Financial Intermediation Networks

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Abstract

We study a dynamic model of financial intermediation in which interbank lending is subject to moral hazard, where intermediaries can divert funds towards inefficient projects. We show that despite the presence of moral hazard, secured lending contracts can discipline the investment choices of all market participants — even those with whom they are not directly contracting — thus partially overcoming market frictions. Our results provide a characterization of the relationship between the intermediation capacity of the system on the one hand, and the extent of moral hazard, the distribution of collateral and the network of interbank relationships on the other. We use this characterization to show that due to the recursive nature of the moral hazard problem, small changes in fundamentals may result in significant drops in the financial system’s intermediation capacity, leading to a complete credit freeze.

Keywords: Financial intermediation, financial networks, secured lending, collateral.

JEL Classification: G01, D85.

Preliminary and Incomplete Comments Welcome

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1 Introduction

In the pre-crisis period, financial markets witnessed a growing reliance on short-term funds raised in wholesale markets. In particular, there was a dramatic rise in the use of repurchase agreements to fund longer-term investment opportunities or to finance inventories of securities held for market-making purposes. Since such funding opportunities were secured by collateral, they were mostly viewed as safe. Since the crisis of 2007–2009, however, the sale and repurchase agreement (repo) market has been viewed as one of the potential sources of fragility in the financial system. As documented by Gorton and Metrick (2012a,b), concerns about the risk and liquidity of the collateral at the onset of the crisis led to a dramatic rise in the inter-dealer bilateral repo haircuts (which captures the extent of overcollateralization), a run in the repo market, and the collapse in short-term lending.

In this paper, we focus on financial institutions’ role as intermediaries between cash lenders (e.g., money market funds) and borrowers (e.g., hedge funds), and study how frictions in a network of intermediaries interlinked via secured lending contracts (such as repurchase agreements) can function as a mechanism for propagation and amplification of risk. In particular, we show that if intermediation is subject to moral hazard, small shocks to the fundamentals can severely impede the financial system’s ability to intermediate funds efficiently, resulting in a potential credit freeze.

More concretely, we focus on an economy consisting of $n$ financial institutions (henceforth, banks) some of which are endowed with excess capital, while other have access to profitable investment opportunities. Due to the presence of trade frictions (such as pairwise commitment problems, adverse selection, or the absence of long-term interbank relationships), however, not all banks may be able to trade with one another, making intermediation necessary to realize the gains from trade. In addition to these trade frictions, financial intermediation in our model is subject to another key friction: any of the intermediaries can divert the funds to invest in potentially riskier, inefficient projects. Therefore, all bilateral transactions in our model are subject to moral hazard. In fact, we assume that not only the investment decisions of the intermediaries are not contractable, but also that lenders cannot write contracts that are contingent on the intricate pattern of trades within the financial system. Rather, they are restricted to writing simple contracts that only specify the interest rate and the haircut they are willing to charge the borrower.

As our first result, we show that as long as there is an abundance of liquid assets to be employed as collateral, bilateral secured lending contracts can overcome all moral hazard frictions within the financial network. More specifically, we show that, despite the absence of contingent contracts, the investment choices of all borrowers in the network can be influenced by increasing the amount of collateral that is required to be posted with the lender. The haircuts thus essentially serves as an instrument that can restore efficiency when borrowers have a private incentive to take excessive risk. This result thus echoes the earlier results by Stiglitz and Weiss (1981) and Bester (1985, 1987) who show that collateralized lending can overcome moral hazard problems. Our results, however, show that simple secured lending contracts not only can overcome moral hazard problems in bilateral lending agreements, but can also discipline the investment choices of all intermediaries in the
network — including the ones with whom they are not directly contracting.

We then show that haircuts increase with the length of the intermediation chains. This is a consequence of the fact that moral hazard problems are more severe in longer chains, as more intermediaries can divert the funds towards risky, inefficient projects. The cumulative nature of the moral hazard friction thus implies that banks have to charge higher haircuts to induce the right behavior throughout the chain. Using a similar reasoning, we also show that equilibrium haircuts increase with the intensity of the moral hazard problem as well as the illiquidity of the collateral.

Even though secured lending contracts can restore first-best efficient outcomes when banks have large endowments of assets that can be used as collateral, the same is no longer true when collateral is scarce. Such a scarcity means that lenders may no longer be able to overcome the moral hazard problems, thus leading to an upper bound on the volume of capital than can be efficiently intermediated through the system. Our results provide a characterization of this upper bound — which we refer to as the financial system’s intermediation capacity — in terms of collateral’s liquidity, volatility and availability. Using this characterization, we show that the distribution of assets that can be employed as collateral among different financial institutions plays a first-order role in determining the system’s intermediation capacity, as it determines the equilibrium allocation of intermediation rents, and hence, the incentives of different institutions to avoid diverting the funds towards inefficient projects.

The fact that the moral hazard problems are cumulative over the financial network also implies that idiosyncratic shocks may be amplified along the chain of intermediaries. In particular, we show that small idiosyncratic shocks (say, to the expected return on the efficient project or the liquidity of the collateral) can lead to a sharp increase in haircuts, and as a result, large drops in the financial system’s intermediation capacity, which may manifest itself as a complete credit freeze. Such events, which are reminiscent of the “repo runs” phenomena, arise not due to the intermediaries’ need for funding liquidity or shocks to expectations, but rather as a response to the build up of moral hazard frictions through the intermediation chain.

As part of our analysis, we also study the role played by other important features of the collateral — beyond its liquidity — in determining the overall intermediation capacity of the financial system. We show that increasing the correlation between the quality of the collateral and the investment opportunities (e.g., by employing real-estate mortgage backed securities as collateral to fund other investments in the housing market) would reduce the intermediation capacity of the financial system as a whole. Intuitively, given that the borrower loses its collateral only in the state of the world in which its project fails, a higher correlation between the investment opportunity and the securities employed as collateral tightens the borrower’s incentive compatibility constraint, thus forcing its corresponding lender to charge a higher haircut. Such overcollateralization, coupled with the cumulative nature of the moral hazard problem, would in turn reduce the intermediation capacity of the financial system, eventually leading to a potential market freeze. Yet another implication of our setting is that when this correlation is positive (negative), an increase in the riskiness of the collateral leads to higher (lower) haircuts, reduces (increases) the intermediation capacity of the market, and
increases (decreases) the possibility of market freezes. The intuition for this result is similar: when positive, any increase in the riskiness of the securities employed as collateral shifts the proceeds from their liquidation from the bad to the good state of the world, thus increasing the incentives of any given borrower to invest in the inefficient project. Consequently, the corresponding lender ends up charging a higher haircut to discipline the borrower.

We remark that even though highly stylized, our setting captures several salient features of the repo market: borrowers can invest in projects with different risk characteristics; the collateral’s value is uncertain and is potentially correlated with the investment’s riskiness; and in case of borrower’s default, the liquidation of the collateral by the lender might be costly. More importantly, however, unlike most of the literature that mainly focuses on a single bilateral transaction, we study an inter-dealer market in which multiple financial institutions can borrow and lend to each other via bilateral repo contracts. Furthermore, in our setup all interest rates and haircuts are jointly determined in equilibrium.

Related Literature Our paper is part of the recent but growing literature that focuses on the role of financial networks in shaping the fragility of the financial system. Started with the seminal works of Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000), this literature studies whether and how the architecture of the financial system can serve as a shock propagation and amplification mechanism. For example, Acemoglu, Ozdaglar, and Tahbaz-Salehi (forthcoming) characterize the extent of contagion in a network of financial institutions linked via unsecured debt contracts, whereas Elliott, Golub, and Jackson (forthcoming) study a model in which different organizations’ cross-holdings of one another’s equity shares may lead to cascading failures.¹ We complement these studies by analyzing a dynamic model of secured lending in which interbank relationships are subject to moral hazard: rather than further lending to the institutions with efficient investment opportunities, a borrowing institution may divert them towards riskier, inefficient projects. Our main results illustrate that, due to the presence of such agency problems, the maximum amount of liquidity that can be intermediated by the system crucially depends on the structure of the financial network and the quality of the assets used as collateral. We then use this characterization to show that small changes in the fundamentals may result in significant drops in the intermediation capacity of the financial system, leading to a complete credit freeze.

