overline{x}_t\Lambda_{_d} + \mu_{_d} + +M_1\rho\mu_{_d}}{\Lambda_{_d} + \gamma\Sigma_{_d} - 2M_2\rho(\mu_{_d}^2 + \Sigma_{_d})}$$

Solving for x_{no} we find:

$$x_{no} = \frac{2\mu\left(\mu^2+\Sigma\right)}{((\mu-1)\mu+\Sigma)\sqrt{\gamma^2\Sigma^2+2\gamma\Lambda\Sigma\left(\rho\left(\mu^2+\Sigma\right)+1\right)+\Lambda^2\left(\rho\left(\mu^2+\Sigma\right)-1\right)^2}+\gamma\Sigma\left(\mu^2+\mu+\Sigma\right)+\Lambda((\mu-1)\mu+\Sigma)\left(\rho\left(\mu^2+\Sigma\right)-1\right)} \tag{94}$$

Note that $x_{no} = \frac{\mu_d}{\gamma\Sigma_d}$ if $\Lambda_d = 0$ or if $\rho = 0$, but otherwise it is different!

Further, even if $x_t = x_{no}$ at some t and thus $u_t = 0$ is optimal, since $\overline{x}_{t+1} = \overline{x}_t R_{t+1}$ in that case, it will become optimal to trade at time $t+1$.

C Calibration of the Simulation Experiment

The RGPs for the characteristics and the factor environments (equations (40) and (41)) are, respectively

$$R_{i,t+1} = \beta_{i,t}^\top (F_{t+1} + \lambda) + \epsilon_{i,t+1}$$

where $\mathsf{E}_t[F_{t+1}] = 0$ and $\mathsf{E}_t[F_{t+1}F_{t+1}^\top] = \Omega$ and

$$R_{i,t+1} = \beta_{i,t}^\top \lambda + \epsilon_{i,t+1},$$

where the factor exposures $\beta_{i,t}$ and premia λ are each $(K,1)$ vectors, and and where the evolution of the factor exposures is given by equation (40):

$$\beta_{i,t+1}^k = (1-\phi_k)\beta_{i,t}^k + \epsilon_{i,t+1},$$

or equivalently:

$$\beta_{i,t}^k = \sum_{s=0}^{\infty} (1-\phi_k)^s \epsilon_{i,t+s}.$$

Taken together, these imply, for either environment, that:

$$\begin{aligned}
\mathsf{E}_t\left[R_{i,t+1}\right] &= \beta_{i,t}^{\top}\lambda \\
&= \sum_{k=1}^{K}\lambda_k\beta_{i,t}^{k} \\
&= \sum_{k=1}^{K}\lambda_k\sum_{s=0}^{\infty}(1-\phi_k)^s\epsilon_{i,t-s}.
\end{aligned}$$

In our simulation experiment in Section 3, we model the return-generating process for equities as consisting of $K = 3$ factors consistent with the short-term-reversal, medium-term-momentum, and long-term-reversal effects. Consistent with the evidence on these three effect, we choose half-lives for these factors of 5 days, 150 days, and 700 days.

To determine the parameters λ and Ω, we calibrate this factor model using the monthly returns of portfolios formed on the basis of momentum, short- and long-term reversal, available on Ken French's website. We use the full sample, 1927:01-2013:12. Note that data is available on both the pre-formation and the post-formation returns of these sets of portfolios. We perform a Fama-MacBeth-like regression of the post-formation returns on the pre-formation returns for each of the three sets of decile portfolios, and use the resulting coefficients to estimate the set of λs, given our assumed set of ϕs.

We characterize the slope coefficients for the three regressions with the formation period return horizons: our notation is that the formation period, for regression $j \in \{str, mom, ltr\}$, runs from time $t - m_j$ to $t - n_j$. For the characteristics model, the (cross-sectional) projection of a one-day return onto a sum of returns from time $t - m_j$ to $t - n_j$ will give, under the assumptions of our model.[28]

$$\begin{aligned}
cov\left(R_{i,t+1}, \sum_{s=n_j}^{m_j}\epsilon_{i,t-s}\right) &= \sigma_\epsilon^2\sum_{k=1}^{3}\lambda_k\beta_{i,t}^{k} \\
&= \sigma_\epsilon^2\sum_{k=1}^{3}\sum_{s=n_j}^{m_j}\lambda_k(1-\phi_k)^s
\end{aligned}$$

