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STRUCTURAL PLANNING UNDER CONTROLLABLE BUSINESS RISK

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I. INTRODUCTION

IN HIS FUNDAMENTAL WORKS [15] and [16] Douglas Vickers integrates the production, investment and financing decisions of the firm into a useful and illuminating model. He deals with uncertainty using risk-adjusted capitalization and interest rates, assumes constant business risk and treats financial risk as a function of leverage. This paper extends Vickers' analysis by allowing business risk to depend on production and investment decisions.[1] In particular, it derives a criterion for investment decisions under general uncertainty and market organization conditions when business risk is subject to control.

The effect of investment decisions on business risk has been studied by several authors (Hamada [4], Mossin [9], and Tuttle and Litzenberger [13]) under the conditions of idealized uncertainty assumed in the Sharpe-Lintner capital market theory. The investment criterion derived in this paper is shown to be generalization of theirs, and of the one obtained by Modigliani and Miller [8] under the constant risk class assumption.

II. MODEL SPECIFICATION

The firm's optimal structural decisions will be analyzed with a multi-product model specified with the following elements:

q = n-output vector
D = debt capital
K = equity capital
$p(q, u)$ = random price vector
$c(q, v)$ = random long-run cost function which includes amortization and maintenance cost of real capital
(u, v) = random disturbances with known subjective first and second moments
$\bar{p}(q)$ = expected price vector
$\bar{c}(q)$ = expected cost function
$r(q, K, D)$ = average rate of interest on debt
$\rho(q, K, D)$ = owners' capitalization rate
$h(q)$ = money capital requirement function

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1. In a related paper Turnovsky [12] reformulates Vickers' model and considers the effects of changing business risk but treats it as exogenously determined and not dependent on production and investment decisions.

The market value of equity is defined as

$$V = \frac{\bar{p}(q)q - \bar{c}(q) - r(q, K, D)D}{\rho(q, K, D)}.$$

The optimal structure of the firm results from maximizing the net present value of equity with respect to the decision variables q, D and K, subject to a money capital constraint. That is,

$$\max_{q, D, K} \quad V - K$$

Subject to

$$K + D - h \geqslant 0, \qquad q, D, K \geqslant 0. \tag{1}$$

Following Vickers we omit the consideration of a profit tax (which will not change the nature of our conclusions), restrict our attention to the case of no money capital saturation and disregard corner solutions. This reduces the Kuhn-Tucker conditions to the usual Lagrange conditions.

This model admits product and factor price as well as technological uncertainty. The money capital requirement is assumed certain in order to avoid a probabilistic financial constraint. h can be allowed to be random by simply eliminating the constraint and one of the decision variables D or K, but the Lagrange multiplier of the constraint has a useful interpretation and facilitates the comparison of our results with those of Vickers.

The capitalization and interest rate functions have q, K and D as arguments. K and D determine the financial risk for a given level of business risk which in turn depends on the output mix and, indirectly, on investment policy. The role of q can be justified by taking the coefficient of variation of operating income as a measure of business risk (see Vickers [15], p. 53 and Van Horne [14], p. 46):

$$\frac{\sigma[p(q, u)q - c(q, v)]}{\bar{p}(q)q - \bar{c}(q)}$$

which, for given technology and markets characteristics, is a function of output. ($\sigma[\cdot]$ stands for the standard deviation of operating income.)[2]

r and ρ are the market supply functions of capital which result from the aggregation of the individual schedules. As Adler [1] has pointed out, if investors are expected utility maximizers, ρ will also depend on the value of the firm V. Since V is a function of q, K and D, we assume for the present purposes that its effect is embedded in the functional form of ρ.