With a few exceptions, such as Babus (2014) and Farboodi (2014), the literature on financial networks mainly studies intermediation while taking the interbank claims and liabilities as exogenous. In contrast, the terms of interbank contracts (such as interest rates and haircuts) as well as the realized paths of credit flow in our model are endogenously determined.

Our work is also related to several recent papers that study the broad question of trade over networks, such as Condorelli and Galeotti (2012), Choi, Galeotti, and Goyal (2014), and Manea (2014).

Condorelli and Galeotti (2012), for example, analyze a sequential model of trade of a single indivisible good over a general network, in which traders have incomplete information about one another’s valuations. Manea (2014), instead, studies a model in which the good is resold via successive bilateral bargaining between linked intermediaries and characterizes the endogenous structure of local monopolies and trading paths that arise in equilibrium. In contrast to these papers, our main focus is on the determinants of the financial system’s intermediation capacity in the presence of moral hazard. We show that equilibrium trading paths and the endogenous allocation of intermediation rents crucially depend on the distribution of collateral assets as well as the extent of agency problems.

A different strand of literature has focused on the emergence of intermediation chains in over-the-counter markets. Glode and Opp (2014) argue that, in the presence of asymmetric information, trading an asset through several heterogeneously informed intermediaries can reduce the extent of adverse selection between counterparties, hence preserving the efficiency of trade. On the other hand, Colliard and Demange (2014) study the incentives of financial institutions to provide liquidity in an OTC market and show that in the presence of search frictions, some institutions emerge as intermediaries by partially reselling the assets they buy, thus leading to the formation of intermediation chains. In contrast to these papers, the key friction in our model is the presence of moral hazard in the intermediation market. Our characterization results show that the quality of the collateral — as measured by its liquidity, volatility and availability — is a key factor in determining the extent of intermediation in equilibrium.

The key role played by the repo market during the panic of 2007–2008 has been highlighted by Gorton and Metrick (2012a,b), who document that haircuts rose dramatically at the onset of the crisis, mainly in response to increasing concerns about the liquidity of markets for bonds used as collateral. Based on these observations, they argue that the bilateral repo market suffered a run, which in turn amplified the financial crisis. Relatedly, Krishnamurthy, Nagel, and Orlov (forthcoming) examine data on the repo lending by money market funds and securities lenders and show that such liquidity providers became less willing to lend against risky or illiquid collateral. These studies highlight the importance of the collateral, and in particular its volume, riskiness and liquidity, in sustaining capital flows between financial intermediaries, all of which are central to our analysis.

On the theoretical side, a recent collection of papers, such as Parlatore Siritto (2012), Martin, Skeie, and von Thadden (2014), and Dang, Gorton, and Holmström (2013) study models of collateralized lending with applications to the repo market. Parlatore Siritto (2012) analyzes the trade-off between selling assets or pledging them as collateral and shows that collateralized debt contracts arise naturally to solve an asymmetric information problem about the borrowers’ ability to repay.4

3On the contrary, Copeland, Martin, and Walker (forthcoming) show that tri-party repo market was stable during the same period.
4Monnet and Narajabad (2012) provide conditions under which repurchase agreements can co-exist with asset sales. In particular, they show that, when borrowers are pairwise matched to lenders, as agents become more uncertain of the value of holding the asset, repos become more prevalent.
Martin, Skeie, and von Thadden (2014) explore the role of liquidity and collateral constraints in determining the possibility of expectations-driven runs in the repo market. On the other hand, Dang, Gorton, and Holmström (2013) argue that overcollateralization arises to overcome the adverse selection problem that lenders may face in case of a borrower’s default. They show that in such an environment, the arrival of public information about the quality of the collateral can lead to repo runs.\(^5\) We, in contrast, focus on a different friction from the aforementioned papers. In particular, we show that due to the cumulative nature of the moral hazard problem over chains of intermediaries, small shocks to the liquidity or availability of collateral assets can lead to large spikes in the haircuts, resulting in the sharp collapse of the systems’ intermediation capacity.

Our paper is also related to the earlier literature that studies the rationales for secured lending. For example, Stiglitz and Weiss (1981) and Bester (1985, 1987) argue that collateralized lending can serve as a screening device in asymmetric information settings or to overcome moral hazard problems. In this paper, we show how such agency problems can build up in a financial network, leading to an excessively fragile financial system.

Finally, the role played by the moral hazard friction in our model is reminiscent of Kim and Shin (2012), who use a multi-layered version of the contracting model of Holmström and Tirole (1997) to highlight that inter-firm credit, such as account receivables and payable, can solve recursive moral hazard problems that may arise in production chains.\(^6\) We, on the other hand, focus on collateralized lending relationships between financial institutions in a general network and show how the presence of moral hazard can lead to a potential market freeze at the face of small shocks.

Outline of the paper The rest of the paper is organized as follows. Section 2 introduces the general model. In Section 3, we focus on a financial intermediation network in which banks are located on a chain, and show that the repo haircuts can discipline the investment choices of all market participants. Section 4 contains our results for the case that the collateral is risky and Section 5 concludes. All proofs are presented in the appendix.

2 Model

Banks and Investments Consider an economy consisting of a collection of risk-neutral financial institutions (henceforth, banks for short), which we denote by \( N = \{1, \ldots, n\} \). The economy lasts for three periods. At \( t = 0 \), each bank is endowed with \( A \) units of a bank-specific illiquid asset, which it can use as collateral for secured borrowing from other banks. If liquidated by its original owner at \( t = 2 \), each unit of the asset leads to one unit of proceeds, whereas banks other than the original owner can only recover a fraction \( \alpha \leq 1 \) of the value of the asset. This parameter captures the

\(^{5}\)More recently, Lee (2014) shows that easier collateral circulation can potentially result in inefficient repo runs, whereas Eren (2014) provides a model in which the demand by dealer banks for funding liquidity determines repo haircuts and interest rates.

\(^{6}\)Gofman (2013) and Kalemli-Özcan, Shin, Kim, Sørensen, and Yesiltas (2013) provide empirical support for the model’s implications.
idea that the collateral might be illiquid and hence, costly for a lender to liquidate.\footnote{In reality, various asset classes with a wide range of liquidities (ranging from the most liquid ones such as U.S. Treasuries to the least liquid ones such as mortgage-backed security or corporate bonds) are used as collateral.} The premature liquidation of the assets at \( t = 1 \) leads to a per-unit proceed of \( \gamma < \alpha \), regardless of the identity of the liquidating bank, a parameter which we assume to be small.

At \( t = 0 \), each bank has access to a constant returns to scale investment opportunity (project), which has a \( t = 1 \) return of \( r_h \) with probability \( 1 - \phi_h \) and zero with probability \( \phi_h \). In addition to this investment opportunity, a randomly chosen bank has access to a low-risk, low-return lumpy project of size \( k \) with rate of return \( r_\ell < r_h \) and failure probability \( \phi_\ell < \phi_h \). We assume that the projects’ expected returns satisfy

\[
\begin{align*}
    r_\ell (1 - \phi_\ell) &> r_h (1 - \phi_h),
\end{align*}
\]  

i.e., it is always efficient to invest in the safer project.

