What Are Asset Demand Tests of Expected Utility Really Testing?

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Abstract

Assuming the classic contingent claim setting, a number of financial asset demand tests of Expected Utility have been developed and implemented in experimental settings. However the domain of preferences of these asset demand tests differ from the mixture space of distributions assumed in the traditional binary lottery laboratory tests of von Neumann-Morgenstern Expected Utility preferences. We derive new sets of axioms for preferences over contingent claims to be representable by an Expected Utility function. We also indicate the additional axioms required to extend the representation to the more general case of preferences over risky prospects.

KEYWORDS. Expected utility, contingent claim demand, additive separability, lottery preferences, contingent claim preferences

JEL CLASSIFICATION. D01, D11, D80.

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1 Introduction

Traditionally, researchers seeking to test whether individuals satisfy the classic von Neumann and Morgenstern (1953) axioms for Expected Utility preferences have often based their experiments on binary choices over a mixture space of lotteries. In recent years, a second approach has evolved which investigates whether financial assets selected by individuals in a laboratory setting are consistent with the maximization of state independent Expected Utility subject to a standard budget constraint.[1] An innovative new experimental design introduced by Choi, et al. (2007a) has facilitated laboratory tests of these theoretical models. Referring to the advantages of their approach versus binary lottery tests, Choi, et al. (2007a, p. 154) argue that "... [our] experimental technique allows us to confront subjects with choice problems that span a broad range of common economic problems, both in theory and in empirical applications, rather than ... stylized choices designed to test a particular theory."

The asset demand tests are based on the classic Arrow-Debreu contingent claim setup, in which it is assumed that there are a finite number of states and agents possess preferences over state contingent consumption. In this setting one typically assumes a fixed set of state probabilities and varying state consumption payoffs. An important variation is to assume that each demand observation corresponds to a different pair of probabilities and prices.[2] Since in these contingent claim settings an individual never chooses between different consumption vectors associated with different state probabilities, we wish to avoid making the overly strong assumption that agents possess preferences over the space of all possible probability distributions. Indeed both von Neumann and Morgenstern (1953, p. 630) and Aumann (1962) argue that the completeness axiom in the lottery model is of "dubious validity". Hence it is natural to ask precisely what axioms guarantee that Expected Utility holds in the choice space corresponding to the particular asset demand model being tested. The diverse collection of asset demand tests differ not only in their assumptions about whether the state probabilities are fixed or variable, but also whether the probabilities are objective or subjective and whether the NM (von Neumann-Morgenstern) index is concave or locally convex.

[1] Non-parametric asset demand tests have been derived by Varian (1983, 1988), Green and Srivastava (1986), Choi, et al. (2007b), Kubler, Selden and Wei (2014), Polisson, Quah and Renou (2015), Echenique and Saito (2015) and implemented in experimental settings by Choi, et al. (2007b) and Polisson, Quah and Renou (2015). Functional form asset demand tests have been developed by Dybvig (1983) and Kubler, Selden and Wei (2014).

[2] See for example, Varian (1983), Green and Srivastava (1986), Kubler, Selden and Wei (2014) and Polisson, Quah and Renou (2015).

One key motivation for the specific sets of axioms we assume is that they in principle allow one to utilize the different experimental tests to determine which set is most consistent with the observed asset demands.

We consider three different choices settings. The first is associated with a single contingent claim space with a fixed set of exogenously given probabilities. We show that the required set of axioms for an Expected Utility representation consists of *Strict Monotonicity*, *Tradeoff Consistency* and *Local Risk Attitude Congruence.*[3] Following Wakker (1989) Strict Monotonicity and Tradeoff Consistency ensure the existence of a subjective Expected Utility (SEU) function where the probabilities are endogenously, and not exogenously, given and the representation is strictly increasing, additively separable and state independent. In order to ensure that the subjective probabilities resulting from the first two axioms match the exogenously given probabilities, we assume Local Risk Attitude Congruence. This axiom generalizes the Risk Aversion axiom of Werner (2005) where the latter guarantees that the NM, or Bernoulli, index is concave which is consistent with the traditional revealed preference tests in Varian (1983, 1988), Green and Srivastava (1986), Choi, et al. (2007b) and Kubler, Selden and Wei (2014). The more general Local Risk Attitude Congruence allows the NM index to have regions of both concavity and convexity which is consistent with the Expected Utility revealed preference tests of Polisson, Quah and Renou (2015).[4] Thus in Theorem 1, we show that Strict Monotonicity, Tradeoff Consistency and Local Risk Attitude Congruence are necessary and sufficient for an Expected Utility representation where the resulting NM index is locally concave or convex. It is natural to wonder if Local Risk Attitude Congruence has substantive empirical content. To see that in fact it does, first note that Polisson, Quah and Renou (2015, pp. 9-11) develop both a test of SEU (Subjective Expected Utility) and a test of Expected Utility with exogenously given objective probabilities where in each case concavity is not required. If asset demands pass the first test but fail the second, then Local Risk Attitude Congruence must be violated and the agent's behavior is consistent with SEU where the subjective and exogenously given objective probabilities do not coincide. Similarly, if one assumes concavity of the utility function and an individual agent's data fails the revealed preference test for Expected Utility in Kubler, Selden and Wei (2014) but passes the revealed preference test for Sub-

[3] This can also be achieved with alternative axioms such as the Sure-thing Principle of Savage (e.g., Werner 2005). Our motivation for imposing Tradeoff Consistency is that it together with the new Modified Tradeoff Consistency axiom provide a very nice bridge between the cases where Expected Utility holds on one contingent claim space and where it holds across multiple spaces.

[4] Although Local Risk Attitude Congruence requires the NM index to be either convex or concave on an open neighborhood, this distinction cannot be tested on a finite data set.

jective Expected Utility in Echenique and Saito (2015), then one can conclude that Strict Monotonicity and Tradeoff Consistency hold but Local Risk Attitude Congruence fails.

The second choice setting considered is a set of contingent claim spaces where each space corresponds to a different set of probabilities. Without additional axioms, although there will be an Expected Utility representation on each space, there is no assurance that the corresponding NM indices will be the same (up to a positive affine transformation). To guarantee that the NM indices are not probability dependent, we introduce a *Modified Tradeoff Consistency* axiom, which generalizes Tradeoff Consistency to this more general choice space. Then in Theorem 2, we show that Strict Monotonicity, Modified Tradeoff Consistency and Local Risk Attitude Congruence are necessary and sufficient for a locally concave or convex Expected Utility representation.

To our knowledge, the idea that NM indices could be probability dependent seems to be new.[5] (See Example 1 for a discussion of the intuition for why an individual's preferences might be representable by a probability dependent utility across contingent claim spaces.) However, there exists some experimental evidence suggesting that individuals may well have different risk preferences on the different contingent claim spaces corresponding to different contingent claim probabilities. First, it can be seen from Tables 1 - 3 in Choi, et al. (2007b) that for preferences exhibiting loss or disappointment aversion, the fitting parameters for these representations are significantly different for the contingent claim spaces associated with the symmetric versus the asymmetric state probabilities. Second, Polisson, Quah and Renou (2015) implement their nonparametric revealed preference tests of Expected Utility and other models on the same data obtained by Choi, et al. (2007b), and find that the critical cost efficiency indices and predictive success measures also vary across symmetric and asymmetric treatments (see Figure 5 and Tables 2 - 4 in Polisson, Quah and Renou 2015). These papers are suggestive that the degree to which asset demands are consistent with Expected Utility maximization does indeed vary across different contingent claim spaces.

[5] In the lottery setting, a similar phenomenon is observed, which is referred to as the "utility evaluation effect" by Machina (1983). McCord and de Neufville (1985), for example, note that

> Using different assessment probabilities...lead to different indifference statements... There is no reason to expect functions assessed with different probabilities to be identical. The systematic differences among functions assessed with different probabilities also appears theoretically compatible with the overvaluing of certainty. (McCord and de Neufville, 1985, p. 282)

However, this claim is somewhat limited by the experimental design, which only varies state probabilities across different treatments of subjects. A more direct and explicit test of the different axioms in Theorems 1 and 2 is given by Polisson, et al. (2016), where crucially state probabilities are allowed to vary within individual subjects. Both parametric and nonparametric procedures are applied to these new data, and evidence is provided supporting the notion that the NM index can be probability dependent. Hence the axioms in Theorem 1 may be satisfied but not necessarily the Modified Tradeoff Consistency axiom in Theorem 2.

The final choice setting considered is the general space of risky prospects where the admissible set of contingent claim distributions includes those for which payoffs are allowed to have different probabilities.[6] We show in Theorem 3 that the *Certainty Uniqueness* axiom is required to extend the Expected Utility representation from a set of contingent claim spaces where the probabilities are fixed and choices are not allowed across the spaces to this more general choice setting. The required Certainty Uniqueness axiom imposes the surprisingly weak requirement that degenerate lotteries with the same payoff on different contingent claim spaces are indifferent. This suggests that the key axiom in extending the Expected Utility on different contingent claim spaces with different state probabilities to the conventional space of risky prospects where choices across contingent claim spaces are allowed is Modified Tradeoff Consistency.

Based on these three sets of axioms, it would seem in principle possible to use laboratory tests (as in Polisson, et al. 2016) to distinguish between asset demands (i) being consistent with maximizing Expected Utility for a single contingent claim space but not for multiple spaces and (ii) being consistent with Expected Utility across multiple spaces. But it should be emphasized that it is impossible to tell from just asset demands whether the consumer's choices are consistent with maximizing Expected Utility over a space of risky prospects and hence whether or not the Certainty Uniqueness axiom holds. This is because asset demands are invariant to a particular class of monotone transforms of Expected Utility functions whereas choices over risky prospects are not (see the discussion of the utility (7) in the next section). For instance, an Expected Utility and a non-Expected Utility representation of preferences over risky prospects can yield exactly the same asset demand functions.[7] To test Certainty Uniqueness, it would seem that the demand

[6] The key difference between the space of risky prospects and the classic distribution space for von Neumann and Morgenstern (1953) and Samuelson (1952) is that the former has a fixed number of states and hence is not a mixture space.

