Habit Formation and Risk Preference Dependence

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Abstract

Reference dependent preferences have been applied in both risky and certainty settings, although little attention has been directed at the relationship between the reference points as well as the loss aversion functions for these models. This paper addresses this relationship for the special case where reference dependence corresponds to habit formation. Multiperiod Expected Utility habit or persistence models have spawned important contributions in asset pricing, life cycle consumption, business cycle analysis and monetary models. Inherence in these models are two very different forms of preference dependence on prior period consumption. First, there is the classic certainty persistence effect of today's consumption on tomorrow's marginal utility of certain consumption. Second, today's consumption also affects tomorrow's choices over distributions of uncertain consumption – a form of risk preference dependence. These two different reference points are confounded in the conventional models which conceals the fact that they can affect asset demand and consumption behavior quite differently. This paper provides a natural generalization of the standard Expected Utility habit model that allows for a separation of certainty persistence and risk preference dependence.

KEYWORDS. Habit formation, risk preference dependence, Expected Utility, consumption-portfolio problem.

JEL CLASSIFICATION. D01, D81, G11.

1 Introduction

In recent years there has been considerable interest in the idea that an individual's preferences depend not just on the level of wealth or consumption, but on the difference from some reference level.[1] This notion of reference dependent preferences has its origins in the concept of loss aversion where consumers are assumed to be more bothered by losses relative to some reference point than pleased by gains (see Rabin 2013, p. 532). Although Kahneman and Tversky originally introduced these ideas in their uncertainty preference modeling of prospect theory (Kahneman and Tversky 1979 and Tversky and Kahneman 1992), they applied them as well in a riskless setting (Tversky and Kahneman 1991). Their key point is that whether the loss is stochastic or deterministic, the deviation from the reference point is critical to the consumer's decision rather than the absolute level of consumption or wealth. It remains an open question what one should assume as a reference point. Tversky and Kahneman (1991) argue that in a certainty setting it should be a status quo value, while Koszegi and Rabin (2009) argue that for risky choices it should be endogenously determined as a function of the decision maker's probabilistic beliefs regarding the choice set he will face and his planned action for each possible choice set. Interestingly, there seems to have been almost no substantive discussion of how the reference point or the structure of the loss aversion function might differ depending on whether the setting is risky or certain.[2]

One special case which can be viewed as an antecedent to reference dependent preferences is habit formation, where the reference point for utility in a given period is based on consumption in prior periods. One important difference in these models is that for standard habit formation preferences, consumption is typically required to exceed the habit reference point whereas in the Tversky and Kahneman reference dependent model the essence is to allow consumption to exceed or fall short of the

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[1] See Munro and Sugden (2003) for a theoretical discussion as well as empirical support for reference dependent preferences and the references cited therein. Blow, Crawford and Crawford (2013) provide an interesting revealed preference perspective on reference dependent utility models.

[2] One exception is the recent study of Gachter, Johnson and Herrmann (2010) in which the authors compare loss aversion in experiments involving a riskless endowment effect and risky lottery choices.

reference point where utility is concave for the former and convex for the latter. In this paper, I propose a simple variant of the classic habit model in which the certainty and uncertainty reference points are allowed to diverge and show that this distinction can shed interesting new insights into the solution to the classic consumption-portfolio problem.[3]

Habit formation (or persistence) was originally introduced as a property of certainty preferences (e.g., Duesenberry 1952, Pollak 1970 and Ryder and Heal 1973). In their simplest form, internal habit formation[4] preferences can be represented by

$$U(c_1, c_2) = u(c_1) + u(c_2 - \alpha c_1), \tag{1}$$

where c_1 and c_2 denote certain period one and two consumption and $\alpha > 0$ is referred to as the persistence parameter. Increasing period one consumption is typically assumed to decrease the marginal utility of c_1 but increase the marginal utility for period two consumption or as Schmitt-Grohe and Uribe (2008) suggest, "the more the consumer eats today, the hungrier he wakes up tomorrow."

About twenty years after the certainty habit model eqn. (1) was introduced, Constantinides (1990) extended it to an uncertainty setting in order to resolve the Equity Premium Puzzle.[5] Constantinides directly incorporated a continuous time version of the certainty persistence structure (1) into an intertemporal NM (von Neumann and Morgenstern) utility function.[6] In its most basic form, this corresponds to Expected Utility preferences over certain period one consumption and random period two consumption $\widetilde{c}_2$ being defined by[7]

$$EW(c_1, \widetilde{c}_2) = w(c_1) + Ew(\widetilde{c}_2 - \gamma c_1), \tag{2}$$

where W is the two-period NM index and $\gamma > 0$ is interpreted as a persistence parameter. His motivation for using this form of utility was two fold. First, there

[3] It should be noted that despite the appeal of the loss aversion feature in many of the reference dependent preference models, this property can result in indifference curve kinks (Tversky and Kahneman 1991) and non-convexities (Munro and Sugden 2003).

[4] In this paper, we focus only on internal versus external habit formation (where in the latter, the consumer's preferences depend on the consumption of others). See Abel (1990).

[5] Also see Sundaresan (1989).

[6] In addition to distinguishing between internal and external habit formation, the literature also considers a ratio form of utility as an alternative to the difference form of eqn. (2) (see the discussion of different models in Chapter 2 in Mehra 2008). In this paper we focus only on the difference form.

[7] Although Constantinides follows the Ryder and Heal (1973) continuous time formulation of habit formation, we simplify the analysis by restating his argument in terms of the discrete time model of Pollak (1970).

was a desire to relax the typical assumption of time separability of the NM utility and to use habit formation as the means for introducing adjacent complementarity between the periods. The NM index in (2) was specialized to a continuous time version of the following[8]

$$W(c_1, c_2) = -\frac{c_1^{-\delta}}{\delta} - \frac{(c_2 - \gamma c_1)^{-\delta}}{\delta} \tag{3}$$

to address the second motivation

> "...to drive a wedge between the...relative risk aversion...coefficient and the inverse of the intertemporal elasticity of substitution" (Constantinides 1990, p. 521).

Inherent in the certainty and uncertainty habit models, there would seem to be two very different forms of dependence on prior period consumption. First, there is the certainty effect of changes in today's consumption on tomorrow's marginal utility of certain consumption mentioned above. Second, there is the effect of changes in today's consumption on tomorrow's preferences over distributions of risky consumption – a form of risk preference dependence.[9] The latter effect has largely been ignored in the literature.[10] While it is well recognized that the Arrow-Pratt risk aversion measures corresponding to eqn. (3) do depend on γ, there has been no attempt to distinguish γ's role as a certainty persistence preference parameter from its role as a risk preference dependence parameter. The standard NM model forces a very strong interdependence between these two different preference effects.

Related to this distinction, the following three questions are addressed:

Q1 Can the NM persistence utility be generalized to allow for separate certainty persistence and risk preference dependence factors that can be varied independently?

[8] The NM persistence utility (3) and other related uncertainty habit representations have been widely employed in studying monetary policy (e.g., Fuhrer 2000), life cycle consumption (e.g., Gupta 2009), the term structure of interest rates (e.g., Buraschi and Jiltsov 2009), business cycles and other macroeconomic applications (e.g., Schmitt-Grohe and Uribe 2008).

[9] These two effects will be seen to correspond to different marginal costs of period one consumption which partially offset one another in the standard Expected Utility formulation.

[10] For an interesting alternative approach to modeling risk preferences changing over time, see Dillenberger and Rozen (2012).

Q2 How do the certainty persistence and risk preference dependence factors separately impact (i) financial asset demands and (ii) the comparative statics of optimal consumption?

Q3 What types of comparative static analyses can be misinterpreted because of the confounding habit and risk preference dependence in the Expected Utility defined by (3)?

To address Q1, the Expected Utility representation corresponding to (3) is generalized to an OCE (ordinal certainty equivalent) representation, where certainty and conditional risk preference can be prescribed independently (see Selden 1978 and 1979). For OCE preferences, the Expected Utility parameter γ can be decomposed into two separate parameters – one corresponding to certainty persistence and the other corresponding to risk preference dependence.

Q2 and Q3 are discussed in the context of the classic consumption-portfolio problem where the consumer chooses optimal period one consumption and a portfolio of risky and risk free assets. Relative to Q2, the certainty persistence and risk preference dependence parameters are shown to have potentially opposite effects on the demand for a risky asset. To introduce Q3, a very simple two-period example is considered in the next section contrasting the behavior of optimal period one consumption in a setting where the only asset is risk free versus one where the consumer can choose between a risky asset and a risk free asset. For the latter case where period two consumption is risky, the Expected Utility function corresponding to (3) is assumed. For the riskless setting, the NM index (3) is assumed as the certainty utility. For the riskless case, when confronting changes in the asset return, habit formation is seen to alter the conditions under which period one consumption increases or decreases. Paradoxically for the Expected Utility corresponding to (3), the conditions for whether period one consumption increases or decreases when asset returns are changed are identical whether habit formation is introduced or not. To resolve this paradox, we introduce a new comparative static result based on a generalized linear constraint and additively separable utility. Given this theorem, one can distinguish the separate roles of certainty persistence and risk preference dependence for a number of comparative static changes such as a pure increase in the risky asset's expected return (or a mean preserving increase in risk). This separation of habit and risk preference dependence effects can be applied to considerably more general preference structures than (3). Certainty preferences need not take the CES (constant elasticity of substitution) form and conditional risk preferences can be extended to other popular members of the HARA (hyperbolic absolute risk aversion) class.

The next section presents the Expected Utility comparative static paradox. Section 3 introduces (i) the OCE generalization of the Expected Utility habit formation model (3) and (ii) a reformulation of the classic consumption-portfolio problem as a two stage optimization in order to more clearly distinguish the effects of certainty persistence from risk preference dependence on the solution. In Section 4, we analyze the separate effects of certainty persistence and risk preference dependence on the marginal costs of consumption and optimal financial asset demands in the classic two-period complete market contingent claim-financial asset setting. In Section 5, we prove the new comparative static result that facilitates the disentangling of the roles of certainty persistence and risk preference dependence on the effects of increasing expected return and risk for the risky asset. The paradox introduced in Section 2 is shown to arise due to a confounding of the roles of persistence and risk preference dependence inherent in the standard Expected Utility habit model (3). Section 6 provides concluding comments. Proofs are provided in the Appendix.

2 Consumption-Portfolio Paradox

In this section, we provide a very simple example in which the consumption-portfolio comparative static properties associated with the NM persistence utility (3) and the analogous certainty persistence utility differ in a surprising way.

First assume certainty preferences are represented by the CES utility

$$U(c_1, c_2) = -\frac{c_1^{-\delta}}{\delta} - \frac{c_2^{-\delta}}{\delta}, \tag{4}$$

where $\delta > -1$. Let I, p_1 and $R_f > 0$ denote respectively period one income or wealth, the price of period one consumption and the risk free (gross) rate of return. Then period two consumption is given by

$$c_2 = (I - p_1 c_1) R_f. \tag{5}$$

Maximizing (4) with respect to c_1 where c_2 is given by (5) yields the following well known necessary and sufficient condition for optimal period one consumption to increase, stay the same or decrease with R_f

$$\frac{\partial c_1}{\partial R_f} \gtreqless 0 \qquad \text{iff} \qquad \delta \gtreqless 0. \tag{6}$$

If one considers exactly the same consumption-savings problem, but replaces the CES utility (4) with the corresponding persistence utility

$$U(c_1, c_2) = -\frac{c_1^{-\delta}}{\delta} - \frac{(c_2 - \alpha c_2)^{-\delta}}{\delta}, \tag{7}$$

where $\alpha > 0$, one obtains the comparative static result

$$\frac{\partial c_1}{\partial R_f} > 0 \qquad \text{if} \qquad \delta \geq \frac{-\alpha}{R_f + \alpha}. \tag{8}$$

With the introduction of persistence, one can only obtain a sufficient condition depending on α instead of the necessary and sufficient condition (6).

Next consider an uncertainty version of the certainty consumption-savings problem. Assume the consumer's preferences over certain-uncertain consumption pairs $(c_1, \widetilde{c}_2)$ are represented by the CES Expected Utility

$$EW\left(c_1, \widetilde{c}_2\right) = -\frac{c_1^{-\delta}}{\delta} - \frac{E\widetilde{c}_2^{-\delta}}{\delta} \tag{9}$$

where $\delta > -1$. Let n and n_f denote, respectively, the number of units of a risky asset with payoff $\widetilde{\xi} > 0$ and the number of units of a risk free asset with payoff $\xi_f > 0$. Let p and p_f be the prices of the risky and risk free assets. Then period two random consumption is defined by

$$\widetilde{c}_2 = \widetilde{\xi} n + \xi_f n_f = pn\widetilde{R} + (I - p_1 c_1 - pn) R_f, \tag{10}$$

where the gross asset rates of return are defined by

$$\widetilde{R} = \frac{\widetilde{\xi}}{p} \qquad \text{and} \qquad R_f = \frac{\xi_f}{p_f}. \tag{11}$$

The consumption-portfolio problem corresponds to maximizing (9) subject to (10) with respect to (c_1, n). As shown in Section 5.2, if one computes the change in optimal period one consumption with respect to a pure increase in $E\widetilde{R}$, the direct analogue to the certainty necessary and sufficient condition (6) is obtained

$$\frac{\partial c_1}{\partial E\widetilde{R}} \gtreqqless 0 \qquad \text{iff} \qquad \delta \gtreqqless 0. \tag{12}$$

Paralleling the certainty analysis, suppose we replace the additively separable Expected Utility (9) with the NM persistence utility

$$EW\left(c_1, \widetilde{c}_2\right) = -\frac{c_1^{-\delta}}{\delta} - \frac{E(\widetilde{c}_2 - \gamma c_1)^{-\delta}}{\delta}, \tag{13}$$

where $\gamma > 0$. Solving the associated consumption-portfolio problem, and considering the comparative static impact on optimal period one consumption corresponding to a pure increase in the expected return on the risky asset yields

$$\frac{\partial c_1}{\partial E\widetilde{R}} \gtreqqless 0 \qquad \text{iff} \qquad \delta \gtreqqless 0. \tag{14}$$

Comparing eqns. (14) and (12), the introduction of persistence appears to play absolutely no role in the effect of an increase in $E\widetilde{R}$ on optimal period one consumption.[11] This is in striking contrast to the certainty case where the introduction of persistence weakens the necessary and sufficient condition and alters the sufficient condition (6) to include a dependence on the persistence parameter α. In Section 5.3 we will consider these comparative static results in detail and show that this paradoxical result is a direct consequence of a confounding of the certainty preferences and conditional risk preferences inherent in the persistence NM utility (3). Properly disentangled, certainty persistence will be seen to generate comparative statics in the uncertain consumption-portfolio problem directly paralleling those in the certainty case, whereas conditional risk preferences will generate a new impact not previously recognized.