III. Optimal Structure of the Firm

Differentiating the Lagrangian of (1) with respect to q_j, $j = 1, \ldots, n$, D, K and the multiplier μ gives the optimum planning conditions:

2. The inclusion of q in the ρ function can be justified even under constant business risk if, as Vickers suggests [15, p. 66], the coefficient of variation of profits is used as argument of ρ.

$$\frac{1}{\rho}\left[\overline{MR}_j - \overline{MC}_j - \left(D\frac{\partial r}{\partial q_j} + V\frac{\partial \rho}{\partial q_j}\right)\right] - \mu\frac{\partial h}{\partial q_j} = 0, j = 1, \ldots, n, \tag{2}$$

$$\frac{1}{\rho}\left(r + D\frac{\partial r}{\partial D} + V\frac{\partial \rho}{\partial D}\right) - \mu = 0, \tag{3}$$

$$\frac{1}{\rho}\left(\rho + D\frac{\partial r}{\partial K} + V\frac{\partial \rho}{\partial K}\right) - \mu = 0, \tag{4}$$

$$K + D - h = 0, \tag{5}$$

where $\overline{MR}_j = \bar{p}_j + q_j\, \partial\bar{p}_j/\partial q_j$ and $\overline{MC}_j = \partial\bar{c}/\partial q_j$.

To interpret the above conditions we note that, at the optimum,

$$\mu = \sum_j \frac{\partial V}{\partial q_j}\frac{dq_j}{d(K + D)},$$

that is, μ measures the marginal value productivity of money capital.

Conditions (3) and (4) are identical to those derived by Vickers. They indicate that the optimal size of the capital budget is attained when the marginal value productivity of money capital equals the full marginal discounted costs of debt and equity, and that the optimal capital structure requires equality of the full marginal costs of debt and equity.

Conditions (2) give the optimal production policy indicating how much to produce of each output. These conditions are more explicit than those of the neoclassical formulation because they relate production policy to the capital budget. In fact, they say that, at the optimum, the net marginal value contribution of each product (the expression in brackets divided by ρ) should equal the marginal value of money capital (the last term). The net marginal contribution of product j to the value of the firm is obtained subtracting from the expected marginal contribution $\overline{MR}_j - \overline{MC}_j$, the marginal variation of money capital costs induced by the output change through its impact on business risk. The implications of this adjustment can be better appreciated by appealing to one of Vickers' [16] basic results: At the optimum, the weighted average cost of capital

$$i = \frac{rD + \rho V}{D + V}$$

is minimum and $i = \rho\mu$. This result can be obtained from the well known equality between minimum average cost and marginal costs, noting that, at the optimum, $\rho\mu$ equals the marginal cost of debt and equity (from conditions (3) and (4)) and $dV/dK = 1$.[3]

Conditions (2) can now be written as follows

$$\frac{1}{i}\left[\overline{MR}_j - \overline{MC}_j - \left(D\frac{\partial r}{\partial q_j} + V\frac{\partial \rho}{\partial q_j}\right)\right] = \frac{\partial h}{\partial q_j}, j = 1, \ldots, n.$$

That is, the adjusted marginal contributions discounted at the average cost of capital should equal the marginal money capital outlay or, in

3. Turnovsky [12] has shown that, when r and ρ are homogeneous of degree zero in D and K, $\rho\mu$ also equals the book value average cost of capital $(rD + \rho K)/(D + K)$. In that case Vickers' result implies that $V = K$ independently of market conditions.

terms of the risk-adjusted discount rates κ_j, $j = 1, \ldots, n$,

$$\frac{\overline{MR}_j - \overline{MC}_j}{\kappa_j} = \frac{\partial h}{\partial q_j}, \qquad j = 1, \ldots, n, \tag{6}$$

where

$$\kappa_j = i + \frac{D\partial r/\partial q_j + V\partial\rho/\partial q_j}{\partial h/\partial q_j}. \tag{7}$$

(6) says that each expected marginal contribution discounted at its risk-adjusted rate κ_j should equal the corresponding marginal money capital outlay. The risk-adjusted rate of discount is specific to each activity or product and results from adding to the minimum average cost of capital the ratio of the output induced marginal variation of money capital costs to the marginal money capital outlay. $\kappa_j \gtreqless i$ as the marginal variation of money capital costs is positive, zero or negative. When $\partial r/\partial q_j = \partial\rho/\partial q_j = 0$, for all j, (6) becomes Vickers' optimum conditions.

