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Beating a moving target: Optimal portfolio strategies for outperforming a stochastic benchmark

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Abstract. We consider the portfolio problem in continuous-time where the objective of the investor or money manager is to exceed the performance of a given stochastic benchmark, as is often the case in institutional money management. The benchmark is driven by a stochastic process that need not be perfectly correlated with the investment opportunities, and so the market is in a sense incomplete. We first solve a variety of investment problems related to the achievement of goals: for example, we find the portfolio strategy that maximizes the probability that the return of the investor's portfolio beats the return of the benchmark by a given percentage without ever going below it by another predetermined percentage. We also consider objectives related to the minimization of the expected time until the investor beats the benchmark. We show that there are two cases to consider, depending upon the relative favorability of the benchmark to the investment opportunity the investor faces. The problem of maximizing the expected discounted reward of outperforming the benchmark, as well as minimizing the discounted penalty paid upon being outperformed by the benchmark is also discussed. We then solve a more standard expected utility maximization problem which allows new connections to be made between some specific utility functions and the nonstandard goal problems treated here.

Key words: Portfolio theory, benchmarking, active portfolio management, constant proportions, growth optimal policy, stochastic control, diffusions

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1 Introduction

In this paper we analyze the optimal portfolio and investment policy for an investor who is concerned about the performance of his wealth relative only to the performance of a particular benchmark. Specifically, we consider the case where a chosen benchmark evolves stochastically over time, and the investor's objective is to exceed the performance of this benchmark (in a sense to be made more precise later) by investing in other stochastic processes. We take as our setting the continuous-time framework pioneered by Merton (1971) and others. The portfolio problem where the objective is to exceed the performance of a selected target benchmark, is sometimes referred to as to *active portfolio management*, see for example Sharpe et al. (1995). It is well known that many professional investors in fact follow this benchmarking procedure: For example, many mutual funds take the Standard and Poors (S&P) 500 Index as a benchmark, commodity funds seek to beat the Goldman Sachs Commodity Index, bond funds try to beat the Lehman Brothers Bond Index, etc. However, benchmarking is not specific to professional investors as many ordinary investors implicitly follow a benchmarking procedure, for example by trying to beat inflation, exchange rates, or other indices.

For the most part we consider some nonstandard objective functions related to the achievement of performance goals and shortfalls. For example, we consider objectives such as maximizing the probability that the investor's wealth achieves a certain performance goal relative to the benchmark, before falling below it to a predetermined shortfall. Other objectives include minimizing the expected time to reach the performance goal, maximizing the expected reward obtained upon reaching the goal, as well as minimizing the expected penalty paid upon falling to the shortfall level. Aside from its intrinsic objective interest, these goal-related objectives are also relevant to the principal-agent problem of a money manager who is judged relative to a benchmark, and as is quite common, receives an incentive (e.g., bonus) upon outperforming a benchmark by a predetermined percentage, but on the other hand, receives a disincentive (e.g., gets fired) should he underperform the benchmark by some other predetermined amount.

We also consider a more standard utility maximizing objective and as a consequence obtain some new relationships between goal-related objectives and certain utility functions. In fact, since the ordinary portfolio problem can be considered a special case of the model treated here with a constant benchmark, our results include some earlier results obtained for the standard portfolio problem as special cases, but since our model is more general we obtain new results as well that were not available previously. The reason for this is that since here we do not require the benchmark to be perfectly correlated with the investment opportunity, the investor cannot completely control his risk, which in turn allows for a more general model in studying risk-return tradeoffs. For the special case where the benchmark is perfectly correlated with the investment opportunities, continuous-time active portfolio management problems over a finite-horizon have been analyzed in Carpenter (1996) and Browne (1996). The former studies a utility maximization objective while the latter studies a probability maximizing objective.

An outline of the remainder of the paper, and a summary of our main results are as follows: In the next section, we provide the model and a description of the problems studied. For the objectives considered here, the relevant state variable is the *ratio* of the investor's wealth to the benchmark. Since the benchmark is not necessarily perfectly correlated with the investment opportunities, in general there is a component of the variance of this ratio process which is completely uncontrollable by the investor's investment strategy. As such, the market is in a sense *incomplete*, and there is no policy under which the investor can relate his wealth to the benchmark with certainty. We provide in Sect. 3 a general theorem in stochastic control for our model, which encompasses all the specific goal-related objectives considered in the sequel as special cases. The upshot of this theorem is that it shows how the optimal value function and associated optimal control function for a general control problem can be obtained as the solution to a particular nonlinear Dirichlet problem. A rigorous proof of this theorem (via a martingale argument) is provided in the appendix. Since each of the specific goal-related problems considered in the sequel are special cases, we need only identify and then solve the appropriate nonlinear Dirichlet problem.

