# Univariate Versus Bivariate Strong Independence

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**Abstract**

As noted by Samuelson in his introduction of the Strong Independence axiom, essentially the same set of axioms rationalize an Expected Utility representation of preferences over lotteries with (i) a scalar payoff such as money and (ii) vector payoffs such as quantities of different commodities. Assume a two-good setting, where an individual's preferences satisfy the Strong Independence axiom for lotteries paying off quantities of each good separately. This paper identifies the incremental axioms required for the preference relation over lotteries paying off the vector of goods to also satisfy the Strong Independence axiom. The key element of this extension is a Coherence axiom which requires a particular "meshing together" of certainty preferences over commodity bundles and preferences over non-degenerate lotteries for individual goods. The Coherence axiom is shown to have interesting theoretical implications for Allais paradox-like behavior when confronting lotteries over multiple goods.

KEYWORDS. Expected Utility, Coherence axiom, Strong Independence axiom, Allais Paradox.

JEL CLASSIFICATION. D01, D11, D80.

# 1 Introduction

It is well known that essentially the same set of axioms rationalize an Expected Utility representation of preferences over lotteries with (i) a scalar payoff such as money and (ii) vector payoffs such as quantities of different commodities. This parallel was first noted in Samuelson (1952) which appeared in a collection of papers in Econometrica discussing the introduction of Samuelson's Strong Independence axiom.[1] The specific context of the multivariate application was an example due to Wold (in Wold, Shackle and Savage 1952) of lottery tickets which give the holder (weekly) quantities of milk and wine consumption. Today more than sixty years later, multivariate applications of Expected Utility in Economics are widespread.

In this paper, I derive the additional axioms which are necessary and sufficient to go from Strong Independence holding on lotteries involving single goods, such as wine or milk, to lotteries paying off multiple goods. To simplify the analysis, it is assumed that there are only two goods. Given the consistent evidence from a large number of laboratory tests over many years challenging the univariate Strong Independence axiom, it would seem quite important in assessing the behavior of individuals when facing multivariate lotteries to differentiate between violations of Strong Independence for individual goods and violations of the incremental axioms. It might, for instance, be the case that the incremental axioms are consistent with laboratory tests and could play a useful role in extending univariate non-Expected Utility to multivariate settings.

One important special case of extending univariate Expected Utility to bivariate choice problems is addressed by Rossman and Selden (1978), where preferences are defined over certain first period consumption and random second period consumption. In their setting because the choice space is not a mixture space, the standard multivariate Expected Utility axioms could not be applied. However in this paper, the quantities of both goods are random and the space of joint distribution functions (corresponding to the set of lotteries) is a mixture space. It is assumed that the risk preferences for each of the two goods conditional on certain quantities of the other good satisfy the Strong Independence axiom. While this is necessary for preferences over lotteries paying off both goods to satisfy the Strong Independence axiom, it is not sufficient. One must add a "Coherence"

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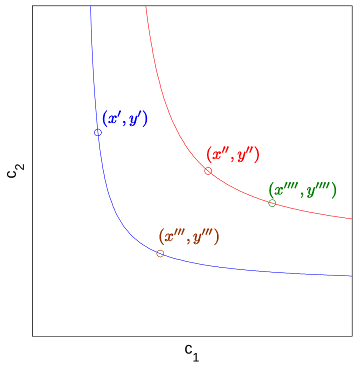
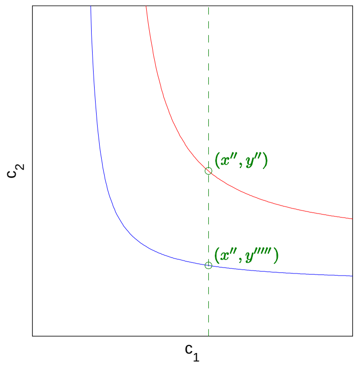
[1] Samuelson (1952) notes that in the von Neumann-Morgenstern (1944) treatment of behavior under uncertainty, no explicit axiom corresponds to his Strong Independence axiom (also see Malinvaud 1952). The first version of this axiom was introduced in Samuelson (1950).




(a) Indifferent lotteries $L_1$ with payoffs $(x', y')$ and $(x'''', y'''')$ and $L_2$ with payoffs $(x''', y''')$ and $(x'', y'')$ before transfer




(b) Indifferent lottery $L_3$ with payoffs $(x'', y''''')$ and $(x'', y'')$ after transfer

Figure 1: Intuition for the Coherence Axiom

axiom, which is illustrated in Figure 1. Let $c_1$ and $c_2$ correspond to quantities of two commodities such as wine and milk and denote by $x'$, $x''$, $x'''$ and $x''''$ different quantities of the first good and $y'$, $y''$, $y'''$ and $y''''$ different quantities of the second good. Assume lottery $L_1$ with payoffs of $(x', y')$ and $(x'''', y'''')$ and lottery $L_2$ with payoffs $(x''', y''')$ and $(x'', y'')$. Coherence requires that if the certain pairs $(x', y')$ and $(x''', y''')$ are indifferent and $(x'', y'')$ and $(x'''', y'''')$ are also indifferent and the payoffs of lotteries $L_1$ and $L_2$ have the same probabilities, then $L_1$ and $L_2$ must be indifferent. If Coherence holds, then both lotteries are indifferent to the same new lottery on a common vertical such as the one passing through the fixed value $x''$ of good one in Figure 1(b). This new lottery $L_3$ has payoffs for the second good of $y''$ and $y'''''$. If risk preferences for lotteries on the vertical conditional on the quantity $x''$ of the first good satisfy the classic univariate Expected Utility axioms, then the initial pair of lotteries $L_1$ and $L_2$ can be thought of as having their payoffs shifted along the certainty indifference curves and then compared using the Expected Utility function conditional on $x''$. Given that this process of transferring lotteries to a common vertical involves using certainty indifference curves, the Coherence axiom can be viewed as requiring a "meshing together" of preferences over risky lotteries on a single vertical, such as $L_3$ in Figure 1(b), and preferences over certain consumption pairs.[2]

[2] In their analysis of comparative risk aversion, Kihstrom and Mirman (1974) observe that in

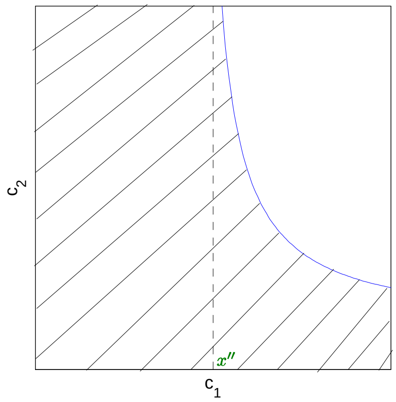



Figure 2: Distinguishing regions of indifference curves intersecting and not intersecting the vertical at x"

The key for the above transfer process to work is that the payoffs of each lottery being compared must lie on a certainty indifference curve that intersects the $x''$-vertical. However suppose one assumes the classic CES (constant elasticity of substitution) utility function characterized by an indifference curve asymptotic to the $x''$-vertical as in Figure 2.[3] Then it is possible that a set of indifference curves in the unshaded region never intersect the vertical corresponding to $x''$. In this case, a comparison of lotteries with payoffs in the unshaded and shaded regions of Figure 2 would not seem possible. However suppose preferences over lotteries for one good conditional on every certain quantity of the other good satisfy the Strong Independence axiom and Coherence holds. Then it is possible to "knit" together the various regions of the consumption space where Strong Independence holds locally and show that Strong Independence holds for the full space of lotteries paying off both goods.

Although Samuelson blew "hot and cold" on the reasonableness of the Strong Independence axiom, he ended up arguing for "independence" based on the fact that lottery tickets must come up "heads or tails" – if one side of the coin comes up, the other cannot. He reasoned that if two lottery tickets, or probability distributions, are indifferent, there is no reason why the combination of the first with a third distribution should contaminate the choice of the second with the same third ticket. Some will no doubt similarly find the Coherence axiom to have an intuitive

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a multivariate setting where the Expected Utility axioms hold, the resulting NM (von Neumann-Morgenstern) index will be a monotonic transform of the representation of certainty preferences. The Coherence axiom is required for the two-argument cardinal utility of the Expected Utility representation and the certainty utility to be ordinally equivalent.

[3]The CES utility function assumed in Figure 2 is $U(c_1, c_2) = -\frac{c_1^{-\delta}}{\delta} - \frac{c_2^{-\delta}}{\delta}$,where $\delta > 0$.

appeal. However suppose an individual's preferences over lotteries on one good conditional on a single value of a second good exhibit Allais paradox-like behavior (Allais 1953, 2008). Then acceptance of the Coherence axiom implies that the individual will also exhibit Allais paradox-like behavior for a specific set of lottery tickets corresponding to every other value of the certain good. Moreover, the individual will also exhibit this behavior for a set of lotteries paying off different quantities of both goods. Several examples are provided illustrating that some but not all of the lotteries implied by Coherence have a similar intuitive appeal as the original Allais set of lotteries. These very simple examples seem to reinforce the point raised above concerning the potential importance of pursuing laboratory tests of both Coherence and univariate Strong Independence. It is interesting to observe that despite the vast literature covering conceptual discussions and laboratory tests associated with the original Allais example,[4] no comparable analysis of choices over lottery tickets with vector payoffs seems to have been undertaken.

In the next section I introduce notation and a formal definition of Coherence. Several examples are introduced to illustrate some of the issues in extending the Strong Independence axiom from a single good to multiple goods. In Section 3, the Coherence axiom is shown to be necessary and sufficient to extend an Expected Utility function defined over lotteries for a single good to a bivariate Expected Utility representation of preferences over the full space of lottery tickets for both goods. Section 4 discusses when an Expected Utility representation in a region of the choice space can be extended to the full space. Section 5 considers the implications of Coherence given the existence of Allais paradox behavior. The last section offers concluding comments.

# 2 Coherence

## 2.1 Preliminaries

Let $c_1 \in C_1$ and $c_2 \in C_2$ denote the quantities of two commodities, where unless stated otherwise $C_1 = C_2 = (0, \infty)$. Define $C = C_1 \times C_2$. It will prove useful to introduce the subsets $C[x] =_{def} \{x\} \times C_2$ where $x \in C_1$ and $C[y] =_{def} C_1 \times \{y\}$ where $y \in C_2$ (see Figure 3). $C[x]$ and $C[y]$, respectively, will be referred to as a "vertical" and "horizontal" in the commodity space $C$. Let $\mathcal{F}_1$ denote the set of c.d.f.s (cumulative distribution functions) defined on $C_1$ and $F_1$ be an element in $\mathcal{F}_1$. Similarly, $\mathcal{F}_2$ is the set of c.d.f.s on $C_2$ and $F_2$ is an element in $\mathcal{F}_2$.