Even though all banks have access to at least one investment opportunity, capital is scarce. In particular, at \( t = 0 \), a randomly chosen bank — which we refer to as the \textit{liquidity provider} — is endowed with \( k \) units of capital, whereas all other banks have no endowments of their own. The liquidity provider can either invest \( k \) in its own project(s), or lend it to other institutions. In view of inequality (1), if the liquidity provider is distinct from the bank with access to the efficient investment opportunity, it is always efficient for the former to (directly or indirectly) lend its excess capital to the latter.

\textbf{Interbank Network} \hspace{1em} Even though there are potential gains from trade, the liquidity provider and the bank with access to the efficient project may not be able to trade with another directly, thus implying that other banks may need to act as intermediaries. The presence of such trade frictions may arise due to asymmetric costs of peer monitoring, adverse selection, absence of long-term interbank relationships, or pairwise commitment problems.\footnote{See Afonso, Kovner, and Schoar (2013) and Di Maggio (2014) for evidence supporting this assumption.}

Formally, we capture the presence of the interbank trade frictions by an undirected, connected network \( G = (V, E) \), where each vertex in \( V \) corresponds to a bank, and an edge \((i, j) \in E\) captures the possibility of trade between banks \( i \) and \( j \).\footnote{A network is said to be \textit{undirected} if \((j, i) \in E\) whenever \((i, j) \in E\). We say the network is \textit{connected}, if there exists a path connecting every bank to every other bank in the network.} Note that the existence of an edge between two banks does not necessarily imply that they would enter into a lending agreement. Rather, as we describe next, the interbank lending and borrowing decisions are determined endogenously.

\textbf{Interbank Lending and Contracts} \hspace{1em} Interbank lending occurs dynamically over \( t = 0 \), which is divided into infinitely many sub-periods. At any given sub-period, one of the banks with liquidity at hand, say bank \( i \), decides whether to invest in the project(s) it has access to or to lend its capital to some bank \( j \) to which it is connected. If it decides to lend to bank \( j \), it offers \( j \) a take-it-or-leave-it
repo contract of the form \((k_{ij}, R_{ij}, h_{ij})\), where \(k_{ij}\) is the size of the loan and \(R_{ij}\) and \(h_{ij} \leq 1\) correspond to the repo rate and haircut, respectively.\(^{10}\) Bank \(j\) then decides whether to accept or reject this contract. In case of rejection, bank \(i\) keeps the capital. If, on the other hand, bank \(j\) accepts the contract, it borrows \(k_{ij}\) from bank \(i\) and transfers \(c_{ij} = k_{ij}/(1 - h_{ij})\) units of its illiquid asset to \(i\) as collateral.\(^{11}\) This process continues until all excess liquidity \(k\) is invested by some bank. As already mentioned, the returns on the investments are realized at \(t = 1\).

The repurchase leg of the repo agreements occur — in the reverse order they have been made — once the investment returns are realized, as each borrower bank \(j\) has to repurchase the collateral at price \(R_{ij}k_{ij}\) from lender \(i\) if it can. If, on the other hand, the borrower is not capable of following through with the repurchase leg of the agreement, the lender has the right to keep possession of the collateral.\(^{12}\) However, recall that the lender can only recover a fraction \(\alpha \leq 1\) of the value of the collateral.

Finally, after the settlement of the repurchase agreements at \(t = 1\), banks liquidate their holdings of the illiquid assets at \(t = 2\) and consumption takes place.

**Discussion of the Model**  The key underlying assumption of our model is that the repo rates and haircuts are *not* contingent on the investment or lending decisions of the borrower bank \(j\), neither do they depend on the decisions of any other bank. This assumption implies that not only all pairwise interbank interactions are subject to moral hazard, but also that the agency problems are cumulative: as more banks act as intermediaries between the liquidity provider and the eventual investors, the agency problems within the financial system intensify. The presence of such *recursive moral hazard problems* lies at the core of our results.

We emphasize that it is fairly realistic to assume that pairwise interbank contracts have no or little contingencies on the potentially complex pattern of trades between all banks within the system, as information on such intricacies may not be available or contractable. Nevertheless, as our following results show, the emergence of haircuts in the pairwise repo contracts ensures that all banks internalize the effect of their decisions, not only on their immediate counterparties, but also on other banks with whom they are not directly contracting.

### 3 Intermediation Chains

In order to present the insights behind our model in the most transparent manner, we first focus on a simple environment in which the financial intermediation network is in the form of a *chain*. This allows us to obtain a parsimonious characterization of the equilibrium and clarify how repo haircuts

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\(^{10}\)The contracts in our model are similar to that of Adrian and Shin (2014).

\(^{11}\)We do not allow for rehypothecation of the collateral.

\(^{12}\)Note that for small enough values of \(\gamma\), the borrower would default on its obligations to \(i\), whenever it does not have enough cash available.
can overcome the recursive moral hazard problem.\textsuperscript{13}

We assume that each bank \(i \in \{2, \ldots, n-1\}\) can only trade with two other banks labeled \(i - 1\) and \(i + 1\), whereas banks 1 and \(n\) can only trade with banks 2 and \(n - 1\), respectively. Figure 1 depicts the corresponding financial network. We further assume that only bank 1 has access to the efficient investment opportunity (with rate of return \(r_\ell\) and failure probability \(\phi_\ell\)), whereas bank \(n\) is the only bank with an excess liquidity of size \(k\). Thus, the excess liquidity available to bank \(n\) can be invested in the efficient project only if banks 2 through \(n - 1\) intermediate between banks 1 and \(n\). As in our general model, we assume that all banks have access to the inefficient project with rate of return \(r_h\) and failure probability \(\phi_h\).

![Figure 1. An intermediation chain: each bank \(i\) can only trade with banks \(i - 1\) and \(i + 1\), whereas banks 1 and \(n\) can only trade with banks 2 and \(n - 1\), respectively. Bank \(n\) is the only bank with excess liquidity, whereas bank 1 is sole bank with access to the efficient investment opportunity.](image)

To characterize the equilibrium in this environment, denote the interest rate and collateral that bank \(i + 1\) charges bank \(i\) by \(R_i\) and \(c_i\), respectively. If the system successfully intermediate the excess liquidity \(k\) available to bank \(n\), then the payoff of bank \(i + 1\) in the good state of the world in which the efficient project has a positive return is equal to \((R_i - R_{i+1})k + A\). On the other hand, if the efficient project fails (and for small enough values of \(\gamma\)), bank \(i\) has no cash available to repurchase the collateral at \(t = 1\). Bank \(i + 1\) thus gets to keep the collateral and obtains a payoff of \(\alpha c_i\) at \(t = 2\), whereas it loses \(c_{i+1}\) to its corresponding lender. Therefore, if the intermediation is successful, the expected payoff of bank \(i + 1\) is given by

\[
\pi_{i+1} = (1 - \phi_\ell)(R_i k - R_{i+1}k + A) + \phi_\ell(\alpha c_i + A - c_{i+1}),
\]

where recall that \(\phi_\ell\) is the failure probability of the efficient investment opportunity.

### 3.1 Equilibrium with No Collateral Constraints

As a benchmark, we first study the case in which banks are not collateral-constrained, in the sense that they can post an arbitrarily large amount of the illiquid asset as collateral, if they chooses to do so. Mathematically, this corresponds to the assumption that the value of \(A\) is large enough.\textsuperscript{14} We have the following result:

\textsuperscript{13}Such chains of intermediaries are very common in several markets, such as interest rates swaps (Viswanathan and Wang, 2004), commodity markets (Weller, 2014) and the interbank market.

\textsuperscript{14}At the end of 2006, the 10-Q filings of the major investment banks of the time, Goldman Sachs, Meryl Lynch, Lehman Brothers, Bear Sterns and J.P. Morgan, revealed that 47\% of their assets were pledged as collateral in repurchase agreements.
Proposition 1. Suppose that banks are not collateral-constrained. Then, there exists \( n^* \) such that the interbank network efficiently intermediates the excess liquidity if and only if \( n \leq n^* \), where \( n^* \) is increasing in \( \alpha \) and satisfies \( \lim_{\alpha \rightarrow 1} n^* = \infty \). Furthermore, all the surplus goes to the liquidity provider bank \( n \).