[7] In Chambers, Liu and Martinez (2016), the authors state "our test is intimately tied to the classic von Neumann-Morgenstern axioms of expected utility". Because of the transformation property of demands, one should not interpret this as suggesting that one can conduct an asset

analysis needs to be augmented by a lottery test, such as in Kim (1996), where lotteries are compared on different contingent claim spaces.

The rest of the paper is organized as follows. In the next section, we compare and contrast the choice spaces and Expected Utility representations associated with (i) contingent claims assuming a fixed set of probabilities, (ii) contingent claims assuming state probabilities vary as parameters and (iii) a set of probability distributions or risky prospects corresponding to the case where both probabilities and consumption vectors are choice variables. Section 3 develops the axiom system for Expected Utility defined over contingent claims, first for the case where probabilities are fixed and then for the case where they vary. In Section 4, we identify the incremental set of axioms required to go from Expected Utility preferences defined over a set of contingent claim spaces to Expected Utility preferences defined over the space of distribution, where the number of states is finite. Proofs of the results are provided in the Appendix.

2 Different Preference Domains

Assume there are $S \geq 2$ states of nature and there is a single consumption good in each state. A typical consumption plan is an S vector $(c_1, c_2, ..., c_S)$ in the consumption space defined by $\mathbb{R}_+^S$. We assume that probabilities are objective and known and denote the probability of state s by π_s. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_S)$, where $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right) = \{\boldsymbol{\pi} \in \mathbb{R}_{++}^S : \sum_{s=1}^S \pi_s = 1\}$. Given this setting, we next define three different choices spaces which we will investigate.

The first preference domain we consider corresponds to the classic Arrow-Debreu contingent claim setup in which for a given value of $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$ a decision maker is assumed to have complete, transitive and continuous preferences over $\mathbb{R}_+^S$ which are denoted $\succeq_{\boldsymbol{\pi}}$. The second preference domain arises if one assumes as in Kubler, Selden and Wei (2014) that the consumer confronts a sequence of independent contingent claim optimizations where probabilities and prices vary. Then corresponding to a set of probability vectors $\{\boldsymbol{\pi}\}$, there will be a set of preference relations $\{\succeq_{\boldsymbol{\pi}}\}$ which need not give the same ordering over consumption vectors. The set of preference orderings is assumed to be representable by a continuous utility function $U(\mathbf{c}|\boldsymbol{\pi}) : \mathbb{R}_+^S \rightarrow \mathbb{R}$ where the notation $U(\mathbf{c}|\boldsymbol{\pi})$ indicates that corresponding to each $\boldsymbol{\pi}$, there will be a potentially different utility. It should be emphasized that for this set of utilities, the probability vector $\boldsymbol{\pi}$ is allowed to change, but only as a parameter. One can view $U(\mathbf{c}|\boldsymbol{\pi})$ as being defined over a series of contingent claim spaces but not on their union.

demand test of the von Neumann-Morgenstern axioms.

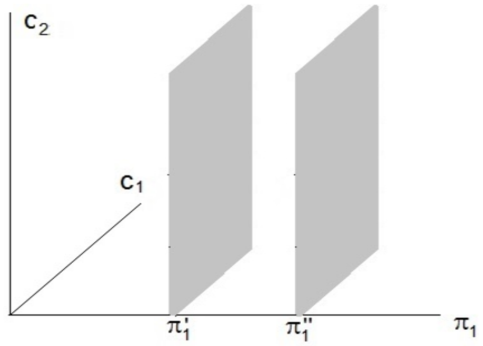


Figure 1:

Therefore although we can use $U(\mathbf{c}|\boldsymbol{\pi})$ to compare lotteries in each given contingent claim space, it cannot be used to compare the lotteries across the different contingent claim spaces. This is expressed geometrically in Figure 1, where two states are assumed. Each shaded plane in the figure corresponds to a contingent claim space with a given π_1. Preferences on the planes corresponding to π_1' and π_1'' are represented respectively by $U(\mathbf{c}|\boldsymbol{\pi}')$ and $U(\mathbf{c}|\boldsymbol{\pi}'')$.

Another motivation, quite different than the one discussed in the introduction, for investigating the axiomatization of Expected Utility where probabilities are allowed to change can be found in the work on speculation and the acquisition and value of information (see Rubinstein 1975, Hirshleifer and Riley 1979 for a classic overview and Schlee 2001 for more a more recent example). The papers in this literature typically utilize the contingent claim setting and continue to assume that risk preferences are representable by the same Expected Utility function as new information is obtained and probabilities vary.[8]

The third choice space we consider is the full set of distributions corresponding to $(\mathbf{c}, \boldsymbol{\pi})$, or the set of "risky prospects". To make this precise, define a risky prospect as a pair of vectors $(\mathbf{c}, \boldsymbol{\pi}) \in \mathbb{R}_+^S \times int\left(\Delta^{S-1}\right)$. Assume that a decision maker has continuous, complete and transitive preferences over $\mathcal{P} = \mathbb{R}_+^S \times int\left(\Delta^{S-1}\right)$, denoted $\succeq_{\mathcal{P}}$. For any fixed $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$ this implies preferences $\succeq_{\boldsymbol{\pi}}$ are well defined. To distinguish the representation of $\succeq_{\mathcal{P}}$ from the representation of $\{\succeq_{\boldsymbol{\pi}}\}$, we use the notation $U(\mathbf{c}, \boldsymbol{\pi})$ instead of $U(\mathbf{c}|\boldsymbol{\pi})$. The former, in contrast to the latter, has both $\mathbf{c}$ and $\boldsymbol{\pi}$ as arguments since one can compare lotteries across different contingent claim spaces, or slices in Figure 1.

For each of the above three preferences cases, we provide in the next two

[8] It is obviously the case that a careful analysis of these issues requires an explicit temporal component of the choice problem. To simplify the analysis, a large part of the literature considers comparative statics experiments where probabilities vary across different static problems

Sections a set of axioms that is necessary and sufficient for preferences to be representable by an Expected Utility function. We next illustrate the difference in the resulting Expected Utilities using the following example[9]

$$U\left(c_1, c_2, c_3 | \pi_1, \pi_2, \pi_3\right) = -\pi_1 \sum_{s=1}^{3} \pi_s \left(\exp\left(-\pi_1 c_s\right) + \exp\left(-\pi_2 c_s\right) + \exp\left(-\pi_3 c_s\right)\right). \tag{1}$$

Note first that if, as in the standard contingent claim case, probabilities are fixed at $\pi_1 = 0.5$, $\pi_2 = 0.3$ and $\pi_3 = 0.2$ (defining a specific slice in Figure 1),[10] eqn. (1) is equivalent up to a positive affine transformation to

$$\begin{aligned} U\left(\mathbf{c}|\boldsymbol{\pi}\right) &= -0.5\left(\exp\left(-0.5c_1\right) + \exp\left(-0.3c_1\right) + \exp\left(-0.2c_1\right)\right) \\ &\quad -0.3\left(\exp\left(-0.5c_2\right) + \exp\left(-0.3c_2\right) + \exp\left(-0.2c_2\right)\right) \\ &\quad -0.2\left(\exp\left(-0.5c_3\right) + \exp\left(-0.3c_3\right) + \exp\left(-0.2c_3\right)\right). \end{aligned} \tag{2}$$

Moreover it can be verified that

$$\left.\frac{\partial U/\partial c_1}{\partial U/\partial c_s}\right|_{c_1=c_s} = \frac{\pi_1}{\pi_s} \quad (s = 2, 3) \tag{3}$$

and the utility (2) passes the Expected Utility test in Dybvig (1983), implying that it can be viewed as an Expected Utility when probabilities are fixed and the NM index is given by

$$v\left(c\right) = -\left(\exp\left(-0.5c\right) + \exp\left(-0.3c\right) + \exp\left(-0.2c\right)\right). \tag{4}$$

However when probabilities are allowed to vary and one considers preferences on different contingent claim spaces, the resulting contingent claim demands cannot pass the tests discussed in Kubler, Selden and Wei (2014). The reason is that when probabilities vary, the NM index associated with the utility (1) will also change. In general, the kind of utility function in (1) takes the form

$$U\left(\mathbf{c}|\boldsymbol{\pi}\right) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{5}$$

where $f\left(\boldsymbol{\pi}, x\right)$ is a monotone transformation of x that can depend on $\boldsymbol{\pi}$ and the NM index $v_{\boldsymbol{\pi}}$ is allowed to depend on probabilities (see footnote 15 below). It should be emphasized that for the utility (1), the NM index

$$v_{\boldsymbol{\pi}}(c) = -\left(\exp\left(-\pi_1 c\right) + \exp\left(-\pi_2 c\right) + \exp\left(-\pi_3 c\right)\right) \tag{6}$$

[9] This utility will be recognized to be a modified version of a representation discussed in Kubler, Selden and Wei (2014).

[10] Obviously for the three state examples considered in this section, the notion of "slices" as in Figure 1 should be viewed as an intuitive proxy for the more complex spaces.

depends on $\boldsymbol{\pi}$ but is state independent and thus is not denoted by $v_{s,\boldsymbol{\pi}}$. The notation $f(\boldsymbol{\pi}, \cdot)$ indicates that on each contingent claim slice corresponding to each probability vector $\boldsymbol{\pi}$, one can consider a different increasing monotonic transform of the Expected Utility $\sum_{s=1}^{S} \pi_s v(c_s)$ and optimal contingent claim demands will not be altered.