[4] See, for instance, Starmer (2000) and Andreoni and Sprenger (2012) and the references cited therein.

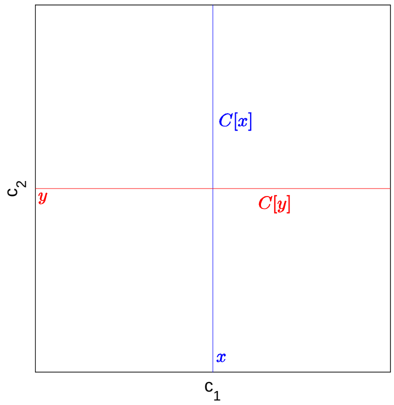


Figure 3: Illustration of the vertical C[x] and horizontal C[y]

The distributions can be viewed as corresponding to lotteries paying off different consumption quantities. The degenerate or one point c.d.f. with saltus or jump point at $c_1 \in C_1$ is denoted $F_1^*(c_1) \in \mathcal{F}_1$ and similarly for $F_2^*(c_2) \in \mathcal{F}_2$. Finally let $\mathcal{J}$ denote the set of joint cumulative distribution functions defined on $C_1 \times C_2$ and $J$ is an element in $\mathcal{J}$. Let $J^*(c_1, c_2) \in \mathcal{J}$ be a degenerate joint distribution with saltus point at $(c_1, c_2) \in C$.[5]

I consider the following three types of preference structures:[6]

1. *Certainty preferences* over consumption pairs in $C_1 \times C_2$ described by the binary relation $\preceq^C$;

2. *Conditional risk preferences* (i) over $\mathcal{F}_2$ and conditioned on each quantity $x \in C_1$ described by the set of binary relations $\{\preceq_x^{\mathcal{F}_2} \mid x \in C_1\}$ and (ii) over $\mathcal{F}_1$ and conditioned on each quantity $y \in C_2$ of the second good described by the set of binary relations $\{\preceq_y^{\mathcal{F}_1} \mid y \in C_2\}$; and

[5] It should be stressed that in this paper, the setting is static. The stochastic structure is very different from the intertemporal (consumption tree) case where the consumer faces random quantities of consumption in two future time periods and the uncertainty is resolved sequentially with the possibility of making choices after the resolution of the first period's outcome.

[6] As is standard in the axiomatic treatment of the Expected Utility hypothesis (see, for example, Machina 2008 and Mas-Colell, Whinston and Green 1995), preferences are defined over alternative possible outcomes or lotteries over these outcomes. In my analysis, as in Samuelson (1952), the outcomes of the lotteries are units of commodity bundles of the form $(c_1, c_2)$. Since the focus of this paper is on the standard representation results, the important distinction between consumption and change in consumption is ignored. For a discussion of the important calibration issues associated with Expected Utility functions defined on absolute consumption or wealth levels and for a simple model for addressing these problems see Bowman, Minehart and Rabin (1999).

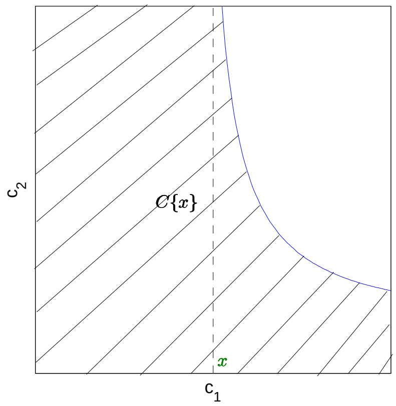


Figure 4: Illustration of the region C{x} where indifference curves intersect the vertical C[x]

3. *Bivariate preferences* over joint distributions in $\mathcal{J}$ described by the binary relation $\preceq^{\mathcal{J}}$ .

It should be stressed that each conditional preference relation can only be used to compare lotteries paying off different quantities of one good with the quantity of the other good being fixed. For instance, they do not describe choices such as between $(x', F_2)$ and $(x'', G_2)$ where $x', x'' \in C_1$, $x' \neq x''$ and $F_2, G_2 \in \mathcal{F}_2$.

It will also be useful to define the set of certain consumption pairs which are indifferent to some pair on a given vertical or horizontal

$$C\{x\} =_{def} \left\{(c_1, c_2) \in C \mid (c_1, c_2) \sim^C (x, c_2'),\ \text{where } (x, c_2') \in C[x]\right\} \tag{1}$$

and

$$C\{y\} =_{def} \left\{(c_1, c_2) \in C \mid (c_1, c_2) \sim^C (c_1', y),\ \text{where } (c_1', y) \in C[y]\right\}. \tag{2}$$

In Figure 4, which assumes the same CES utility as in Figure 2, the subset $C\{x\}$ of $C$ corresponds to the shaded area below a boundary indifference curve which is asymptotic to the vertical $C[x]$. Every point in the shaded area lies on some indifference curve intersecting $C[x]$ and every point northeast of the boundary indifference curve lies on a curve not intersecting the vertical. (See Figure 7 and the related discussion in Subsection 3.1 for the more general case.) It will prove convenient to denote the support of $J$ as *supp* $J$ and define the following subsets of $\mathcal{J}$

$$\mathcal{S}[x] =_{def} \{J \in \mathcal{J} \mid \textit{supp } J \subset C[x],\ \text{where } x \in C_1\}, \tag{3}$$

$$\mathcal{S}[y] =_{def} \{J \in \mathcal{J} \mid supp\ J \subset C[y],\ \text{where } y \in C_2\}, \tag{4}$$

$$\mathcal{S}\{x\} =_{def} \{J \in \mathcal{J} \mid supp\ J \subset C\{x\},\ \text{where } x \in C_1\} \tag{5}$$

and

$$\mathcal{S}\{y\} =_{def} \{J \in \mathcal{J} \mid supp\ J \subset C\{y\},\ \text{where } y \in C_2\}. \tag{6}$$

In terms of Figure 3, the set $\mathcal{S}[x]$ includes each joint distribution function characterized by each of its saltus points lying on the vertical $C[x]$. The set of joint distribution functions $\mathcal{S}\{x\}$ can be illustrated in terms of Figure 4, where any c.d.f. in $\mathcal{S}\{x\}$ is characterized by its support lying in the shaded region $C\{x\}$.

Throughout, the following are assumed to hold.

**Assumption 1** *The set of joint cumulative distribution functions $\mathcal{J}$ corresponds to some (topological) subspace of the space of all (countably additive) joint probability measures $M(C_1 \times C_2)$,[7] which is endowed with the topology of weak convergence. The set of one-point joint c.d.f.s (supported by the domain of $\mathcal{J}$), $\mathcal{J}^*$, is a subset of $\mathcal{J}$.*

**Assumption 2** *There exists a complete preference preordering on $\mathcal{J}$, $\preceq^{\mathcal{J}}$.*

**Assumption 3** $\preceq^{\mathcal{J}}$ *is representable by a continuous "Bernoulli index" $\Psi : \mathcal{J} \to \mathbb{R}$.*[8,9]

**Assumption 4** $\forall \mathbf{c}, \mathbf{c}' \in C,\ \mathbf{c} \preceq^{C} \mathbf{c}' \Longleftrightarrow J^*(\mathbf{c}) \preceq^{\mathcal{J}} J^*(\mathbf{c}')$.

**Assumption 5** *(Monotonicity)* $\forall (c_1, c_2), (c_1, c_2') \in C,\ J^*(c_1, c_2) \preceq^{\mathcal{J}} \left(\prec^{\mathcal{J}}\right) J^*(c_1, c_2') \Leftrightarrow c_2 \leq (<)\, c_2'$ *and* $\forall (c_1, c_2), (c_1', c_2) \in C,\ J^*(c_1, c_2) \preceq^{\mathcal{J}} \left(\prec^{\mathcal{J}}\right) J^*(c_1', c_2) \Leftrightarrow c_1 \leq (<)\, c_1'$.

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[7] $M(C_1 \times C_2)$ is defined on the measurable space $(C_1 \times C_2, B(C_1 \times C_2))$ where $C_1 \times C_2$ is clearly a metric space and $B(C_1 \times C_2)$ is its Borel $\sigma$-field.

[8] I follow Grandmont (1972) in using the term "Bernoulli index" to refer to any real-valued order-preserving representation $\Psi$. If this representation takes the very special Expected Utility form, I refer to the utility defined on the commodities as the NM index. It should be noted that Machina (2008) uses similar terminology referring to the representations on distributions as $\Psi$ and the utility defined on payoffs as the NM utility. When discussing only Expected Utility preferences, it is more common to refer to the utility defined on distributions as the NM utility and refer to the utility defined on the payoffs as the Bernoulli utility (see Mas-Colell, Whinston and Green 1995).

[9] Instead of assuming $\Psi$, one could prove its existence by placing topological restrictions on $\mathcal{J}$ and conditions on $\preceq^{\mathcal{J}}$ following Theorem 1 in Grandmont (1972).

Given Assumption 4, define the natural embedding $\iota : C_1 \times C_2 \to \mathcal{J}^*$ such that

$$\iota\left(c_1, c_2\right) = J^*(c_1, c_2), \quad \forall (c_1, c_2) \in C_1 \times C_2, \tag{7}$$

where $J^*(c_1, c_2)$ has a saltus point at $(c_1, c_2) \in C$. With the above assumptions, we have the following Lemma.

**Lemma 1** *Assumptions 1-5 hold. The certainty preferences $\preceq^C$ over $C_1 \times C_2$ can be represented by a continuous, strictly increasing ordinal index $U : C_1 \times C_2 \to \mathbb{R}$ with $(c_1, c_2) \preceq^C (c_1', c_2')$ iff $U(c_1, c_2) \leq U(c_1', c_2')$.*

**Proof.** Defining

$$U\left(c_1, c_2\right) = \Psi\left(\iota\left(c_1, c_2\right)\right) = \Psi\left(J^*(c_1, c_2)\right), \tag{8}$$

it can be seen that $U : C_1 \times C_2 \to \mathbb{R}$ with $(c_1, c_2) \preceq^C (c_1', c_2')$ iff $U(c_1, c_2) \leq U(c_1', c_2')$. Next we argue that $U$ is continuous. Note that $U = \Psi \circ \iota$ will be continuous if $\Psi$ and $\iota$ are. Since $\Psi$ is continuous, we need only establish the continuity of $\iota$. But given that $\mathcal{J}$ is endowed with the weak topology, it is clear that $\iota$ will be continuous. Finally, Assumption 5 clearly implies that $U$ is strictly increasing in each of its argument. ■

Following Samuelson (1952), the univariate Strong Independence axiom is defined as follows.