Thus, any intermediation chain of length \( n \leq n^* \) can be sustained in equilibrium, in the sense that the excess liquidity available to bank \( n \) is intermediated via banks 2 through \( n - 1 \) and is eventually invested in the efficient project by bank 1. More importantly, this is despite the fact that banks are restricted to writing simple repo contracts whose terms are not contingent on the lending or investment decisions of other institutions. Nevertheless, by charging a haircut, each lender bank \( i \) not only ensures that bank \( i - 1 \) does not divert the funds to the inefficient project, but also overcomes all moral hazard problems further down the chain. In other words, haircuts effectively "complete" the space of contracts.

The intuition behind this result is simple: by charging enough collateral, bank \( i \) can force its corresponding borrower, bank \( i - 1 \), to be more exposed to the downside risk of its investment decisions, hence making excessive risk-taking less appealing. In fact, bank \( i \) charges an interest rate and an haircut that simultaneously bind \( i - 1 \)'s incentive compatibility and participation constraints, thus essentially eliminating the agency problem.

On the flip side, however, Proposition 1 also establishes that if the investment opportunity is too far from the bank with excess liquidity, that is \( n > n^* \), then the interbank network is incapable of efficient intermediation: capital remains in the hands of bank \( n \) and is invested in the inefficient, riskier project. Note that even though bank \( i \) can always bring bank \( i - 1 \) all the way to its indifference point by charging a high enough haircut, the former can only recover a fraction \( \alpha < 1 \) of the value of the collateral. Thus, as the length of the chain (and hence, the severity of the recursive moral hazard problem) increases, the collateral that bank \( n \) needs to charge may be so large that it finds it optimal to simply invest in the inefficient project, as opposed to lending to \( n - 1 \).

The above argument also highlights that the maximum intermediation length that can be sustained in equilibrium crucially depends on the liquidity of the collateral, captured via parameter \( \alpha \). In fact as Proposition 1 shows, as the collateral becomes perfectly liquid, the equilibrium coincides with the first-best outcome regardless of the value of \( n \). This result implies that when intermediaries employ highly liquid assets (such as Treasuries) as collateral, capital flows are efficient, even in the presence of agency problems. In contrast, if intermediaries’ balance sheets are flooded by asset-backed securities and corporate bonds, then efficient investment opportunities are lost.

Finally, we remark that, in addition to the value of \( \alpha \), the maximum intermediation length also depends on the severity of the moral hazard problem. In particular, \( n^* \) is decreasing in the difference in returns of the inefficient and efficient projects, \( \Delta r \). As the riskier investment becomes more attractive for the borrowing institutions, the agency problem between lenders and borrowers is exacerbated, shortening the intermediation chain that can be sustained in equilibrium.

Our next result provides an explicit characterization of the equilibrium interest rate and haircut
charged by the banks along the chain.

**Proposition 2.** Suppose that banks are not collateral-constrained and that \( n \leq n^* \). The equilibrium interest rate and haircut that bank \( i + 1 \) charges bank \( i \) are given by

\[
R_i = r_h \left( 1 - \frac{\phi_h}{\zeta^{n^*-i}} \right)
\]

and

\[
h_i = 1 - \frac{\zeta^{n^*-i}}{r_h(1 - \phi_h)},
\]

respectively, where \( \zeta = 1 + (1 - \alpha)\phi_\ell(1 - \phi_h)/\Delta \phi \).

Figure 2 depicts the equilibrium contracts along the chain. The key observation is that the equilibrium interest rates decrease as we move further away from the bank with the efficient investment opportunity, whereas the haircuts are increasing over the chain. Intuitively, the further away the excess liquidity is from the investment opportunity, the larger is the size of the cumulative moral hazard problem, thus requiring an increase in haircuts over the chain. On the other hand, given that each lender can discipline its corresponding borrower with both the interest rate and the haircut, it is natural that the two instruments function as substitutes. Thus, the interest rates are reduced as the haircuts are increased along the chain.

Proposition 2 can also be used for the purpose of comparative statics. More specifically, we have the following result:

**Corollary 1.** The equilibrium haircuts are decreasing in \( \alpha \) and increasing in \( \Delta r \), whereas the interest rates are increasing in \( \alpha \) and decreasing in \( \Delta r \).
Thus, as the assets pledged as collateral become more illiquid (i.e., as $\alpha$ goes down), the haircuts increase while at the same time banks charge lower interest rates. This suggests that the emergence of haircut in our model not only depends on the severity of the agency problems, but also on the presence of frictions in the secondary market for the collateral. As expected, each lender is willing to lend less against a less liquid collateral. Furthermore, Corollary 1 shows that increasing $\Delta r$ also leads to higher haircuts across the board. Note that a higher $\Delta r$ increases the borrowers’ incentives to deviate and invest in the inefficient project, which in equilibrium makes it optimal for the lender to charge a higher haircut. Such an increase in haircuts in turn implies that the interest rates are optimally set to lower levels.

A key implication of our results is that small shocks can propagate throughout the chain, with possibly significant implications for the terms of equilibrium contracts and hence, the efficiency of the outcome. In particular, a slight decrease, say, in the return of the safer project available to bank 1 not only affects the investment incentives of that bank, but also intensifies the agency problems between any pair of banks over the chain. More importantly, since the moral hazard problems are cumulative, such a small shock may lead to large spikes in equilibrium haircuts charged by banks further down the chain. In fact, given that $n^*$ is decreasing in $\Delta r$, a small shock to bank 1’s investment opportunity may in fact lead to a total market freeze, in the sense that bank $n$ refrains from lending its excess liquidity altogether, and instead invests in the inefficient project.

### 3.2 Equilibrium with Collateral Constraints

We now analyze the case in which the collateral constraint of the intermediaries may bind; that is, there is an upper limit $A$ to the value of assets that each bank can pledge as collateral.

The presence of such a limit in conjunction with the recursive moral hazard problem means that there is a maximum amount of liquidity that can be intermediated by the system. More specifically, we define:

**Definition 1.** The *intermediation capacity* of the financial system, $k_{\text{max}}$, is the maximum size of capital at the hands of the liquidity provider that can be eventually invested in the efficient project.

The intermediation capacity arises due to the fact that bank $n$ faces a trade-off in how to allocate its excess liquidity between the efficient project (which requires intermediation) and the riskier investment opportunity. On one hand, it has an incentive to invest in the more efficient project by pushing more of its capital through the chain of intermediaries. On the other hand, due to the recursive moral hazard problem within the system, a large investment in the efficient project implies that at a certain point (say, from bank $m$ onwards), the financial institutions’ collateral constraints start to bind. This is due to the fact that banks cannot pledge more than the amount of assets they have on their balance sheets. Consequently, as depicted in Figure 3, for $i > m$, the haircuts $h_i$ cannot be increased above $h_{\text{max}} = 1 - k/A$. Thus, in order to make the capital flow possible, each lender bank $i > m$ needs to leave some rents to its corresponding borrower bank $i - 1$ by cutting the interest rate $R_{i-1}$ above and beyond what it would have been in the absence of collateral constraints. Eventually,
Bank index $i$

**Figure 3.** The equilibrium contract $(k, R_i, h_i)$ charged by bank $i + 1$ to bank $i$, in the presence of collateral constraints. The red and blue lines, respectively, depict the interest rate $R_i$ and haircut $h_i$ as a function of $i$. Bank $m$ is the first bank whose collateral constraint binds. The dashed red line depicts the interest rates had the collateral constraints not been binding.

The cumulative size of these rents would become so large that bank $n$ finds it optimal to reduce the investment in the efficient project and instead, divert some of its excess liquidity to the less efficient project.