[28] I calculate the betas using returns rather than residals, so this is an approximation. However, given that most of the variance of returns is idiosyncratic as opposed to expected return variation, something which is unambiguously true in the data, particularly at short horizons.

and

$$var\left(\sum_{s=n_j}^{m_j}\epsilon_{i,t-s}\right) = (m_j - n_j + 1)\sigma_\epsilon^2.$$

and finally

$$\begin{aligned}
\beta_j &= \frac{cov\left(R_{i,t+1}, \sum_{s=n_j}^{m_j}\epsilon_{i,t-s}\right)}{var\left(\sum_{s=n_j}^{m_j}\epsilon_{i,t-s}\right)} = \sum_{k=1}^{3}\lambda_k \frac{1}{(m_j - n_j + 1)}\sum_{s=n_j}^{m_j}(1-\phi_k)^s. \\
&= \sum_{k=1}^{3}\left(\frac{(1-\phi_k)^{n_j} - (1-\phi_k)^{m_j+1}}{\phi_k(m_j - n_j + 1)}\right)\lambda_k \\
&= \sum_{k=1}^{3} a_{j,k}\lambda_k
\end{aligned}$$

where

$$a_{j,k} = \left(\frac{(1-\phi_k)^{n_j} - (1-\phi_k)^{m_j+1}}{\phi_k(m_j - n_j + 1)}\right) \tag{95}$$

We find the three values of λ_k by solving the set of linear equations (for the three empirically estimated β_js).

$$\begin{bmatrix} \beta_{str} \\ \beta_{mom} \\ \beta_{ltr} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

λ Estimation:

The Fama-MacBeth regressions yield (average) coefficients of:

$$\begin{bmatrix} \beta_{str} \\ \beta_{mom} \\ \beta_{ltr} \end{bmatrix} = \begin{bmatrix} -0.00116273 \\ 0.00044366 \\ -0.00010126 \end{bmatrix}$$

The resulting λ estimates are:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -0.093482 \\ 0.001484 \\ -0.000400 \end{bmatrix}$$

Ω Calibration:

The goal in the Ω calibration is to come up with an upper bound on the magnitude of the covariance matrix. We employ the following procedure to estimate the 3×3 factor covariance matrix Ω using the three sets of decile portfolio returns: *str*, *mom*, and *ltr*.

First, we use only the excess returns of the zero-investment portfolios formed by going long the top decile and short the bottom decile (*i.e.*, the $10-1$ portfolios). The factor loadings for these excess return portfolios are (from equation (40)

$$\beta_{j,k,t}^{10\text{–}1} = \sum_{s=0}^{\infty}(1-\phi_k)^s \epsilon_{j,t-s}^{10\text{–}1}$$

Here, $j \in \{str, mom, ltr\}$ is French's portfolio formation method; $k \in \{1,2,3\}$ is the factor identifier, and t is the time (end-of-period) at which we are measuring the factor loading. As in the preceding section, $t-n_j$ and $t-m_j$ are the starting and ending times for the period over which the pre-formation returns are measured for portfolio j.

We are going to make several assumptions to allow the calculation of the factor loadings for each of these three portfolios. First, since portfolio j is formed on the basis of individual firm returns from $t-m_j$ to $t-n_j$, we'll assume that the residual returns for the portfolios are zero outside of

that time range. This means that:

$$\beta_{j,k,t}^{10\text{–}1} = \sum_{s=n_j}^{m_j} (1-\phi_k)^s \epsilon_{j,t-s}^{10\text{–}1}$$

Second, note that French only provides the formation period return on an annual basis. So, for example, for the LHR portfolios we have their cumulative return from t-60 months through t-12 months. So what we do is to assume that the average return was earned equally over each day in the 48 month period. If we denote the total pre-formation return as R^{pre}, I assume that the daily return, for each day in the 4 year period, was $R^{pre}/(4 \cdot 252)$. In general, given a $10-1$ differential pre-formation return for strategy j in year y of $R_{j,y}^{pre,10\text{–}1}$, I calculated the each daily return over the formation period as:

$$R_{j,s}^{pre,10\text{–}1} = \frac{R_{j,y}^{pre,10\text{–}1}}{(m_j - n_j + 1)}$$

for each day s between $t - m_j$ and $t - n_j$, and zero outside of the formation period. This means that the factor loading for portfolio $10-1$ portfolio j on factor k is:

$$\begin{aligned}
\beta_{j,k,t}^{10\text{–}1} &= \frac{R_{j,y}^{pre,10\text{–}1}}{(m_j - n_j + 1)} \sum_{s=n_j}^{m_j} (1-\phi_k)^s \quad \forall t \in y \\
&= \left(\frac{(1-\phi_k)^{n_j} - (1-\phi_k)^{m_j+1}}{\phi_k (m_j - n_j + 1)} \right) R_{j,y}^{pre,10\text{–}1} \quad \forall t \in y \\
&= a_{j,k} R_{j,y}^{pre,10\text{–}1}
\end{aligned}$$

where $a_{j,k}$ is defined in equation (95).