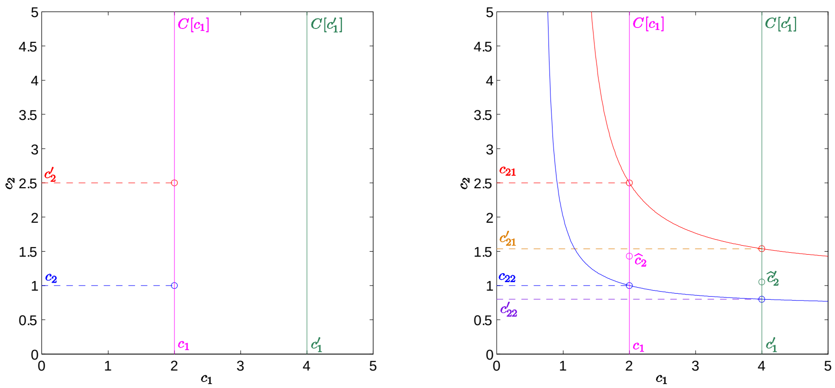
3 General Habit-RPD Setting

This section first introduces a preference model in which a certainty persistence utility and a risk preference dependent utility can be prescribed independently. To most clearly distinguish the separate roles of the two utilities, the standard consumption-portfolio problem is decomposed into a two stage optimization. In the first stage, the consumer solves a one-period portfolio allocation problem conditional on an assumed level of period one consumption. In the second stage, optimal period one consumption is determined by solving a simple certainty consumption-savings problem.

3.1 General Preference Setting

Let (c_1, F) be an element in the product set $S =_{def} C_1 \times \mathcal{F}$, where $C_1 = (0, \infty)$ is the domain for period one consumption and $\mathcal{F}$ is the set of c.d.f.s (cumulative distribution functions) on $C_2 = (0, \infty)$. The c.d.f. F corresponds to random period

[11] An analogous conclusion will also be shown to hold in Section 5 for a mean preserving increase in risk associated with the payoff variable $\widetilde{\xi}$.



(a) (b)

Figure 1:

two consumption, $\widetilde{c}_2$. Assume that there exists a complete preordering $\preceq$ which is defined on the choice space S. Let $C =_{def} C_1 \times C_2$. The relation $\preceq$ induces a complete preordering $\preceq |C$ on certain consumption pairs $(c_1, c_2) \in C$, which is representable by $U : C \to \mathbb{R}$. The certainty choice space corresponds to the positive orthant in Figure 1. The certain-uncertain choice space S can be thought of as being comprised of a set of slices or verticals

$$C[c_1] =_{def} \{c_1\} \times \mathcal{F}, \tag{15}$$

where the set of c.d.f.s $\mathcal{F}$ is defined on each vertical. Two verticals conditioned on c_1 and c_1' are portrayed in Figure 1(a). The first period consumption c_1 and the two point distribution on $C[c_1]$ with payoffs c_2 and c_2' and probabilities π and $1-\pi$ can be thought of as corresponding to the pair (c_1, F). The relation $\preceq$ induces conditional risk preferences on each vertical $C[c_1]$, which are assumed to be NM representable where for any $F, G \in \mathcal{F}$

$$(c_1, F(c_2)) \preceq (c_1, G(c_2)) \qquad \text{iff} \qquad \int_{C_2} V_{c_1}(c_2) dF(c_2) \leq \int_{C_2} V_{c_1}(c_2) dG(c_2) \tag{16}$$

and V_{c_1} is the conditional NM index. Each V_{c_1} is assumed to be strictly monotonically increasing. The set of conditional risk preferences is defined by $V(c_1, c_2)$. Then given the pair $(U(c_1, c_2), V(c_1, c_2))$, $\preceq$ is said to be OCE representable in that for any $(c_1, F), (c_1', G) \in S$,

$$(c_1, F) \preceq (c_1', G) \Leftrightarrow U(c_1, \widehat{c}_2(F)) \leq U(c_1', \widehat{c}_2(G)), \tag{17}$$

where

$$\widehat{c}_2(F) = V_{c_1}^{-1} \int_{C_2} V(c_1, c_2) dF(c_2) \qquad \text{and} \qquad \widehat{c}_2(G) = V_{c_1'}^{-1} \int_{C_2} V(c_1', c_2) dG(c_2). \quad (18)$$

(See Selden 1978 for the corresponding axiomatic development and representation theorem.[12]) The application of OCE preferences can be illustrated using Figure 1(b). Two discrete distributions $F =< c_{21}, c_{22}; \pi_{21}, \pi_{22} >$ and $G =< c_{21}', c_{22}'; \pi_{21}', \pi_{22}' >$ are specified respectively on the verticals $C[c_1]$ and $C[c_1']$. Let V_{c_1} and $V_{c_1'}$ denote the conditional NM indices on $C[c_1]$ and $C[c_1']$. The indifference curves correspond to the certainty utility $U(c_1, c_2)$. Using eqn. (18) to determine the certainty equivalents $\widehat{c}_2$ and $\widehat{c}_2'$ and then using (17), one can ascertain whether $(c_1, \widehat{c}_2)$ or $(c_1', \widehat{c}_2')$ lies on a higher certainty indifference curve and is preferred.

The preference relation $\preceq$ is said to be representable by a two-period Expected Utility function if $\forall (c_1, F), (c_1', G) \in S$,

$$(c_1, F(c_2)) \preceq (c_1', G(c_2)) \qquad \text{iff} \qquad \int_{C_2} W(c_1, c_2) dF(c_2) \leq \int_{C_2} W(c_1', c_2) dG(c_2), \quad (19)$$

where W is the two-period NM index. The key distinction between the two–period OCE and Expected Utility representations is that for the former each V_{c_1} can be prescribed independently from other members of $V(c_1, c_2)$ and from the certainty representation U. The two-period Expected Utility is a special case of OCE preferences where W is affinely equivalent to V and is a monotonic transform of the certainty utility U.[13] The dependence of conditional risk preferences on period one consumption is characterized by the following definition.

Definition 1 *OCE conditional risk preferences will be said to exhibit risk preference independence (RPI) if and only if each* $V_{c_1'}, V_{c_1''} \in \{V_{c_1} | \forall c_1 \in C_1\}$ *satisfies*

$$V_{c_1''} = a + bV_{c_1'}, \quad (20)$$

where where a *and* $b > 0$ *are arbitrary constants. If any pair* $V_{c_1'}, V_{c_1''} \in \{V_{c_1} | \forall c_1 \in C_1\}$ *fails to satisfy eqn. (20), conditional preferences will be said to exhibit risk preference dependence (RPD).*

[12] Kreps and Porteus (1978) developed a dynamic version of essentially the same model. Also see Epstein and Zin (1989).

[13] See Rossman and Selden (1978) for a formal treatment of the relation between the OCE and two-period Expected Utility representations. Kihlstrom and Mirman (1974) were the first to stress that multi-attribute Expected Utility preferences induce a representation of certainty preferences U which is ordinally equivalent to the NM index W. They use this relationship to characterize the notion of one agent being more risk averse than another.

Since for OCE preferences $U(c_1, c_2)$ and $V(c_1, c_2)$ can be prescribed independently, one can assume certainty preferences are defined by the persistence utility $U(c_1, c_2 - \alpha c_1)$ and conditional risk preferences are defined by the RPD NM index $V(c_2 - \beta c_1)$ where $\alpha \neq \beta$. This OCE generalization addresses question Q1. For the two-period Expected Utility special case, the fact that U and W must be ordinally equivalent implies that $\alpha = \beta = \gamma$ and W takes the persistence form (3).

3.2 Consumption-Portfolio Problem: Two Stage Optimization

Without loss of generality assume there is one risky and one risk free asset, where the former has payoff $\widetilde{\xi}$, a random variable that takes the value ξ_{21} with probability π_{21} and ξ_{22} with the probability $\pi_{22} = 1 - \pi_{21}$.[14] Let $\xi_{21} > \xi_{22} > 0$. The risk free asset has payoff $\xi_f > 0$. Let n and n_f denote the number of units of the risky asset and risk free asset, respectively, where positive holdings of the latter are not required. Denote by p and p_f the price of the risky and risk free asset, respectively. We follow the conventional assumption that $\frac{E\widetilde{\xi}}{p} > \frac{\xi_f}{p_f}$, which guarantees $n > 0$. To ensure that there is no arbitrage opportunity, it is assumed that $\frac{\xi_{21}}{p} > \frac{\xi_f}{p_f} > \frac{\xi_{22}}{p}$.

As is standard, the asset demand model can be embedded in a contingent claim framework and complete markets are assumed. (For a more complete characterization of the dual contingent claim and financial asset setting and the associated demand properties, see Kubler, Selden and Wei 2013 and 2014.) Let c_{21} and c_{22} denote the contingent claims for period two consumption and p_{21} and p_{22} be the corresponding prices. I is period one income or initial wealth. The utilities $U(c_1, c_2 - \alpha c_1)$ and $V(c_2 - \beta c_1)$ are three times continuously differentiable in c_1 and c_2. $U(\cdot, \cdot)$ is strictly quasiconcave and strictly increasing in each of its arguments and $V(\cdot)$ satisfies $V' > 0$ and $V'' < 0$. Moreover it is assumed for U and V, respectively, that $c_2 > \alpha c_1$ and $c_2 > \beta c_1$.

The general OCE optimization problem can be expressed as

$$\max_{c_1, c_{21}, c_{22}} U\left(c_1, V_{c_1}^{-1}\left(\pi_{21} V_{c_1}(c_{21}) + \pi_{22} V_{c_1}(c_{22})\right)\right) \tag{21}$$

subject to

$$p_1 c_1 + p_{21} c_{21} + p_{22} c_{22} \leq I, \tag{22}$$

[14]When there are multiple risky assets, the two stage optimization process described below can still be followed to solve for optimal consumption and asset demands.

where

$$c_{21} = \xi_{21} n + \xi_f n_f, \quad c_{22} = \xi_{22} n + \xi_f n_f \tag{23}$$

and

$$p_{21} = \frac{\xi_f p - \xi_{22} p_f}{(\xi_{21} - \xi_{22})\xi_f} \quad \text{and} \quad p_{22} = \frac{\xi_{21} p_f - \xi_f p}{(\xi_{21} - \xi_{22})\xi_f}. \tag{24}$$

To distinguish γ's separate roles as a persistence parameter and a risk preference dependence parameter in the Expected Utility (2), it will prove convenient to utilize the OCE functions $U(c_1, c_2 - \alpha c_1)$ and $V(c_2 - \beta c_1)$ to solve an equivalent two stage problem. The first stage portfolio problem conditional on c_1 is given by maximizing the conditional Expected Utility

$$(c^o_{21}(c_1), c^o_{22}(c_1)) = \arg\max_{c_{21}, c_{22}} \pi_{21} V(c_{21} - \beta c_1) + \pi_{22} V(c_{22} - \beta c_1) \tag{25}$$

subject to

$$p_{21} c_{21} + p_{22} c_{22} \leq I - p_1 c_1. \tag{26}$$

The second stage consumption-savings problem corresponds to[15]

$$\max_{c_1} U(c_1, \widehat{c}^o_2(c_1) - \alpha c_1), \tag{27}$$

where $\widehat{c}^o_2(c_1)$ is defined by

$$V(\widehat{c}^o_2(c_1) - \beta c_1) = \pi_{21} V(c^o_{21}(c_1) - \beta c_1) + \pi_{22} V(c^o_{22}(c_1) - \beta c_1). \tag{28}$$

For the NM persistence utility (2), the parameter $\gamma = \alpha = \beta$ clearly plays different roles in each stage. The OCE generalization facilitates a very intuitive geometric interpretation of the two disparate roles, even for the Expected Utility special case.

If the $\widehat{c}_2(c_1)$ constraint for the second stage consumption-savings optimization is linear in c_1, the analysis can be significantly simplified.[16] We next show that this will be the case if and only if the period two conditional NM index V is a member of the HARA class.[17,18]

[15] It should be noted that due to the no bankruptcy restriction, we always require that $c_{21}, c_{22} \geq \beta c_1$ which can impose a restriction on β given other parameters in the optimization problem.

[16] For an analysis of the consumption-savings problem where the $\widehat{c}_2(c_1)$ constraint need not be linear (although certainty persistence and risk preference dependence are not considered) see Kimball and Weil (2009).

[17] See Gollier (2001) for a description of the HARA family of utility functions.

[18] It should be noted that Proposition 1 can be applied for the case of multiple risky assets even when markets are incomplete. The reason is that when markets are incomplete, since V takes the

Proposition 1 *Assume* $\{\preceq |C[c_1], \forall c_1 \in C_1\}$ *are NM representable where* $V(c_1, c_2)$ *is the NM index. Then* $\widehat{c}_2$ *is a linear function of* c_1 *if and only if* V *takes the following form*

$$V(c_1, c_2) = f(c_1)\, h(c_2 - \zeta c_1) + g(c_1), \tag{29}$$

where $f(\cdot) > 0$ *and* $g(\cdot)$ *are arbitrary functions of* c_1, $h(\cdot)$ *is a member of the HARA class of utility functions and* ζ *is an arbitrary constant.*

For the specific RPD form of the NM index assumed in this paper, condition (29) of the Proposition becomes

$$V(c_1, c_2) = f(c_1)\, V(c_2 - \beta c_1) + g(c_1). \tag{30}$$

4 CES-HARA Preferences

In order to address question Q2, it will prove useful to specialize the OCE preference formulation defined by $U(c_1, c_2 - \alpha c_1)$ and $V(c_2 - \beta c_1)$ as follows

$$U(c_1, c_2 - \alpha c_1) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{(c_2 - \alpha c_1)^{-\delta_1}}{\delta_1}, \quad \text{and} \quad V(c_2 - \beta c_1) = -\frac{(c_2 - \beta c_1)^{-\delta_2}}{\delta_2}, \tag{31}$$

where $\delta_1, \delta_2 > -1$, $\alpha > 0$ is a pure certainty persistence parameter and $\beta > 0$ is a pure risk preference dependence parameter.[19] In Subsection 4.4 below, V is allowed to take the form of other HARA NM indices. Clearly (31) converges to the NM persistence representation (3) if and only if $\delta_1 = \delta_2 = \delta$ and $\alpha = \beta = \gamma$. It should be emphasized that if $\delta_1 = \delta_2 = \delta$, the NM axioms which require U and W to be equivalent up to an increasing monotonic transform make it impossible to have an NM representation where $\alpha \neq \beta$.[20] Thus using NM preferences, one cannot investigate the implications

HARA form, following Rubinstein (1974), the markets are effectively complete. Thus the financial asset setting can be transformed into the contingent claim setting naturally. Since each conditional contingent claim demand c_{2s} $(s = 1, 2, ..., S)$ is linear in c_1, the certainty equivalent function $\widehat{c}_2$ is also linear in c_1.