**Definition 1** *For a given $x \in C_1$, the conditional risk preference relation $\preceq_x^{\mathcal{F}_2}$ satisfies the Strong Independence axiom iff $\forall F_2, G_2, H_2, I_2 \in \mathcal{F}_2$ and $\forall \alpha \in [0, 1]$*

$$F_2 \sim_x^{\mathcal{F}_2} (\succ_x^{\mathcal{F}_2}) G_2,\ H_2 \sim_x^{\mathcal{F}_2} (\succ_x^{\mathcal{F}_2}) I_2 \Rightarrow \alpha_1 F_2 + (1-\alpha) H_2 \sim_x^{\mathcal{F}_2} (\succ_x^{\mathcal{F}_2}) \alpha G_2 + (1-\alpha) I_2. \tag{9}$$

Then we can make the following assumption.

**Assumption 6** *For a given $x \in C_1$, the conditional risk preference relation $\preceq_x^{\mathcal{F}_2}$ satisfies the Strong Independence axiom.*

It should be noted that Assumption 6 imposes no restrictions on the relationship between the elements of $\{\preceq_x^{\mathcal{F}_2}\}$ or of $\{V_x\}$. The Lemma below directly follows from Grandmont (1972).

**Lemma 2** *Assumptions 1-6 hold. For a given $x \in C_1$, the conditional risk preferences $\preceq_x^{\mathcal{F}_2}$ are representable according to the Expected Utility principle where*

*$V_x : C_2 \to \mathbb{R}$ is a continuous, strictly increasing NM (von Neumann-Morgenstern) index such that, for all $F_2, G_2 \in \mathcal{F}_2$, $F_2 \preceq_x^{\mathcal{F}_2} G_2$ iff* [10]

$$\int_{C_2} V_x(c_2)\, dF_2(c_2) \leq \int_{C_2} V_x(c_2)\, dG_2(c_2). \tag{10}$$

While Definition 1, Assumption 6 and Lemma 2 are based on lotteries with payoffs on a given vertical $C[x]$, analogous statements can be made for lotteries with payoffs on a horizontal $C[y]$.

Finally, I define the bivariate Strong Independence axiom on all of $\mathcal{J}$.

**Definition 2** *The preference relation $\preceq^{\mathcal{J}}$ satisfies the Strong Independence axiom iff $\forall J_1, J_2, J_3, J_4 \in \mathcal{J}$ and $\forall \alpha \in [0,1]$*

$$J_1 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) J_2,\ J_3 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) J_4 \Rightarrow \alpha J_1 + (1-\alpha) J_3 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) \alpha J_2 + (1-\alpha) J_4. \tag{11}$$

Since each lottery corresponds to a joint probability measure which is equivalent to a joint c.d.f., for notational simplicity lotteries will be referred to as elements in $\mathcal{J}$.

Following from Grandmont (1972), we have Lemma 3.

**Lemma 3** *Assumptions 1-5 hold and $\preceq^{\mathcal{J}}$ satisfies the Strong Independence axiom. Then $\preceq^{\mathcal{J}}$ is representable according to the Expected Utility principle where $W : C_1 \times C_2 \to \mathbb{R}$ is a continuous, strictly increasing NM index such that, for all $J_1, J_2 \in \mathcal{J}$, $J_1 \preceq^{\mathcal{J}} J_2$ iff* [11]

$$\int_{C_1} \int_{C_2} W(c_1, c_2)\, dJ_1(c_1, c_2) \leq \int_{C_1} \int_{C_2} W(c_1, c_2)\, dJ_2(c_1, c_2). \tag{12}$$

It should be stressed that rather than assuming Strong Independence holds on all of $\mathcal{J}$, I seek to identify what in addition to Assumptions 1-5 and the Strong Independence Assumption 6 is necessary and sufficient for Strong Independence to hold on all of $\mathcal{J}$ and for $\preceq^{\mathcal{J}}$ to be representable by an Expected Utility function as in Lemma 3.

[10] Note that in both Lemma 2 and Lemma 3, the NM index is defined up to a positive affine transformation as is well-known.

[11] For any $J \in \mathcal{J}$, $dJ$ is defined by

$$dJ(c_1, c_2) =_{def} \frac{\partial^2 J(c_1, c_2)}{\partial c_1 \partial c_2} dc_1 dc_2.$$

## 2.2 Motivating Example

Consider the following simple example which illustrates that assuming Strong Independence holds for lotteries with payoffs on each vertical and on each horizontal does not guarantee that Strong Independence holds for lotteries with payoffs corresponding to c.d.f.s in all of $\mathcal{J}$.

**Example 1** *Assume the bivariate preferences* $\preceq^{\mathcal{J}}$ *are represented by*

$$\Psi(J) = \int_{C_1} V_1(c_1)\, dF_1(c_1) \int_{C_2} V_2(c_2)\, dF_2(c_2), \tag{13}$$

*where* $V_i > 0$, $V_i' > 0$ *and* $V_i'' < 0$ $(i = 1, 2)$[12] *and* $F_1$ *and* $F_2$ *are defined respectively by*

$$F_1(x) = \int_0^x \int_{C_2} dJ(c_1, c_2) \tag{14}$$

*and*

$$F_2(y) = \int_0^y \int_{C_1} dJ(c_1, c_2). \tag{15}$$

*In this case,* $\preceq^{\mathcal{J}}$ *induces the certainty preference relation* $\preceq^{C}$ *which is representable by the strictly quasiconcave utility*

$$U(c_1, c_2) = V_1(c_1)\, V_2(c_2). \tag{16}$$

*It can be easily verified that* $\forall x \in C_1$*, if* $J \in \mathcal{S}[x]$*, then*

$$\Psi(J) = \int_{C_2} V_1(x)\, V_2(c_2)\, dF_2(c_2), \tag{17}$$

*and similarly,* $\forall y \in C_2$*, if* $J \in \mathcal{S}[y]$*,*

$$\Psi(J) = \int_{C_1} V_1(c_1)\, V_2(y)\, dF_1(c_1). \tag{18}$$

*Thus,* $\Psi(J)$ *takes the Expected Utility form for any lottery with* $J$ *in* $\mathcal{S}[x]$ *or* $\mathcal{S}[y]$*. However to see that the representation in (13) does not in general satisfy Strong Independence over* $\mathcal{J}$ *assume that*

$$V_1(c_1) = \sqrt{c_1} \quad \text{and} \quad V_2(c_2) = \sqrt{c_2}. \tag{19}$$

*Consider the following two lotteries* $L_1 =< (1, 1), (4, 1/4); 50\%, 50\% >$ *and* $L_2 =< (1, 64/25), (9/4, 1/25); 50\%, 50\% >$*. Since*

$$\Psi(L_1) = \left(\frac{1}{2} + 1\right) \times \left(\frac{1}{2} + \frac{1}{4}\right) = \frac{9}{8} \tag{20}$$

[12] It should be noted that in eqn. (13), one has considerable freedom in choosing $V_1$ and $V_2$.

*and*

$$\Psi\left(L_2\right)=\left(\frac{1}{2}+\frac{3}{4}\right)\times\left(\frac{4}{5}+\frac{1}{10}\right)=\frac{9}{8}, \tag{21}$$

*we have* $L_1 \sim^{\mathcal{J}} L_2$. *Now consider the degenerate lottery* $L_3=\left(\frac{1}{4},\frac{1}{4}\right)$. *If Strong Independence holds over all of* $\mathcal{J}$, *then it must be that* $0.5L_1+0.5L_3 \sim^{\mathcal{J}} 0.5L_2+0.5L_3$. *Noticing that*

$$0.5L_1+0.5L_3=<\left(\frac{1}{4},\frac{1}{4}\right),(1,1),(4,\frac{1}{4});50\%,25\%,25\%> \tag{22}$$

*and*

$$0.5L_2+0.5L_3=<\left(\frac{1}{4},\frac{1}{4}\right),(1,\frac{64}{25}),(\frac{9}{4},\frac{1}{25});50\%,25\%,25\%>, \tag{23}$$

*we have*

$$\Psi\left(0.5L_1+0.5L_3\right)=\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right)\times\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{8}\right)=\frac{5}{8} \tag{24}$$

*and*

$$\Psi\left(0.5L_2+0.5L_3\right)=\left(\frac{1}{4}+\frac{1}{4}+\frac{3}{8}\right)\times\left(\frac{1}{4}+\frac{2}{5}+\frac{1}{20}\right)=\frac{49}{80}, \tag{25}$$

*implying that* $0.5L_1+0.5L_3 \succ^{\mathcal{J}} 0.5L_2+0.5L_3$ *and hence the Strong Independence axiom does not hold over all of* $\mathcal{J}$.