Next consider the utility function

$$U(c_1, c_2, c_3 | \pi_1, \pi_2, \pi_3) = -\pi_1 \sum_{s=1}^{3} \pi_s \left(\exp(-0.5c_s) + \exp(-0.3c_s) + \exp(-0.2c_s)\right). \tag{7}$$

If one ignores the π_1 in front, this is a standard Expected Utility with the same NM index on each contingent claim slice

$$v(c) = -\left(\exp(-0.5c) + \exp(-0.3c) + \exp(-0.2c)\right). \tag{8}$$

More generally, the utility (7) takes the form

$$U(\mathbf{c}|\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{9}$$

where $f(\boldsymbol{\pi}, x)$ is a monotone transformation of x that can depend on $\boldsymbol{\pi}$ but the NM index v is independent of probabilities $\boldsymbol{\pi}$. Since (7) is an Expected Utility on each contingent claim slice in Figure 1 and the NM index is the same on each slice, it will result in demands that pass the tests in Kubler, Selden and Wei (2014). From observing optimal contingent claim demands, one can never distinguish ordinal transformations in the utility function corresponding to $f(\boldsymbol{\pi}, \cdot)$. However when considering comparisons over lotteries, the utility function defined in (7) (and more generally (9)) cannot be viewed as an Expected Utility function. To see this, consider the following two lotteries

$$L_1 =< 1, 2, 3; 0.2, 0.3, 0.5 > \text{ and } L_2 =< 2, 1, 3; 0.3, 0.2, 0.5 >, \tag{10}$$

where the payoffs in L_1 and L_2, respectively, are given by $1, 2, 3$ and $2, 1, 3$ and the probabilities by $0.2, 0.3, 0.5$ and $0.3, 0.2, 0.5$. Clearly for any Expected Utility maximizer, L_1 and L_2 will be indifferent. However for the utility function (7) since $\pi_1 = 0.2$ for L_1 and $\pi_1 = 0.3$ for L_2, we have

$$U(L_1) < U(L_2). \tag{11}$$

Hence from the lottery point of view, the transformation $f(\boldsymbol{\pi}, x) = \pi_1 x$ affects the consumer's choice whereas it does not in a demand optimization. Because of the transformation, the probabilities do not enter into the utility function linearly and

(7) is not an Expected Utility function. The probability weighting function for state i ($i = 1, 2, 3$) is $\pi_1 \pi_i$. From this perspective, this utility form can be viewed as being more analogous to a Prospect Theory form (see Kahneman and Tversky 1979) than Expected Utility.

Finally for the third choice space where preferences over lotteries (for a finite number of states S), $\succeq_{\mathcal{P}}$, are represented by an Expected Utility function, the representation will take the form

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{12}$$

where f is a continuous, monotone transformation independent of probabilities and the continuous NM index v is also independent of probabilities. For instance in terms of the examples considered above, $U(\mathbf{c}, \boldsymbol{\pi})$ can be any probability independent monotone transform of

$$-\sum_{s=1}^{3} \pi_s \left(\exp\left(-0.5c_s\right) + \exp\left(-0.3c_s\right) + \exp\left(-0.2c_s\right)\right). \tag{13}$$

3 Preferences over Contingent Claims

In this section, we derive Expected Utility representations assuming preferences are defined over a single or set of contingent claim spaces conditioned on state probabilities. For the set of state probabilities $int\left(\Delta^{S-1}\right)$, suppose that the corresponding set $\{\succeq_{\boldsymbol{\pi}}\}$ exists and is representable by $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$. We first give the representation result over each contingent claim space, where $\boldsymbol{\pi}$ is specified. Then we investigate the incremental axioms which are necessary and sufficient for the Expected Utility representation for each preference relation in the set $\{\succeq_{\boldsymbol{\pi}}\}$ to have the same NM index v, up to a positive affine transform, on each slice. We compare and contrast axioms in our risky setting with related axioms in the SEU (Subjective Expected Utility) setting.

3.1 Representation over Each Contingent Claim Space

In this subsection, we first consider the standard contingent claim setting where for a fixed $\boldsymbol{\pi}$, $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ takes the state independent Expected Utility form as in (5). We provide the necessary and sufficient conditions for this to be the case.

It will prove convenient to introduce the following natural Strict Monotonicity axiom first.

Axiom 1 *(Strict Monotonicity) For any given* $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, $\mathbf{c} \succeq_{\boldsymbol{\pi}} \mathbf{c}'$ *whenever* $c_s \geq c'_s$ *for all* $s \in \{1, 2, ..., S\}$ *and* $\mathbf{c} \succ_{\boldsymbol{\pi}} \mathbf{c}'$ *whenever* $c_s \geq c'_s$ *for all* $s \in \{1, 2, ..., S\}$ *and there exists at least one* $i \in \{1, 2, ..., S\}$ *such that* $c_i > c'_i$.

Based on the SEU literature, a natural candidate axiom for $U(\mathbf{c}|\boldsymbol{\pi})$ to be an Expected Utility is the following version of the Tradeoff Consistency axiom introduced by Wakker (1989).[11]

Axiom 2 *(Tradeoff Consistency) For any given* $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, *if* $\mathbf{c}_{-s}x \sim_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y$, $\mathbf{c}'_{-s}w \sim_{\boldsymbol{\pi}} \mathbf{c}_{-s}z$ *and* $\mathbf{c}'''_{-s'}y \sim_{\boldsymbol{\pi}} \mathbf{c}''_{-s'}x$, *then* $\mathbf{c}'''_{-s'}w \sim_{\boldsymbol{\pi}} \mathbf{c}''_{-s'}z$, *where* $\mathbf{c}_{-s}x$ *denotes the consumption vector* $\mathbf{c}$ *with consumption* c_s *in state* s *replaced by* x *and* $x, y, z, w \in \mathbb{R}_+$.

It follows from Köbberling and Wakker (2003, Theorem 5) that Axiom 2 implies the Sure-Thing Principle

$$\mathbf{c}_{-s}x \succeq_{\boldsymbol{\pi}} \mathbf{c}'_{-s}x \Leftrightarrow \mathbf{c}_{-s}y \succeq_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y \tag{14}$$

and the Thomsen-Blaschke condition when $S = 2$,

$$(c_1, c_2) \sim_{\boldsymbol{\pi}} (c'_1, c'_2), (c_1, c'_2) \sim_{\boldsymbol{\pi}} (c'_1, c''_2), (c''_1, c'_2) \sim_{\boldsymbol{\pi}} (c_1, c''_2) \Rightarrow (c''_1, c_2) \sim_{\boldsymbol{\pi}} (c_1, c''_2). \tag{15}$$

Therefore Axiom 2 implies that the utility function is additively separable. Moreover, Wakker (1984, Theorem 3.1) proves that Axioms 1 and 2 imply that there exists a SEU representation

$$U(\mathbf{c}|\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \omega_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{16}$$

where f and $v_{\boldsymbol{\pi}}$ are continuous and $\boldsymbol{\omega} = (\omega_1, ..., \omega_S)$ is the unique endogenously determined probability vector.[12] However, there is no guarantee that for each $s \in \{1, 2, ..., S\}$, ω_s coincides with the exogenously given probability π_s. Therefore, in order to obtain the representation (5), we need another axiom. Before introducing this axiom, we first define Local Risk Attitude Congruence.

Definition 1 *For any given* $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, *the agent is locally risk attitude congruent if and only if there exists a* $c > 0$ *such that there is an open neighborhood*

[11] The SEU setting is considered in the seminal book of Savage (1954) and further investigated in an extensive literature including the important papers of Anscombe and Aumann (1963) and Wakker (1989). For a more complete discussion of the SEU framework and associated axioms, see, for example, Wakker (1989), Nau (2011) and Karni (2013).

[12] It should be noted that in eqn. (16), ω_s is allowed to depend on $\boldsymbol{\pi}$.

$B(\mathbf{c})$ in the contingent claim space around the point $\mathbf{c} = (c, c, .., c) \in \mathbb{R}^S_{++}$ such that for every $\mathbf{c}_0 = (c_0, c_0, .., c_0) \in B(\mathbf{c})$, either $\mathbf{c}' \succeq_{\boldsymbol{\pi}} \mathbf{c}_0$ or $\mathbf{c}_0 \succeq_{\boldsymbol{\pi}} \mathbf{c}'$ holds for all $\mathbf{c}' \in B(\mathbf{c})$ with $\sum_{s=1}^{S} \pi_s c'_s = c_0$.

Then we can assume the following axiom.

Axiom 3 *(Local Risk Attitude Congruence) For any given $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, the agent is locally risk attitude congruent.*

The Local Risk Attitude Congruence axiom has two distinct implications on behavior. First, it requires that there is an open neighborhood on which cardinal utility is either concave or convex. This should be seen as a regularity condition. The condition is equivalent to requiring that there is an open neighborhood on which cardinal utility is twice continuously differentiable.[13] While it is possible that cardinal utility is nowhere twice continuously differentiable, we view this part of the axiom as a regularity assumption which has little empirical content (clearly it cannot be falsified from any finite set of observations). Second, and importantly, the axiom requires that in regions where the agents' utility is concave (convex), he is actually risk-averse (or risk-loving) with respect to the objective probabilities. This forces the objective probabilities to agree with the subjective probabilities. To see this more explicitly, consider the following example. Assume that the individual's risk preferences implied by Tradeoff Consistency are represented by

$$U(\mathbf{c}|\boldsymbol{\omega}) = \omega_1 \ln c_1 + \omega_2 \ln c_2, \tag{17}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2)$ is the subjective probability vector. Let $\boldsymbol{\pi} = (\pi_1, \pi_2)$ denote the objective probability vector. Now assume that $\boldsymbol{\omega} \neq \boldsymbol{\pi}$, i.e.,

$$\boldsymbol{\omega} = (\omega_1, \omega_2) = (0.6, 0.4) \quad \text{and} \quad \boldsymbol{\pi} = (\pi_1, \pi_2) = (0.5, 0.5). \tag{18}$$

Choose a consumption vector $(c_1, c_2) = (1, 1)$ and consider its open neighborhood

$$\{(c_1, c_2) \mid 0.8 < c_1 < 1.2 \text{ and } 0.8 < c_2 < 1.2\}. \tag{19}$$

Choose two consumption streams in this open neighborhood $(c'_1, c'_2) = (1.1, 0.9)$ and $(c''_1, c''_2) = (0.9, 1.1)$. Based on the objective probability $\boldsymbol{\pi}$, both distributions

[13] It should be noted that if differentiability is assumed, then Axiom 3 can be weakened to redefine Local Risk Attitude Congruence as follows. For a given $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, there exists at least one $c_0 > 0$ such that there is an open neighborhood $B(c_0)$ in the contingent claim space around the point $\mathbf{c}_0 = (c_0, c_0, .., c_0) \in \mathbb{R}^S_{++}$ such that either $\mathbf{c} \succeq_{\boldsymbol{\pi}} \mathbf{c}_0$ or $\mathbf{c}_0 \succeq_{\boldsymbol{\pi}} \mathbf{c}$ holds for all $\mathbf{c} \in B(c_0)$ with $\sum_{s=1}^{S} \pi_s c_s = c_0$. Similarly, Axiom 4 can be weakened from assuming risk aversion for every $\mathbf{c} \in \mathbb{R}^S_{+}$ to assuming risk aversion for a given $\mathbf{c} \in \mathbb{R}^S_{+}$.