[19] Unless stated otherwise, these restrictions on δ_1, δ_2, α and β hold throughout the paper.

[20] To see this, first assume $W(c_1, c_2) = -\frac{c_1^{-\delta}}{\delta} - \frac{(c_2 - \gamma c_1)^{-\delta}}{\delta}$ and U is given by eqn. (31). If there exists an increasing transformation T such that $T \circ U = W$, then any two points (c_1, c_2) and (c_1', c_2') on the same indifference curve corresponding to U should also be on the same indifference curve corresponding to W. Therefore, we have

$$c_2' = \left(c_1^{-\delta} + (c_2 - \gamma c_1)^{-\delta} - c_1'^{-\delta}\right)^{-1/\delta} + \gamma c_1' = \left(c_1^{-\delta} + (c_2 - \alpha c_1)^{-\delta} - c_1'^{-\delta}\right)^{-1/\delta} + \alpha c_1'.$$

for asset demand and period one consumption of assuming just certainty persistence where $\beta = 0$ or just risk preference dependence where $\alpha = 0$.

4.1 Distinguishing the Persistence and RPD Parameters

For the certainty persistence utility U in (31) in addition to requiring $c_2 > \alpha c_1$, the assumption of positive marginal utility for period one consumption implies that

$$c_2 > (\alpha + \alpha^{\frac{1}{\delta_1+1}})c_1. \tag{32}$$

Computing the reciprocal of the intertemporal elasticity of substitution η yields

$$\frac{1}{\eta} =_{def} -\frac{d\ln(\frac{U_2}{U_1})}{d\ln\frac{c_2}{c_1}} = (\delta_1+1)\frac{\frac{c_2}{c_1}}{\frac{c_2}{c_1} - \alpha\left(1 + \left(\frac{c_2}{c_1} - \alpha\right)^{-\delta_1}\right)}, \tag{33}$$

where $U_1 = \frac{\partial U(c_1,c_2)}{\partial c_1}$ and $U_2 = \frac{\partial U(c_1,c_2)}{\partial c_2}$. Following the standard interpretation, period one and two consumption are said to be Fisherian complements if $1/\eta > 1$. The following shows that the introduction of the persistence parameter α into the CES utility increases the degree of intertemporal complementarity.

Proposition 2 *For the certainty persistence utility U defined by (31),*

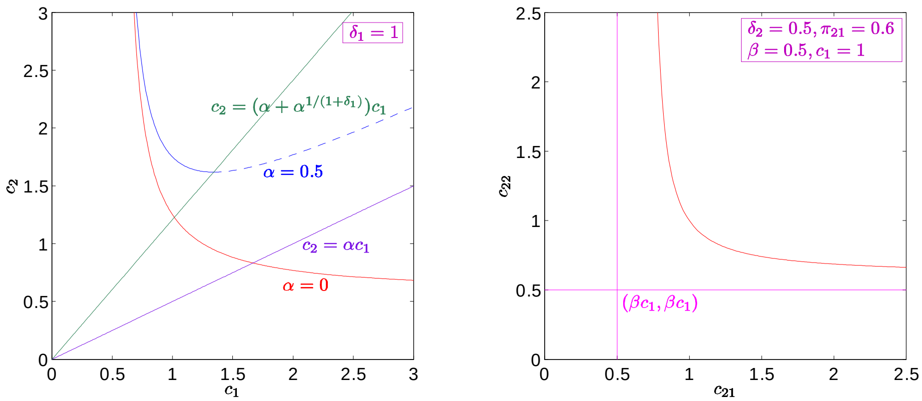
$$\frac{\partial\frac{1}{\eta}}{\partial\alpha} = (1+\delta_1)\frac{\frac{c_2}{c_1}\left(1 + \left(\frac{c_2}{c_1} - \alpha\right)^{-1-\delta_1}\left(\alpha\delta_1 + \frac{c_2}{c_1} - \alpha\right)\right)}{\left(\frac{c_2}{c_1} - \alpha\left(1 + \left(\frac{c_2}{c_1} - \alpha\right)^{-\delta_1}\right)\right)^2} > 0. \tag{34}$$

These properties are illustrated in Figure 2(a). The two indifference curves plotted correspond respectively to the persistence factors $\alpha = 0$ and $\alpha = 0.5$ and converge to same point $(c_1, c_2) = (0.5, \infty)$. It is clear from Figure 2(a) that persistence increases the curvature of the certainty indifference curves or the complementarity between c_1 and c_2. Condition (32) ensures that for a given α, optimal consumption pairs lie northwest of the ray[21]

$$\frac{c_2}{c_1} = \alpha + \alpha^{\frac{1}{\delta_1+1}}. \tag{35}$$

Since this equation holds for any c_1', the coefficients in front of c_1' should be the same, implying $\alpha = \gamma$.

[21] It should be noted that since the right hand side of eqn. (35) is increasing in α, it follows that for any given $\frac{c_2}{c_1}$, a $\alpha_{\max} > 0$ can be found from (35) such that corresponding to the $\frac{c_2}{c_1}$ value and $\alpha_{\max}$, we have $\partial U(c_1, c_2)/\partial c_1 = 0$.



(a) (b)

Figure 2:

To understand the very different economic intuition associated with the RPD utility parameter β versus the certainty persistence parameter α in (31), first consider the following contingent claim version of the RPD utility in (31)

$$EV(\widetilde{c}_2 - \beta c_1) = -\frac{\pi_{21}\left(c_{21} - \beta c_1\right)^{-\delta_2} + \pi_{22}\left(c_{22} - \beta c_1\right)^{-\delta_2}}{\delta_2}. \tag{36}$$

One can view βc_1 as the period two subsistence level which depends on consumption in period one.[22] An example of the associated Expected Utility indifference curves is plotted in Figure 2(b). Computing the Arrow-Pratt relative risk aversion measure for $V(c_1, c_2) = V(c_2 - \beta c_1)$ in (31), one obtains

$$\tau_R =_{def} -\frac{c_2 V_{22}}{V_2} = (\delta_2 + 1)\frac{c_2}{c_2 - \beta c_1}, \tag{37}$$

where $V_2 = \frac{\partial V(c_1,c_2)}{\partial c_2}$ and $V_{22} = \frac{\partial^2 V(c_1,c_2)}{\partial c_2^2}$. It follows immediately for this RPD case that $\partial \tau_R / \partial c_1 > 0$. Hence a natural interpretation of βc_1 is that the more I eat today, the more I want to avoid risk in my consumption tomorrow and the larger my subsistence requirement. Because β controls the impact of today's consumption on tomorrow's risk aversion, it is reasonable to refer to β as the risk preference

[22] The subsistence interpretation was first used in the context of the certainty linear expenditure demand system (e.g., Pollak 1970) and then by Rubinstein (1976) for NM utilities. Also see Detemple and Zapatero (1991) and Meyer and Meyer (2005).

dependence parameter. It should be stressed that this interpretation of β is very different from the more I eat today the hungrier I become tomorrow associated with the certainty persistence parameter α.

4.2 Marginal Costs of Period One Consumption

In the Expected Utility habit model (3), the requirement that period two consumption must exceed the minimum subsistence requirement γc_1 can be thought of as inducing a cost of period one consumption in addition to $p_1 c_1$.[23] However for the OCE formulation defined by (31), it will be shown that different costs are associated with the certainty requirement αc_1 and the RPD requirement βc_1 and that an element of each cancels out in the Expected Utility special case. One obstacle to deriving the cost elements associated with αc_1 and βc_1 is that they enter into the consumption-savings problem (27) - (28) differently. The former enters via the certainty utility U and the latter enters via the $\widehat{c}_2(c_1)$ constraint. However it is possible to migrate the αc_1 effect into the $\widehat{c}_2(c_1)$ constraint to facilitate the comparison. To see this difference, we proceed in two steps. First, the case where $\beta = 0$ and there is only the certainty persistence effect is considered. Then it is assumed that $\alpha = 0$, and the RPD effect is examined.

Solving the conditional portfolio problem (25) - (26) assuming the CRRA (constant relative risk aversion) NM index corresponding to $\beta = 0$ in (31) yields

$$c_{21}^o = \frac{I - p_1 c_1}{p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}} \qquad \text{and} \qquad c_{22}^o = \frac{k^{\frac{1}{1+\delta_2}} (I - p_1 c_1)}{p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}}, \tag{38}$$

where

$$k = \frac{\pi_{22} p_{21}}{\pi_{21} p_{22}}. \tag{39}$$

Computing the certainty equivalent $\widehat{c}_2 = V^{-1} EV(\widetilde{c}_2)$, the second stage consumption-savings problem becomes

$$\max_{c_1} U\left(c_1, \widehat{c}_2^o\right) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{\left(\widehat{c}_2^o - \alpha c_1\right)^{-\delta_1}}{\delta_1} \tag{40}$$

[23] The fact that the cost for c_1 consists of two parts was first noted for the certainty case in Spinnewyn (1979). It was also observed for the uncertainty case assuming Expected Utility habit preferences by Detemple and Zapatero (1991), although the authors seemed unaware of Spinnewyn's prior contribution and used a different definition of the cost of consumption.

subject to

$$\widehat{c}_2^o(c_1) = (I - p_1 c_1)\,\widehat{R}, \tag{41}$$

where

$$\widehat{R} = \frac{\left(\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22}\right)^{-\frac{1}{\delta_2}}}{p_{21} + k^{\frac{1}{1+\delta_2}}p_{22}}. \tag{42}$$

The problem (40) - (41) can be transformed into an equivalent problem where certainty preferences are CES and the persistence effect is transferred to the certainty equivalent constraint by defining[24]

$$\widehat{c}_2^{new} =_{def} \widehat{c}_2^o - \alpha c_1, \tag{43}$$

where $\widehat{c}_2^{new}$ is defined relative to the stage one conditionally optimal (c_{21}^o, c_{22}^o). The new problem becomes

$$\max_{c_1, \widehat{c}_2^{new}} U\left(c_1, \widehat{c}_2^{new}\right) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{\left(\widehat{c}_2^{new}\right)^{-\delta_1}}{\delta_1}, \tag{44}$$

subject to

$$I = \left(p_1 + \frac{\alpha}{\widehat{R}}\right)c_1 + \frac{\widehat{c}_2^{new}}{\widehat{R}}. \tag{45}$$

The coefficients of c_1 and $\widehat{c}_2^{new}$ in (45) can be interpreted as the marginal costs of consumption

$$q_1 = p_1 + \frac{\alpha}{\widehat{R}} \quad \text{and} \quad q_2 = \frac{1}{\widehat{R}}. \tag{46}$$

The marginal cost of period one consumption q_1 is the per unit price p_1 plus the per unit present value cost of ensuring that period two certainty equivalent consumption exceeds the certainty persistence amount αc_1. The marginal cost or quasi (discounted) price of $\widehat{c}_2^{new}$ is denoted by q_2. Because preferences are defined in terms of the certainty equivalent $\widehat{c}_2^{new}$, it is natural to use the certainty equivalent return $\widehat{R}$ as the (gross) discount rate. Expressing the certainty equivalent in (41) in terms of $\widehat{c}_2^{new}$, it is clear that in the $c_1 - \widehat{c}_2$ plane the marginal cost component $\frac{\alpha}{\widehat{R}}$ results in a downward rotation of the constraint $\widehat{c}_2 = (I - p_1 c_1)\,\widehat{R}$ anchored at $(0, I\widehat{R})$.

Next to determine the marginal cost associated with the period two RPD requirement βc_1, assume that $\alpha = 0$ in eqn. (31). Again first solving the conditional portfolio problem yields

$$c_{21}^o = \beta c_1 + \frac{I - \left(p_1 + \frac{\beta p_f}{\xi_f}\right)c_1}{p_{21} + k^{\frac{1}{1+\delta_2}}p_{22}} \tag{47}$$

[24] A similar transformation was introduced for the certainty habit case by Spinnewyn (1979, 1981).

and

$$c_{22}^{o} = \beta c_1 + \frac{\left(I - \left(p_1 + \frac{\beta p_f}{\xi_f}\right) c_1\right) k^{\frac{1}{1+\delta_2}}}{p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}}. \tag{48}$$

Computing $\widehat{c}_2^{o}(c_1)$, the second stage consumption-savings problem becomes

$$\max_{c_1} U\left(c_1, \widehat{c}_2^{o}\right) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{\left(\widehat{c}_2^{o}\right)^{-\delta_1}}{\delta_1} \tag{49}$$

subject to[25]

$$\widehat{c}_2^{o}(c_1) = \widehat{R}\left(I - \left(p_1 + \beta\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right)\right) c_1\right), \tag{50}$$

where $\widehat{R}$ is defined by eqn. (42), R_f is the rate on a risk free portfolio comprised of one unit each of c_{21} and c_{22} implying

$$R_f = \frac{1}{p_{21} + p_{22}}, \tag{51}$$

and $\widehat{R} > R_f$.[26]

Once again it is possible to define the marginal costs of c_1 and $\widehat{c}_2^{o}$, respectively,

$$q_1 = p_1 + \beta\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right) \qquad \text{and} \qquad q_2 = \frac{1}{\widehat{R}}, \tag{52}$$

[25]It should be emphasized that $\widehat{R}$ appearing in the $\widehat{c}_2$ constraint (50) does not include the risk preference dependence parameter β. In other words, no matter whether there is one or two fund separation in terms of the portfolio of financial assets n and n_f, $\widehat{R}$ is defined by (42) and is always the certainty equivalent of the return from the risky mutual fund.