Our next result formalizes the above argument and characterizes the intermediation capacity of the financial system.

**Proposition 3.** Suppose that $n \leq n^*$. The intermediation capacity of the chain is

$$k_{\text{max}} = \begin{cases} \frac{A \cdot \zeta^{n^*-m}}{r_h(1 - \phi_h)} & \text{if } m \geq 1 \\ \infty & \text{otherwise} \end{cases} \quad (4)$$

where

$$m = \left\lceil \frac{n \log \omega - n^* \log \zeta}{\log \omega - \log \zeta} \right\rceil - 1 \quad (5)$$

is the first bank on the chain whose collateral constraint binds and $\omega = (1 - \phi \ell) / \Delta \phi$.

The above result highlights the importance of collateralizable assets in the financial institutions’ role as intermediaries: the shortage of such assets reduces the ability of lenders to discipline potential deviations by the borrowers. In fact, as predicted by Proposition 3, the intermediation capacity of the financial system is increasing in $A$.

It is important to note that even though the financial system can efficiently intermediate $k_{\text{max}}$ units of capital, not all the surplus would go to the liquidity provider bank $n$. Rather, any bank
The intermediation capacity \( k_{\text{max}} \) as a function of the collateral’s illiquidity, measured in terms of \( 1 - \alpha \). The intermediation capacity decreases as the collateral becomes more illiquid.

\[ \frac{A \Delta \phi / \Delta r}{(1-\phi_h)(1-\phi_e)} \]

Figure 4.

\( i \in \{m + 1, \ldots, n - 1\} \) would also get a portion of the surplus as rents. In fact, it is only because of such rents that the system can efficiently intermediate \( k_{\text{max}} \). Furthermore, the cumulative nature of the agency problems within the chain implies that banks that are closer to the liquidity provider \( n \) get a larger share of the surplus as rents. The above result highlights that, unlike Farboodi (2014), the allocation of surplus between different parties are determined endogenously in our model. Finally, note that if the excess liquidity available to bank \( n \) exceeds the intermediation capacity \( k_{\text{max}} \), it invests the difference \( k - k_{\text{max}} \) in the inefficient project.

We have the following corollary to Proposition 3.

**Corollary 2.** Suppose that \( n \leq n^* \). The intermediation capacity of the chain, \( k_{\text{max}} \), is decreasing in \( \Delta r \) and \( n \), and increasing in \( \alpha \).

Intuitively, increasing \( \Delta r \) increases the borrowers’ incentives to deviate and invest in the riskier project, hence intensifying the agency problem between any pair of banks. Similarly, the larger the distance between the liquidity provider and the eventual borrower is, the more severe the recursive moral hazard problem within the chain becomes. Thus, as the above result shows, increasing \( \Delta r \) and \( n \) would reduce the intermediation capacity of the system.

Corollary 2 also shows that the intermediation capacity is higher the more liquid the collateral asset is, as each bank is willing to lend more against such assets. Figure 4 depicts \( k_{\text{max}} \) as a function of \( 1 - \alpha \). The key observation is that not only the intermediation capacity of the system decreases as the collateral becomes more illiquid, but also that it changes discontinuously as a function of \( 1 - \alpha \). Therefore, a slight change in the collateral asset’s liquidity may result in significant drops in the system’s intermediation capacity. Furthermore, a sufficient reduction in \( \alpha \) may lead to a complete market freeze, whereby the intermediation capacity of the system collapses to zero.

The emergence of discontinuities in the system’s intermediation capacity is due to the fact that for all banks \( i \geq m \), the corresponding lender bank \( i + 1 \) can overcome the moral hazard problem
only if it leaves some rents with \( i \). Therefore, as the collateral asset becomes less liquid, at some point the collateral constraint of bank \( m - 1 \) would also start to bind, reducing the marginal benefit of lending one more unit of capital to bank \( m - 1 \) (as now a fixed fraction of the surplus has to be left with \( m - 1 \)). This discontinuous change in the marginal benefit of intermediation leads to the a discontinuous drop in the system’s overall intermediation capacity. Finally, recall from Proposition 1 that the maximum length of the intermediation chain that can be sustained in equilibrium, i.e., \( n^* \), is increasing in \( \alpha \). Thus, a sufficient decrease in the collateral asset’s liquidity may bring \( n^* \) below \( n \), thus leading to a complete market freeze.

4 Risky Collateral

Our results so far depended on the assumption that each unit of the illiquid assets used as collateral in the secured lending contracts has a fixed, deterministic value. In reality, however, an important concern for the lenders is the riskiness of the securities pledged to them as collateral. In this section, we show how the presence of such risks affects the equilibrium and the intermediation capacity of the system. More specifically, we assume that the liquidation value of the collateral not only depends on the timing of liquidation and the identity of the liquidating bank, but also on the aggregate state of the world.

To capture this idea formally, suppose that the liquidation value of the collateral by its original owner at \( t = 2 \) is a random variable \( z \), with standard deviation \( \sigma \) and expected value normalized to 1. As before, we assume that any other bank can only recover a fraction \( \alpha \leq 1 \) of the value of the asset. Furthermore, let \( \rho \) denote the correlation between \( z \) and the aggregate state of the world, defined as the success or failure of the inefficient project. Throughout, we assume that the returns of the efficient and inefficient projects are independent.

This extension of the basic setup aims to capture two important features of the repo arrangements between financial institutions. First, the riskiness of the collateral captures the necessity for the lenders to protect themselves from the risk of holding assets of lower than expected quality. Second, the correlation \( \rho \) might capture several possibilities. For instance, the borrowing institutions might invest in the housing market, by lending directly to risky households or purchasing collateralized debt obligations (CDO), and pledging as collateral in repo transactions mortgage-backed securities already held in portfolio. In this way, the lending institution might ended up significantly more exposed to the housing market through the investment of the borrowing institution, which would increase his counterparty risk, and through the collateral, which would decrease his proceeds in the case of borrower’s default. That is, a negative shock to the housing market, and the resulting default of the borrower, is even more problematic for the lender as the collateral will turn out to be less valuable. These concerns seem to have been particularly important at the onset of the financial crisis, with liquidity providers pulling out from their lending relationships due to an increase in the perceived risk of the collateral, for example for the cases of ABCP and ABS, and due to an increase in their exposure to a declining housing market.
We have the following result:

**Proposition 4.** Consider the chain financial network and suppose that $\rho \sigma \leq \left( \frac{\Delta \phi}{1-\phi_h} \right) / \sqrt{\phi_h (1-\phi_h)}$.

(a) If banks are not collateral-constrained, then there exists $n^*$ decreasing in $\rho \sigma$, such that the interbank network efficiently intermediates the excess liquidity if and only if $n \leq n^*$.

(b) All interbank interest rates are decreasing in $\rho \sigma$, whereas haircuts increase with $\rho \sigma$.

(c) Suppose that $n \leq n^*$ and that each bank is endowed with $A$ units of the illiquid asset. Then, the intermediation capacity of the market is decreasing in $\rho \sigma$.

The above result thus generalizes Propositions 1–3 to the case in which the returns of the securities used as collateral are potentially risky.\(^{15}\) It shows that when banks are not collateral-constrained, an increase in the correlation between the collateral’s liquidation value and the inefficient project’s returns not only increases the equilibrium haircuts, but may also lead to a complete market freeze. In particular, increasing $\rho$ may reduce $n^*$ to such an extent that the liquidity provider would refrain from lending altogether. Similarly, when banks are constrained by how much collateral they can put up, increasing the correlation between the returns of the securities and the projects would reduce the intermediation capacity of the financial system as a whole.