$(c_1', c_2'; \pi_1, \pi_2)$ and $(c_1'', c_2''; \pi_1, \pi_2)$ are mean preserving spreads of $(c_1, c_2; \pi_1, \pi_2)$. Therefore, if the individual's preferences are represented by

$$U(\mathbf{c}|\boldsymbol{\pi}) = \pi_1 \ln c_1 + \pi_2 \ln c_2, \tag{20}$$

we always have

$$U(c_1, c_2|\boldsymbol{\pi}) > U(c_1', c_2'|\boldsymbol{\pi}) \quad \text{and} \quad U(c_1, c_2|\boldsymbol{\pi}) > U(c_1'', c_2''|\boldsymbol{\pi}) \tag{21}$$

This can be also confirmed by the following calculations

$$U(c_1, c_2|\boldsymbol{\pi}) = \pi_1 \ln 1 + \pi_2 \ln 1 = 0, \tag{22}$$

$$U(c_1', c_2'|\boldsymbol{\pi}) = \pi_1 \ln 1.1 + \pi_2 \ln 0.9 \approx -0.005 \tag{23}$$

and

$$U(c_1'', c_2''|\boldsymbol{\pi}) = \pi_1 \ln 0.9 + \pi_2 \ln 1.1 \approx -0.005. \tag{24}$$

However, since subjective probabilities and objective probabilities diverge, based on the representation (17), the risk attitudes are not consistent in this open neighborhood. Especially, we have

$$U(c_1, c_2|\boldsymbol{\omega}) = \omega_1 \ln 1 + \omega_2 \ln 1 = 0, \tag{25}$$

$$U(c_1', c_2'|\boldsymbol{\omega}) = \omega_1 \ln 1.1 + \omega_2 \ln 0.9 \approx 0.015 \tag{26}$$

and

$$U(c_1'', c_2''|\boldsymbol{\omega}) = \omega_1 \ln 0.9 + \omega_2 \ln 1.1 \approx -0.025, \tag{27}$$

implying that

$$U(c_1, c_2|\boldsymbol{\omega}) < U(c_1', c_2'|\boldsymbol{\omega}) \quad \text{and} \quad U(c_1, c_2|\boldsymbol{\pi}) > U(c_1'', c_2''|\boldsymbol{\omega}). \tag{28}$$

As we show in Theorem 1 below Local Risk Attitude Congruence excludes all possible subjective beliefs that do not coincide with the given objective probabilities.

It should be noted that the following Risk Aversion axiom in Werner (2002, 2005) is a special case of Axiom 3.[14]

[14] Since we assume Tradeoff Consistency proposed by Wakker instead of the Sure-Thing Principle used by Werner (2005), we automatically obtain the SEU representation (16) instead of the additively separable form

$$U(\mathbf{c}|\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} v_s(\boldsymbol{\pi}, c_s)\right).$$

For Werner (2005), since Sure-Thing Principle can only ensure this latter representation, to obtain a state-independent objective Expected Utility representation globally, he needs to prove that at every point $x \in \mathbb{R}_{++}$, $v_s(\boldsymbol{\pi},x) = \frac{\pi_s}{\pi_1} v_1(\boldsymbol{\pi},x)$ and hence he requires a global Risk Aversion axiom. For us, once the form (16) is obtained, it is enough to show that at one point $\omega_s = \pi_s$ and hence we only need a local axiom.

Axiom 4 *(Risk Aversion) For any given* $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$ *and every given* $\mathbf{c} \in \mathbb{R}_{+}^{S}$,

$$E_{\boldsymbol{\pi}}\left(\mathbf{c}\right) \succeq_{\boldsymbol{\pi}} \mathbf{c}, \tag{29}$$

where $E_{\boldsymbol{\pi}}(\mathbf{c})$ *denotes the* S*-vector* $\overline{\mathbf{c}}$ *for which* $\bar{c}_s = \sum_{i=1}^{S} \pi_i c_i$ *for each* s.

Then we have the following result.

Theorem 1 *For any given* $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ *takes the following functional form*

$$U\left(\mathbf{c}|\boldsymbol{\pi}\right) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{30}$$

where $f\left(\boldsymbol{\pi}, x\right)$ *is a continuous function that can depend on* $\boldsymbol{\pi}$ *and is strictly increasing in* $\sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)$ *and* $v_{\boldsymbol{\pi}}\left(c\right)$ *is a continuous and strictly increasing function where there exists a* $c \in \mathbb{R}_{++}$ *and an* $\epsilon\left(c\right) > 0$ *such that* $v_{\boldsymbol{\pi}}\left(c\right)$ *is either concave or convex in the positive open interval* $\left(c - \epsilon\left(c\right), c + \epsilon\left(c\right)\right)$, *if and only if Axioms 1, 2 and 3 hold.*

Remark 1 *Since we assume in Section 2 that* U *is strictly increasing, we can take* $v(\cdot)$ *and* $f(\boldsymbol{\pi}, \cdot)$ *to be strictly increasing.*[15] *It should also be noted that the use of Axiom 3 in Theorem 1 implies that the indifference (hyper)surfaces of the utility* $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ *need not to be concave or convex. In contrast, if one instead assumes Axiom 4 holds, then the NM index* $v_{\boldsymbol{\pi}}$ *is guaranteed to be concave and hence* $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ *is quasiconcave. Both of these observations also apply to Theorems 2 - 4 below.*

It will be noted that each NM index $v_{\boldsymbol{\pi}}$ is allowed to depend on probabilities. This is consistent with the utility (1) discussed in Section 2, which takes the form of $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ in Theorem 1. Indeed it can readily be verified that (1) satisfies Axioms 1, 2 and 3 for each given probability vector. Whereas Theorem 1 clearly allows for the possibility of probability dependent NM indices across slices, the following example provides some economic intuition for how this might arise.

Example 1 *Consider the following representation*

$$U\left(c_1, c_2 \middle| \pi_1, \pi_2\right) = \pi_1 c_1^{f(\pi_1,\pi_2)} + \pi_2 c_2^{f(\pi_1,\pi_2)}, \tag{31}$$

[15] Note that if $v_{\boldsymbol{\pi}}$ is strictly decreasing and $f(\boldsymbol{\pi}, x)$ is also strictly decreasing in x, $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ is still a strictly increasing function. However, this will not yield a new $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ or alter demands. In the context of solving for optimal demands in the classic contingent claim setting, the NM index $v_{\boldsymbol{\pi}}$ is required to be increasing and the assumption that $U\left(\mathbf{c}|\boldsymbol{\pi}\right)$ is strictly increasing implies that $f(\boldsymbol{\pi}, x)$ must be strictly increasing in x. The case where $v_{\boldsymbol{\pi}}$ and $f(\boldsymbol{\pi}, x)$ are strictly decreasing is ignored throughout this paper.

where $f(\pi_1, \pi_2) = a_0 |\pi_2 - \pi_1| + a_1$ *and* $a_0 < 0$ *and* a_1 *are some constants. Clearly this utility satisfies the conditions in Theorem 1 for a fixed set of objective probabilities. In this case, the utility takes the popular CRRA (constant relative risk aversion) form. But when* (π_1, π_2) *varies, moving from one contingent claim slice to another, the NM index* $v_{\boldsymbol{\pi}}(c) = c^{f(\pi_1, \pi_2)}$ *varies (by more than a positive affine transform) and hence preferences across slices are not Expected Utility representable. Since* $a_0 < 0$*, increasing the difference* $|\pi_2 - \pi_1|$ *decreases exponents of* c_1 *and* c_2 *and hence increases the Arrow-Pratt relative risk aversion. But can an intuitive argument be given for why an individual's risk aversion should increase as the difference in probabilities increases when one moves across slices? In the contingent claim setting, since the payoff in each contingent claim state is unknown before prices are given and one solves for optimal demands, it would seem reasonable to view the contingent claim slice associated with the probability* $\boldsymbol{\pi} = \left(\frac{1}{8}, \frac{7}{8}\right)$ *or* $\left(\frac{7}{8}, \frac{1}{8}\right)$ *as being more risky than the contingent claim slice associated with the probability* $\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$*. The worry would be that the low (high) probability state might occur when the consumer faces a low (high) price. That is, since the consumer does not know which contingent claim vector will ultimately be selected, the risk is that the contingent claim that she desires most in the optimization might have a very low chance of occurring. Ex ante, it seems reasonable to suppose that she would prefer each state to be equally likely. Following this logic, an individual would have higher risk aversion on the contingent claim slice associated with* $\boldsymbol{\pi}$ *equaling* $\left(\frac{1}{8}, \frac{7}{8}\right)$ *or* $\left(\frac{7}{8}, \frac{1}{8}\right)$ *versus on the contingent claim slice associated with* $\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$*. This argument would suggest that risk aversion increases with the dispersion between the probabilities and that there is no a priori reason to distinguish between cases like* $\left(\frac{1}{8}, \frac{7}{8}\right)$ *and* $\left(\frac{7}{8}, \frac{1}{8}\right)$*, which is consistent with* $f(\pi_1, \pi_2) = a_0 |\pi_2 - \pi_1| + a_1$*.*

As noted above in the discussion of the SEU representation (16) since in our setting probabilities are given exogenously and not endogenously determined, the Strict Monotonicity and Tradeoff Consistency axioms can not ensure that the endogenously determined $\boldsymbol{\omega}$ matches the exogenously given $\boldsymbol{\pi}$. To see this more explicitly, consider the following two examples. The first one can be viewed as a variant of a Prospect Theory representation and the second one can be viewed a state dependent utility. Both examples satisfy Strict Monotonicity and Tradeoff Consistency.