[26]To prove that $\widehat{R} > R_f$, first note that

$$\frac{\pi_{21} + k^{-1}\pi_{22}}{\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22}} = \frac{p_{21} + p_{22}}{p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}},$$

where the right-hand side can be obtained by using the using the definition of k given by eqn. (39). Let x denote the random variable with payoffs $(1, k^{\frac{-\delta_2}{1+\delta_2}})$ and probabilities π_{21} and π_{22}. Note that $f(x) = x^{\frac{1+\delta_2}{\delta_2}}$ is a convex function. Using Jensen's inequality, we have $E[f(x)] \geq f(E[x])$ implying $\pi_{21} + k^{-1}\pi_{22} \geq (\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22})^{\frac{1+\delta_2}{\delta_2}}$ or

$$\frac{\pi_{21} + k^{-1}\pi_{22}}{\left(\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22}\right)^{\frac{1+\delta_2}{\delta_2}}} \geq 1.$$

Solving the first equation in this footnote for $1/(p_{21} + p_{22})$ and multiplying that expression by the left hand side of the second equation above, one obtains $\widehat{R}$ given by eqn. (42). It then follows that $\widehat{R} \geq \frac{1}{p_{21}+p_{22}}$ where the equality is reached when and only when $\delta_2 \to \infty$ or $k \to 1$.

and rewrite the constraint (50) as follows in terms of the quasi prices

$$I = q_1 c_1 + q_2 \widehat{c}_2^o. \tag{53}$$

To provide intuition for why in addition to p_1 there are the two components for the RPD marginal cost β/R_f and $-\beta/\widehat{R}$, consider the following rearrangement of (50)

$$\frac{\widehat{c}_2^o(c_1) - \beta c_1}{\widehat{R}} = I - p_1 c_1 - \frac{\beta c_1}{R_f}. \tag{54}$$

The $\frac{\beta c_1}{R_f}$ term on the right hand side corresponds to the conditional portfolio investment in a risk free portfolio funding the subsistence requirement. The $\frac{\beta c_1}{\widehat{R}}$ on the left hand side corresponds to the requirement in the NM index $V(\widehat{c}_2 - \beta c_1)$ that $\widehat{c}_2 > \beta c_1$, paralleling the requirement for the certainty U that $\widehat{c}_2 > \alpha c_1$. In (52), these two elements are offset. Finally, it is clear from (50) that the RPD marginal cost element $\beta\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right)$, like the certainty persistence marginal cost element $\frac{\alpha}{\widehat{R}}$, results in a downward rotation in the $c_1 - \widehat{c}_2$ plane of the constraint $\widehat{c}_2 = (I - p_1 c_1)\,\widehat{R}$ anchored at $(0, I\widehat{R})$.

Comparing the certainty persistence and RPD marginal costs of period one consumption in eqns. (46) and (52), respectively, it follows that if $\alpha = \beta$

$$\frac{\alpha}{\widehat{R}} \gtreqless \beta\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right) \Leftrightarrow \widehat{R} \lesseqgtr 2R_f. \tag{55}$$

It is interesting to note that following the above process for the OCE case where both the certainty persistence and RPD effects are present, one obtains[27]

$$q_1 = p_1 + \left(\frac{\beta}{R_f} - \frac{\beta - \alpha}{\widehat{R}}\right) \qquad \text{and} \qquad q_2 = \frac{1}{\widehat{R}}. \tag{56}$$

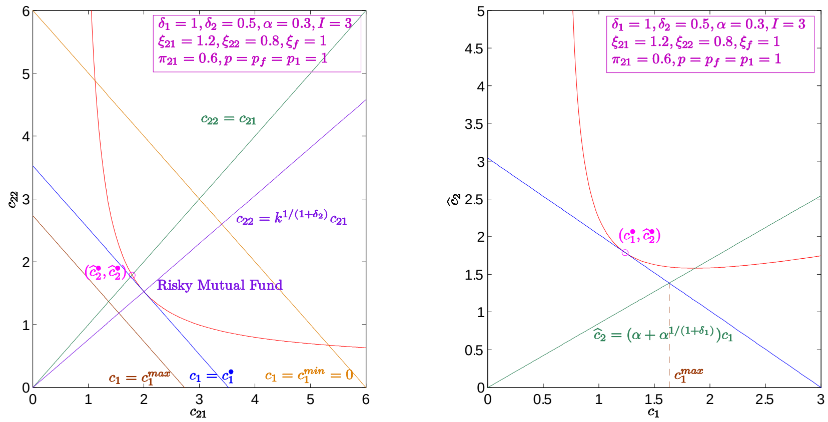
For the Expected Utility special case where $\gamma = \alpha = \beta$, the marginal cost of period one consumption simplifies to

$$q_1 = p_1 + \frac{\gamma}{R_f}, \tag{57}$$

where the OCE elements $\alpha/\widehat{R}$ and $\beta/\widehat{R}$ cancel each other out.

The marginal costs of the consumption q_1 and q_2 derived in this section will play a critical role in our application of Theorem 1 in Section 5.1 to uncertainty comparative statics. In particular, the separate effects of alpha and beta on the form of the $\widehat{c}_2(c_1)$ constraint strongly affect how optimal period one consumption varies with changes in the risky asset's expected return or risk.

[27]See Case 4 in Subsection 5.3.



(a) (b)

Figure 3:

4.3 Competing Effects of α and β on Asset Demands

Since changes in certainty persistence and risk preference dependence will be seen to have very different effects on the consumer's investment decisions, the focus of this subsection is on the demand for risky and risk free assets.

First, consider the presence of just habit persistence where $\alpha \neq 0$ but $\beta = 0$. It follows from eqns. (23), (24) and (38) that the solution to the dual financial asset conditional portfolio problem is given by

$$\frac{c_{22}^o}{c_{21}^o} = k^{\frac{1}{1+\delta_2}} \Leftrightarrow \frac{n_f^o}{n^o} = \frac{\xi_{21} k^{\frac{1}{1+\delta_2}} - \xi_{22}}{\left(1 - k^{\frac{1}{1+\delta_2}}\right) \xi_f}, \tag{58}$$

where as noted above $n^o > 0$. The fact that the asset ratio is independent of c_1 and I implies classic one fund separation. The two stage optimization process is shown in Figure 3. In Figure 3(a), consistent with one fund separation, the linear contingent claim expansion path, defined by $c_{22} = k^{\frac{1}{1+\delta_2}} c_{21}$, passes through the origin below the 45° certainty ray. The set of feasible budget constraints is bounded by $I - p_1 c_1^{\min}$ and $I - p_1 c_1^{\max}$, where $c_1^{\min} = 0$ and $c_1^{\max}$ is defined by the requirement that U satisfies the positive marginal utility condition

$$\widehat{c}_2 > \left(\alpha + \alpha^{\frac{1}{1+\delta_1}}\right) c_1, \tag{59}$$

(see Figure 3(b)). Because certainty persistence does not affect asset allocation, its impact on asset demands is straightforward.

Proposition 3 *Assume OCE preferences are defined by eqn. (31), where conditional risk preferences exhibit RPI, i.e., $\beta = 0$. Then*

$$\frac{\partial n}{\partial \alpha} > 0 \qquad \textit{and} \qquad \frac{\partial |n_f|}{\partial \alpha} > 0. \tag{60}$$

This result follows directly from (i) the fact that optimal period one consumption always decreases with α and the demands for the risky and risk free assets increase[28] and (ii) the fact that the optimal composition of the asset portfolio is unaffected by an increase in α. Hence the demand for the risky asset, which the consumer holds long, always increases with α. Since shorting of the risk free asset is possible, the absolute value of the risk free asset demand always increases with α.

Figure 3(b) illustrates the second stage process for finding the optimal c_1. Intuitively in terms of the constraint (41), each $\widehat{c}_2$ value corresponding to a given $c_1 \in [0, c_1^{\max})$ can be thought of as being obtained by finding the contingent claim (or financial asset) optimal point in Figure 3(a) and then determining the $\widehat{c}_2$ value by finding the point $\widehat{c}_2 = c_{21} = c_{22}$ where the indifference curve tangent to the contingent claim budget line intersects the 45° ray.

Next, consider the presence of just risk preference dependence where $\beta \neq 0$ and $\alpha = 0$. It follows from the contingent claim demands (47) and (48) that

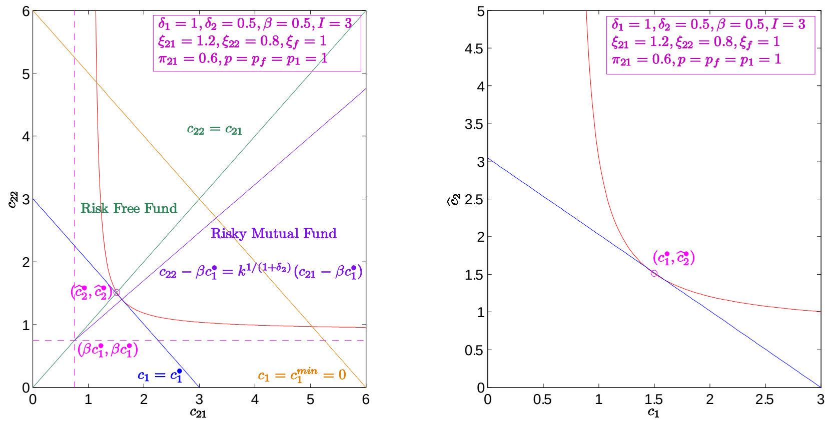
$$\frac{c_{22}^o - \beta c_1}{c_{21}^o - \beta c_1} = k^{\frac{1}{1+\delta_2}} \Leftrightarrow \frac{n_f^o - \beta c_1/\xi_f}{n^o} = \frac{\xi_{21} k^{\frac{1}{1+\delta_2}} - \xi_{22}}{\left(1 - k^{\frac{1}{1+\delta_2}}\right) \xi_f}. \tag{61}$$

The consumer invests $p_f \beta c_1/\xi_f$ in a risk free subsistence fund[29] and the remainder of her income $I - p_1 c_1 - p_f \beta c_1/\xi_f$ in a risky fund, which has exactly the same $\frac{n_f}{n}$ asset mix (58) as when OCE preferences are characterized by $\alpha \neq 0$ and $\beta = 0$. The consumer's total risk free investment $p_f^o n_f^o$ consists of two parts, the holdings $p_f \beta c_1/\xi_f$ and the fixed percentage of the risky fund determined by $k^{\frac{1}{1+\delta_2}}$.

This optimization process is illustrated in Figure 4(a). The investment in c_{21}

[28] Given that $U(c_1, \widehat{c}_2^{new})$ defined by (44) is additively separable, it follows that period one consumption is a normal good, implying that $\partial c_1/\partial q_1 < 0$. Noticing that $\partial q_1/\partial \alpha > 0$, we always have $\partial c_1/\partial \alpha < 0$.

[29] Note that the subsistence fund needs to deliver βc_1 in terms of period two consumption, which is accomplished by investing $n_f \xi_f = \beta c_1$. To do so, the consumer needs to purchase $n_f = \beta c_1/\xi_f$ units of the risk free asset.



(a) (b)

Figure 4:

and c_{22} consists two parts $c_{2i} = c_{2i}^{(1)} + c_{2i}^{(2)}$ $(i = 1, 2)$. The first part corresponds to the risk free subsistence fund $c_{21}^{(1)} = c_{22}^{(1)} = \beta c_1$ defined along the 45° ray. The second part corresponds to the ray $c_{22}^{(2)} = k^{\frac{1}{1+\delta_2}} c_{21}^{(2)}$, which has the same slope as in Figure 3(a). The key difference is that in Figure 4(a), the ray starts from the point $(c_{21}, c_{22}) = (\beta c_1, \beta c_1)$ rather than the origin. The second stage optimization is indicated in Figure 4(b).

In contrast to the Proposition 3 result that increasing α always increases the demand for both the risky asset and the absolute value of the risk free asset, increasing β decreases the demand for the risky asset and increases the demand for the risk free asset if $\delta_1 \geq 0$, implying that period one and two consumption are intertemporal (Fisherian) independents or complements, i.e., $1/\eta = 1 + \delta_1 \geq 1$.

Proposition 4 *Assume OCE preferences are defined by eqn. (31), where certainty preferences do not exhibit persistence, i.e., $\alpha = 0$. Then if $\delta_1 \geq 0$, we have*

$$\frac{\partial n}{\partial \beta} < 0 \qquad \text{and} \qquad \frac{\partial n_f}{\partial \beta} > 0. \tag{62}$$

The intuition for this result can best be understood in terms of a competing allocation effect and an investment effect. Noting that

$$n = \frac{\left(I - \left(p_1 + \frac{\beta p_f}{\xi_f}\right) c_1\right)\left(1 - k^{\frac{1}{1+\delta_2}}\right)}{\left(p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}\right)(\xi_{21} - \xi_{22})}, \tag{63}$$

it follows that

$$\frac{\partial n}{\partial \beta} = J_1 + J_2, \tag{64}$$

where

$$J_1 = -\frac{p_f c_1 \left(1 - k^{\frac{1}{1+\delta_2}}\right)}{\left(p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}\right) (\xi_{21} - \xi_{22})\, \xi_f} \quad \text{and} \quad J_2 = -\frac{\left(p_1 + \frac{\beta p_f}{\xi_f}\right) \left(1 - k^{\frac{1}{1+\delta_2}}\right)}{\left(p_{21} + k^{\frac{1}{1+\delta_2}} p_{22}\right) (\xi_{21} - \xi_{22})} \frac{\partial c_1}{\partial \beta}. \tag{65}$$

Note that $J_1 < 0$ and since $\frac{\partial c_1}{\partial \beta} < 0$, $J_2 > 0$.[30] J_1 represents the allocation effect, since for a fixed c_1, increasing β (and hence τ_R) will shift the investment in the risky asset to the risk free asset and thus decrease the risky asset holdings. J_2 represents an investment effect, since increasing β reduces period one consumption and thus increases the investment in the risky asset. Although J_1 and J_2 always affect the risky asset holdings in opposite ways, if $\delta_1 \geq 0$, the allocation effect J_1 always dominates J_2 and thus increasing β reduces the risky asset holdings. The critical role of δ_1 can be understood in terms of the elasticity of substitution. When $\delta_1 \geq 0$, the consumer can be thought of as viewing period one and two consumption as independents or complements and resisting the revision of her optimal $c_1/\widehat{c}_2$ ratio when changes in β alter the slope of the $\widehat{c}_2$ constraint. As a result, the allocation effect always dominates. On the other hand, when $\delta_1 \to -1$, the consumer becomes very substitute oriented and hence very responsive to changes in the slope of the $\widehat{c}_2$ constraint. This means that it is possible corresponding to a change in β that the positive investment effect can dominate and n will actually increase with β.[31]

Returning to Question 2 in Section 1, we see that the parameter γ in the NM utility (3) plays both the role of a certainty persistence parameter and a RPD parameter. With regard to optimal period one consumption, these roles are not in conflict as c_1 decreases with both α and β. However the two roles affect asset demands differently. First when $\gamma = \beta$, it induces two fund separation and impacts the portfolio composition. Second when $\gamma = \alpha$, it alters the allocation between c_1 and the certainty equivalent $\widehat{c}_2$.

Given the difference in risky asset demand behavior with respect to α and β, we next extend Propositions 3 and 4 to the case where both preference effects are present.

[30] Given that $\alpha = 0$ and U is additively separable, it follows that period one consumption is a normal good, implying that $\partial c_1/\partial q_1 < 0$. Noticing from eqn. (56) that $\partial q_1/\partial \beta > 0$ since $\widehat{R} > R_f$, we always have $\partial c_1/\partial \beta < 0$.

[31] More specifically in terms of the proof of Proposition 4, when $\delta_1 \to -1$, since $\frac{p_1 \widehat{R}}{\beta^{1+\delta_1}}$ in eqn. (140) is almost a constant, both eqns. (139) and (140) will reverse signs resulting in $\frac{\partial n}{\partial \beta} > 0$.