**Remark 1** *It is natural to wonder whether the utility function (13) satisfies first order stochastic dominance. It follows from Crawford (2005) that bivariate first order stochastic dominance implies first order dominance in the marginal distributions, i.e.,* $\forall J_1,\ J_2\in\mathcal{J}$, *if* $J_1$ *dominates* $J_2$, *then*

$$\forall x\in C_1,\quad \int_0^x\int_{C_2} dJ_1\left(c_1,c_2\right)\le\int_0^x\int_{C_2} dJ_2\left(c_1,c_2\right) \tag{26}$$

*and*

$$\forall y\in C_2,\quad \int_0^y\int_{C_1} dJ_1\left(c_1,c_2\right)\le\int_0^y\int_{C_1} dJ_2\left(c_1,c_2\right), \tag{27}$$

*implying that* $\Psi\left(J_1\right)\ge\Psi\left(J_2\right)$ *in Example 1. Therefore, the utility function (13) satisfies first order stochastic dominance.*[13]

## 2.3 Transfer Maps

Before formally defining the Coherence axiom, it will prove useful to introduce the notion of a transfer map. Assume without loss of generality one confronts a lottery with the payoffs $(x',y')$ and $(x'',y'')$ as in Figure 5, where the associated joint

[13]In addition to Crawford (2005), a detailed discussion of first order stochastic dominance in the multivariate case can be found in Levhari, Paroush and Peleg (1975).

distribution is denoted by $J$.[14] Further suppose that conditional risk preferences are defined on the vertical $C[x]$ and both payoffs are in $C\{x\}$. Define the vertical transfer $\gamma_x : C\{x\} \rightarrow C[x]$ by

$$\gamma_x = U_x^{-1} U(c_1, c_2), \tag{28}$$

which maps the two points $(x', y')$ and $(x'', y'')$ respectively into $(x, y''')$ and $(x, y'''')$ on the same vertical $C[x]$ by the "sliding the points along the certainty indifference curves".[15] Define the vertical "induced" transfer $\gamma_x : \mathcal{S}\{x\} \rightarrow \mathcal{S}[x]$ which is also given by (28) (the same symbol is used for both mappings). A new certain quantity of the first good and c.d.f. for the second good $(x, F_2) \in \{x\} \times \mathcal{F}_2$ is obtained by (1) finding the two jump points $\{(x, y'''), (x, y'''')\}$ on $C[x]$ by applying the the transfer $\gamma_x$ to $(x', y')$ and $(x'', y'')$ where $(x', y')$ and $(x, y''')$ lie on the same indifference curve and $(x'', y'')$ and $(x, y'''')$ lie on the same indifference curve and (2) assuming the same "probability structure" as $J$, i.e., $J(x', y') = F_2(y''')$ and $J(x'', y'') = F_2(y'''')$.[16]

**Definition 3** *(Vertical Induced Transfer Mapping) The vertical transfer mapping $\gamma_x : C\{x\} \rightarrow C[x]$ is characterized by the relation $c \sim^C \gamma_x c$ for each $c \in C\{x\}$. The vertical induced transfer mapping $\gamma_x : \mathcal{S}\{x\} \rightarrow \mathcal{S}[x]$ associates a joint distribution $J$ to the pair $(x, F_2)$ where $\forall\, (c_1, c_2) \in C\{x\}$, $J(c_1, c_2) = F_2(y)$ if $\gamma_x(c_1, c_2) = (x, y)$.*

Whereas the induced transfer defined in Definition 3 always maps a lottery into a specific vertical, it is also possible to define a map which transfers any joint distribution to distributions on a set of "diagonal" linear rays which are neither vertical nor horizontal.

Suppose that the Strong Independence axiom holds for all lotteries on diagonal rays and on all verticals and horizontals, is this sufficient for Strong Independence to hold on all of $\mathcal{J}$? A concrete example is provided next which demonstrates that this is not the case. First it will prove convenient to give the following definitions. For any $a, b \in \mathbb{R}$,

$$\mathcal{S}[x = ay + b] = \{J \in \mathcal{J} \mid \text{support of } J \in C[x = ay + b]\}, \tag{29}$$

[14] This discussion can readily be extended to lotteries with more than two payoffs.

[15] The analysis of a transfer map for the case of the horizontal $C[y]$ is analogous to that of the vertical $C[x]$. Since discussing them simultaneously is notationally cumbersome, throughout the rest of the paper only the vertical case is considered.

[16] The vertical induced transfer map eqn. (28) generalizes the map $\gamma_{x'x} = U_x^{-1} U_{x'}(c_2)$ introduced in Rossman and Selden (1978), where the latter requires that the support of the distribution before the transfer be in the single vertical $C[x']$ for the fixed $x'$.

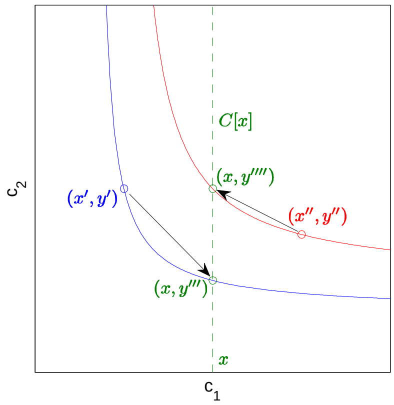



Figure 5: Using the vertical induced transfer to map the lottery with payoffs (x',y') and (x",y") to the indifferent lottery on the C[x] vertical

where

$$C[x = ay{+}b] = \{(c_1, c_2) \in C_1 \times C_2 \mid (c_1, c_2) = (ay + b, y), (ay + b, y) \in C_1 \times C_2\}. \tag{30}$$

**Example 2** *Assume the bivariate preferences* $\preceq^{\mathcal{J}}$ *are represented by*

$$\Psi(J) = \sqrt{\int_{C_1} \sqrt{c_1} dF_1(c_1) \int_{C_2} \sqrt{c_2} dF_2(c_2)}, \tag{31}$$

*which will be recognized to be a special case of the representation (13) in Example 1.* $\preceq^{\mathcal{J}}$ *induces the certainty preference relation* $\preceq^{C}$ *which is representable by*

$$U(c_1, c_2) = (c_1 c_2)^{\frac{1}{4}}. \tag{32}$$

*Then* $\forall x \in C_1$*, if* $J \in \mathcal{S}[x]$*,*

$$\Psi(J) = \sqrt{\int_{C_2} \sqrt{c_1 c_2} dF_2(c_2)}, \tag{33}$$

*which is ordinally equivalent to*

$$\Psi(J) = \int_{C_2} \sqrt{c_1 c_2} dF_2(c_2) \tag{34}$$

*and similarly* $\forall y \in C_2$*, if* $J \in \mathcal{S}[y]$*,*

$$\Psi(J) = \sqrt{\int_{C_1} \sqrt{c_1 c_2} dF_1(c_1)}, \tag{35}$$

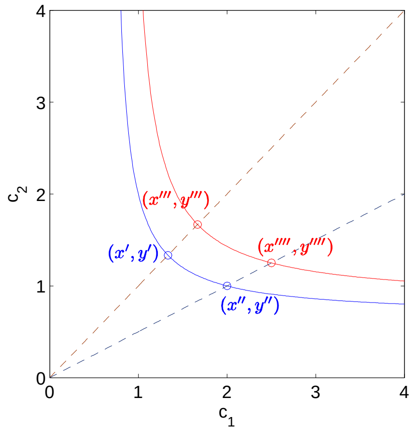



Figure 6: Indifferent lotteries along two diagonals

*which is ordinally equivalent to*

$$\Psi(J) = \int_{C_1} \sqrt{c_1 c_2} dF_1(c_1). \tag{36}$$

*Thus for any* $J$ *in* $\mathcal{S}[x]$ *or* $\mathcal{S}[y]$, $\Psi(J)$ *takes the Expected Utility form and hence for both cases the univariate Strong Independence axiom holds. Conditional on all rays going through the origin, i.e.,* $J \in \mathcal{S}[x = ay]$ $(a > 0)$, *(see Figure 6) we have*

$$\Psi(J) = \sqrt{\int_{C_1} \sqrt{ac_2} dF_2(c_2) \int_{C_2} \sqrt{c_2} dF_2(c_2)} = \sqrt{a} \int_{C_2} \sqrt{c_2} dF_2(c_2), \tag{37}$$

*which is also an Expected Utility representation and hence the Strong Independence axiom holds. Thus in terms of Figure 6, the relative preference for the two lotteries* $< (x', y'), (x''', y'''); \pi, 1-\pi >$ *and* $< (x'', y''), (x'''', y''''); \pi, 1-\pi >$ *on the different diagonals can be evaluated using the Expected Utility (37). However, since (31) is a special case of the utility in Example 1, the Strong Independence axiom does not hold over all of* $\mathcal{J}$.

## 2.4 Coherence Axiom

From Example 2, it is apparent that the Strong Independence axiom holding for each vertical, horizontal and set of rays through the origin is not enough to ensure that it holds for all of $\mathcal{J}$. To extend the Strong Independence axiom, the following Coherence property, which is based on utilizing the vertical induced transfer map $\gamma_x$, is key.

**Definition 4** *For any given $x \in C_1$, the preference relation $\preceq^{\mathcal{J}}$ exhibits Coherence over $\mathcal{S}\{x\}$ iff*

$$\forall J_1, J_2 \in \mathcal{S}\{x\}, \; J_1 \sim^{\mathcal{J}} J_2 \Leftrightarrow \gamma_x J_1 \sim^{\mathcal{J}} \gamma_x J_2. \tag{38}$$

**Assumption 7** *For a given $x \in C_1$, the preference relation $\preceq^{\mathcal{J}}$ exhibits Coherence over $\mathcal{S}\{x\}$.*

**Remark 2** *When Coherence holds, the vertical induced transfer map $\gamma_x$ can be given an interesting compensation interpretation. Consider the case in Figure 5. Using certainty preferences $\preceq^{C}$, the change in the consumption values for the first good from $x'$ to $x$ and $x''$ to $x$ is exactly compensated by adjusting the consumption levels of the second good from $y'$ to $y'''$ and $y''$ to $y''''$, where the probability structure is held fixed. In this process the joint distribution is deformed into a pair comprised of certain consumption for one good and a univariate distribution for the second good, $(x, F_2)$. Notice that since $\gamma_x J$ is already on the $x$-vertical, applying $\gamma_x(\gamma_x J)$ will not affect it. Hence*

$$\gamma_x J \sim^{\mathcal{J}} \gamma_x \circ \gamma_x J \qquad \forall J \in \mathcal{S}\{x\}, \tag{39}$$

*and thus it can be easily seen that Coherence is equivalent to the following invariance property*

$$J \sim^{\mathcal{J}} \gamma_x J \qquad \forall J \in \mathcal{S}\{x\}, \tag{40}$$

*which is consistent with the geometry in Figure 5 where the lotteries $< (x', y'), (x'', y'')\,; \pi, 1-\pi >$ and $< (x, y'''), (x, y'''')\,; \pi, 1-\pi >$ are indifferent.*

# 3 Bivariate Expected Utility

In this section assuming an individual satisfies the Strong Independence axiom for lotteries over one good, I identify the additional axiom structure which is both necessary and sufficient for the Strong Independence axiom to hold for lotteries over two goods and for the existence of an Expected Utility representation of preferences over the full space of joint distributions $\mathcal{J}$.

## 3.1 Strong Independence Axiom on $\mathcal{S}\{x\}$

As I next show, the Coherence axiom and the Strong Independence axiom holding on $\{x\} \times \mathcal{F}_2$ together are equivalent to the Strong Independence axiom holding over the subspace $\mathcal{S}\{x\}$.