The intuition underlying this result is simple: given that the borrower loses its collateral only in the bad state of the world, increasing $\rho$ tightens its incentive compatibility constraint, thus forcing its corresponding lender to charge a higher haircut. Furthermore, due to the cumulative nature of the moral hazard problem in our model, a higher $\rho$ also reduces the intermediation capacity of the system and may eventually lead to a market freeze. Note that this result is in contrast to that of Parlatore Siritto (2012), who shows that keeping the (unconditional) expected return of the asset fixed, an increase in the correlation between the success of the investment made by the banks and the future dividends paid by the asset increases the asset’s debt capacity.

Yet another implication of Proposition 4 is that when $\rho$ is positive (negative), an increase in the riskiness of the collateral leads to higher (lower) haircuts, increases (decreases) the possibility of market freezes, and reduces (increases) the intermediation capacity of the market. The intuition for this result is identical: when $\rho > 0$, any increase in the riskiness of the securities, i.e., a higher $\sigma$, shifts the proceeds from liquidating the collateral from the bad to the good state of the world, thus increasing the incentives of any given borrower to invest in the inefficient project. Consequently, the lender ends up charging a higher haircut to discipline the borrower.

5 Conclusions

In this paper, we study a dynamic model of financial intermediation in which interbank lending is subject to moral hazard and show that in the presence of such agency problems, small shocks can

\(^{15}\)We provide the closed-form expression of the intermediation capacity of the system in the Appendix.
lead to large spikes in the haircuts and a sudden collapse in the intermediation capacity of the financial system. In particular, we argue that due to the cumulative nature of the moral hazard problems over chains of intermediaries, a slight change in the perception about the collateral’s liquidity or the expected returns of different projects may lead to a complete credit freeze. We also show how the quality of collateral — in particular its liquidity, volatility and availability — as well as its allocation throughout the financial system plays a central role in determining the system's capacity to allocate funds efficiently.
Appendix

A Proofs

Proof of Proposition 1

Note that in all subgame perfect equilibria of the game, no bank other than \( n \) would invest in the inefficient project. This is due to the fact that in any candidate equilibrium in which some other bank invests in the inefficient project, bank \( n \) can deviate by investing in the inefficient project directly and obtaining a strictly higher payoff. Thus, there are only two possible lending patterns that are consistent with equilibrium: either (i) bank \( n \) invests directly in the inefficient project; or (ii) for all \( i \in \{2, \ldots, n\} \), bank \( i \) lends to bank \( i - 1 \) and bank 1 eventually invests in the efficient asset. Note that in the latter equilibrium, the expected profit of bank \( i + 1 \) is equal to

\[
\pi_{i+1} = (1 - \phi) (R_i - R_{i+1}) k + \alpha (\alpha c_i - c_{i+1}) + A,
\]

where \( R_i \) and \( c_i \) are the interest rates and collateral that \( i + 1 \) charges \( i \), with the convention that \( R_0 = r \) and \( R_n = c_0 = c_n = 0 \). The first term on the right-hand side is the payoff of the bank in the good state of the world. On the other hand, in the bad state of the world, the bank obtains \( \alpha c_i \) from liquidating the collateral posted by \( i \), while at the same time losing \( c_{i+1} \), as bank \( i + 2 \) would not return its collateral.\(^{16}\)

We now derive conditions under which full lending over the chain can be sustained as an equilibrium. Consider the subgame in which bank \( i + 1 \) obtains the liquidity. As already mentioned above, this is consistent with equilibrium if the contract \((R_{i+1}, c_{i+1})\) offered by bank \( i + 2 \) induces \( i + 1 \) to subsequently lend the liquidity to bank \( i \) — as opposed to investing it in the inefficient project. Thus, in any such subgame, bank \( i + 1 \) choose the contract \((R_i, c_i)\) offered to \( i \) that solves the following problem:

\[
\max_{R_i, c_i} \pi_{i+1}
\]

s.t.
\[
\pi_i \geq (1 - \phi) (r_h - R_i) k - \phi c_i + A
\]
\[
\pi_i \geq A.
\]

Denote the Lagrange multipliers corresponding to the incentive compatibility and participation constraints in the above problem with \( \lambda_i \) and \( \mu_i \), respectively. The first-order conditions imply

\[
\mu_i = 1 - \phi + \alpha \phi
\]
\[
\lambda_i = \phi (1 - \phi) (1 - \alpha) / \Delta \phi
\]

guaranteeing that both constraints bind in the optimal solution. The fact that the participation constraints of all banks bind implies that \( \pi_i = A \) for \( i \neq n \). Hence, summing equation (6) over \( i \) leads

\[^{16}\text{Note that as long as } c_i \leq R_i k / \gamma, \text{ bank } \ i \text{ chooses to default on its commitment to } i + 1 \text{ as opposed to repurchasing the collateral. Hence, for small enough } \gamma, \text{ all banks default on their obligations in the bad state of the world.}\]
Thus, to determine bank $n$’s payoff when it lends to bank $n - 1$, it is sufficient to determine the size of the collateral demanded by each lender bank from its respective borrower. On the other hand, the fact that both constraints in the above problem bind, it is immediate that the pair $(R_i, c_i)$ is the solution to the system of equations

$$
(1 - \phi_\ell)(R_{i-1} - R_i)k + \phi_\ell(\alpha c_{i-1} - c_i) = 0 \tag{9}
$$
$$
(1 - \phi_h)(r_h - R_i)k - \phi_h c_i = 0. \tag{10}
$$

Eliminating the interest rates form the above equations implies that the optimal collateral values are given by the following recursion

$$
c_i = \zeta c_{i-1}, \tag{11}
$$

with the initial condition

$$
c_1 = (1 - \phi_h)(1 - \phi_\ell)\Delta rk/\Delta \phi, \tag{12}
$$

where

$$
\zeta = 1 + \frac{(1 - \alpha)\phi_\ell(1 - \phi_h)}{\Delta \phi}.
$$

Substituting the above in (8) implies that the expected profit of bank $n$ is equal to

$$
\pi_n = (1 - \phi_\ell)r_\ell k - \phi_\ell(1 - \alpha) \sum_{i=1}^{n-1} c_i + A. \tag{8}
$$

Thus, regardless of the value of $k$, bank $n$ prefers to lend to bank $n - 1$ if and only if

$$
(1 - \phi_\ell)r_\ell - (1 - \phi_\ell) (\zeta^{n-1} - 1) \Delta rk \geq (1 - \phi_h)r_h
$$

which holds as long as $n \leq n^*$, where $n^*$ satisfies\textsuperscript{17}$

$$
\zeta^{n^*} - 1 = \frac{r_h \Delta \phi}{(1 - \phi_\ell)\Delta r}. \tag{13}
$$

Given that $\lim_{\alpha \to 1} \zeta = 1$, it is immediate that $n^*$ becomes arbitrarily large as $\alpha$ converges to 1. \hfill \Box

\textsuperscript{17}To simplify derivations, we assume that the solution to this equation is an integer. All our results and their economic insights would remain valid if the solution is not an integer.
Proof of Proposition 2

From (11) in the proof of Proposition 1, we already know that \( c_i = \zeta c_{i-1} \), which implies that \( c_i = \zeta^{i-1} c_1 \). This observation along with (12) imply that the size of the collateral posted by bank \( i \) with bank \( i - 1 \) is equal to

\[
c_i = \zeta^{i-1} (1 - \phi_h)(1 - \phi_\ell) \Delta r_k / \Delta \phi.
\]

Using the fact that \( h_i = 1 - k/c_i \) and the definition of \( n^* \) in (13) leads to (3).