Proposition 5 *Assume OCE preferences are defined by eqn. (31). Then we have*

$$\frac{\partial n}{\partial \alpha} > 0. \tag{66}$$

If we assume that $\delta_1 \geq 0$ *and* $\beta \geq \alpha$*, then we have*

$$\frac{\partial n}{\partial \beta} < 0 \qquad \text{and} \qquad \frac{\partial n_f}{\partial \beta} > 0. \tag{67}$$

Assuming $\alpha = \beta = \gamma$, we can obtain the following Corollary.

Corollary 1 *Assume OCE preferences are defined by eqn. (31). Then if* $\alpha = \beta = \gamma$*,*

$$\frac{\partial n}{\partial \gamma} \gtreqless 0 \Leftrightarrow \delta_1 \lesseqgtr 0 \tag{68}$$

and

$$\frac{\partial n_f}{\partial \gamma} > 0 \quad \text{if } \delta_1 \geq 0. \tag{69}$$

Corollary 1 clearly holds for the NM persistence utility special case where $\delta_1 = \delta_2 = \delta$.

Remark 1 *In general, if* $\delta_1 < 0$*, the demand for the risk free asset need not be monotone in* β *or* γ*. For example, in Figure 5, we plot the optimal demands for* c_1, n *and* n_f *versus* γ *when* $\delta_1 = \delta_2 = \delta < 0$*. Clearly although we have* $\frac{\partial c_1}{\partial \gamma} < 0$ *and* $\frac{\partial n}{\partial \gamma} > 0$*, the* n_f *curve is not monotone in* γ*.*

Remark 2 *Lupton (2001) assumes a dynamic setting with Expected Utility preferences, a risk free asset and a single risky asset where price changes are assumed to follow a Gauss-Wiener process. He concludes that the demand for the risky asset is reduced by a habit liability (present value of increases in future consumption induced by habit formation) both in level and as a share of net wealth. Despite our different setting, we similarly find that the existence of a risk free subsistence fund reduces the demand for the risky asset when* $\delta\,(= \delta_1 = \delta_2) > 0$*. However it is clear from Proposition 5 that this decrease in demand is attributable to risk preference dependence and not certainty persistence which is not discernible by Lupton given his use of Expected Utility preferences.*[32]

[32] Detemple and Zapatero (1991) and Constantinides (1990) consider similar dynamic settings where preferences are Expected Utility and risky asset returns follow specific stochastic processes. However these authors focus on asset prices and do not explicitly address the comparative statics of asset demand behavior with respect to changes in the habit formation parameter γ.

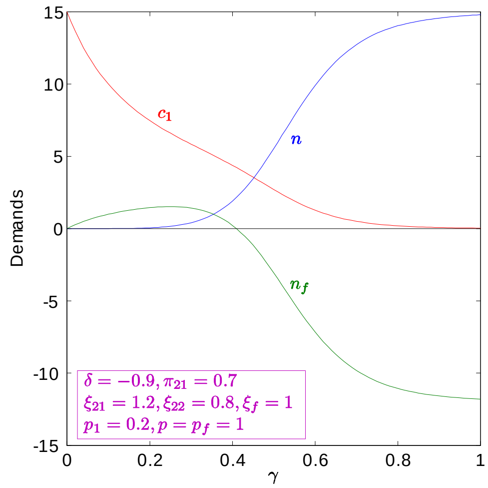


Figure 5:

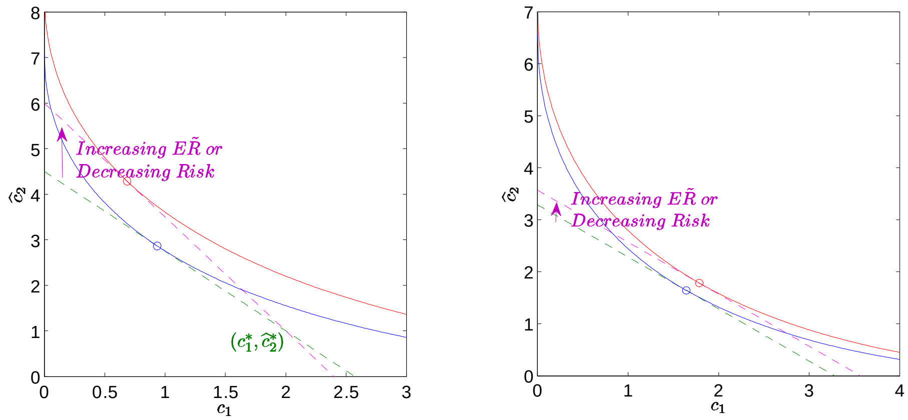
4.4 Selected HARA Conditional Utilities

For the asset return comparative static analyses considered in Section 5, the $\widehat{c}_2(c_1)$ constraint is required to be linear. Following Proposition 1, this will be the case if and only if the conditional NM index $V_{c_1}(c_2)$ is a member of the HARA family. Without loss of generality, the $\widehat{c}_2$ constraint can be expressed as

$$I = q_1 c_1 + q_2 \widehat{c}_2 + \Delta, \tag{70}$$

where Δ is independent of c_1 and $\widehat{c}_2$ and can be thought of as an adjustment to I.

In Table 1, expressions for q_1, q_2 and Δ are provided for popular members of the HARA family. Note that the negative exponential utility is RPI even though βc_1 enters into the utility function, since the βc_1 term can be eliminated via a positive affine transformation without affecting optimal asset holdings. As will become clear in the next section, changes in the risky asset's expected return or risk cause the linear $\widehat{c}_2$ constraint to rotate or shift depending on the particular HARA utility assumed. For instance for the CRRA case (line 1 in Table 1), the constraint in Figure 6(a) rotates around a fixed point $(c_1^*, \widehat{c}_2^*)$. For the negative exponential case (line 3 in Table 1), the constraint in Figure 6(b) makes a parallel shift. This difference is due to the fact that for negative exponential utility, optimal risky asset demand is independent of conditional income $I - p_1 c_1$.



(a) (b)

Figure 6:

$V(c_1, c_2) = -\frac{(c_2 - \beta c_1)^{-\delta_2}}{\delta_2}$	$q_1 = p_1 + \frac{\beta}{R_f} - \frac{\beta}{\widehat{R}},\ q_2 = \frac{1}{\widehat{R}},\ \Delta = 0$
$V(c_1, c_2) = -\frac{(c_2 - a - \beta c_1)^{-\delta_2}}{\delta_2}$	$q_1 = p_1 + \frac{\beta}{R_f} - \frac{\beta}{\widehat{R}},\ q_2 = \frac{1}{\widehat{R}},\ \Delta = \frac{a}{R_f} - \frac{a}{\widehat{R}}$
$V(c_1, c_2) = -\frac{\exp(-\kappa(c_2 - \beta c_1))}{\kappa}$	$q_1 = p_1,\ q_2 = \frac{1}{R_f},\ \Delta = \frac{\ln\left(\pi_{21} k^{R_f p_{22}} + \pi_{22} k^{-R_f p_{21}}\right)}{\kappa R_f}$
$V(c_1, c_2) = (b - c_2 + \beta c_1)^2$	$q_1 = p_1 + \frac{\beta}{R_f} - \frac{\beta}{\widehat{R}},\ q_2 = \frac{1}{\widehat{R}},\ \Delta = \frac{b}{R_f} - \frac{b}{\widehat{R}}$

Table 1:

In the table, it is assumed that $\delta_2 > -1, \alpha, \beta, b, \kappa > 0$ and $a >$ or < 0.

Remark 3 *It should be noted that for the quadratic case (line 4 in Table 1), $\widehat{R}$ is defined by eqn. (42) with $\delta_2 = -2$. For the negative exponential case (line 3 in Table 1), it can be seen from the formula for the $\widehat{c}_2$ constraint implied by q_1, q_2 and Δ that a certainty equivalent return $\widehat{R}$ does not exist.*

5 General Linear Constraint Comparative Statics

In this section, we first prove a new comparative static result for general linear constraints and then show how it can be applied to the consumption-portfolio problem when preferences take the OCE form defined by the certainty persistence U in eqn. (31) and the conditional HARA utilities $V(c_1, c_2)$ in Table 1. Because certainty persistence and risk preference dependence parameters are reflected differently in the $\widehat{c}_2$

linear constraint, it is possible to isolate their respective effects on the comparative statics of pure increases in the risky asset's return and risk. Finally, this isolation facilitates a very simple explanation of the paradox introduced in Section 2.

5.1 General Theorem

We next state our main comparative static Theorem and Corollary and discuss the intuition using a very simple geometric argument.

Theorem 1 *Assume the optimization problem*

$$\max_{c_1,c_2} U(c_1,c_2) = u_1(c_1) + u_2(c_2) \tag{71}$$

$$S.T.\ c_2 - c_2^* = (c_1^* - c_1)\, R, \tag{72}$$

where U is defined up to an increasing transformation, $u_i' > 0$, $u_i'' < 0$ $(i = 1, 2)$, $-R$ is the slope of the constraint and (c_1^*, c_2^*) is the anchor point of the constraint when changing R. Then

(a) *if* $c_2^* = 0$, *then* $\frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1$,

(b) *if* $c_2^* > 0$, *then* $\frac{\partial c_1}{\partial R} < 0$ *if* $-\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \leq 1$,

(c) *if* $c_2^* < 0$, *then* $\frac{\partial c_1}{\partial R} > 0$ *if* $-\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \geq 1$.

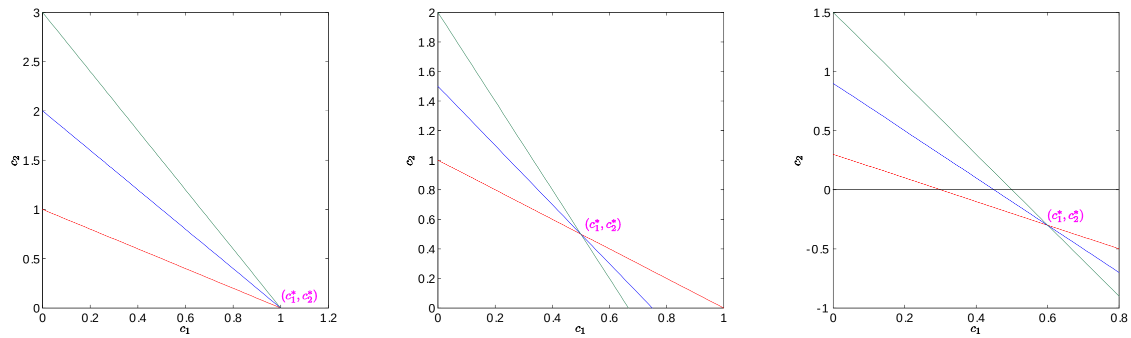
Cases (a), (b) and (c) in Theorem 1 correspond to Figures 7(a), (b) and (c), respectively. When changing R, (c_1^*, c_2^*) remains fixed and the constraint is always anchored at this point. Therefore in applying Theorem 1, if a given utility function can be transformed into the additively separable form (71) and the constraint can be rewritten in the form of eqn. (72), then it is possible to determine the sign of $\frac{\partial c_1}{\partial R}$ from the sign of c_2^*.

Remark 4 *To apply Theorem 1, one must ensure that the parameters resulting in changes in R do not enter into c_1^* or c_2^*. Suppose the constraint takes the form*

$$I = q_1 c_1 + q_2 c_2 + \Delta, \tag{73}$$

where in general q_1, q_2 and Δ are functions of the parameters $\vartheta_1, \vartheta_2, ..., \vartheta_n$, and can be transformed into (72), then

$$R = \frac{q_1}{q_2}. \tag{74}$$



(a) (b) (c)

Figure 7:

To obtain (72), one must solve for c_2^* *and* c_1^*. *Assume that changing* ϑ_1 *results in the different constraints as shown in Figure 7(a), (b) or (c). It should be stressed that* c_1^* *or* c_2^* *cannot be a function of* ϑ_1. *Otherwise, the constraint (73) cannot be transformed into (72). Noticing that at* $c_2 = c_2^*$, c_1 *will not change with* ϑ_1 *and one can obtain* c_2^* *as a function of* ϑ_i $(i \in \{2, 3, ..., n\})$ *from*

$$\left.\frac{\partial c_1}{\partial \vartheta_1}\right|_{c_2=c_2^*} = \left.\frac{\partial\left(\frac{I-\Delta-q_2 c_2}{q_1}\right)}{\partial \vartheta_1}\right|_{c_2=c_2^*} = 0. \tag{75}$$

Similarly, c_1^* *as a function of* ϑ_i $(i \in \{2, 3, ..., n\})$ *can be obtained from*

$$\left.\frac{\partial c_2}{\partial \vartheta_1}\right|_{c_1=c_1^*} = \left.\frac{\partial\left(\frac{I-\Delta-q_1 c_1}{q_2}\right)}{\partial \vartheta_1}\right|_{c_1=c_1^*} = 0. \tag{76}$$

A simple geometric intuition can be given for Theorem 1. Define the marginal rate of substitution and minus the slope of the constraint (72), respectively, by

$$m_1 =_{def} \frac{u_1'(c_1)}{u_2'(c_2)} \qquad \text{and} \qquad m_2 =_{def} -\frac{c_2 - c_2^*}{c_1 - c_1^*}. \tag{77}$$

In Figure 8, consider the two constraint lines anchored at the common point (c_1^*, c_2^*). At the tangency between the lower constraint and indifference curve, $m_1 = m_2$. Increasing R in eqn. (72) corresponds to a rotation of the lower constraint line

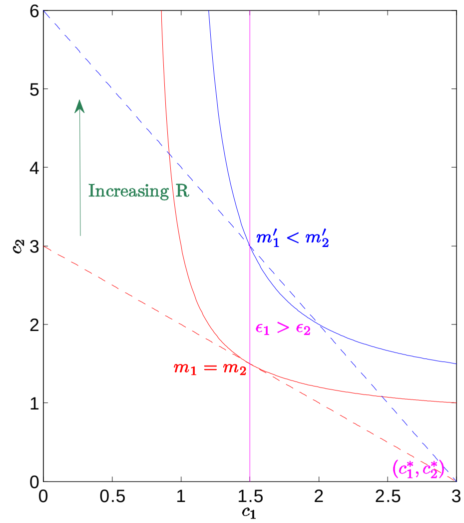


Figure 8:

upward to the right and is equivalent to changing c_2 for a fixed c_1. The elasticities of the two slope changes with respect to c_2 are given by

$$\epsilon_1 =_{def} \frac{\partial \ln m_1}{\partial \ln c_2} = -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \qquad \text{and} \qquad \epsilon_2 =_{def} \frac{\partial \ln m_2}{\partial \ln c_2} = \frac{c_2}{c_2 - c_2^*}. \tag{78}$$

If $\epsilon_1 \gtreqless \epsilon_2$ for $\forall c_2$ then we have $\frac{\partial c_1}{\partial R} \gtreqless 0$. Noticing that

$$c_2^* \lesseqgtr 0 \Leftrightarrow \epsilon_2 \lesseqgtr 1, \tag{79}$$

the results in Theorem 1 follow immediately. Returning to the case in Figure 8, because $\epsilon_1 > \epsilon_2$ in response to an increase in R, the higher indifference curve intersects the shifted constraint at the initial optimal c_1, implying that the tangent to the indifference curve is steeper than the shifted constraint. Therefore, the new optimal c_1 is to the right of the initial c_1-value, implying that c_1 increases with R.