Therefore, we have shown that the risk-adjusted average cost of capital can be used to obtain optimal structural planning decisions. This is correct as long as the firm simultaneously determines the optimal production, investment and financing decisions, but, as Vickers has pointed out, the use of the average cost of capital is inappropriate at infra-optimum level of equity. Using condition (3), we can write the risk-adjusted rate when equity is fixed as follows

$$\kappa_j = \left(r + D\frac{\partial r}{\partial D} + V\frac{\partial\rho}{\partial D}\right) + \frac{D\partial r/\partial q_j + V\partial\rho/\partial q_j}{\partial h/\partial q_j},$$

where the term in parentheses is the full marginal cost of debt and the risk-adjustment is the same as in (7).

IV. Optimal Factor Use

The optimal combination of production factors was implicit in the previous specification of the planning problem. This section deals with production factors explicitly. For that we introduce the production function

$$\bar{q} = f(x),$$

where q is the n-expected output vector, and x is the m-input vector which has associated an expected input factor cost vector $\bar{\gamma}$. Thus, the capitalization and interest rate functions can be written as follows

$$\rho = \rho(x, K, D), \qquad r = r(x, K, D).$$

Also, the money capital requirement function is now written in terms of factor use as h(x).

The optimum production conditions expressed in terms of factor use are:

$$\frac{1}{\rho}\left[\left(\sum_{j}\overline{\mathrm{MR}}_j\frac{\partial \bar{q}_j}{\partial x_k}\right) - \bar{\gamma}_k - \left(D\frac{\partial r}{\partial x_k} + V\frac{\partial \rho}{\partial x_k}\right)\right] - \mu\frac{\partial h}{\partial x_k} = 0, \qquad k = 1, \ldots, m. \quad (8)$$

Conditions (8) say that the surplus marginal revenue product of each factor, adjusted by the marginal variation of money capital costs induced by the input change through its impact on business risk, and discounted at the owners' capitalization rate, should equal the corresponding marginal value of money capital. Moreover, (8) can be written as

$$\frac{1}{\kappa'_k}\left[\left(\sum_{j}\overline{\mathrm{MR}}_j\frac{\partial \bar{q}_j}{\partial x_k}\right) - \bar{\gamma}_k\right] = \frac{\partial h}{\partial x_k},\ k = 1, \ldots, m, \tag{9}$$

where

$$\kappa'_k = i + \frac{D\partial r/\partial x_k + V\partial \rho/\partial x_k}{\partial h/\partial x_k}. \tag{10}$$

(9) and (10) have similar interpretations to (5) and (6).

It is interesting to characterize the expansion path under changing business risk. From (8), maintaining all outputs but the s^{th} fixed, we have

$$\frac{\partial \bar{q}_s/\partial x_k}{\partial \bar{q}_s/\partial x_l} = \frac{\bar{\gamma}_k + i\partial h/\partial x_k + (D\partial r/\partial x_k + V\partial \rho/\partial x_k)}{\bar{\gamma}_l + i\partial h/\partial x_l + (D\partial r/\partial x_l + V\partial \rho/\partial x_l)}.$$

In this formulation the relevant marginal factor costs are composed of the direct unit cost, the money capital cost and the business risk effect. That is, the expansion path is not only affected by the explicit introduction of money capital costs which includes a risk premium in i, as in Vickers' analysis, but also by the changing business risk effect. It is not difficult to illustrate the importance of this consideration for the choice of production techniques. For instance, flexible techniques will tend to have a low, possibly negative business risk effect, while specialized techniques and factors of unreliable supply will tend to have a high positive effect.