**Theorem 1** *Assumptions 1-5 hold. The Strong Independence axiom is satisfied over $\mathcal{S}\{x\}$ for a given $x \in C_1$ iff Assumptions 6 and 7 are satisfied.*

**Proof.** First prove necessity. Since the Strong Independence axiom is satisfied on $\mathcal{S}\{x\}$, $\forall J_1, J_2, J_3, J_4 \in \mathcal{S}\{x\}$ and $\forall \alpha \in [0,1]$

$$J_1 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) J_2,\ J_3 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) J_4 \Rightarrow \alpha J_1 + (1-\alpha) J_3 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) \alpha J_2 + (1-\alpha) J_4. \quad (41)$$

Choosing $J_1, J_2, J_3, J_4 \in \mathcal{S}[x]$, it is obvious that the Strong Independence axiom condition (9) in Definition 1 holds. For the Coherence axiom, it follows from Remark 2 that we only need to prove

$$J \sim^{\mathcal{J}} \gamma_x J \qquad \forall J \in \mathcal{S}\{x\}. \quad (42)$$

Without loss of generality, consider a two state lottery. Assume the first state payoff is denoted by $c^{(1)} \in C_1 \times C_2$ and has probability $\pi = \alpha$. The second state payoff is $c^{(2)} \in C_1 \times C_2$ with the probability $\pi = 1 - \alpha$. If we use $J_1^*$ to denote the degenerate c.d.f. with the saltus point at $c^{(1)}$ and $J_2^*$ to denote the degenerate c.d.f. with the saltus point at $c^{(2)}$, then

$$J = \alpha J_1^* + (1-\alpha) J_2^*. \quad (43)$$

Since the general transfer $\gamma_x$ is an affine mapping, i.e., $\gamma_x(aJ_1 + bJ_2) = a\gamma_x J_1 + b\gamma_x J_2$ where $a, b \geq 0$ and $a + b = 1$ (see Rossman and Selden 1978, p. 74), we have

$$\gamma_x J = \alpha \gamma_x J_1^* + (1-\alpha) \gamma_x J_2^*. \quad (44)$$

Since $\gamma_x J_1^* \sim^{\mathcal{J}} J_1^*$ and $\gamma_x J_2^* \sim^{\mathcal{J}} J_2^*$, it follows from eqn. (41) that $J \sim^{\mathcal{J}} \gamma_x J$. Next prove sufficiency. By Assumption 7,

$$J_1 \sim^{\mathcal{J}} J_2 \Leftrightarrow \gamma_x J_1 \sim^{\mathcal{J}} \gamma_x J_2 \text{ and } J_3 \sim^{\mathcal{J}} J_4 \Leftrightarrow \gamma_x J_3 \sim^{\mathcal{J}} \gamma_x J_4. \quad (45)$$

Since $\gamma_x J_1$, $\gamma_x J_2$, $\gamma_x J_3$ and $\gamma_x J_4$ are defined on $\mathcal{S}[x]$, following Assumption 6,

$$\gamma_x J_1 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) \gamma_x J_2 \text{ and } \gamma_x J_3 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) \gamma_x J_4 \quad (46)$$

implies that

$$\alpha \gamma_x J_1 + (1-\alpha) \gamma_x J_3 \sim^{\mathcal{J}} (\succ^{\mathcal{J}}) \alpha \gamma_x J_2 + (1-\alpha) \gamma_x J_4. \quad (47)$$

Since the general transfer $\gamma_x$ is affine mapping, we have

$$\alpha \gamma_x J_1 + (1-\alpha) \gamma_x J_3 \sim^{\mathcal{J}} \gamma_x (\alpha J_1 + (1-\alpha) J_3) \quad (48)$$

and

$$\alpha \gamma_x J_2 + (1-\alpha) \gamma_x J_4 \sim^{\mathcal{J}} \gamma_x (\alpha J_2 + (1-\alpha) J_4). \quad (49)$$

By Assumption 7, $J \sim^{\mathcal{J}} \gamma_x J$ ($\forall J \in \mathcal{S}\{x\}$). Hence combining the above two equations with eqn. (47) immediately yields eqn. (41). ■

**Remark 3** *Comparing Definitions 2 and 4, it is important to observe that for lotteries in $\mathcal{S}\{x\}$ the Coherence axiom follows from an application of the Strong Independence axiom to pairs of indifferent degenerate lotteries not necessarily defined on the same vertical and use of the natural embedding mapping (7). This relationship is used in the necessity proof of Theorem 1. However, as Example 1 demonstrates, Coherence is not implied by the Strong Independence axiom holding on each vertical (Assumption 6) since this imposes no restriction on the certainty preferences $\preceq^C$. Thus Coherence as defined, can be applied to settings where one or both goods in the lottery are random.*

Given that Coherence enables the Strong Independence on a given vertical $C[x]$ to be inherited on $\mathcal{S}\{x\}$, will this be true as well for the full space of joint distributions $\mathcal{J}$? Depending on the form of the certainty $U$, $C\{x\}$ can take one of the four possible forms in Figure 7.[17] It can be seen that for the class of utility functions corresponding to Figure 7(a), $C\{x\} = C$ and hence $\mathcal{S}\{x\} = \mathcal{J}$. For the other cases in Figure 7, it is clear that one cannot apply the transfer map to points in the unshaded regions. Thus it would seem that Coherence cannot be used to ensure that Strong Independence holds for c.d.f.s outside the subset $\mathcal{S}\{x\}$ of $\mathcal{J}$.

## 3.2 Expected Utility Representation Theorem on $\mathcal{J}$

In this subsection, I show what additional axiom structure is required to overcome the limitation of Theorem 1 only holding for $\mathcal{S}\{x\}$ and provide the necessary and sufficient conditions for the existence of an Expected Utility representation for $\preceq^{\mathcal{J}}$

$$\Psi(J) = \int\int_C W(c_1, c_2) dJ(c_1, c_2), \tag{50}$$

as introduced in Lemma 3.

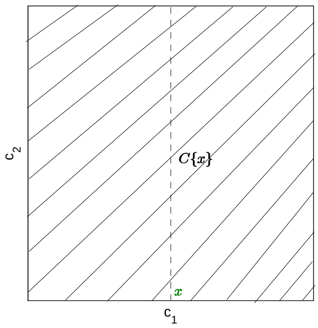
Consider the following modifications of Assumptions 6 and 7.

**Assumption 6′** *For each $x \in C_1$, the conditional risk preference relation $\preceq_x^{\mathcal{F}_2}$ satisfies the Strong Independence axiom.*

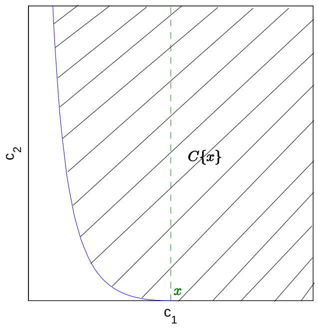
**Assumption 7′** *For each $x \in C_1$, the preference relation $\preceq^{\mathcal{J}}$ exhibits Coherence over $\mathcal{S}\{x\}$.*

**Theorem 2** *Assumptions 1-5 together with Assumptions 6′ and 7′ are necessary and sufficient for the existence of a real-valued continuous, strictly increasing NM*

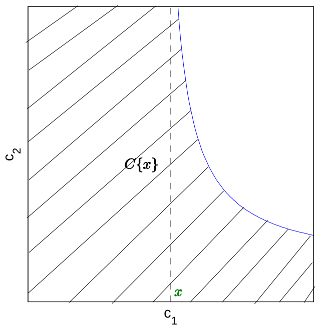
[17] See Rossman and Selden (1978).



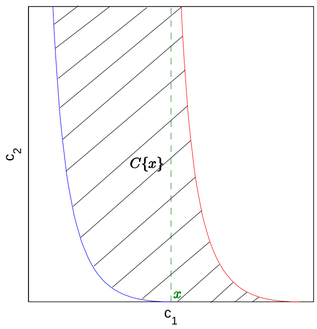

(a) Type 1




(b) Type 2




(c) Type 3




(d) Type 4

Figure 7: Types of utility function

*utility function $W$ defined on $C_1 \times C_2$, such that the Bernoulli index $\Psi$ defined on $\mathcal{J}$ takes the Expected Utility form $\Psi(J) = \int\int_C W(c_1, c_2)dJ(c_1, c_2)$, where $W$ is ordinally equivalent to the representation $U$ of $\preceq^C$ and is defined up to a positive affine transform.*

**Proof.** Necessity is obvious. We only need to prove sufficiency. The outline of the proof is as follow. First based on Theorem 1, we know that one can extend Strong Independence on a given $x$-vertical to the subspace $\mathcal{S}\{x\}$. Then we will argue that there exists a series of $x$-values such that the union of $\mathcal{S}\{x\}$ can cover the whole space $\mathcal{J}$ (This is be shown in Lemma 4 below). Using the uniqueness of the Bernoulli index $\Psi$, we can merge the representation on each subspace $\mathcal{S}\{x\}$ to obtain an Expected Utility form. The distributions considered here are more general than those in the proof of Theorem 3 in Rossman and Selden (1978). As a result, a number technical issues related to merging different representations must be addressed. First, the following will be used to define a sequence of subsets of $C$ which can be used to spread the representation defined on $\mathcal{S}\{x\}$ to all of $\mathcal{J}$.