Applying Theorem 1 to CES preferences, yields the following Corollary.

Corollary 2 *Assume the following optimization problem*

$$\max_{c_1, c_2} U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{c_2^{-\delta_1}}{\delta_1} \tag{80}$$

$$S.T.\ c_2 - c_2^* = (c_1^* - c_1) R, \tag{81}$$

where U is defined up to an increasing transformation, $-R$ is the slope of the constraint and (c_1^*, c_2^*) is the anchor point of the constraint when changing R. Then

(a) if $c_2^* = 0$, then $\frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0$,

(b) if $c_2^* > 0$, then $\frac{\partial c_1}{\partial R} < 0$ if $\delta_1 \leq 0$,

(c) if $c_2^* < 0$, then $\frac{\partial c_1}{\partial R} > 0$ if $\delta_1 \geq 0$.

The following Example shows that Theorem 1 may apply even if $u_2(c_2)$ does not take the form of utility in Corollary 2.

Example 1 *Assume the additively separable form of U and constraint in Theorem 1. Let $u_2(c_2)$ be given by*

$$u_2(c_2) = -\frac{c_2^{-(\delta+e)}}{\delta+e} - \frac{c_2^{-(\delta-e)}}{\delta-e}, \tag{82}$$

where $\delta > -1$ and $0 < e \leq 1+\delta$. We show that it is possible to determine whether $-\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtrless 1$ for all c_2 and hence use Theorem 1 to determine the sign of $\partial c_1/\partial R$. Note that u_2 is increasing and concave and can be thought of as being a perturbation of the power utility where δ is increased and decreased by e. In contrast to the CES case where the elasticity $\epsilon_1 = 1+\delta$, for the utility (82) we have

$$\epsilon_1 = -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} = 1 + \delta + \left(\frac{1-c_2^{2e}}{1+c_2^{2e}}\right) e \tag{83}$$

which is bounded by $1+\delta \pm e$ since

$$1+\delta-e < 1+\delta+\left(\frac{1-c_2^{2e}}{1+c_2^{2e}}\right) e < 1+\delta+e. \tag{84}$$

If $\delta - e \geq 0$ ($\delta + e \leq 0$), then it follows that $-\frac{c_2 u_2''(c_2)}{u_2'(c_2)} > (<) 1$.

Remark 5 *It should be emphasized that Theorem 1 sets restrictions only on $u_2(c_2)$. The function $u_1(c_1)$ can be completely general and $u_2(c_2)$ can take more complicated forms than the power utility in (80), such as in Example 1. One can still apply Theorem 1 to obtain comparative statics for c_1 even though it may not be possible to derive an analytic expression for optimal c_1.*

5.2 Certainty Persistence Comparative Statics

In this subsection and the next, four cases are considered which illustrate the application of Corollary 2. The first case assumes the certainty persistence U and budget constraint considered in Section 2. The process outlined in Remark 4 is illustrated in some detail, allowing for a more abbreviated analysis in Cases 2-4 below.

Case 1 *Assume the following certainty habit persistence optimization problem*

$$\max_{c_1} U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{(c_2 - \alpha c_1)^{-\delta_1}}{\delta_1}, \alpha > 0 \qquad S.T.\ c_2 = (I - p_1 c_1) R_f. \tag{85}$$

Transform the problem into the equivalent

$$\max_{c_1} \ -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{(c_2^{new})^{-\delta_1}}{\delta_1} \qquad S.T.\ c_2^{new} = c_2 - \alpha c_1 = (I - p_1 c_1) R_f - \alpha c_1. \tag{86}$$

In order to apply Corollary 2, the c_2^{new} constraint needs to be rewritten in the form of (81). First observing that the marginal costs $q_1 = p_1 + \alpha/R_f$, $q_2 = 1/R_f$ and

$$R = \frac{q_1}{q_2} = p_1 R_f + \alpha, \tag{87}$$

the risk free rate of interest R_f can be viewed as the parameter ϑ_1 in Remark 4. Next it is necessary to solve for $(c_2^{new})^*$ and c_1^* and then verify that each is independent of R_f. To find $(c_2^{new})^*$, express c_1 as a function of R_f and c_2^{new}

$$c_1 = \frac{I R_f - c_2^{new}}{p_1 R_f + \alpha} \tag{88}$$

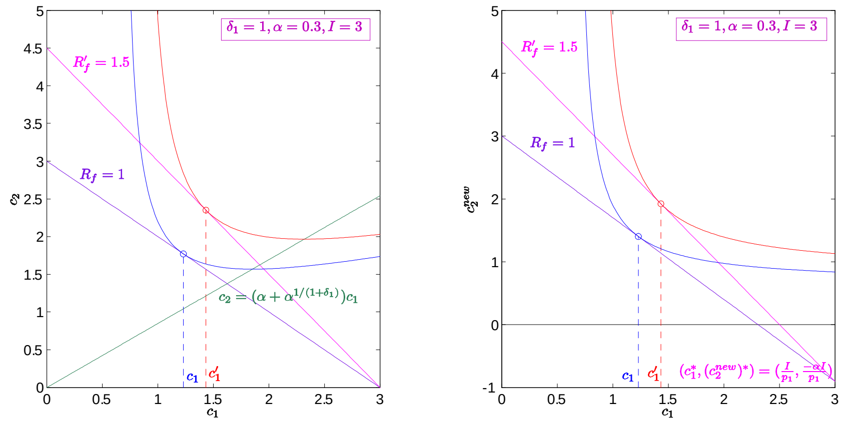
and then differentiate with respect to R_f holding $c_2^{new} = (c_2^{new})^*$. Since we are at the anchor point, it follows that

$$\begin{aligned} \left.\frac{\partial c_1}{\partial R_f}\right|_{c_2^{new} = \left(c_2^{new}\right)^*} &= 0 = \frac{\alpha I + p_1 (c_2^{new})^*}{(p_1 R_f + \alpha)^2} \\ &\Rightarrow (c_2^{new})^* = -\frac{\alpha I}{p_1} < 0. \end{aligned} \tag{89}$$

Analogously differentiating the c_2^{new} constraint with respect to R_f holding $c_1 = c_1^*$, yields

$$\begin{aligned} \left.\frac{\partial c_2^{new}}{\partial R_f}\right|_{c_1 = c_1^*} &= 0 = I - p_1 c_1^* \\ &\Rightarrow c_1^* = \frac{I}{p_1}. \end{aligned} \tag{90}$$

Clearly c_1^* and $(c_2^{new})^*$ satisfy the requirement of being independent of R_f. Therefore, it follows from Corollary 2(c) that $\delta_1 \geq 0$ implies $\frac{\partial c_1}{\partial R_f} > 0$. The geometry for this Case is shown in Figure 9. Figure 9(a) illustrates that for the optimization of the initial untransformed problem, c_1 increases with R_f when $\delta_1 = 1 > 0$. However, since



(a) (b)

Figure 9:

the indifference curves do not correspond to CES preferences, Corollary 2 cannot be applied directly. In Figure 9(b), the same problem is considered in the new coordinate system, $c_1 - c_2^{new}$*, utilizing standard CES indifference curves. Since* $(c_2^{new})^* < 0$*, it follows that* c_1 *increases with* R_f *when* $\delta_1 = 1 > 0$*.*

Remark 6 *When comparing Figures 9(a) and (b), it is natural to wonder why for considering changes in* R_f *there is a shift in the anchor point from* $(I/p_1, 0)$ *to* $(I/p_1, -\alpha I/p_1)$*. This follows immediately from the fact that when the utility is transformed from* $U(c_1, c_2 - \alpha c_1)$ *to* $U(c_1, c_2^{new})$*, since*

$$c_2^{new} = c_2 - \alpha c_1, \tag{91}$$

the c_2*-component of the anchor point shifts from* $c_2^* = 0$ *to* $(c_2^{new})^* = -\alpha c_1^*$*.*

5.3 Uncertainty Comparative Statics

By using Corollary 2, it is possible to resolve the paradox introduced in Section 2 concerning the effects on optimal period one consumption of a pure increase in $E\widetilde{R}$ (and a mean preserving increase in the risk associated with $\widetilde{R}$). Three cases based on the certainty persistence utility and a generalization of the RPD utility in (31) are considered. They yield a clear separation of the roles of certainty persistence and risk preference dependence. In order to derive these comparative static results, it is necessary to express the certainty equivalent return $\widehat{R}$ in terms of the risky asset's payoffs ξ_{21} and ξ_{22}. Substituting the expressions for p_{21} and p_{22} from (24) into the formula (42) for $\widehat{R}$, it is then possible to prove the following two results.

Result 1 *Assume the conditional NM index $V_{c_1}(c_2)$ is a member of the HARA family*[33] *and the bivariate asset return is given by*

$$R_{2s}(\theta) = R_{2s} + \theta, \tag{92}$$

where $R_{2s} = \frac{\xi_{2s}}{p}$ $(s = 1, 2)$ and $\theta > 0$ is a pure expected return shift parameter. Then it follows that

$$\frac{\partial \widehat{R}}{\partial E\widetilde{R}} = \frac{\partial \widehat{R}}{\partial \theta} > 0, \tag{93}$$

where $E\widetilde{R} = \sum_{s=1}^{2} \pi_{2s} R_{2s}$.

Result 2 *Assume the conditional NM index $V_{c_1}(c_2)$ is a member of the HARA family and the bivariate asset return is given by*

$$R_{21}(\lambda) = \lambda R_{21} - (\lambda - 1)E\widetilde{R} \qquad \textit{and} \qquad R_{22}(\lambda) = \lambda R_{22} - (\lambda - 1)E\widetilde{R}, \tag{94}$$

where $R_{2s} = \frac{\xi_{2s}}{p}$ $(s = 1, 2)$, $E\widetilde{R} = \sum_{s=1}^{2} \pi_{2s} R_{2s}$ and $\lambda > 0$ is a risk spread parameter. Then corresponding to a mean preserving increase in risk associated with λ,

$$\frac{\partial \widehat{R}}{\partial \lambda} < 0. \tag{95}$$

Although these results are based on the assumption of a two asset complete market setting, they can easily be extended to the more general cases of multiple risky assets and incomplete markets if one makes the natural modifications to the definition of $\widehat{R}$. The extension to incomplete markets follows from the fact that $\widehat{R}$ is defined only for the HARA class of conditional risk preferences, and in this case the incomplete markets are effectively complete as discussed above in connection with Proposition 1.[34]

[33] It should be noted that the negative exponential utility cannot be assumed in Results 1 and 2, since following Remark 3 no $\widehat{R}$ exists for this utility.

[34] For example, suppose V takes the CRRA form and there are m risky assets and one risk free asset. Then $\widehat{c}_2 = \widehat{R}(I - p_1 c_1)$. If $I - pc_1$ is fixed and $E\widetilde{R}_i$ for risky asset i ($i \in \{1, ..., m\}$) increases, $\widetilde{c}_2 = \sum_{i=1}^{m} n_i \widetilde{\xi}_{2i} + n_f \xi_f$ must increase implying that $\widehat{c}_2$ and $\widehat{R}$ increase as well. Therefore, $\widehat{R}$ is an increasing function of the expected return for any risky asset. A similar argument can be made for other members of the HARA class. Also, the argument extends to changes in risk.

For each of the subsequent uncertainty cases it is possible to characterize the sign of $\partial c_1/\partial \widehat{R}$, where utilizing Results 1 and 2 the increase in $\widehat{R}$ can be viewed as arising from an either an increase in $E\widetilde{R}$ or a mean preserving reduction in risk.

Case 2 *Consider the OCE setting with certainty persistence, where*

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{(c_2 - \alpha c_1)^{-\delta_1}}{\delta_1}, \alpha > 0 \qquad \textit{and} \qquad V(c_2) = -\frac{c_2^{-\delta_2}}{\delta_2}. \tag{96}$$

The period one consumption-savings problem (40) - (42) can be transformed into

$$\max_{c_1} \; -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{(\widehat{c}_2^{new})^{-\delta_1}}{\delta_1} \qquad S.T.\; \widehat{c}_2^{new} = \widehat{c}_2 - \alpha c_1 = (I - p_1 c_1)\widehat{R} - \alpha c_1. \tag{97}$$

Proceeding as in Case 1, it can be verified that

$$(\widehat{c}_2^{new})^* = -\frac{\alpha I}{p_1} < 0 \qquad \textit{and} \qquad R = p_1 \widehat{R} + \alpha. \tag{98}$$

Then based on Corollary 2(c), $\delta_1 \geq 0$ implies $\frac{\partial c_1}{\partial \widehat{R}} > 0$.

Case 3 *Consider the OCE setting with risk preference dependence*

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{c_2^{-\delta_1}}{\delta_1} \qquad \textit{and} \qquad V_{c_1}(c_2) = -\frac{(c_2 - \beta c_1)^{-\delta_2}}{\delta_2}, \beta > 0. \tag{99}$$

It can be verified that the constraint can be written as[35]

$$\widehat{c}_2 = (I - p_1 c_1)\widehat{R} - c_1 \beta \left(\frac{\widehat{R}}{R_f} - 1 \right). \tag{100}$$

Since

$$\widehat{c}_2^* = \frac{\beta I}{p_1 + \beta/R_f} > 0 \qquad \textit{and} \qquad R = \left(p_1 + \frac{\beta}{R_f} \right) \left(\widehat{R} - \frac{\beta}{p_1 + \beta/R_f} \right), \tag{101}$$

it follows from Corollary 2(b) that $\delta_1 \leq 0$ implies $\frac{\partial c_1}{\partial \widehat{R}} < 0$. The geometry is shown in Figure 10. Since the preferences are already CES, it is not necessary to transform U and create a new coordinate system. It follows from $\widehat{c}_2^* > 0$ that $\delta_1 = -0.5 < 0$ implies $\frac{\partial c_1}{\partial \widehat{R}} < 0$.