Example 2 *Assume that*

$$U(\mathbf{c}|\boldsymbol{\pi}) = \pi_1^2 v(c_1) + \pi_2^2 v(c_2) + \pi_3^2 v(c_3). \tag{32}$$

Note that this representation is affinely equivalent to the SEU representation[16]

$$U(\mathbf{c}) = \sum_{s=1}^{3} \omega_s v(c_s), \tag{33}$$

where

$$\omega_s = \frac{\pi_s^2}{\pi_1^2 + \pi_2^2 + \pi_3^2}. \tag{34}$$

Clearly the utility (32) satisfies Strict Monotonicity and Tradeoff Consistency and is a SEU function. Although the utility satisfies the state independence requirement of Theorem 1, it does not satisfy the requirement that the probabilities enter into the utility function linearly.

Example 3 *Assume that*

$$U(\mathbf{c}|\boldsymbol{\pi}) = \pi_1 v(c_1) + 2\pi_2 v(c_2) + 3\pi_3 v(c_3). \tag{35}$$

Note that (35) is affinely equivalent to the state independent SEU

$$U(\mathbf{c}) = \sum_{s=1}^{3} \omega_s v(c_s), \tag{36}$$

where

$$\omega_1 = \frac{\pi_1}{\pi_1 + 2\pi_2 + 3\pi_3}, \omega_2 = \frac{2\pi_2}{\pi_1 + 2\pi_2 + 3\pi_3} \text{ and } \omega_3 = \frac{3\pi_3}{\pi_1 + 2\pi_2 + 3\pi_3} \tag{37}$$

and (35) satisfies Strict Monotonicity and Tradeoff Consistency. However in our setting, the probability vector $\boldsymbol{\pi}$ *is exogenous and fixed and cannot be transformed into* $\boldsymbol{\omega}$*. To see that this implies the utility (35) is not state independent, observe that it can be written as*

$$U(\mathbf{c}|\boldsymbol{\pi}) = \sum_{s=1}^{3} \pi_s s v(c_s), \tag{38}$$

where the NM index in each state is given by

$$v_s(c_s) = s v(c_s), \tag{39}$$

which is clearly state dependent and is inconsistent with the representation in Theorem 1. Thus, the Strict Monotonicity and Tradeoff Consistency axioms in the SEU setting do not imply state independence in our setting, where probabilities are exogenous.

[16] Since the SEU axioms imply the existence of a v and a $\boldsymbol{\omega}$, we use in (33) the notation $U(\mathbf{c})$ rather than $U(\mathbf{c}|\boldsymbol{\omega})$ to reflect the fact that $\boldsymbol{\omega}$ should not be viewed as a parameter that can be changed like our exogenously given $\boldsymbol{\pi}$.

Remark 2 *Wakker and Zank (1999, Theorem 7) argue that Strict Monotonicity and Tradeoff Consistency can guarantee the existence of an objective Expected Utility representation if the objective probabilities are known. This would seem to contradict our discussion above. To see why this is not the case, first note that (i) they assume preferences are defined over risky prospects, (ii) all risky prospects are available to be chosen, (iii) their definition of Tradeoff Consistency is based on preferences over risky prospects and (iv) preferences satisfy probabilistic sophistication (see the discussion of Axiom 7 below). In this setup, they prove that the objective and subjective probabilities are always the same. However as can be seen very clearly in Exercise 2.3.1 in Wakker (2010), the argument hinges on changing the probability structure of the lotteries by modifying the number of states and then using probabilistic sophistication. But in the contingent claim setting, the number of states and the probability of each state are fixed and neither can be changed. As a result, we use the Local Risk Attitude Congruence Axiom 3 to ensure that the subjective and objective probabilities are the same.*

3.2 Representation over Sequence of Contingent Claim Spaces

Suppose rather than allowing the NM index v in Theorem 1 to vary as the state probabilities change, one wants to ensure that the set of preference relations $\{\succeq_{\boldsymbol{\pi}}\}$ are representable by a common Expected Utility function across contingent claim slices as in Figure 1. As shown in eqn. (1), even if $U(\mathbf{c}|\boldsymbol{\pi})$ takes the Expected Utility form in each contingent claim space, it may not be an Expected Utility with respect to the set of preference relations $\{\succeq_{\boldsymbol{\pi}}\}$. Interestingly, this additional requirement can be achieved by simply modifying the Tradeoff Consistency Axiom 2 to be applicable to the case of multiple slices and multiple probability vectors.

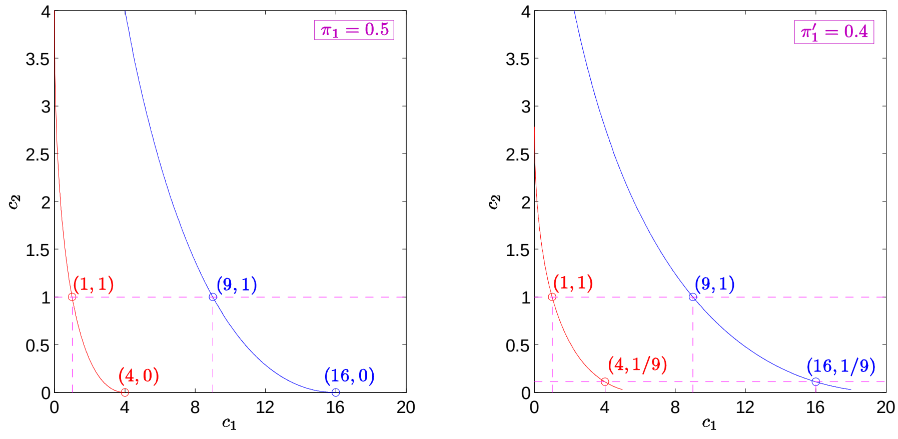
Axiom 5 *(Modified Tradeoff Consistency) For each $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, assuming $\mathbf{c}_{-s}x \sim_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y$ and $\mathbf{c}'_{-s}w \sim_{\boldsymbol{\pi}} \mathbf{c}_{-s}z$ then for any $\boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right)$, if $\mathbf{c}'''_{-s'}y \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}x$, we have $\mathbf{c}'''_{-s'}w \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}z$, where $x, y, z, w \in \mathbb{R}_+$.*

To provide some intuition for Axiom 5, assume $S = 2$ and consider the following consumption pairs

$$\mathbf{c} = (c_1, 1), \mathbf{c}' = (c'_1, 0), \mathbf{c}'' = (c''_1, 1) \text{ and } \mathbf{c}''' = \left(c'''_1, \frac{1}{9}\right). \tag{40}$$

Consider two contingent claim slices corresponding to

$$\pi_1 = 0.5 \text{ and } \pi'_1 = 0.4 \tag{41}$$



(a) (b)

Figure 2:

and the consumption values

$$x = 1, y = 4, w = 16, z = 9. \tag{42}$$

Axiom 5 implies that if

$$(1,1) \sim_{\boldsymbol{\pi}} (4,0), (9,1) \sim_{\boldsymbol{\pi}} (16,0) \text{ and } (1,1) \sim_{\boldsymbol{\pi}'} \left(4, \frac{1}{9}\right), \tag{43}$$

then we must have

$$(9,1) \sim_{\boldsymbol{\pi}'} \left(16, \frac{1}{9}\right). \tag{44}$$

The chain of indifferent consumption pairs, (43) and (44), are shown respectively in Figures 2(a) and (b), where we assume the Expected Utility representation

$$U(\mathbf{c}|\boldsymbol{\pi}) = \pi_1\sqrt{c_1} + \pi_2\sqrt{c_2}. \tag{45}$$

Axiom 5 is clearly satisfied.[17] In the SEU setting, since the probabilities are endogenously determined, one only considers the case with a fixed probability structure like Figure 2(a). Our contribution here is to assume Tradeoff Consistency holds where the probability structure changes as in Figure 2(b).

Then we have the following theorem.

Theorem 2 *For all* $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, $U(\mathbf{c}|\boldsymbol{\pi})$ *takes the following functional form*

$$U(\mathbf{c}|\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{46}$$

[17] Figure 2 is similar to Figure 4.5.1 in Köbberling and Wakker (2004).

where $f(\boldsymbol{\pi}, x)$ is a continuous function that can depend on $\boldsymbol{\pi}$ and is strictly increasing in $\sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)$ and $v(c)$ is a continuous and strictly increasing function where there exists a $c \in \mathbb{R}_{++}$ and an $\epsilon(c) > 0$ such that $v(c)$ is either concave or convex in the positive open interval $(c - \epsilon(c), c + \epsilon(c))$, if and only if Axioms 1, 3 and 5 hold for all $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$.

4 Preferences over Lotteries

In the previous section preferences were assumed to be defined over contingent consumption, and probabilities entered only as parameters. In this section, we suppose instead that a decision maker faces choices over different "risky prospects" or lotteries, which are defined as vectors $(\mathbf{c}, \boldsymbol{\pi}) \in \mathbb{R}_{+}^{S} \times int\left(\Delta^{S-1}\right)$. As described in Section 2, we assume a continuous, complete and transitive preference ordering over $\mathcal{P} = \mathbb{R}_{+}^{S} \times int\left(\Delta^{S-1}\right)$ denoted by $\succeq_{\mathcal{P}}$. What additional axioms beyond those in Theorem 2 are required to extend the Expected Utility representation of $\{\succeq_{\boldsymbol{\pi}}\}$ to $\succeq_{\mathcal{P}}$? Maintaining Axioms 1, 3 and 5, the following turns out to be necessary and sufficient.