**Lemma 4** *(Rossman and Selden 1978, Lemma 3) There is a subset $\{x_n \mid n = 0, \pm 1, \pm 2, ...\}$ of $C_1$, such that (i) $C\{x_n\} \cap C\{x_{n+1}\} \neq \varnothing$ for each $n$, (ii) $x_n < x_{n+1}$ for each $n$ and (iii) $\cup_n C\{x_n\} = C$.*

Following this Lemma we only need to prove that for $N = 0, 1, 2, ...$ there is an affine index $\lambda_N$, which is affine in the probabilities, representing $\preceq^{\mathcal{J}}$ over $\mathcal{S}(\cup_{-N}^{N} C\{x_n\})$[18] such that

$$\lambda_{N+1}|_{\text{domain } \lambda_N} = \lambda_N. \tag{51}$$

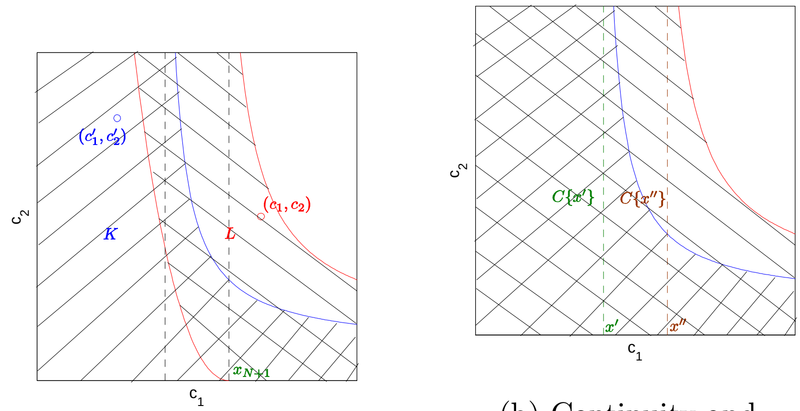
Since the Coherence axiom and the Strong Independence axiom are satisfied on $\mathcal{S}\{x_0\}$, Theorem 1 and Lemma 3 together generate $\lambda_0$ as the starting index of the induction. Proceed Inductively. Let $K = \cup_{-N}^{N} C\{x_n\}$ and $L = C\{x_{N+1}\}$. By Theorem 1 and Lemma 3, there is an affine index $\lambda$ representing preferences over $L$. Noticing that both $\lambda_N$ and $\lambda$ provide affine indices representing $\preceq^{\mathcal{J}}$ over $\mathcal{S}(K \cap L)$, by the classical NM uniqueness result, there is an affine transform $\tau$ such that

$$\tau \circ \lambda|_{\mathcal{S}(K \cap L)} = \lambda_N. \tag{52}$$

For any $(Z + V)$-state lottery $J \in \mathcal{S}(K \cup L)$,[19] suppose the first $V$ states are in $K$ and the remaining $Z$ states are in $L \setminus K$. Adjusting the probabilities in the

[18] If $K \subset C_1 \times C_2$, then $\mathcal{S}(K)$ denotes the class of all $J \in \mathcal{J}$ whose support lies in $K$. With this notation, $\mathcal{S}[x] = \mathcal{S}(C[x])$ and $\mathcal{S}\{x\} = \mathcal{S}(C\{x\})$.

[19] Although for simplicity a finite state lottery is assumed here, this assumption is not essential. For lotteries with infinite number of states, one can adjust the probability function similarly.




(a) Two lotteries for Type 4 utility that need to be compared

(b) Continuity and uniqueness ensure that there exists an NM index $W$ on $C\{x'\} \cup C\{x''\}$

Figure 8: Using continuity and uniqueness to ensure an Expected Utility representation over the whole space

following way

$$\pi'_v = \frac{\pi_v}{\sum\limits_{v \in V} \pi_v} \ (\forall v \in V) \quad \text{and} \quad \pi'_z = \frac{\pi_z}{\sum\limits_{z \in Z} \pi_z} \ (\forall z \in Z), \tag{53}$$

then we obtain two new lotteries $J^V$ and $J^Z$, where $J^V$ has $V$ states with the adjusted probability $\pi'_v$ and $J^Z$ has $Z$ states with the adjusted probability $\pi'_z$. Consequently, $\lambda_{N+1}$ may be defined by

$$\lambda_{N+1}(J) = \lambda_N\left(J^V\right) \sum_{v \in V} \pi_v + \tau \circ \lambda\left(J^Z\right) \sum_{z \in Z} \pi_z. \tag{54}$$

The intuition for why we define $\lambda_{N+1}(J)$ in this way can be illustrated as follows. For simplicity, consider a two-state lottery $< (c_1, c_2), (c'_1, c'_2); \pi, 1-\pi >$. If $\{(c_1, c_2), (c'_1, c'_2)\} \subset K$, then $\lambda_{N+1}(J) = \lambda_N(J)$. If $\{(c_1, c_2), (c'_1, c'_2)\} \subset L \setminus K$, then $\lambda_{N+1}(J) = \tau \circ \lambda(J)$. If $(c_1, c_2) \in L \setminus K$ and $(c'_1, c'_2) \in K$ as shown in Figure 8(a),[20]

$$\lambda_{N+1}(J) = (1-\pi)\lambda_N(\iota(c'_1, c'_2)) + \pi\tau \circ \lambda(\iota(c_1, c_2)), \tag{55}$$

[20]In Figure 8(a), the type of certainty utility in Figure 7(d) is assumed for the most general case. If the certainty utility function is the type in Figures 7(c) and in 8(b), then $J \in \mathcal{S}(K \cup L) = \mathcal{S}(L)$ always and $\lambda_{N+1}(J) = \tau \circ \lambda(J)$. Noticing that $\tau \circ \lambda|_{S(K \cap L)} = \lambda_N$, this definition is equivalent to (54).

where $\iota$ is the natural embedding defined by eqn. (7).[21] Two points must be verified: (1) $\lambda_{N+1}$ represents $\preceq^{\mathcal{J}}$ over $\mathcal{S}(K \cup L)$, and (2) $\lambda_{N+1}$ is affine. To show point (1), consider $\forall J, J' \in \mathcal{S}(K \cup L)$. Suppose $J \prec^{\mathcal{J}} J'$. There must exist $\mathbf{c}, \mathbf{c}' \in K \cup L$, where $\mathbf{c} \prec^{C} \mathbf{c}'$, such that $J \sim \iota\mathbf{c}$ and $J' \sim \iota\mathbf{c}'$. Since we need not consider the cases where $\{\mathbf{c}, \mathbf{c}'\} \subset K$ or $\{\mathbf{c}, \mathbf{c}'\} \subset L$, we may assume $\mathbf{c} \in K$ and $\mathbf{c}' \in L \setminus K$. Then $\lambda_{N+1}(J) = \lambda_N(\iota\mathbf{c})$ and $\lambda_{N+1}(J') = \tau \circ \lambda(\iota\mathbf{c}')$. Notice that

$$\lambda_N(\iota\mathbf{c}) < \sup\{\lambda_N(\iota\mathbf{c}'') \mid \mathbf{c}'' \in K \cap L\} \tag{56}$$

and

$$\tau \circ \lambda(\iota\mathbf{c}') > \sup\{\tau \circ \lambda(\iota\mathbf{c}'') \mid \mathbf{c}'' \in K \cap L\}. \tag{57}$$

Combining the above two equations with eqn. (52) yields

$$\lambda_{N+1}(J) = \lambda_N(\iota\mathbf{c}) < \tau \circ \lambda(\iota\mathbf{c}') = \lambda_{N+1}(J'). \tag{58}$$

If $J \sim^{\mathcal{J}} J'$, then $\mathbf{c} \sim^{C} \mathbf{c}'$. Thus $\{\mathbf{c}, \mathbf{c}'\} \subset K \cap L$ and clearly $\lambda_{N+1}(J) = \lambda_{N+1}(J')$. Now we have proved that $\forall J, J' \in \mathcal{S}(K \cup L)$, $J \preceq^{\mathcal{J}} J'$ iff $\lambda_{N+1}(J) \leq \lambda_{N+1}(J')$. Therefore, $\lambda_{N+1}$ represents $\preceq^{\mathcal{J}}$ over $\mathcal{S}(K \cup L)$. To show point (2), we use the classical uniqueness result once again to show that $\lambda_{N+1}$ coincides with an affine transform over $\mathcal{S}\{x\}$ for every $x$ such that $\mathcal{S}\{x\} \subset \mathcal{S}(K \cup L)$. First, note that there is some affine index $\varphi$ over $\mathcal{S}\{x\}$ ($\forall x \in C_1$). But then $\varphi$ and $\lambda_{N+1}$ give affine indices over $\mathcal{S}\{x\} \cap \mathcal{S}(K)$. So

$$\lambda_{N+1}|_{\mathcal{S}\{x\} \cap \mathcal{S}(K)} = \tau_1 \circ \varphi|_{\mathcal{S}\{x\} \cap \mathcal{S}(K)}, \tag{59}$$

where $\tau_1$ is an arbitrary affine transform. Similarly, we have

$$\lambda_{N+1}|_{\mathcal{S}\{x\} \cap \mathcal{S}(L)} = \tau_2 \circ \varphi|_{\mathcal{S}\{x\} \cap \mathcal{S}(K)}, \tag{60}$$

where $\tau_2$ is an arbitrary affine transform. Consequently,

$$\tau_1 \circ \varphi|_{\mathcal{S}\{x\} \cap \mathcal{S}(K \cap L)} = \tau_2 \circ \varphi|_{\mathcal{S}\{x\} \cap \mathcal{S}(K \cap L)}. \tag{61}$$

[21] Rossman and Selden (1978) define $\lambda_{N+1}$ as

$$\lambda_{N+1}(J) = \begin{cases} \lambda_N(J) & J \in \mathcal{S}(K) \\ \tau \circ \lambda(J) & J \in \mathcal{S}(L) \end{cases}$$

when proving their Lemma 5 (Rossman and Selden 1978, p. 82). This is appropriate in their setting where one good is certain since if $J \in \mathcal{S}(K \cup L)$ and $J \notin \mathcal{S}(K)$, then $J \in \mathcal{S}(L)$ and hence $\lambda_{N+1}$ is defined for all the possible distributions over $K \cup L$. However in the current setting where both goods are random, if a lottery has two states corresponding to the two points in Figure 8, then although $J \in \mathcal{S}(K \cup L)$, $J \notin \mathcal{S}(K)$ and $J \notin \mathcal{S}(L)$. To solve this problem, we define $\lambda_{N+1}$ in another way as in eqn. (54). When $J \in \mathcal{S}(K)$ or $J \in \mathcal{S}(L)$, it is easy to see that the two definitions converge.