[35] As noted in footnote 25, $\widehat{R}$ is the certainty equivalent of the return from the risky mutual fund only and the term $c_1\beta\left(\frac{\widehat{R}}{R_f} - 1\right)$ in the $\widehat{c}_2$ constraint is a risk preference dependence perturbation term incorporating the effect of the risk free subsistence fund.

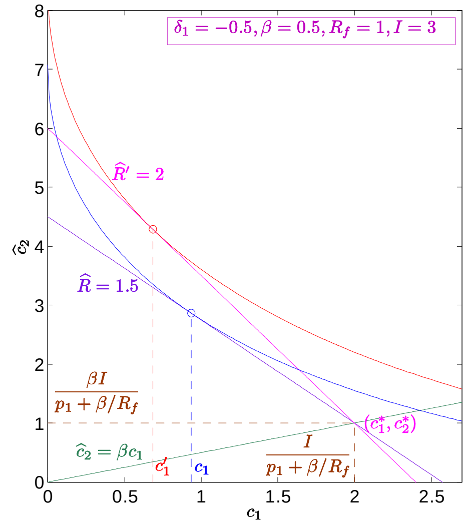


Figure 10:

Remark 7 *It is clear from Figure 10 that for the RPD case, the anchor point* (c_1^*, c_2^*) *is determined by the intersection of the ray corresponding to the requirement that* $\widehat{c}_2 > \beta c_1$ *and the* $\widehat{c}_2$ *constraint. Equating eqn. (100) and* $\widehat{c}_2 = \beta c_1$ *and solving for period one consumption yields*

$$c_1^* = \frac{I}{p_1 + \frac{\beta}{R_f}} \qquad \text{and} \qquad \widehat{c}_2^* = \beta c_1^*. \tag{102}$$

Comparing the certainty persistence and RPD cases, the anchor points satisfy $\widehat{c}_2^* = \alpha c_1^*$ *and* $\widehat{c}_2^* = \beta c_1^*$*, respectively. The two cases differ in the value of* c_1^**. The RPD term* $p_1 + \frac{\beta}{R_f}$ *will be recognized to be the marginal direct cost of* c_1 *plus the indirect cost of the risk free subsistence fund (see the discussion of eqn. (54) in Subsection 4.2).*

Case 4 *Consider the OCE setting with both certainty persistence and risk preference dependence*

$$U(c_1, c_2) = -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{(c_2 - a - \alpha c_1)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V_{c_1}(c_2) = -\frac{(c_2 - a - \beta c_1)^{-\delta_2}}{\delta_2}, \tag{103}$$

where a *is allowed to be positive, zero or negative,* $c_2 > a + \alpha c_1$ *and* $c_2 > a + \beta c_1$*. It can be verified that the constraint can be written as*

$$\widehat{c}_2 = (I - p_1 c_1)\widehat{R} - (\beta c_1 + a)\,\widehat{R}\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right). \tag{104}$$

Defining $\widehat{c}_2^{new} = \widehat{c}_2 - \alpha c_1$, *the transformed optimization problem becomes*

$$\max_{c_1} \; -\frac{c_1^{-\delta_1}}{\delta_1} - \frac{\left(\widehat{c}_2^{new}\right)^{-\delta_1}}{\delta_1} \;\; S.T. \; \widehat{c}_2^{new} = (I - p_1 c_1)\widehat{R} - (\beta c_1 + a)\,\widehat{R}\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right) - \alpha c_1. \tag{105}$$

Since

$$q_1 = p_1 + \left(\frac{\beta}{R_f} - \frac{\beta - \alpha}{\widehat{R}}\right) \qquad \textit{and} \qquad q_2 = \frac{1}{\widehat{R}}, \tag{106}$$

it follows that

$$R = \frac{q_1}{q_2} = \left(p_1 + \frac{\beta}{R_f}\right)\left(\widehat{R} - \frac{\beta - \alpha}{p_1 + \beta/R_f}\right). \tag{107}$$

Moreover,

$$\left.\frac{\partial \widehat{c}_2^{new}}{\partial \widehat{R}}\right|_{c_1 = c_1^*} = I - \frac{a}{R_f} - p_1 c_1^* - \frac{\beta}{R_f} c_1^* = 0 \Rightarrow c_1^* = \frac{I R_f - a}{p_1 R_f + \beta}. \tag{108}$$

Noticing that

$$c_1 = -\frac{\widehat{c}_2^{new} + a\left(\frac{\widehat{R}}{R_f} - 1\right) - I\widehat{R}}{p_1\widehat{R} + \frac{\beta\widehat{R}}{R_f} - \beta + \alpha}, \tag{109}$$

one can obtain

$$\left.\frac{\partial c_1}{\partial \widehat{R}}\right|_{\widehat{c}_2^{new} = \left(\widehat{c}_2^{new}\right)^*} = -\frac{R_f\left((\beta - \alpha) R_f I + a(\alpha + p_1 R_f) - (\beta + p_1 R_f)\left(\widehat{c}_2^{new}\right)^*\right)}{\left(\beta\left(\widehat{R} - R_f\right) + \left(\alpha + p_1\widehat{R}\right) R_f\right)^2} = 0, \tag{110}$$

implying that

$$\left(\widehat{c}_2^{new}\right)^* = \frac{a(p_1 R_f + \alpha) + (\beta - \alpha) R_f I}{p_1 R_f + \beta}. \tag{111}$$

Following Corollary 2, one can conclude that

(i) when $a > 0$,

$$\alpha \le \beta: \qquad \frac{\partial c_1}{\partial \widehat{R}} < 0 \;\; \textit{if} \;\; \delta_1 \le 0; \tag{112}$$

(ii) when $a = 0$,

$$\alpha = \beta: \qquad \frac{\partial c_1}{\partial \widehat{R}} \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0, \tag{113}$$

$$\alpha < \beta: \qquad \frac{\partial c_1}{\partial \widehat{R}} < 0 \;\; \textit{if} \;\; \delta_1 \le 0, \tag{114}$$

$$\alpha > \beta: \qquad \frac{\partial c_1}{\partial \widehat{R}} > 0 \;\; \textit{if} \;\; \delta_1 \ge 0; \tag{115}$$

(iii) when $a < 0$,

$$\alpha \ge \beta: \qquad \frac{\partial c_1}{\partial \widehat{R}} > 0 \;\; \textit{if} \;\; \delta_1 \ge 0. \tag{116}$$

We can now readily resolve the paradox introduced in Section 2. There it was observed that on the one hand the presence of the persistence parameter γ in the NM index (3) does not affect the necessary and sufficient for condition for $\partial c_1/\partial E\widetilde{R} \gtreqqless 0$ and on the other hand the presence of the persistence parameter in the analogous CES certainty utility does affect the condition for $\partial c_1/\partial R_f \gtreqqless 0$. Assuming $a = 0$, it is immediately clear from eqn. (111) that $(c_2^{new})^* = 0$ if $\alpha = \beta = \gamma$ whether or not the common value is equal to 0. As a result, the Expected Utility expression (113) holds and using Result 1, the condition for $\partial c_1/\partial E\widetilde{R} \gtreqqless 0$ is unaffected by the persistence parameter. Effectively the negative $(c_2^{new})^*$ component of the anchor point for certainty persistence (Case 2) is exactly offset by the positive $\widehat{c}_2^*$ component for risk preference dependence (Case 3), resulting in Figure 7(a) applying whether or not γ is assumed to equal 0. For the certainty case only the situation without persistence corresponds to Figure 7(a), with Figure 7(c) holding in the presence of persistence.

Given the strong comparative static consequences of assuming $\alpha = \beta$ together with the observations in Subsection 4.1 that certainty persistence and risk preference dependence reflect very different preference attributes, it seems quite reasonable to allow the two parameters to be different. But then, can a case be made for assuming $\alpha > \beta$ or $\beta > \alpha$? (To simplify the discussion, assume $a = 0$.) First, note that $c_{22} > \beta c_1$ ensures no bankruptcy. Since βc_1 is the minimum subsistence level that will be tolerated when facing random period two consumption, a consumer will not allow consumption in any state to fall below this level no matter how small the probability π_{22} or how attractive c_{21} is. Second, $\widehat{c}_2 > \left(\alpha + \alpha^{\frac{1}{1+\delta_1}}\right) c_1$ guarantees positive marginal utility with αc_1 establishing the minimum level for the certainty equivalent $\widehat{c}_2$. This certainty persistence restriction can allow for a very low level of c_{22} so long as the overall $\widehat{c}_2$ based on consumption in both states is above $\left(\alpha + \alpha^{\frac{1}{1+\delta_1}}\right) c_1$. Hence it may not be unreasonable to suppose that βc_1 rather than αc_1 establishes the bare minimum consumption, suggesting that $\alpha > \beta$.

Remark 8 *Suppose V takes the negative exponential form as in line 3 of Table 1. Then changing $E\widetilde{R}$ or risk results in parallel shifts of the $\widehat{c}_2$ constraint as in Figure 6(b) rather than a rotation around a fixed anchor point $(c_1^*, \widehat{c}_2^*)$. Such a change in the constraint is inconsistent with the characterization in (72) of Theorem 1 and as reflected in Figure 7. Hence Theorem 1 cannot be applied for this member of the HARA family.*

6 Conclusion

In this paper, I have shown that the classic Expected Utility habit formation model confounds the different meanings and consequences of certainty persistence and risk preference dependence. The former has the interpretation that the more I eat today the hungrier I become tomorrow, while the latter is associated with the more I eat today, the more I want to avoid risk in my consumption tomorrow and the larger my subsistence requirement. In addition to being associated with different preference interpretations, these two properties can have opposite effects on risky asset demand and generate potentially opposite responses to increases in the risky asset's expected return and risk. This confounding parallels that of the Arrow-Pratt risk aversion measure and the reciprocal of the intertemporal elasticity of substitution in CES Expected Utility preferences that I first noted in Selden (1978, 1979). Today it is generally viewed that assuming $\delta_1 = \delta_2$ or equivalently $\tau_R = 1/\eta$ is an overly strong restriction to impose on preferences.

Given the important role that the Expected Utility habit formation model has played in asset pricing and macroeconomics, the results in this paper suggest a number of potentially interesting questions for future research. One natural extension would be to dynamic settings. Another very interesting but quite different area for future work would involve the application of the new laboratory methodologies for testing different uncertainty preference hypotheses in terms of observed asset demand behavior by subjects in an experimental environment (e.g., Choi, et. al. 2007). In a two period setting such as assumed in this paper, do individuals' certainty preferences exhibit certainty persistence and do their risk preferences exhibit risk preference dependence? If both are verified, then can one show based on observed choices that the persistence and risk preference dependence effects are different?[36]

Finally as noted in Section 1, there is a clear connection between the habit formation and reference dependent preference models. For the latter, there also exist different strands of research which consider certainty and risky settings. What would be the implications of assuming that the reference point and loss aversion functions differ between certainty and uncertainty environments in multiperiod applications such as considered in this paper?

Appendix

[36] Indeed still another possibility not considered in this paper is that the parameters α and β are neither identical nor independent, but rather exhibit some functional relationship.

A Proof of Proposition 1

First prove necessity. $\widehat{c}_2$ is a linear function of c_1 only if each state c_{2i} is a linear function of c_1. It follows from Pollak (1971) that the NM index must be a HARA member.[37] Next prove sufficiency. If h is a HARA member, it can be easily verified that (Gollier 2001)

$$-\frac{\partial^2 h/\partial c_2^2}{\partial h/\partial c_2} = \frac{1}{a + b\left(c_2 - \zeta c_1\right)} = \frac{1}{\left(a - b\zeta c_1\right) + bc_2}, \tag{117}$$

where a and b are arbitrary constants. Since for the conditional problem, c_1 is fixed, it follows from Selden (1980, Corollary, p. 440) that $\widehat{c}_2$ is a linear function of c_1. Since ζc_1 will not affect linearity of the $\widehat{c}_2$ constraint and an affine transformation of V based on c_1 will not change the $\widehat{c}_2$ constraint, we can conclude that if

$$V_{c_1}\left(c_2\right) = f\left(c_1\right) h\left(c_2 - \zeta c_1\right) + g\left(c_1\right), \tag{118}$$

the $\widehat{c}_2$ constraint is linear in c_1.

B Proof of Proposition 2

The computation of $\frac{\partial(1/\eta)}{\partial\alpha}$ is straightforward as shown in eqn. (34). This proof establishes the sign of the derivative. To simplify the notation, let $x = \frac{c_2}{c_1}$. It follows from (34) that

$$\frac{\partial \eta}{\partial \alpha} \gtreqless 0 \Leftrightarrow f\left(\alpha\right) \lesseqgtr 0, \tag{119}$$

where

$$f\left(\alpha\right) = 1 + \left(x - \alpha\right)^{-1-\delta_1}\left(\alpha\delta_1 + x - \alpha\right). \tag{120}$$

For any fixed x,

$$0 < \alpha < \alpha_{\max}, \tag{121}$$

where $\alpha_{\max}$ satisfies

$$\alpha_{\max} + \alpha_{\max}^{\frac{1}{1+\delta_1}} = x. \tag{122}$$

When $\delta_1 \geq 0$, we have

$$\alpha\delta_1 + x - \alpha > \alpha\delta_1 \geq 0 \Rightarrow f\left(\alpha\right) > 0 \Rightarrow \frac{\partial\eta}{\partial\alpha} < 0. \tag{123}$$

[37] It should be noted that the modified Bergson family defined in Pollak (1971) corresponds to the HARA class of NM indices (Gollier 2001 and Rubinstein 1976).

When $\delta_1 < 0$, it can be verified that

$$f'(\alpha) = \delta_1 (x - \alpha)^{-2-\delta_1} (2x + (\delta_1 - 1)\alpha). \tag{124}$$

Since $-1 < \delta_1 < 0$, we have

$$2x + (\delta_1 - 1)\alpha > 2(x - \alpha) > 0. \tag{125}$$

Therefore, $f'(\alpha) < 0$, implying $f(\alpha)$ is a monotonically decreasing function of α. Moreover when $\alpha = \alpha_{\max}$, it follows that

$$f(\alpha) = 1 + \alpha_{\max}^{-1}\left(\alpha_{\max}\delta_1 + \alpha_{\max}^{\frac{1}{1+\delta_1}}\right) = 1 + \delta_1 + \alpha_{\max}^{-\frac{\delta_1}{1+\delta_1}} > 0. \tag{126}$$

Therefore we always have $f(\alpha) > 0$, $\partial\eta/\partial\alpha < 0$ and $\partial(1/\eta)/\partial\alpha > 0$.

C Proof of Proposition 3

To prove this result, we first introduce the following Lemma.