Axiom 6 *(Certainty Uniqueness) For any certain consumption $\mathbf{c} = (c_1, c_2, ..., c_S)$, where $c_s = \bar{c}$ is a constant for each state s,*

$$(\mathbf{c}, \boldsymbol{\pi}) \sim_{\mathcal{P}} (\mathbf{c}, \boldsymbol{\pi}') \quad \forall \boldsymbol{\pi}, \boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right). \tag{47}$$

Certainty Uniqueness suggests that an individual's preferences are "slice independent". She evaluates a lottery based just on its distribution function independent of the contingent slice on which the lottery is located. In fact, this axiom is quite natural when choosing over degenerate lotteries. For example, consider the certain consumption vector $\bar{\mathbf{c}} = (\bar{c}, ..., \bar{c})$. Assume that the Certainty Uniqueness axiom does not hold and the preferences over a set of contingent claim slices are represented by

$$U(\mathbf{c}|\boldsymbol{\pi}) = \left(\sum_{s=1}^{S} \pi_s \sqrt{c_s}\right)^{\pi_1}. \tag{48}$$

It can be easily verified that

$$U(\bar{\mathbf{c}}|\boldsymbol{\pi}) = \bar{c}^{\frac{\pi_1}{2}}. \tag{49}$$

This implies that the certain consumption vector $(\bar{c}, ..., \bar{c})$ in the contingent claim slice with the higher π_1 is preferred to that in the contingent claim slice with the lower π_1. This seems counter intuitive since if one views the consumption vector $(\bar{c}, ..., \bar{c})$ as a degenerate lottery, it has the same payoffs on each contingent claim slice. When considering an optimization problem in the contingent claim setting rather than making binary choices over lotteries, one never compares consumption vectors across different contingent claim slices and hence one cannot say $(\bar{c},...,\bar{c})$ in every contingent claim slice is indifferent. Therefore although Certainty Uniqueness seems indisputable in a lottery setting, it is essentially irrelevant in the contingent claim setting. Suppose one assumes that (i) consumption vectors on each contingent claim slice can be always viewed as lotteries and (ii) the individual's preferences over consumption vectors are naturally extendable to preferences over lotteries. Then Certainty Uniqueness would seem to hold automatically. In other words, if the preferences are Expected Utility representable with the same NM index on each contingent claim slice and Assumptions (i) and (ii) hold, then one would expect that preferences over risky prospects would also be Expected Utility representable with the same NM index and this is precisely what Certainty Uniqueness guarantees.

Axiom 6 can be illustrated in Figure 1, where it would correspond to an individual being indifferent to the same $\mathbf{c}$ point along the 45° rays on the slices characterized by $\boldsymbol{\pi}'$ and $\boldsymbol{\pi}''$. Together, Axiom 6 and the assumption that $U(\mathbf{c}|\boldsymbol{\pi})$ is strictly increasing suggest that for any certain consumption vectors $\mathbf{c} = (c_1, c_2, ..., c_S)$ and $\mathbf{c}' = (c'_1, c'_2, ..., c'_S)$ where $c_s = \bar{c}$ and $c'_s = \bar{c}'$ $(\forall s \in \{1, 2, ..., S\})$ and for any $\boldsymbol{\pi}, \boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right)$, $(\mathbf{c}, \boldsymbol{\pi}) \succeq_{\mathcal{P}} (\mathbf{c}', \boldsymbol{\pi}')$ if and only if $\bar{c} \geq \bar{c}'$. Therefore, $U(\mathbf{c}, \boldsymbol{\pi})$ is also strictly increasing in $\mathbf{c}$.

Then we have the following theorem.

Theorem 3 *$U(\mathbf{c}, \boldsymbol{\pi})$ representing $\succeq_{\mathcal{P}}$ takes the following functional form*

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{50}$$

where $f(x)$ is a continuous and strictly increasing function and $v(c)$ is a continuous and strictly increasing function where there exists a $c \in \mathbb{R}_{++}$ and an $\epsilon(c) > 0$ such that $v(c)$ is either concave or convex in the positive open interval $(c - \epsilon(c), c + \epsilon(c))$, if and only if Axioms 1, 3, 5 and 6 hold for all $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$.

Remark 3 *Comparing Theorems 2 and 3, it is clear that the only difference is the assumption of the Certainty Uniqueness Axiom 6 for the latter. However in our opinion, this axiom appears to be quite weak and it would be very surprising if laboratory tests revealed that individuals systematically violated Certainty Uniqueness. Thus the significant restrictions in the contingent claim setting derive from the Tradeoff Consistency and Modified Tradeoff Consistency axioms. The former axiom together with Strict Monotonicity restricts the consumer's utility to take* *the Expected Utility form on subsets of the larger space of risky prospects $\mathcal{P}$ corresponding contingent claim slices. The latter requires the NM index to be the same for each subset of $\mathcal{P}$ corresponding to the set of slices. If this seemingly strong restriction fails to hold for all slices, then the consumer's choices over lotteries from different slices will fail to be representable by an Expected Utility function with the same NM index and Theorem 3 will fail to hold as well. The fact that an individual's preferences might be representable by an Expected Utility function over some by not other subsets of the general space of lotteries is not new. Albeit in a quite different context, there is considerable laboratory analysis suggesting that individuals behave quite differently when lottery choices include versus do not include degenerate lotteries. In his analysis of this "certainty effect", Conlisk (1989), for instance, finds that the fraction of Expected Utility violations drops from about 50 percent to about 32 percent depending on whether the choice set includes or excludes degenerate lotteries.*

Comparing the representations (46) and (50) in Theorems 2 and 3, respectively, Axiom 6 is necessary and sufficient for the transformation f to be independent of $\boldsymbol{\pi}$. For example, the introduction of Axiom 6 rules out eqn. (1) in Section 2 as a possible representation of $\succeq_{\mathcal{P}}$. It should be stressed that the form of utility in Theorem 3 is not verifiable at the demand level since whether or not the transformation f depends on probabilities cannot be determined from the contingent claim demand functions.

It is natural to wonder whether it is enough to use the Tradeoff Consistency Axiom 2 instead of the modified version Axiom 5 together with Axioms 1, 3 and 6 to obtain the desired result in Theorem 3. The following example shows that this is not the case.

Example 4 *Assume that*

$$U(\mathbf{c},\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right) = \left(\pi_1 c_1^{\frac{\pi_1}{2}} + \pi_2 c_2^{\frac{\pi_1}{2}} + \pi_3 c_3^{\frac{\pi_1}{2}}\right)^{\frac{1}{\pi_1}}, \tag{51}$$

where

$$v_{\boldsymbol{\pi}}(c_s) = c_s^{\frac{\pi_1}{2}} \quad \text{and} \quad f(\boldsymbol{\pi}, x) = x^{\frac{1}{\pi_1}}. \tag{52}$$

If consumption in each of the states is the same, $c_s = \overline{c}$, then

$$U(\mathbf{c},\boldsymbol{\pi}) = \left(\pi_1 \overline{c}^{\frac{\pi_1}{2}} + \pi_2 \overline{c}^{\frac{\pi_1}{2}} + \pi_3 \overline{c}^{\frac{\pi_1}{2}}\right)^{\frac{1}{\pi_1}} = \sqrt{\overline{c}}, \tag{53}$$

which is independent of probabilities and hence Axiom 6 holds. For each fixed probability $\boldsymbol{\pi}$, (51) is clearly an Expected Utility function. Therefore, Axioms 1, 2 and 3 hold. But obviously (51) does not take the form of (50) in Theorem 3.

Assuming Axioms 1, 2 (in place of Axiom 5) and 3 hold, is it possible to replace Axiom 6 by another axiom which ensures that U takes the form in (50)? Before introducing a new axiom, we define some additional notation. For any $(\mathbf{c}, \boldsymbol{\pi})$, where $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, assuming $(\mathbf{c}, \boldsymbol{\pi})$ corresponds to the random variable X, the cumulative distribution function is

$$F_X(z) = \sum_{s=1}^{S} \pi_s I(c_s \leq z), \tag{54}$$

where

$$I(c_s \leq z) = \begin{cases} 1 & (c_s \leq z) \\ 0 & (c_s > z) \end{cases}. \tag{55}$$

Axiom 7 *For any pair of random variables X and Y corresponding, respectively, to $(\mathbf{c}, \boldsymbol{\pi})$ and $(\mathbf{c}', \boldsymbol{\pi}')$, where $\boldsymbol{\pi}, \boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right)$, if $F_X(z) = F_Y(z)$, then*

$$(\mathbf{c}, \boldsymbol{\pi}) \sim_{\mathcal{P}} (\mathbf{c}', \boldsymbol{\pi}'). \tag{56}$$

The intuition for this axiom is that for any pair of lotteries defined on different contingent claim spaces, if their respective cumulative distribution functions are the same, then the lotteries will be indifferent. This is consistent with both the NM index v and the transformation f being independent of $\boldsymbol{\pi}$. It is clear that Axiom 7 implies Axiom 6.

Remark 4 *Axiom 7 will be recognized to be similar to the probabilistic sophistication property introduced by Machina and Schmeidler (1992) in an SEU setting (also see Grant, Özsoy and Polak 2008). Because this property is based on subjective probabilities, it is necessary to introduce axiomatic structure to ensure that the endogenous probabilities satisfy probabilistic sophistication. However in the case of Axiom 7, the probabilities are given exogenously and the axiom can be directly assumed.*

We next show that Axiom 7 together with Axioms 1, 2 and 3 are necessary and sufficient for $\succeq_{\mathcal{P}}$ to be representable by an Expected Utility function where the NM index does not depend on probabilities in contrast to the case of Example 4.

Theorem 4 *When $S > 2$, $U(\mathbf{c}, \boldsymbol{\pi})$ representing $\succeq_{\mathcal{P}}$ takes the following functional form*

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{57}$$

where $f(x)$ is a continuous and strictly increasing function and $v(c)$ is a continuous and strictly increasing function where there exists a $c \in \mathbb{R}_{++}$ and an $\epsilon(c) > 0$ such that $v(c)$ is either concave or convex in the positive open interval $(c - \epsilon(c), c + \epsilon(c))$, if and only if Axioms 1, 2, 3 and 7 hold for all $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$.

Finally, it is natural to inquire into the relationship between Theorem 4 and the conventional Expected Utility representation result based on the Strong Independence axiom (e.g., Samuelson 1952 and Grandmont 1972). First let $\mathcal{F}$ denote the set of all cumulative distribution functions defined on the consumption space $(0, \infty)$. Assume preferences are defined over $\mathcal{F}$, which is a mixture space. Since $\mathcal{F}$ consists of all possible distributions, it is not restricted to S states. Indeed the Strong Independence axiom typically assumed for preferences over $\mathcal{F}$ holds for any mixture of lotteries where the maximum number of states of the lotteries is S. Therefore, the only difference between the set of risky prospects $\mathcal{P}$ assumed in this section and $\mathcal{F}$ is that for the former the number of the states are fixed at S and for $\mathcal{F}$, there is no restriction to the number of states.