V. Optimal Structure Under Sharpe-Lintner Idealized Uncertainty

For a perfect capital market with a constant borrowing-lending interest rate and risk-averse investors having preferences based on mean and variance and holding homogeneous expectations, Sharpe [10] and Lintner [5], [6] have shown that the risk-adjusted owners' capitalization rate applying to the firm is

$$\rho = r^* + b r_{fm}\sigma_f, \tag{11}$$

where $b = (\bar{R}_m - r^*)/\sigma_m$, and r^* is the riskless rate of interest, r_{fm} the correlation coefficient between the firm's and the market portfolio rates of return, σ_f the standard deviation of the firm's rate of return, and $\bar{R}_m$

and σ_m the expected value and standard deviation of the rate of return on the market portfolio R_m.[4]

It is well known that Modigliani-Miller Proposition I is valid under weaker assumptions than those leading to (11) (see Mossin [9] and Stiglitz [11]). That is, under idealized uncertainty V + D, the total value of the firm, and i, the average cost of capital, are independent of financial leverage. This implies that the net present value of equity V − K is independent of financial structure since

$$V - K = V + D - h(q). \tag{12}$$

Therefore, the optimal planning problem is reduced to the determination of the optimal production-investment decisions. The optimum conditions are obtained from (2) and (5) taking into account that $r = r^*$:

$$\frac{1}{\rho}\left[\overline{MR}_j - \overline{MC}_j\right] - V\frac{\partial\rho}{\partial q_j} - \mu\frac{\partial h}{\partial q_j} = 0,\ j = 1, \ldots, n,$$
$$K + D - h = 0, \tag{13}$$

where, since the partial derivative of (12) with respect to q_j must be zero at the optimum,

$$\mu = \frac{\partial V/\partial q_j}{\partial h/\partial q_j} = 1. \tag{14}$$

Conditions (13) can be expressed in terms of risk-adjusted discount rates as in (6) where now, taking into account (11) and (14),

$$\kappa_j = \rho + \frac{V\partial\rho/\partial q_j}{\partial h/\partial q_j} = r^* + br_{fm}\sigma_f + \frac{Vb\partial(r_{fm}\sigma_f)/\partial q_j}{\partial h/\partial q_j},\ j = 1, \ldots, n. \tag{15}$$

In order to reduce (15) to a simpler expression we write the undiversifiable risk of the firm as

$$r_{fm}\sigma_f = \frac{1}{\sigma_m}\text{cov}\left(\frac{\pi}{V}, R_m\right) = \frac{1}{\sigma_m V}\text{cov}\,(pq - c, R_m),$$

where $\pi = pq - c - rD$, differentiate it with respect to q_j and obtain

$$\frac{\partial(r_{fm}\sigma_f)}{\partial q_j} = \frac{1}{\sigma_m V}\text{cov}\,(MR_j - MC_j, R_m) - \frac{1}{V}r_{fm}\sigma_f\frac{\partial V}{\partial q_j}, \tag{16}$$

since $\partial R_m/\partial q_j$ and $\partial\sigma_m/\partial q_j$ are negligible.

Substituting (16) into (15) and noting from (12) that $\partial V/\partial q_j = \partial h/\partial q_j$, we get

$$\kappa_j = r^* + \frac{b}{\sigma_m}\text{cov}\left(\frac{MR_j - MC_j}{\partial h/\partial q_j}, R_m\right) = r^* + br_{jm}\sigma_j, \tag{17}$$

where r_{jm} is the correlation coefficient of the rate of return of the margin-

4. Fama [3] has extended the above results to include all symmetric stable distributions with finite means. The analysis and conclusions of this section can be easily reformulated in terms of Fama's model.

al output $(MR_j - MC_j)/(\partial h/\partial q_j)$ with the rate of return of the market portfolio, and σ_j is the standard deviation of the rate of return of the marginal output.

Therefore, under idealized uncertainty, our optimum production conditions and risk-adjusted discount rates reduce to the investment criterion proposed by Hamada [4], Mossin [9], and Tuttle and Litzenberger [13]. (6) and (17) are Tuttle and Litzenberger's criterion which, as Litzenberger and Budd [7] have shown, is equivalent to the criteria of Hamada and Mossin.[5] Moreover, our alternative derivation of (17) establishes the needed link between capital market theory and the theory of the firm and shows that the expression (7) for the risk-adjusted rate can be interpreted as a generalization of (17) for non-ideal non-competitive conditions.