Since $\mathcal{S}\{x\} \cap \mathcal{S}(K \cap L)$ contains at least two elements which are not indifferent, the affine transforms $\tau_1$ and $\tau_2$ must be identical, i.e., $\tau = \tau_1 = \tau_2$. Therefore,

$$\lambda_{N+1}|_{\mathcal{S}\{x\}\cap\mathcal{S}(K\cup L)} = \tau \circ \varphi|_{\mathcal{S}\{x\}\cap\mathcal{S}(K\cup L)}, \tag{62}$$

implying that $\lambda_{N+1}|_{\mathcal{S}\{x\}} = \tau \circ \varphi$. Since both $\tau$ and $\varphi$ are affine, $\lambda_{N+1}$ must also be affine in $\mathcal{S}(K \cup L)$. ■

The intuition for Theorem 2 can be explained very simply in terms of Figure 8(b). Consider the two verticals $C[x']$ and $C[x'']$ where according to Assumption $6'$ the Strong Independence holds on each vertical. Based on the modified Coherence Assumption $7'$, there exists an Expected Utility representation $\int\int_{C\{x'\}} W(c_1, c_2)dJ(c_1, c_2)$ for lotteries where $J \in \mathcal{S}\{x'\}$ and an Expected Utility representation $\int\int_{C\{x''\}} W(c_1, c_2)dJ(c_1, c_2)$ for lotteries where $J \in \mathcal{S}\{x''\}$. Noting that $C\{x'\} \cap C\{x''\} \neq \varnothing$ or $\mathcal{S}\{x'\} \cap \mathcal{S}\{x''\} \neq \varnothing$, due to the uniqueness of the representation $\Psi$ defined on the intersection region $\mathcal{S}\{x'\} \cap \mathcal{S}\{x''\}$, we can conclude that the NM indices $W_{x'}$ and $W_{x''}$ must be affinely equivalent. In other words, there exists a unique $W$ for $C\{x'\} \cup C\{x''\}$. Since we can find a series of $x$ such that $C = \cup C\{x\}$, by induction, we will have a unique (up to an affine transformation) NM index $W$ such that $\int\int_C W(c_1, c_2)dJ(c_1, c_2)$ represents the preference relation $\preceq^{\mathcal{J}}$ over the whole space $\mathcal{J}$.[22]

Given Theorem 2, we immediately have the following Corollary.

**Corollary 1** *Let Assumptions 1-5 hold. Then the Strong Independence axiom is satisfied on all of* $\mathcal{J}$ *iff Assumptions* $6'$ *and* $7'$ *are satisfied.*

[22] Although Example 1 satisfies Assumption $6'$, it violates Assumption $7'$ and hence $\preceq^{\mathcal{J}}$ is not representable by an Expected Utility in general. The fact that Assumption $7'$ is violated can be seen as follows. Assume that the certain pairs $(x', y')$ and $(x'', y'')$ are indifferent as are the pairs $(x', y''')$ and $(x'', y'''')$, i.e., given the representation $U$ defined by (16)

$$V_1(x') V_2(y') = V_1(x'') V_2(y'')$$

and

$$V_1(x') V_2(y''') = V_1(x'') V_2(y'''').$$

It follows that if Assumption $7'$ holds, the pairs $L_1 = <(x', y'), (x', y'''); \pi, 1-\pi>$ and $L_2 = <(x', y'), (x'', y''''); \pi, 1-\pi>$ must be indifferent. However, it can be easily verified that

$$\Psi(L_1) = V_1(x') (\pi V_2(y') + (1-\pi) V_2(y'''))$$

and

$$\Psi(L_2) = (\pi V_1(x') + (1-\pi) V_1(x'')) (\pi V_2(y') + (1-\pi) V_2(y'''')).$$

Since $\Psi(L_1) \neq \Psi(L_2)$, Assumption $7'$ does not hold.

As discussed above, Theorem 1 and Lemma 3 together can only guarantee an Expected Utility representation on $\mathcal{S}\{x\}$. However, if $\mathcal{S}\{x\} = \mathcal{J}$ for a given $x$, then Theorem 2 can be weakened by replacing Assumption 6′ by the simpler requirement that the conditional risk preference relation $\preceq_x^{\mathcal{F}_2}$ satisfies the Strong Independence axiom for the corresponding $x$ (Assumption 6). For the case in Figure 7(a), any $x \in C_1$ can be chosen.

# 4 Local Expected Utility

In Example 1, it was demonstrated that although preferences satisfy the Expected Utility hypothesis locally over the subsets $\cup_x \mathcal{S}[x] =_{def} \mathcal{S}[x]$ $(\forall x \in C_1)$ and $\cup_y \mathcal{S}[y] =_{def} \mathcal{S}[y]$ $(\forall y \in C_2)$, this provides no guarantee that there exists an Expected Utility representation over the rest of the space of joint distributions $\mathcal{J}^o$, where

$$\mathcal{J}^o = \mathcal{J} \setminus ((\cup_x \mathcal{S}[x]) \cup (\cup_y \mathcal{S}[y])) . \tag{63}$$

However as the following Theorem shows, if there exists an Expected Utility representation over $\mathcal{J}^o$, then due to the continuity of $\Psi$, one must have an Expected Utility representation over $\cup_x \mathcal{S}[x]$ and $\cup_y \mathcal{S}[y]$. It should be noted that $\mathcal{J}^* \subset \cup_x \mathcal{S}[x]$, $\mathcal{J}^* \subset \cup_y \mathcal{S}[y]$ and $\mathcal{J}^* \cap \mathcal{J}^o = \varnothing$, where $\mathcal{J}^*$ is the full set of degenerate distributions.

**Theorem 3** *Assumptions 1-5 hold. Furthermore assume that* $\forall J \in \mathcal{J}^o$

$$\Psi(J) = \int_{C_1} \int_{C_2} W(c_1, c_2)\, dJ(c_1, c_2) . \tag{64}$$

*Then $\preceq^{\mathcal{J}}$ induces the certainty preference relation $\preceq^C$ which is representable by $U(c_1, c_2)$ which up to a monotone transformation is equivalent to $W(c_1, c_2)$. When $J \in \cup_x \mathcal{S}[x]$,*

$$\Psi(J) = \int_{C_2} W(c_1, c_2)\, dF_2(c_2) \tag{65}$$

*and when $J \in \cup_y \mathcal{S}[y]$,*

$$\Psi(J) = \int_{C_1} W(c_1, c_2)\, dF_1(c_1) , \tag{66}$$

*where $F_1$ and $F_2$ are defined respectively by (14) and (15).*[23]

[23] It should be noted that from the continuity of $\Psi$, it follows that the NM indices in (64), (65) and (66) must be identical.

**Proof.** First given Assumption 1 that $\mathcal{J}$ corresponds to the space of probability measures $M$ that is endowed with the weak convergent topology, $\forall J \in \cup_x \mathcal{S}[x]$, it always possible to find a set of joint distributions $J_i$ $(i = 1, 2, ...)$ in $\mathcal{J}^o$ that weakly converge to $J$. Since $J_i \in \mathcal{J}^o$,

$$\Psi(J_i) = \int_{C_1}\int_{C_2} W(c_1, c_2)\, dJ_i(c_1, c_2). \tag{67}$$

Due to the continuity of $\Psi$ and the definition of weak convergence, one must have

$$\Psi(J) = \lim_{J_i \to J} \Psi(J_i) = \lim_{J_i \to J} \int_{C_1}\int_{C_2} W(c_1, c_2)\, dJ_i(c_1, c_2) = \int_{C_2} W(x, c_2)\, dF_2(c_2). \tag{68}$$

The case when $J \in \cup_y \mathcal{S}[y]$ can be discussed similarly. ■

The critical role played by the continuity assumption of $\Psi$ in Theorem 3 is illustrated by the following example. The assumed form of utility can be viewed as a type of bivariate extension of the $u - V$ non-Expected Utility preference model introduced by Schmidt (1998) and further analyzed in Andreoni and Sprenger (2012).[24] Without continuity, the assumption that preferences on $\mathcal{J}^o$ are representable by an Expected Utility function does not imply the same for preferences on verticals and horizontals.

**Example 3** *Assume bivariate preferences, $\preceq^{\mathcal{J}}$ are represented by*

$$\Psi(J) = \begin{cases} U(c_1, \widehat{c}_2) & (J \in \cup_x \mathcal{S}[x]) \\ U(\widehat{c}_1, c_2) & (J \in \cup_y \mathcal{S}[y]) \\ \int_{C_1}\int_{C_2} W(c_1, c_2)\, dJ(c_1, c_2) & (J \in \mathcal{J}^o) \end{cases}. \tag{69}$$

*where*

$$\widehat{c}_1 = \left(V_{c_2}^{(1)}\right)^{-1} \int_{C_1} V_{c_2}^{(1)}(c_1)\, dF_1(c_1) \quad \text{and} \quad \widehat{c}_2 = \left(V_{c_1}^{(2)}\right)^{-1} \int_{C_2} V_{c_1}^{(2)}(c_2)\, dF_2(c_2), \tag{70}$$

*and $V_{c_2}^{(1)}$ and $V_{c_1}^{(2)}$ are conditional NM indices. $\preceq^{\mathcal{J}}$ induces the certainty preference relation $\preceq^C$ which is representable by $U(c_1, c_2)$. Clearly, the representation (69)*

[24]Schmidt (1998) created this specific preference model to facilitate measuring the certainty effect associated with the widely documented laboratory violations of Expected Utility preferences. This $u - V$ model is based on an axiom system involving choices over univariate lotteries where the utility $u$ is used to evaluate choices involving degenerate lotteries (associated with certain outcomes) and $V$ is the NM index of an Expected Utility representation used to evaluate non-degenerate lotteries. Andreoni and Sprenger (2012) conducted lab tests and found that "Expected Utility performs well away from certainty, but fails primarily near certainty", which is consistent with the $u - V$ model. However it should be noted that if $u$ and $V$ are not ordinally equivalent to each other, Schmidt's $u - V$ model is not continuous.

*will be continuous only if the indices* $W(c_1, c_2)$*,* $U(c_1, c_2)$*,* $V_{c_2}^{(1)}(c_1)$ *and* $V_{c_1}^{(2)}(c_2)$ *are the same.* $\forall x \in C_1$*, if* $J \in \mathcal{S}[x]$*, then*

$$\Psi(J) = U\left(x, \left(V_{c_1}^{(2)}\right)^{-1} \int_{C_2} V_{c_1}^{(2)}(c_2)\, dF_2(c_2)\right), \tag{71}$$

*and similarly,* $\forall y \in C_2$*, if* $J \in \mathcal{S}[y]$*, then*

$$\Psi(J) = U\left(\left(V_{c_2}^{(1)}\right)^{-1} \int_{C_1} V_{c_2}^{(1)}(c_1)\, dF_1(c_1), y\right). \tag{72}$$

*If* $U$*,* $V_{c_2}^{(1)}$ *and* $V_{c_1}^{(2)}$ *are not ordinally equivalent, then (71) and (72) are not Expected Utility representations*[25] *even though when* $J \in \mathcal{J}^o$*,* $\Psi(J)$ *takes the Expected Utility form.*

**Remark 4** *The representation (69) can be viewed as an extension of the Schmidt's* $u - V$ *model in the following sense. When comparing degenerate and non-degenerate lotteries in* $\mathcal{J}^*$ *and* $\mathcal{J}^o$*, respectively, one uses* $U$ *for the former and the Expected Utility representation defined by* $W$ *for the latter. However, when comparing lotteries on* $\mathcal{S}[x]$ *for a given* $x$ *(or on* $\mathcal{S}[y]$ *for a given* $y$*), the utility does not take the Schmidt form since the classic single argument Expected Utility representation is assumed.*

# 5 Allais Paradox

As mentioned in Section 1, despite the intuitive appeal of Samuelson's (Samuelson 1952) mutually exclusive argument for the univariate Strong Independence axiom, there is extensive laboratory evidence over many years of violations of the axiom. Many of these experiments replicate versions of the famous Allais Paradox (Allais 1953 and 2008). Since the Coherence axiom is essential to the bivariate form of Strong Independence holding but at the same time is a totally independent axiom, it is natural to wonder whether a persuasive case can be made for Coherence. Absent direct laboratory tests on this axiom, the following two examples suggest that the case for Coherence may be less than fully compelling.