It should be noted that preference properties such as first order stochastic dominance relating to the shape of the indifference curves in the probability triangle proposed by Marschak (1950) and extended by Machina (1982) fail to be distinguishable at the corresponding contingent claim demand level. In fact, the set of lotteries in the probability triangle can be viewed as orthogonal to the set of lotteries in the contingent claims spaces parameterized by $\boldsymbol{\pi}$. The existence of an Expected Utility representation for lotteries defined in the contingent claim space cannot ensure an Expected Utility representation over lotteries corresponding to the probability triangle and vice versa.

5 Conclusion

In this paper, axiom systems are presented for an Expected Utility representation in three different subspaces of the full distribution space. The first subspace is a single contingent claim space. The axioms for this space are consistent with the revealed preference tests assuming known fixed probabilities. The second subspace is a set of contingent claim spaces. The axioms for this space are consistent with the revealed preference tests assuming known variable probabilities. The third subspace is the space of risky prospects with fixed number of states. Although the axiom system for this space is not testable based on the optimal demand data $(\boldsymbol{\pi}^i, \mathbf{c}^i, \mathbf{p}^i)_{i=1}^n$, it provides a bridge for connecting the contingent claim space and the lottery space. We identify the additional axioms required to have an

Expected Utility representation over risky prospects when one has an Expected Utility representation for each contingent claim space.

For future research, it would seem potentially interesting to extend the analysis in this paper to non-Expected Utility preferences. Indeed there have recently been efforts to apply more general preference models such as Loss/Disappointment Aversion, RDU (Rank-Dependent Expected Utility) and Cumulative Prospect Theory to the contingent claim demand setting (see, for example, Choi, et al. 2007b and Carlier and Dana 2011). In each of these cases, a fixed set of state probabilities are assumed. What modifications of the preference axioms in the single contingent claim space are required when extending the choice spaces of these non-Expected Utility models to a set of contingent claim spaces and a set of risky prospects? In these alternative choice domains do analogues of Local Risk Attitude Congruence, Modified Tradeoff Consistency and Certainty Uniqueness play a role?

Appendix

A Proof of Theorem 1

For each $\boldsymbol{\pi} \in int\left(\Delta^{S-1}\right)$, given our assumption that $\succeq_{\boldsymbol{\pi}}$ is complete, transitive and continuous, it follows from Wakker (1984, Theorem 3.1) that there exists a SEU representation

$$U\left(\mathbf{c}|\boldsymbol{\pi}\right) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \omega_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{A.1}$$

where f and $v_{\boldsymbol{\pi}}$ are continuous if and only if Axioms 1 and 2 hold. Therefore, we only need to show that $\boldsymbol{\omega} = \boldsymbol{\pi}$ if and only if Axiom 3 holds. The proof of necessity is almost directly from Werner (2002) – we include it for completeness: For a given $\mathbf{c} = (c, c, .., c) \in \mathbb{R}^S_{++}$, consider the open neighborhood $B\left(\mathbf{c}\right) = \left(c - \epsilon\left(c\right), c + \epsilon\left(c\right)\right)^S$. Then for each point $\mathbf{c}_0 = (c_0, c_0, .., c_0) \in B\left(\mathbf{c}\right)$, consider the following optimization problems

$$\max_{\mathbf{c}' \in B(c)} f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c'_s)\right) \quad S.T. \ \sum_{s=1}^{S} \pi_s c'_s = c_0 \tag{A.2}$$

if $v_{\boldsymbol{\pi}}$ is concave and

$$\min_{\mathbf{c}' \in B(c)} f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c'_s)\right) \quad S.T. \ \sum_{s=1}^{S} \pi_s c'_s = c_0 \tag{A.3}$$

if $v_{\boldsymbol{\pi}}$ is convex. If $v_{\boldsymbol{\pi}}$ is concave in $B(\mathbf{c})$, then it follows from Jensen's inequality that

$$\sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s') \leq v_\pi \left(\sum_{s=1}^{S} \pi_s c_s' \right) = v_\pi (c_0) = \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_0). \tag{A.4}$$

Therefore, $\mathbf{c}_0$ is an optimal solution to the optimization problem (A.2), implying that $\mathbf{c}_0 \succeq_{\boldsymbol{\pi}} \mathbf{c}'$ holds for all $\mathbf{c}' \in B(\mathbf{c})$ with $\sum_{s=1}^{S} \pi_s c_s' = c_0$. Similarly, one can argue if $v_{\boldsymbol{\pi}}$ is convex, then $\mathbf{c}' \succeq_{\boldsymbol{\pi}} \mathbf{c}_0$ holds for all $\mathbf{c}' \in B(\mathbf{c})$ with $\sum_{s=1}^{S} \pi_s c_s' = c_0$. Since this conclusion holds for every $\mathbf{c}_0 \in B(\mathbf{c})$, Axiom 3 holds. Next prove sufficiency. If Axiom 3 holds, for a given $c \in \mathbb{R}_{++}$ and each $\mathbf{c}_0 = (c_0, c_0, .., c_0) \in B(\mathbf{c})$, we define $\widetilde{\boldsymbol{\varepsilon}}(\mathbf{c}_0) \in \mathbb{R}^S$ by $\varepsilon_t = \varepsilon(c_0)$, $\varepsilon_s = -\pi_t \varepsilon(c_0)/\pi_s$ and $\varepsilon_k = 0$ for all $k \neq t, s$ such that $\mathbf{c}_0 + \widetilde{\boldsymbol{\varepsilon}}(\mathbf{c}_0), \mathbf{c}_0 + \boldsymbol{\varepsilon}(\mathbf{c}_0), \mathbf{c}_0 + \boldsymbol{\varepsilon}(\mathbf{c}_0) - \widetilde{\boldsymbol{\varepsilon}}(\mathbf{c}_0) \in B(\mathbf{c})$, where $\boldsymbol{\varepsilon}(\mathbf{c}_0) = (\varepsilon(c_0), \varepsilon(c_0), ..., \varepsilon(c_0))$. Since $\mathbf{c}_0 \succeq_{\boldsymbol{\pi}} \mathbf{c}_0 + \widetilde{\boldsymbol{\varepsilon}}(\mathbf{c}_0)$ or $\mathbf{c}_0 + \widetilde{\boldsymbol{\varepsilon}}(\mathbf{c}_0) \succeq_{\boldsymbol{\pi}} \mathbf{c}_0$, one of the following two inequalities holds

$$\omega_s v_{\boldsymbol{\pi}}(c_0 + \varepsilon(c_0)) + \omega_t v_{\boldsymbol{\pi}} \left(c_0 - \frac{\pi_t \varepsilon(c_0)}{\pi_s} \right) \leq \omega_s v_{\boldsymbol{\pi}}(c_0) + \omega_t v_{\boldsymbol{\pi}}(c_0) \tag{A.5}$$

or

$$\omega_s v_{\boldsymbol{\pi}}(c_0 + \varepsilon(c_0)) + \omega_t v_{\boldsymbol{\pi}} \left(c_0 - \frac{\pi_t \varepsilon(c_0)}{\pi_s} \right) \geq \omega_s v_{\boldsymbol{\pi}}(c_0) + \omega_t v_{\boldsymbol{\pi}}(c_0). \tag{A.6}$$

Without loss of generality, assume that inequality (A.5) holds. Then we also have $\mathbf{c}_0 + \boldsymbol{\varepsilon}(\mathbf{c}_0) \succeq_{\boldsymbol{\pi}} \mathbf{c}_0 + \boldsymbol{\varepsilon}(\mathbf{c}_0) - \widetilde{\boldsymbol{\varepsilon}}(\mathbf{c}_0)$, implying that

$$\omega_s v_{\boldsymbol{\pi}}(c_0) + \omega_t v_{\boldsymbol{\pi}} \left(c_0 + \left(1 + \frac{\pi_t}{\pi_s} \right) \varepsilon(c_0) \right) \leq \omega_s v_{\boldsymbol{\pi}}(c_0 + \varepsilon(c_0)) + \omega_t v_{\boldsymbol{\pi}}(c_0 + \varepsilon(c_0)). \tag{A.7}$$

Adding separately the left and right hand sides of eqns. (A.5) and (A.7) and rearranging terms yields

$$\omega_t v_{\boldsymbol{\pi}} \left(c_0 + \left(1 + \frac{\pi_t}{\pi_s} \right) \varepsilon(c_0) \right) + \omega_t v_{\boldsymbol{\pi}} \left(c_0 - \frac{\pi_t \varepsilon(c_0)}{\pi_s} \right) \leq \omega_t v_{\boldsymbol{\pi}}(c_0) + \omega_t v_{\boldsymbol{\pi}}(c_0 + \varepsilon(c_0)). \tag{A.8}$$

Setting

$$x = c_0 + \varepsilon(c_0)/2, \; h = \left(1 + 2\frac{\pi_t}{\pi_s} \right) \frac{\varepsilon(c_0)}{2} \text{ and } \lambda = \frac{1}{1 + 2\frac{\pi_t}{\pi_s}}, \tag{A.9}$$

eqn. (A.8) can be rewritten as

$$v_{\boldsymbol{\pi}}(x + h) + v_{\boldsymbol{\pi}}(x - h) \leq v_{\boldsymbol{\pi}}(x + \lambda h) + v_{\boldsymbol{\pi}}(x - \lambda h). \tag{A.10}$$

Using the above inequality repeatedly n times, one obtains

$$v_{\boldsymbol{\pi}}(x + h) + v_{\boldsymbol{\pi}}(x - h) \leq v_{\boldsymbol{\pi}}(x + \lambda^n h) + v_{\boldsymbol{\pi}}(x - \lambda^n h). \tag{A.11}$$

Since $\lambda < 1$, taking the limit $n \to \infty$ yields

$$\frac{1}{2}\left(v_{\boldsymbol{\pi}}\left(x+h\right)+v_{\boldsymbol{\pi}}\left(x-h\right)\right) \leq v_{\boldsymbol{\pi}}\left(x\right). \tag{A.12}$$

Since $v_{\boldsymbol{\pi}}(c)$ is continuous, following Jensen (1906), midpoint concavity is equivalent to concavity. Therefore, eqn. (A.12) implies that $v_{\boldsymbol{\pi}}(c)$ is concave in the open interval corresponding to $B(\mathbf{c})$. Since $v_{\boldsymbol{\pi}}(c)$ is continuous and concave in the open interval corresponding to $B(\mathbf{c})$, it is differentiable except for at most countable points in the open interval corresponding to $B(\mathbf{c})$. Thus $\sum_{s=1}^{S} \omega_s v_{\boldsymbol{\pi}}(c_s)$ is also differentiable except for at most countable points in the open interval corresponding to $B(\mathbf{c})$. It follows from the first order condition that for every point $\mathbf{c}_0$ in the open interval corresponding to $B(\mathbf{c})$ except at most countable points

$$\frac{\omega_s v'_{\boldsymbol{\pi}}(c_0)}{\omega_1 v'_{\boldsymbol{\pi}}(c_0)} = \frac{\omega_s}{\omega_1} = \frac{\pi_s}{\pi_1} \quad (s = 2, 3, ..., S). \tag{A.13}$$

Since

$$\sum_{s=1}^{S} \omega_s = \sum_{s=1}^{S} \pi_s = 1, \tag{A.14}$$

we have

$$\omega_s = \pi_s \quad (s = 1, 2, ..., S). \tag{A.15}$$

Therefore,

$$U(\mathbf{c}|\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{A.16}$$

which completes the proof for sufficiency.