We now relate the optimum production-investment conditions to Modigliani and Miller's investment criterion. In accordance with these authors [8] let us require that marginal production-investment changes keep the firm within the same risk class such that $\partial i/\partial q_j = 0$ for all j, and disregard the assumptions of mean-variance preferences and constant borrowing-lending interest rate which are now superfluous. Given the irrelevance of financial structure (due to Modigliani-Miller Proposition I), which implies (14), conditions (2) now become

$$\frac{\overline{MR}_j - \overline{MC}_j}{\partial h/\partial q_j} = \rho + \frac{D\partial r/\partial q_j + V\partial \rho/\partial q_j}{\partial h/\partial q_j}, \; j = 1, \ldots, n,$$

but, since differentiating i partially with respect to q_j yields

$$\frac{\partial i}{\partial q_j} = \frac{1}{D + V}\left(\rho\frac{\partial V}{\partial q_j} + V\frac{\partial \rho}{\partial q_j} + D\frac{\partial r}{\partial q_j} - i\frac{\partial V}{\partial q_j}\right),$$

we have that $\partial i/\partial q_j = 0$ and (14) imply

$$i = \rho + \frac{D\partial r/\partial q_j + V\partial \rho/\partial q_j}{\partial h/\partial q_j}$$

at the optimum. Therefore, under the stated assumptions, the optimum conditions (6) and (7) can be written as follows

$$\frac{\overline{MR}_j - \overline{MC}_j}{\partial h/\partial q_j} = i = \frac{\bar{p}q - \bar{c}}{V + D}, \; j = 1, \ldots, n,$$

which say that the expected return on marginal investments should equal the current expected return on the firm's total money capital. This is Modigliani-Miller Proposition III.

5. These criteria are based on the same assumptions as those used to obtain (17) with one exception which has led to a minor discrepancy. In fact, while (17) indicates that the project risk-adjusted discount rate is independent of the firm which undertakes the investment, Mossin's result explicitly shows that the investment criterion is different from each firm. Without entering into details, let us note that the reason for this is that Mossin maintains some of the terms disregarded in [13], [7] and in the present paper (equation (16)). The exact form of Mossin's criterion can be easily obtained from (15).

VI. Conclusions

Production-investment decisions determine the productive structure of the firm and, thus, its degree of business risk. We have introduced this effect into the theory of the firm developed by Vickers and explored its immediate implications. Our main conclusions are:

(1) Under general uncertainty and market organization conditions each investment project should be discounted at its own risk-adjusted rate which takes into account its specific effect on business risk. This rate is obtained adding to the minimum average cost of capital, which already includes a risk premium, the project induced variation of money capital costs per dollar invested.

(2) This risk-adjusted rate is valid only at what Vickers has called the optimum optimorum position of the firm. It requires an optimal financial structure. Under conditions of equity capital rationing, the average cost of capital is irrelevant and for planning purposes should be replaced by the full marginal cost of debt.

(3) The surplus marginal revenue product of each factor should be adjusted by its own money capital cost and by the factor induced variation of the firm's total money capital costs.

(4) Under the idealized type of uncertainty assumed in the Sharpe-Lintner capital asset pricing model, our optimum production-investment conditions become the investment criterion derived in alternative ways by Hamada, Mossin, and Tuttle and Litzenberger, which is shown to be a particular case of our more general criterion. This also establishes an important link between the theory of the firm and capital market theory.

(5) Our investment criterion becomes Modigliani-Miller Proposition III when the firm stays in the same risk class.

A final comment is in order. The fact that the results of this paper include as particular cases well known investment criteria—obtained in the literature assuming ideal competitive markets—is in itself interesting. It should not hide, however, the intrinsic relevance of the generalized optimum planning conditions derived in Section III, which also apply to non-ideal non-competitive markets. Those conditions show the power of Vickers' fundamental formulation and provide a tractable structure upon which meaningful and empirically relevant comparative statics analysis can be based.[6]

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