Lemma 1 *For the consumption-portfolio problem in the OCE setting where U and V are given in (31), optimal period one consumption satisfies*

$$\frac{\partial c_1}{\partial \alpha} < 0 \text{ and } \frac{\partial c_1}{\partial \beta} < 0. \tag{127}$$

Proof. Solving the consumption-portfolio problem, it can be verified that

$$c_1 = \frac{I}{p_1 + \alpha/\widehat{R} + \beta\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right) + \left(p_1\widehat{R} + \alpha + \beta\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{\frac{1}{1+\delta_1}}/\widehat{R}}. \tag{128}$$

Obviously,

$$\frac{\partial c_1}{\partial \alpha} < 0. \tag{129}$$

Since $\widehat{R} > R_f$, it also follows that

$$\frac{\partial c_1}{\partial \beta} < 0. \tag{130}$$

■

Next we prove the result. Since Lemma 1 still applies when $\beta = 0$, it follows from

$$p_1\frac{\partial c_1}{\partial \alpha} + p\frac{\partial n}{\partial \alpha} + p_f\frac{\partial n_f}{\partial \alpha} = 0 \tag{131}$$

and

$$\frac{\partial c_1}{\partial \alpha} < 0 \tag{132}$$

that

$$p\frac{\partial n}{\partial \alpha} + p_f\frac{\partial n_f}{\partial \alpha} > 0. \tag{133}$$

It follows from (58) that

$$\frac{\partial \left(\frac{n_f}{n}\right)}{\partial \alpha} = 0 \tag{134}$$

and remembering that $n > 0$, we can immediately obtain

$$\frac{\partial n}{\partial \alpha} > 0 \qquad \text{and} \qquad \frac{\partial |n_f|}{\partial \alpha} > 0. \tag{135}$$

D Proof of Proposition 4

Since Lemma 1 still applies when $\alpha = 0$, we have

$$\frac{\partial c_1}{\partial \beta} < 0. \tag{136}$$

It follows from substituting the optimal period one consumption from (128) where $\alpha = 0$ into the conditional demand function (63) that

$$n = \frac{I\left(1 - k^{\frac{1}{1+\delta_2}}\right)\left(\left(p_1\widehat{R} + \beta\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{\frac{1}{1+\delta_1}} - \beta\right)}{\widehat{R}\left(p_1 + \beta\left(\frac{1}{R_f} - \frac{1}{\widehat{R}}\right) + \left(\left(p_1\widehat{R} + \beta\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{\frac{1}{1+\delta_1}}\right)/\widehat{R}\right)\left(p_{21} + k^{\frac{1}{1+\delta_2}}p_{22}\right)(\xi_{21} - \xi_{22})}, \tag{137}$$

which can be rewritten as

$$n = \frac{I\left(1 - k^{\frac{1}{1+\delta_2}}\right)\left(1 - \left(\frac{p_1\widehat{R}}{\beta^{1+\delta_1}} + \frac{1}{\beta^{\delta_1}}\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{-\frac{1}{1+\delta_1}}\right)}{\left(\left(p_1\widehat{R} + \beta\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{\frac{\delta_1}{1+\delta_1}} + 1\right)\left(p_{21} + k^{\frac{1}{1+\delta_2}}p_{22}\right)(\xi_{21} - \xi_{22})}. \tag{138}$$

If $\delta_1 \geq 0$,

$$\frac{\partial \left(p_1\widehat{R} + \beta\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{\frac{\delta_1}{1+\delta_1}}}{\partial \beta} \geq 0 \tag{139}$$

and

$$\frac{\partial \left(\frac{p_1\widehat{R}}{\beta^{1+\delta_1}} + \frac{1}{\beta^{\delta_1}}\left(\frac{\widehat{R}}{R_f} - 1\right)\right)^{-\frac{1}{1+\delta_1}}}{\partial \beta} > 0. \tag{140}$$

Since $n \geq 0$, it follows that

$$\frac{\partial n}{\partial \beta} < 0. \tag{141}$$

Noticing that

$$p_1 \frac{\partial c_1}{\partial \beta} + p \frac{\partial n}{\partial \beta} + p_f \frac{\partial n_f}{\partial \beta} = 0, \tag{142}$$

if

$$\frac{\partial c_1}{\partial \beta} < 0 \quad \text{and} \quad \frac{\partial n}{\partial \beta} < 0, \tag{143}$$

it follows from the budget constraint that

$$\frac{\partial n_f}{\partial \beta} > 0. \tag{144}$$

E Proof of Proposition 5

For the consumption-portfolio problem where the OCE utilities U and V are given in (31), conditional optimal asset demands satisfy eqn. (61),[38] implying that

$$\frac{\partial \left(\frac{n_f - \beta c_1 / \xi_f}{n} \right)}{\partial \alpha} = 0. \tag{145}$$

Following a similar argument as in eqns. (131) - (133), we can conclude that

$$\frac{\partial n}{\partial \alpha} > 0. \tag{146}$$

It follows from substituting the optimal period one consumption from (128) into the conditional demand function (63) that

$$n = \frac{I \left(1 - k^{\frac{1}{1+\delta_2}}\right) \left(\left(p_1 \widehat{R} + \alpha + \beta \left(\frac{\widehat{R}}{R_f} - 1 \right) \right)^{\frac{1}{1+\delta_1}} + \alpha - \beta \right)}{\left(p_1 \widehat{R} + \alpha + \beta \left(\frac{\widehat{R}}{R_f} - 1 \right) + \left(p_1 \widehat{R} + \alpha + \beta \left(\frac{\widehat{R}}{R_f} - 1 \right) \right)^{\frac{1}{1+\delta_1}} \right) \left(p_{21} + k^{\frac{1}{1+\delta_2}} p_{22} \right) (\xi_{21} - \xi_{22})}, \tag{147}$$

which can be rewritten as

$$n = \frac{I \left(1 - k^{\frac{1}{1+\delta_2}}\right) \left(1 - \frac{1 - \alpha/\beta}{\left(\frac{p_1 \widehat{R} + \alpha}{\beta^{1+\delta_1}} + \frac{1}{\beta^{\delta_1}} \left(\frac{\widehat{R}}{R_f} - 1 \right) \right)^{\frac{1}{1+\delta_1}}} \right)}{\left(\left(p_1 \widehat{R} + \alpha + \beta \left(\frac{\widehat{R}}{R_f} - 1 \right) \right)^{\frac{\delta_1}{1+\delta_1}} + 1 \right) \left(p_{21} + k^{\frac{1}{1+\delta_2}} p_{22} \right) (\xi_{21} - \xi_{22})}. \tag{148}$$

[38] It should be noted that eqn. (61) holds no matter whether $\alpha = 0$ or not.

If $\delta_1 \geq 0$, we have

$$\frac{\partial\left(p_1\widehat{R}+\alpha+\beta\left(\frac{\widehat{R}}{R_f}-1\right)\right)^{\frac{\delta_1}{1+\delta_1}}}{\partial\beta} \geq 0, \tag{149}$$

$$\frac{\partial\left(1-\alpha/\beta\right)}{\partial\beta} > 0 \tag{150}$$

and

$$\frac{\partial\left(\frac{p_1\widehat{R}+\alpha}{\beta^{1+\delta_1}}+\frac{1}{\beta^{\delta_1}}\left(\frac{\widehat{R}}{R_f}-1\right)\right)^{-\frac{1}{1+\delta_1}}}{\partial\beta} > 0. \tag{151}$$

Therefore, when assuming $\beta \geq \alpha$, we have

$$\frac{\partial n}{\partial\beta} < 0 \quad \text{if} \quad \delta_1 \geq 0. \tag{152}$$

Finally, since

$$\frac{\partial c_1}{\partial\beta} < 0 \quad \text{and} \quad \frac{\partial n}{\partial\beta} < 0, \tag{153}$$

it follows from the budget constraint that

$$\frac{\partial n_f}{\partial\beta} > 0. \tag{154}$$

F Proof of Corollary 1

If $\alpha = \beta = \gamma$, it follows from eqn. (148) that

$$n = \frac{I\left(1-k^{\frac{1}{1+\delta_2}}\right)}{\left(\left(p_1\widehat{R}+\gamma\frac{\widehat{R}}{R_f}\right)^{\frac{\delta_1}{1+\delta_1}}+1\right)\left(p_{21}+k^{\frac{1}{1+\delta_2}}p_{22}\right)\left(\xi_{21}-\xi_{22}\right)}. \tag{155}$$

It can be easily seen that

$$\frac{\partial n}{\partial\gamma} \gtreqless 0 \Leftrightarrow \delta_1 \lesseqgtr 0. \tag{156}$$

When $\delta_1 \geq 0$, we have

$$\frac{\partial c_1}{\partial\gamma} < 0 \quad \text{and} \quad \frac{\partial n}{\partial\gamma} \leq 0, \tag{157}$$

implying that

$$\frac{\partial n_f}{\partial\gamma} > 0. \tag{158}$$

G Proof of Theorem 1

Differentiating the first order condition

$$\frac{u_1'(c_1)}{u_2'(c_2)} = R \tag{159}$$

with respect to R, yields

$$u_1''(c_1)\frac{\partial c_1}{\partial R} = Ru_2''(c_2)\frac{\partial c_2}{\partial R} + u_2'(c_2). \tag{160}$$

Differentiating the constraint

$$c_2 - c_2^* = (c_1^* - c_1)R \tag{161}$$

with respect to R, it follows that

$$\frac{\partial c_2}{\partial R} = (c_1^* - c_1) - R\frac{\partial c_1}{\partial R}. \tag{162}$$

Substituting eqn. (162) into (160) yields

$$(u_1''(c_1) + Ru_2''(c_2))\frac{\partial c_1}{\partial R} = u_2''(c_2)(c_1^* - c_1) + u_2'(c_2). \tag{163}$$

Since we require that the optimal point given by the first order condition is a local maximum, the second order condition ensures that

$$u_1''(c_1) + R^2 u_2''(c_2) < 0. \tag{164}$$

Therefore,

$$\frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow Ru_2''(c_2)(c_1^* - c_1) + u_2'(c_2) \lesseqgtr 0. \tag{165}$$

Notice that

$$Ru_2''(c_2)(c_1^* - c_1) + u_2'(c_2) \lesseqgtr 0 \Leftrightarrow -\frac{(c_2 - c_2^*)u_2''(c_2)}{u_2'(c_2)} \gtreqless 1. \tag{166}$$

If $c_2^* = 0$, then we have

$$\frac{\partial c_1}{\partial R} \gtreqless 0 \Leftrightarrow -\frac{c_2 u_2''(c_2)}{u_2'(c_2)} \gtreqless 1. \tag{167}$$

If $c_2^* > 0$, then

$$-\frac{c_2 u_2''(c_2)}{u_2'(c_2)} < 1 \Rightarrow -\frac{(c_2 - c_2^*)u_2''(c_2)}{u_2'(c_2)} < 1 \Rightarrow \frac{\partial c_1}{\partial R} < 0. \tag{168}$$

If $c_2^* < 0$, then

$$-\frac{c_2 u_2''(c_2)}{u_2'(c_2)} > 1 \Rightarrow -\frac{(c_2 - c_2^*)u_2''(c_2)}{u_2'(c_2)} > 1 \Rightarrow \frac{\partial c_1}{\partial R} > 0. \tag{169}$$

H Proof of Corollary 2

Applying Theorem 1 to the special CES utility function, we can immediately get the result.

I Proof of Result 1

Consider the bivariate distribution defined by eqn. (92) and define

$$k = \frac{\pi_{22}\left(\xi_f p - \xi_{22}(\theta) p_f\right)}{\pi_{21}\left(\xi_{21}(\theta) p_f - \xi_f p\right)}, \tag{170}$$

where

$$\xi_{21}(\theta) = \xi_{21} + \theta p, \ \xi_{22}(\theta) = \xi_{22} + \theta p. \tag{171}$$

It can be easily seen that

$$\frac{\partial k}{\partial \theta} < 0 \Leftrightarrow \frac{\partial k}{\partial E\widetilde{\xi}} < 0 \Leftrightarrow \frac{\partial k}{\partial E\widetilde{R}} < 0. \tag{172}$$

Notice that

$$(p_{21} + p_{22})\widehat{R} = \frac{\pi_{21} + k^{-1}\pi_{22}}{\left(\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22}\right)^{\frac{1}{\delta_2}+1}}. \tag{173}$$

Since $p_{21} + p_{22} = \frac{p_f}{\xi_f} = const$ and

$$\frac{\partial \frac{\pi_{21}+k^{-1}\pi_{22}}{\left(\pi_{21}+k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22}\right)^{\frac{1}{\delta_2}+1}}}{\partial k} = \frac{k^{-2+\frac{2\delta_2}{1+\delta_2}}\pi_{21}\pi_{22}\left(\pi_{21} + k^{\frac{-\delta_2}{1+\delta_2}}\pi_{22}\right)^{-\frac{1}{\delta_2}}\left(k^{\frac{1}{1+\delta_2}} - 1\right)}{\left(k^{\frac{\delta_2}{1+\delta_2}}\pi_{21} + \pi_{22}\right)^2} < 0, \tag{174}$$

we can conclude that

$$\frac{\partial \widehat{R}}{\partial E\widetilde{R}} > 0. \tag{175}$$

J Proof of Result 2

Defining

$$\xi_{21}(\lambda) = \lambda\xi_{21} - (\lambda - 1)E\widetilde{\xi} \qquad \text{and} \qquad \xi_{22}(\lambda) = \lambda\xi_{22} - (\lambda - 1)E\widetilde{\xi}. \tag{176}$$

and noting that

$$\begin{aligned}
&\pi_{21}\xi_{21}(\lambda) + \pi_{22}\xi_{22}(\lambda) = \lambda\left(\pi_{21}\xi_{21} + \pi_{22}\xi_{22}\right) - (\lambda - 1)E\widetilde{\xi} \\
= \;& \lambda E\widetilde{\xi} - (\lambda - 1)E\widetilde{\xi} = E\widetilde{\xi},
\end{aligned} \tag{177}$$

it is clear that (176) corresponds to a mean preserving spread. Since

$$p_{21} = \frac{\xi_f p - \xi_{22}(\lambda)p_f}{(\xi_{21}(\lambda) - \xi_{22}(\lambda))\xi_f} \quad \text{and} \quad p_{22} = \frac{\xi_{21}(\lambda)p_f - \xi_f p}{(\xi_{21}(\lambda) - \xi_{22}(\lambda))\xi_f}, \tag{178}$$

it can be easily verified that

$$\frac{\partial p_{22}}{\partial \lambda} = -\frac{\partial p_{21}}{\partial \lambda} < 0, \tag{179}$$

implying that k defined by (39) satisfies

$$\frac{\partial k}{\partial \lambda} > 0. \tag{180}$$

From the proof of Result 1, we have

$$\frac{\partial \widehat{R}}{\partial k} < 0, \tag{181}$$

implying that

$$\frac{\partial \widehat{R}}{\partial \lambda} < 0. \tag{182}$$

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