B Proof or Theorem 2

Necessity is clear. Next prove sufficiency. Taking $\boldsymbol{\pi}' = \boldsymbol{\pi}$, it follows from Theorem 1 that Axioms 1, 3 and 5 imply that

$$U(\mathbf{c}|\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right). \tag{B.1}$$

Suppose that $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$. $\mathbf{c}_{-s}x \sim_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y$ and $\mathbf{c}'_{-s}w \sim_{\boldsymbol{\pi}} \mathbf{c}_{-s}z$ imply that

$$v_{\boldsymbol{\pi}}(x) - v_{\boldsymbol{\pi}}(y) = v_{\boldsymbol{\pi}}(z) - v_{\boldsymbol{\pi}}(w). \tag{B.2}$$

Similarly, $\mathbf{c}'''_{-s'}y \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}x$ and $\mathbf{c}'''_{-s'}w \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}z$ imply that

$$v_{\boldsymbol{\pi}'}(x) - v_{\boldsymbol{\pi}'}(y) = v_{\boldsymbol{\pi}'}(z) - v_{\boldsymbol{\pi}'}(w). \tag{B.3}$$

Moreover, in eqns. (B.2) and (B.3), x, y, z can be freely chosen, i.e., for any x, y, due to continuity, one can always find a consumption stream $\mathbf{c}'_{-s}y$ on the same indifference (hyper)surface as $\mathbf{c}_{-s}x$. Since Axiom 5 implies that for any x, y, z, if eqn. (B.2) holds, then (B.3) also holds. Therefore, $v_{\boldsymbol{\pi}}$ and $v_{\boldsymbol{\pi}'}$ must be affinely equivalent, i.e., for any $\boldsymbol{\pi} \neq \boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right)$, we must have

$$v_{\boldsymbol{\pi}} = a v_{\boldsymbol{\pi}'} + b, \tag{B.4}$$

where $a > 0$ and b are some constants. Since the NM index is defined up to a positive affine transformation, we can conclude that

$$U\left(\mathbf{c}|\boldsymbol{\pi}\right) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{B.5}$$

which completes the proof.

C Proof of Theorem 3

Necessity is obvious. Next we prove sufficiency. It follows from Theorem 2 that Axioms 1, 3 and 5 are equivalent to a utility representation of the form

$$U\left(\mathbf{c}, \boldsymbol{\pi}\right) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{C.1}$$

where $v(c_s)$ is a strictly increasing function. It follows from Axiom 6 that $\forall \mathbf{c} = (\overline{c}, \overline{c}, ..., \overline{c}) \in \mathbb{R}_+^S$ and $\forall \boldsymbol{\pi}, \boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right)$, we have

$$f\left(\boldsymbol{\pi}, v\left(\overline{c}\right)\right) = f\left(\boldsymbol{\pi}', v\left(\overline{c}\right)\right), \tag{C.2}$$

implying that $f(\cdot, \cdot)$ must be independent of probabilities.

D Proof of Theorem 4

Necessity is obvious. Next we prove sufficiency. It follows from Theorem 1 that Axioms 1, 2 and 3 are equivalent to a utility representation of the form

$$U\left(\mathbf{c}, \boldsymbol{\pi}\right) = g\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right). \tag{D.1}$$

Assuming that $c_1 = c_2 = c_3 = ... = c_S = \overline{c}$, Axiom 7 implies that

$$U(\mathbf{c}, \boldsymbol{\pi}) = g\left(\boldsymbol{\pi}, v_{\boldsymbol{\pi}}(\overline{c})\right) \tag{D.2}$$

is independent of probabilities and hence

$$g_{\boldsymbol{\pi}}\left(v_{\boldsymbol{\pi}}(\overline{c})\right) = f\left(\overline{c}\right), \tag{D.3}$$

where f is independent of probabilities. It follows that $\forall \overline{c}$

$$g_{\boldsymbol{\pi}}\left(\overline{c}\right) = g_{\boldsymbol{\pi}}\left(v_{\boldsymbol{\pi}} \circ v_{\boldsymbol{\pi}}^{-1}(\overline{c})\right) = f \circ v_{\boldsymbol{\pi}}^{-1}\left(\overline{c}\right), \tag{D.4}$$

implying that

$$g_{\boldsymbol{\pi}} = f \circ v_{\boldsymbol{\pi}}^{-1}. \tag{D.5}$$

If $c_1 \neq c_2 = c_3 = ... = c_S = \overline{c}$ then it follows from Axiom 7 that

$$U(\mathbf{c}, \boldsymbol{\pi}) = f \circ v_{\boldsymbol{\pi}}^{-1}\left(\left(1 - \pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1)\right) \tag{D.6}$$

is independent of π_s $(s > 1)$, or equivalently,

$$\frac{\partial v_{\boldsymbol{\pi}}^{-1}\left(\left(1 - \pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1)\right)}{\partial \pi_s} = 0 \quad \left(\forall s = 2, 3, ..., S\right). \tag{D.7}$$

Holding π_1 fixed, consider two different profiles of probabilities $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ with associated NM indices $v_{\boldsymbol{\pi}}$ and $v_{\boldsymbol{\pi}'}$. It follows from (D.7) that there exists a $\eta(c_1, \overline{c})$ such that

$$\eta(c_1, \overline{c}) = v_{\boldsymbol{\pi}}^{-1}\left(\left(1 - \pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1)\right) \tag{D.8}$$

and

$$\eta(c_1, \overline{c}) = v_{\boldsymbol{\pi}'}^{-1}\left(\left(1 - \pi_1\right) v_{\boldsymbol{\pi}'}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}'}(c_1)\right), \tag{D.9}$$

implying that

$$v_{\boldsymbol{\pi}}\left(\eta(c_1, \overline{c})\right) = \left(1 - \pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1) \tag{D.10}$$

and

$$v_{\boldsymbol{\pi}'}\left(\eta(c_1, \overline{c})\right) = \left(1 - \pi_1\right) v_{\boldsymbol{\pi}'}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}'}(c_1). \tag{D.11}$$

Therefore,

$$\begin{aligned} v_{\boldsymbol{\pi}}\left(\eta(c_1, \overline{c})\right)) &= h(v_{\boldsymbol{\pi}'}\left(\eta(c_1, \overline{c})\right)) \\ &= \pi_1 h(v_{\boldsymbol{\pi}'}(c_1)) + (1 - \pi_1) h(v_{\boldsymbol{\pi}'}(\overline{c})) \\ &= h\left(\left(1 - \pi_1\right) v_{\boldsymbol{\pi}'}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}'}(c_1)\right). \end{aligned} \tag{D.12}$$

Since eqn. (D.12) holds for any given π_1, c_1 and $\overline{c}$, we can conclude that h is a linear function which can depend on π_s and π_s' $(s = 2, 3, ..., S)$. Since $\forall \boldsymbol{\pi}, \boldsymbol{\pi}' \in int\left(\Delta^{S-1}\right)$ with the same π_1, there always exists a linear function $h_{\boldsymbol{\pi},\boldsymbol{\pi}'}(\cdot)$ such that $v_{\boldsymbol{\pi}}(c) = h_{\boldsymbol{\pi},\boldsymbol{\pi}'}(v_{\boldsymbol{\pi}'}(c))$, we can conclude that

$$v_{\boldsymbol{\pi}}(c) = \kappa_1'\left(\pi_1, \pi_2, ..., \pi_{S-1}\right) v_{\pi_1}\left(c\right) + \kappa_2'\left(\pi_1, \pi_2, ..., \pi_{S-1}\right) \quad \left(\forall c\right), \tag{D.13}$$

where κ_1' and κ_2' are arbitrary coefficients. Assuming $c_2 \neq c_1 = c_3 = ... = c_S = \overline{c}$ and following the similar argument, we can also show that

$$v_{\boldsymbol{\pi}}(c) = \kappa_1''(\pi_1, \pi_2, ..., \pi_{S-1})\, v_{\pi_2}(c) + \kappa_2''(\pi_1, \pi_2, ..., \pi_{S-1}) \quad (\forall c)\,, \tag{D.14}$$

where κ_1'' and κ_2'' are arbitrary coefficients. Combining eqn. (D.13) with (D.14) yields

$$v_{\boldsymbol{\pi}}(c) = \kappa_1(\pi_1, \pi_2, ..., \pi_{S-1})\, v(c) + \kappa_2(\pi_1, \pi_2, ..., \pi_{S-1}) \quad (\forall c)\,, \tag{D.15}$$

where κ_1 and κ_2 are arbitrary coefficients and v is independent of probabilities. Therefore we have

$$U(\mathbf{c}, \boldsymbol{\pi}) = g\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right) = f \circ v_{\boldsymbol{\pi}}^{-1}\left(\sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right). \tag{D.16}$$

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