To derive the equilibrium interest rates, recall from the proof of Proposition 1 that the incentive compatibility and participation constraints both bind at the optimal solution, implying that the equilibrium interest rate and collateral that bank \( i + 1 \) charges bank \( i \) satisfy the system of equations (9) and (10). Eliminating \( c_i \) and solving for \( R_i \) imply that the interest rates are determined via the recursion

\[
R_i = \zeta R_{i-1} - (1 - \alpha)(1 - \phi_h) \phi_\ell r_h / \Delta \phi
\]

for \( i \geq 2 \) with the initial condition

\[
R_1 = r_h - \phi_h (1 - \phi_\ell) \Delta r / \Delta \phi.
\]

Solving for the above recursion implies that \( R_i = r_h - \zeta^{i-1}(r_h - R_1) \) and hence,

\[
R_i = r_h - \left( \frac{\phi_h (1 - \phi_\ell) \Delta r}{\Delta \phi} \right) \zeta^{i-1}
\]

which coincides with (2) once one replaces for \( \zeta^{n^*-1} \) from (13).

Proof of Corollary 1

Recall from the proof of Proposition 2 that the interest rate charged by bank \( i + 1 \) is given by

\[
R_i = r_h - \left( \frac{\phi_h (1 - \phi_\ell) \Delta r}{\Delta \phi} \right) \zeta^{i-1}.
\]

Given that \( \zeta \) is decreasing in \( \alpha \) it is immediate that \( R_i \) is increasing in \( \alpha \). To obtain the comparative statics with respect to \( \Delta r \), note that increasing \( r_\ell \) increases \( R_i \). On the other hand,

\[
\frac{\partial R_i}{\partial r_h} = 1 - \frac{\phi_h (1 - \phi_\ell)}{\Delta \phi} \zeta^{i-1}.
\]

Given that both \( \zeta \) and \( \phi_h (1 - \phi_\ell) / \Delta \phi \) are greater than 1 it is immediate that the right-hand side of the above equality is negative, implying that \( R_i \) is decreasing in \( r_h \). Consequently, \( R_i \) is decreasing in \( \Delta r \).

As for the haircuts, recall from the proof of Proposition 2 that the collateral demanded by bank \( i + 1 \) is given by

\[
c_i = \zeta^{i-1} (1 - \phi_h)(1 - \phi_\ell) \Delta r_k / \Delta \phi.
\]

Given that \( \zeta \) is decreasing in \( \alpha \), it is immediate that \( c_i \) is also decreasing in \( \alpha \). The above equality also implies that \( c_i \) is increasing in \( \Delta r \), completing the proof.
Proof of Proposition 3

Consider a subgame perfect equilibrium of the game in which bank \( n \) lends \( k \) units of capital to bank \( n - 1 \) which is the intermediated through the chain until bank 1 eventually invests it in the efficient project. At any point on the chain, bank \( i + 1 \) chooses the interest rate and collateral it charges bank \( i \) as solutions to the following problem:

\[
\begin{align*}
\max_{R_i,c_i} & \quad \pi_{i+1} \\
\text{s.t.} & \quad \pi_i \geq (1 - \phi_h)(r_h - R_i)k - \phi_h c_i + A \\
& \quad \pi_i \geq A \\
& \quad c_i \leq A,
\end{align*}
\]

where the last inequality captures the fact that the collateral bank \( i + 1 \) charges bank \( i \) cannot exceed the \( i \)'s endowment of the illiquid asset.

Two observations are in order. First, note that the solution to the above problem is independent of the values of \( R_{i+1} \) and \( c_{i+1} \) that bank \( i + 2 \) charges bank \( i + 1 \) (and in fact more generally, it is independent of the terms of the contracts offered by all bank \( j > i \)). Second, recall from equation (3) in Proposition 2 that \( c_i \) is increasing in \( i \). Therefore, the collateral constraint (17) does not bind for all bank \( i < m \) where \( m \) is smallest integer that satisfies

\[
\zeta^{m-1}(1 - \phi_h)(1 - \phi_e)\Delta r k/\Delta \phi \geq A. \tag{18}
\]

In other words, bank \( m \) is the bank with the smallest index for which the collateral constraint binds. Note that in the above argument we are assuming that \( k \) is large enough so that the collateral constraints start to bind for a bank \( m < n \).

Now consider a bank \( i > m \). The collateral constraint of such bank binds, that is \( c_i = A \). This, in addition with the fact that the incentive compatibility constraints have to bind implies that the interest rate that bank \( i + 1 \) charges bank \( i \) satisfies

\[
(1 - \phi_e)(R_{i-1} - R_i)k - \phi_e(1 - \alpha)A = (1 - \phi_h)(r_h - R_i)k - \phi_h A.
\]

Thus, the interbank interest rates solve the recursion

\[
R_i = \left( \frac{1 - \phi_e}{\Delta \phi} \right) R_{i-1} + (1 + \alpha \phi_e/\Delta \phi)(A/k) - r_h(1 - \phi_h)/\Delta \phi,
\]

for \( i > m \) with the initial condition

\[
R_m = r_h + A/k - (1 - \phi_e)\zeta^{m-1} \left( \frac{\Delta r}{\Delta \phi} \right),
\]

where we are using the fact that bank \( m \)'s collateral constraint binds. Therefore, the interest rate that bank \( n \) charges bank \( n - 1 \) is given by

\[
R_{n-1} = r_h - \left( \frac{\alpha \phi_e + \Delta \phi}{1 - \phi_h} \right) A/k + \left( \frac{1 - \phi_e}{\Delta \phi} \right)^{n-m-1} \left[ \frac{A/k}{1 - \phi_h} (1 - \phi_e + \alpha \phi_e) - (1 - \phi_e)\zeta^{m-1} \left( \frac{\Delta r}{\Delta \phi} \right) \right].
\]
and consequently, its expected profit $\pi_n = (1 - \phi_\ell)R_{n-1}k + (\alpha\phi_\ell + 1)A$ is equal to

$$\pi_n = r_h (1 - \phi_\ell)k - (\alpha\phi_\ell + \Delta \phi) \left( \frac{1 - \phi_\ell}{1 - \phi_h} \right) A + \omega^{n-m} \left[ \frac{A\Delta \phi}{1 - \phi_h} (1 - \phi_\ell + \alpha\phi_\ell) - (1 - \phi_\ell) \zeta^{m-1}\Delta r_k \right] + (\alpha\phi_\ell + 1)A,$$

where recall that $\omega = (1 - \phi_\ell) / \Delta \phi$. Note that bank $n$’s opportunity cost of lending out $k$ units of capital to the rest of the banking system is equal to $(1 - \phi_h)r_h k$. Therefore, the bank chooses $k$ that maximizes $\Delta \pi = \pi_n - (1 - \phi_h)r_h k - A$; that is,

$$k_{\text{max}} = \arg\max_k \left\{ r_h (1 - \phi_\ell)k + \omega^{n-m} \left[ A(1 - \phi_\ell + \alpha\phi_\ell) - (1 - \phi_\ell)(1 - \phi_h) \left( \frac{\Delta r}{\Delta \phi} \right) \zeta^{m-1}k \right] \right\}, \quad (19)$$

where $m$ is itself a function of $k$, determined as the smallest integer that satisfies (18). To solve for the above maximization problem, define the decreasing sequence of numbers $\{k_1, k_2, \ldots\}$ such that

$$k_s = A\zeta^{1-s} \left( \frac{\Delta \phi / \Delta r}{1 - \phi_h} \right), \quad (20)$$

for $s \geq 1$ with the convention that $k_0 = \infty$. By equation (18), if $k \in [k_s, k_{s-1})$, then it must be the case that $m = s$. Now we separately maximize the objective function in (19) over the interval $k \in [k_s, k_{s-1})$ for all $s \in \{1, 2, \ldots\}$.

Note that over the interval $k \in [k_s, k_{s-1})$, the objective function is weakly increasing in $k$ if and only if

$$r_h \Delta \phi \geq (1 - \phi_h)\Delta r \zeta^{s-1} \omega^{n-s},$$

which in view of (13) can be rewritten as

$$\left( \frac{\omega}{\zeta} \right)^s \geq \frac{\omega^n}{\zeta^{n^*}}. \quad (21)$$

Consequently, the maximum of the objective function in (19) is obtained at $k = k_{s-1}$ where $s$ is the smallest integer for which inequality (21) is satisfied.