Next, we assume that, since these are relatively well diversified portfolios, the residual risk (σ_ϵ^2) is zero and further assume that all of the systematic risk comes from the three factors. These two assumptions imply that the covariance matrix for the time $t+1$ returns of the three 10–1 portfolios, which we denote $\mathbf{\Sigma}_t$, is given by:

$$\mathbf{\Sigma}_t = \boldsymbol{\beta}_t \Omega_t \boldsymbol{\beta}_t^\top$$

where

$$\boldsymbol{\beta}_t = \begin{bmatrix} \beta^{10\text{–}1}_{str,1,t} & \beta^{10\text{–}1}_{str,2,t} & \beta^{10\text{–}1}_{str,3,t} \\ \beta^{10\text{–}1}_{mom,1,t} & \beta^{10\text{–}1}_{mom,2,t} & \beta^{10\text{–}1}_{mom,3,t} \\ \beta^{10\text{–}1}_{ltr,1,t} & \beta^{10\text{–}1}_{ltr,2,t} & \beta^{10\text{–}1}_{ltr,3,t} \end{bmatrix}$$

Note that this system is just identified, and Ω is given by:

$$\Omega = \left(\boldsymbol{\beta}_t^\top \boldsymbol{\beta}_t\right)^{-1} \boldsymbol{\beta}_t^\top \boldsymbol{\Sigma}_t \boldsymbol{\beta}_t \left(\boldsymbol{\beta}_t^\top \boldsymbol{\beta}_t\right)^{-1}$$

We can estimate this either using the full sample covariance and the average pre-formation returns, or year-by-year and averaging the results.

Over the full-sample the average daily volatility of the daily $10-1$ portfolio returns are (in basis points):

$$\begin{bmatrix} \sigma_{str} \\ \sigma_{mom} \\ \sigma_{lhr} \end{bmatrix} = \begin{bmatrix} 28.464 \\ 37.817 \\ 30.367 \end{bmatrix}$$

and the correlation matrix of the returns is:

$$\begin{bmatrix} 1 & 0.250744 & 0.087098 \\ 0.250744 & 1 & 0.333539 \\ 0.087098 & 0.333539 & 1 \end{bmatrix}$$

The factor loading matrix for these three portfolios is:

$$\mathbf{B} = \begin{bmatrix} 0.007291874 & 0.2927041 & 0.3146322 \\ 1.974574 \times 10^{-05} & 0.6481128 & 1.0529 \\ 1.061207 \times 10^{-28} & 0.2732635 & 2.100848 \end{bmatrix} \tag{96}$$

giving an estimated $\hat{\Omega}$ of:

$$\hat{\Omega} = \begin{bmatrix} 0.1655572 & -0.001041718 & 0.000119914 \\ -0.001041718 & 4.898553 \times 10^{-05} & -7.10805 \times 10^{-06} \\ 0.000119914 & -7.10805 \times 10^{-06} & 3.109768 \times 10^{-06} \end{bmatrix}$$

Or, decomposing this, the (daily) factor volatilities are:[29]

$$\hat{\sigma}_f = \begin{bmatrix} 0.4068872 \\ 0.0069990 \\ 0.0017635 \end{bmatrix}$$

and the correlation matrix of the factors is estimated to be:

$$\hat{\rho} = \begin{bmatrix} 1 & -0.3657987 & 0.1671214 \\ -0.3657987 & 1 & -0.5759073 \\ 0.1671214 & -0.5759073 & 1 \end{bmatrix}$$

[29]Note that the first factor has a large volatility (40%/day). This is a result of the way that we define the factor loadings in equation (40), where a firm's factor loading is an exponentially weighted sum of past residual returns. When ϕ^k is large, as it is for $k = 1$, the dispersion in factor loadings across firms in the economy will be small. This is apparent in equation (96). Thus, a large factor volaility is required to explain the the volatility of the long-short str volatility of only 28 bp/days.