**Example 4** *Let* $c_1$ *and* $c_2$ *denote lottery payoffs in periods one and two, respectively. For simplicity, assume linear certainty indifference curves corresponding to*

$$U(c_1, c_2) = c_1 + c_2 \tag{73}$$

[25] The specific forms of $\Psi$ in (71) and (72) will be recognized to be OCE (Ordinal Certainty Equivalent) representations introduced in Selden (1978).

*with no discounting. Consider the following set of lotteries which pay off both a fixed certain dollar value of period one generalized consumption,* $c_1 = 3$ *million dollars, and a distribution of dollar values of period two consumption* $c_2$ *(for both this example and the next one, the units are in millions of dollars).*

$$L_1 : \;\; < (3, 1);\; 100\% >, \tag{74}$$

$$L_2 : \;\; < (3, 1)\,, (3, 5)\,, (3, 0)\,;\; 89\%, 10\%, 1\% >, \tag{75}$$

$$L_3 : \;\; < (3, 1)\,, (3, 0)\,;\; 11\%, 89\% > \tag{76}$$

*and*

$$L_4 : \;\; < (3, 5)\,, (3, 0)\,;\; 10\%, 90\% > . \tag{77}$$

*Each lottery can be viewed as being defined on the* $x$*-vertical, where* $x = 3$*. The lottery payoffs are not part of an optimization problem and must be consumed. They cannot be shifted between periods.*[26] *The pattern of* $c_2$*-payoffs will be recognized to mimic the distributions in the classic univariate Allais Paradox. As a result, it is natural to expect that many individuals based on* $\preceq^{\mathcal{J}}$ *will prefer* $L_1$ *to* $L_2$ *and* $L_4$ *to* $L_3$*, whereas bivariate Expected Utility preferences would require that* $L_3$ *is preferred to* $L_4$*.*[27,28] *In order to investigate the implications of the Coherence axiom holding, consider the transfer map (28) corresponding to (73) given by*

$$c_2' = \gamma_{xx'}(c_2) = \;\; U_{x'}^{-1} U_x(c_2) = x - x' + c_2, \tag{78}$$

*where* $(x', c_2') \sim^C (x, c_2)$*. The transfer maps* $c_2$*-payoff points on the* $x = 3$ *vertical into points on an* $x'$*-vertical. Thus the transferred set of lotteries (74) – (77) on the* $x'$*-vertical are given by*

$$L_1' : \;\; < (x', \gamma_{xx'}(1));\; 100\% >, \tag{79}$$

$$L_2' : \;\; < (x', \gamma_{xx'}(1))\,, (x', \gamma_{xx'}(5))\,, (x', \gamma_{xx'}(0))\,;\; 89\%, 10\%, 1\% >, \tag{80}$$

$$L_3' : \;\; < (x', \gamma_{xx'}(1))\,, (x', \gamma_{xx'}(0))\,;\; 11\%, 89\% > \tag{81}$$

*and*

$$L_4' :< (x', \gamma_{xx'}(5))\,, (x', \gamma_{xx'}(0))\,;\; 10\%, 90\% > . \tag{82}$$

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[26] Alternatively, one could think of $c_1$ and $c_2$ as corresponding to non-tradeable dollar values of permanent housing services and vacation housing services.

[27] For the classic univariate argument, see Mas-Colell, Whiston and Green (1995).

[28] There is no reason to suppose *a priori* that the common fixed payment of \$3 million for $c_1$ would reduce the distaste for the $c_2$-payoff of \$0 frequently exhibited in univariate laboratory tests. Indeed for different certainty representations corresponding to $U$, the $c_2$-payoff of \$0 might be viewed as being even more unacceptable. Of course, these observations invite direct laboratory tests.

*If Coherence holds, then an individual who prefers* $L_1$ *to* $L_2$ *and* $L_4$ *to* $L_3$ *must also prefer* $L_1'$ *to* $L_2'$ *and* $L_4'$ *to* $L_3'$*. Thus Coherence will transfer Allais paradox-like (non-Expected Utility) behavior occurring on an* $x$*-vertical to all other* $x'$*-verticals. But are these preferences on the* $x'$*-verticals always compelling? Suppose that* $x' = 1$*. Then it follows from (79)-(82) that the transferred lotteries on the* $x'$*-vertical are given by*

$$L_1' : \ < (1,3); \ 100\% >, \tag{83}$$

$$L_2' : \ < (1,3), (1,7), (1,2); \ 89\%, 10\%, 1\% >, \tag{84}$$

$$L_3' : \ < (1,3), (1,2); \ 11\%, 89\% > \tag{85}$$

*and*

$$L_4' : \ < (1,7), (1,2); \ 10\%, 90\% > . \tag{86}$$

*Now the Coherence axiom requires that an individuals who behaves consistently with the traditional Allais paradox prefers* $L_1'$ *to* $L_2'$ *and* $L_4'$ *to* $L_3'$*. Since* 3 *million is not very different from* 2 *million and* 7 *million is more than double of* 3 *million, it seems possible that many individuals may actually prefer* $L_2'$ *to* $L_1'$ *and* $L_4'$ *to* $L_3'$*, which contradicts to the conclusion required by Coherence.*[29]

The next example considers the implications of Coherence for lotteries where the transferred payoffs are on the 45° ray rather than a vertical.

**Example 5** *The same assumptions and notation are employed as in Example 4. Consider the following set of lotteries which pay off both a fixed certain dollar value of period one generalized consumption,* $c_1 = 2$ *million dollars, and a distribution of dollar values of period two consumption which again mimics the Allais paradox distributions*

$$L_1 : \ < (2,1); \ 100\% >, \tag{87}$$

$$L_2 : \ < (2,1), (2,5), (2,0); \ 89\%, 10\%, 1\% >, \tag{88}$$

$$L_3 : \ < (2,1), (2,0); \ 11\%, 89\% > \tag{89}$$

[29] It is natural to wonder if the argument against assuming Coherence in this example is independent of the assumed form of $U$. On the one hand, an analogous argument against Coherence does not seem unreasonable if the classic CES form $U(c_1, c_2) = c_1^{\frac{2}{3}} + c_2^{\frac{2}{3}}$ is assumed. In this case, the transferred lotteries $L_1'$, $L_2'$, $L_3'$ and $L_4'$ become respectively $< (1,3); 100\% >$, $< (1,3), (1,8), (1,1.1); 89\%, 10\%, 1\% >$, $< (1,3), (1,1.1); 11\%, 89\% >$ and $< (1,8), (1,1.1); 10\%, 90\% >$. It still seems reasonable to prefer $L_2'$ to $L_1'$ and $L_4'$ to $L_3'$. However on the other hand if certainty preferences are defined by $U(c_1, c_2) = \min(c_1, c_2)$, the Allais paradox behavior on one $x$-vertical can very reasonably be assumed to transfer to all other verticals resulting in $L_1'$ being preferred to $L_2'$ and $L_4'$ being preferred to $L_3'$.

*and*

$$L_4: \quad <(2,5),(2,0);\ 10\%,90\%> . \tag{90}$$

*Again it is natural, based on* $\preceq^{\mathcal{J}}$*, that* $L_1$ *will be preferred to* $L_2$ *and* $L_4$ *will be preferred to* $L_3$*, whereas Expected Utility preferences would require that* $L_3$ *is preferred to* $L_4$*. Transferring the above lotteries along the certainty indifference curve to the* $45^\circ$ *ray yields*

$$L_1': \quad <(1.5,1.5);\ 100\%>, \tag{91}$$

$$L_2': \quad <(1.5,1.5),(3.5,3.5),(1,1);\ 89\%,10\%,1\%>, \tag{92}$$

$$L_3': \quad <(1.5,1.5),(1,1);\ 11\%,89\%> \tag{93}$$

*and*

$$L_4': \quad <(3.5,3.5),(1,1);\ 10\%,90\%> . \tag{94}$$

*If Coherence holds, then those individuals who prefer* $L_1$ *to* $L_2$ *and* $L_4$ *to* $L_3$ *must prefer* $L_1'$ *to* $L_2'$ *and* $L_4'$ *to* $L_3'$*. However, since* $(1,1)$ *is close to* $(1.5,1.5)$ *and* $(3.5,3.5)$ *is more than twice of* $(1.5,1.5)$*, it seems reasonable that many individuals will actually prefer* $L_2'$ *to* $L_1'$ *and* $L_4'$ *to* $L_3'$*, which contradicts to the conclusion required by Coherence.*

# 6 Conclusion

For preferences over lotteries paying off two goods, the classic Strong Independence axiom has been shown roughly speaking to be equivalent to Strong Independence holding for one good and bivariate preferences satisfying a Coherence axiom. Examples 3 - 5 suggest several avenues for potentially interesting future research. First, given the extensive evidence from laboratory experiments challenging the predictive ability of assuming that univariate preferences satisfy Strong Independence, a number of alternative preference models not requiring this axiom have been developed and tested against the Expected Utility benchmark. Most of this work has assumed univariate preferences. Following the results in Example 3 which focuses on a bivariate extension of the $u-V$ non-Expected Utility model, it is natural to ask whether Coherence can be used to extend other non-Expected Utility models to multivariate settings and to investigate what implications this might have for the properties of the overall ordering. Second, Examples 4 and 5 suggest that the Coherence axiom might result in a spreading of localized violations of univariate preference axioms to larger regions of the choice space. It would seem to be of considerable interest to examine both the related theoretical issues as well as the predictive ability of the Coherence axiom in standard laboratory experiments.

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