To summarize, if inequality (21) is satisfied for $s = 1$, then $k_{\text{max}} = k_0 = \infty$. If, on the other hand, the smallest integer $s$ that satisfies inequality (21) is greater than 1, then let

$$m = \left\lceil \frac{n \log \omega - n^* \log \zeta}{\log \omega - \log \zeta} \right\rceil - 1$$

which by assumption satisfies $m \geq 1$. Consequently, $s = m + 1$ is the smallest integer that satisfies (21), implying that $k_{\text{max}} = k_m$ defined in (20). Using (13) one more time completes the proof.

**Proof of Corollary 2**

From (5), we have

$$n^* - m = \left\lceil \frac{(n^* - n) \log \omega}{\log(\omega/\zeta)} \right\rceil.$$
On the other hand, recall from (13) that \( n^* \) is decreasing in \( \Delta r \). Given that \( \omega > \zeta \), it is immediate that \( n^* - m \) is non-increasing in \( \Delta r \). Thus, by equation (4), increasing \( \Delta r \) decreases \( k_{\text{max}} \). Similarly, it is immediate from the above equation that \( n^* - m \) is decreasing in \( n \). Therefore, by (4), increasing \( n \) reduces the intermediation capacity \( k_{\text{max}} \).

Finally, we show that \( k_{\text{max}} \) is increasing in \( \alpha \). Note that we can rewrite (4) as

\[
k_{\text{max}} = \frac{A/\Delta r}{\omega(1-\phi_h)\zeta^{m-1}}.
\]

On the other hand, equation (5) implies that

\[
(m - 1) \log \zeta = \log \zeta \left\lfloor \frac{B}{\log(\omega/\zeta)} \right\rfloor,
\]

where \( B \) is a constant independent from \( \alpha \), and we are using the fact that \( (n^* - 1) \log \zeta \) is independent of \( \zeta \). It is immediate to verify that the numerator and denominator of the right-hand side above are increasing and decreasing in \( \zeta \), respectively. Thus, \( (m - 1) \log \zeta \) is increasing in \( \zeta \). On the other hand, from the proof of Proposition 1 we know that \( \zeta \) is decreasing in \( \alpha \). Consequently, \( \zeta^{m-1} \) is also decreasing in \( \alpha \), implying that \( k_{\text{max}} \) increases with \( \alpha \). \( \square \)

**Proof of Proposition 4**

Denote the expected liquidation value of the asset conditional on the failure and the success of the inefficient project by \( \tilde{z} \) and \( \bar{z} \), respectively. It is thus immediate that \( \phi_h \tilde{z} + (1 - \phi_h)\bar{z} = 1 \). Furthermore, one can show that

\[
\tilde{z} = 1 + \rho \sigma \sqrt{\frac{\phi_h}{1 - \phi_h}},
\]

\[
\bar{z} = 1 - \rho \sigma \sqrt{\frac{1 - \phi_h}{\phi_h}}.
\]

**Proof of part (a)** An argument identical to the proof of Proposition 1 implies that when banks are not collateral-constrained, then the chain efficiently intermediates the excess liquidity if and only if \( n \leq n^* \) where \( n^* \) is the largest integer satisfying

\[
\tilde{z}^{n-1} \leq \frac{r_h \Delta \phi}{(1 - \phi_\ell)\Delta r},
\]

in which

\[
\tilde{z} = 1 + \frac{(1 - \alpha)\phi_\ell(1 - \phi_h)}{\phi_h(1 - \phi_\ell)\bar{z} - \phi_\ell(1 - \phi_h)}.
\]

(22)

It is immediate that the above expression is decreasing in \( \bar{z} \), which in turn is decreasing in \( \rho \sigma \). Thus, \( n^* \) is decreasing in \( \rho \sigma \). \( \square \)
Proof of part (b) A recursive argument similar to that of Propositions 1 and 2 implies that the equilibrium collateral and interest rates charged by bank $i + 1$ to bank $i$ are given by

$$c_i = (1 - \phi_h)\tilde{\omega}\tilde{\zeta}^{i-1}\Delta r k$$

and

$$R_i = r_h - \tilde{\omega}\phi_h\tilde{\zeta}^{i-1}\Delta r,$$

respectively, where $\tilde{\zeta}$ is given by (22) and

$$\tilde{\omega} = \frac{1 - \phi_\ell}{\phi_h(1 - \phi_\ell)\tilde{\zeta} - \phi_\ell(1 - \phi_h)}.$$

Note that both $\tilde{\omega}$ and $\tilde{\zeta}$ are decreasing in $\tilde{\zeta}$, which in turn is decreasing in $\rho\sigma$. It is thus immediate that $c_i$ is increasing in $\rho\sigma$. On the other hand, note that $\tilde{\omega}\phi_h\tilde{\zeta}$ is decreasing in $\tilde{\zeta}$, thus implying that $R_i$ is increasing in $\tilde{\zeta}$. Consequently, all interest rates are decreasing in $\rho\sigma$.

Proof of part (c) As in the proof of Proposition 3, suppose that bank $n$ lends $k$ units of capital to bank $n - 1$ which is then intermediated through the chain until bank 1 eventually invests it in the efficient project. Also, let $m$ denote the first bank for which the collateral constraint binds, that is,

$$(1 - \phi_h)\tilde{\omega}\tilde{\zeta}^m\Delta r k \geq A > (1 - \phi_h)\tilde{\omega}\tilde{\zeta}^{m-2}\Delta r k.$$

Once again, an argument similar to the proof of Proposition 3 shows that the interest rate that bank $n$ charges bank $n - 1$ is given by

$$R_{n-1} = r_h + \left(\frac{1 - \phi_\ell}{\Delta \phi}\right)^{n-m-1} \left[(1 - \phi_\ell + \alpha\phi_\ell)\Delta \phi + \phi_h(1 - \phi_\ell)(\tilde{\zeta} - 1)(A/k) - (1 - \phi_\ell)\tilde{\zeta}^{m-1}\left(\frac{\Delta r}{\Delta \phi}\right)\right] - \left(\frac{\alpha\phi_\ell + \Delta \phi + \phi_h(\tilde{\zeta} - 1)}{1 - \phi_h}\right) A/k.$$

Given that bank $n$’s opportunity cost of lending out $k$ units of capital to the rest of the banking system is equal to $(1 - \phi_h)r_h k$, the intermediation capacity of the market is given by

$$k_{\text{max}} = \arg \max_k \left\{ r_h k - (1 - \phi_\ell) \left(\frac{\Delta r}{\Delta \phi}\right) \omega^{n-m}\tilde{\zeta}^{m-1} k \right\},$$

where the domain over which the maximization is obtained is such that $k$ and $m$ satisfy (24). An argument similar to that of proof of Proposition 3 thus implies that $m$ is given by

$$m = \left[ \frac{n \log \omega - n^* \log \tilde{\zeta}}{\log \omega - \log \tilde{\zeta}} \right] - 1,$$

Thus, the intermediation capacity of the chain is given by

$$k_{\text{max}} = \frac{A/\Delta r}{(1 - \phi_h)\tilde{\omega}\tilde{\zeta}^{m-1}}.$$

We can now perform the comparative statics of the above expression as a function of $\rho\sigma$. Recall from the proof of part (b) that $\tilde{\omega}$ and $\tilde{\zeta}$ are increasing in $\rho\sigma$. Furthermore, from the definition of $m$, it is immediate that $m$ is also increasing in $\rho\sigma$, thus implying that $k_{\text{max}}$ decreases as $\rho\sigma$ is increased. □
References