In Sect. 4 we consider a probability maximizing problem. Specifically, we find the optimal portfolio strategy for maximizing the probability that the investor outperforms the benchmark by a predetermined percentage, before falling below it by another predetermined percentage. For all relevant parameter values, this problem has a solution, and we provide an explicit formula for it as well as for the optimal policy. (A related finite-horizon problem is studied in Browne 1996 for the special case where the investment opportunity is perfectly correlated with the benchmark). In Sect. 5, we consider the problem of minimizing the expected time until the benchmark is beaten by a predetermined percentage as well as the related problem of maximizing the expected time until being beaten by the benchmark. Contrary to the probability maximizing problem of the previous section, whether these problems have solutions depends on the value of a certain *favorability parameter*: If the favorability parameter is positive, then the ordinary optimal growth, or log-optimal, policy for the case where there is no benchmark (see e.g. Merton 1990, Chapter 6) is also optimal in our setting for minimizing the expect time to beat the benchmark. If the favorability parameter is negative, then this optimal growth policy is optimal once again, but for the objective of maximizing the expected time until the investor's wealth is beaten by the benchmark. This result allows us to then generalize some classical results about favorable and unfavorable games of chance. In Sect. 6, we consider the case of maximizing the expected discounted reward of beating the benchmark and the related problem of minimizing the expected discounted penalty of being beaten by the benchmark. These last two problems are connected to the probability maximizing problem of Sect. 4 and expected (linear) time problems of Sect. 5. Finally, in Sect. 7 we consider a utility maximization problem which allows us to obtain new equivalences between power utility functions and the various objective criteria considered previously.

The optimal policies we obtain here are all *constant proportion*, or constant mix, portfolio allocation strategies, whereby the portfolio is continuously rebalanced so as to always keep a constant proportion of wealth in the various asset classes, regardless of the level of wealth. (Observe that this rebalancing requires selling an asset when its price rises relative to the other prices, and conversely, buying an asset when its price drops relative to the others.) Such policies have a variety of optimality properties associated with them for the ordinary portfolio problem (see e.g., Merton (1990), or Browne (1998) for surveys) and are widely used in asset allocation practice (see Perold and Sharpe (1988) and Black and Perold (1992)). Since our model is a generalization of the ordinary portfolio problem, our results provide some extended optimality properties for these policies, however, except for the minimal/maximal expected time problem of Sect. 5, the constants we obtain depend heavily on *all* the parameters in the model (i.e., on the benchmark parameters as well as the investment opportunity parameters) and as such do not follow from earlier results on the standard portfolio problem.

2 The model

The model under consideration here consists of $k + 2$ underlying processes: k (correlated) risky assets or stocks $S^{(1)}, \ldots, S^{(k)}$, a riskless asset B called a bond, and a stochastic benchmark Y. The investor may invest only in the risky stocks and the bond, whose price processes will be denoted, respectively, by $\{S_t^{(i)}, t \geq 0\}_{i=1}^k$ and $\{B_t, t \geq 0\}$.

The probabilistic setting is as follows: we are given a filtered probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathsf{P}\right)$, supporting $k + 1$ independent standard Brownian motions, $\left(W^{(1)}, \ldots, W^{(k+1)}\right)$, where $\mathscr{F}_t$ is the P-augmentation of the natural filtration $\mathscr{F}_t^W := \sigma\{W_s^{(1)}, W_s^{(2)}, \ldots, W_s^{(k+1)}; 0 \leq s \leq t\}$.

It is assumed that the first k of these Brownian motions generates the prices of the k risky stocks, with the remaining Brownian motion being a component of only the benchmark.

Specifically, following Merton (1971) and many others, we will assume that the risky stock prices are correlated geometric Brownian motions, i.e., $S_t^{(i)}$ satisfies the stochastic differential equation

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sum_{j=1}^{k} \sigma_{ij} S_t^{(i)} dW_t^{(j)}, \quad \text{for } i = 1, \ldots, k \tag{1}$$

where $\{\mu_i : i = 1, \ldots, k\}$ and $\{\sigma_{ij} : i, j = 1, \ldots, k\}$ are constants, and $W_t^{(j)}$ denotes a standard independent Brownian motion, for $j = 1, \ldots, k$. The price of the risk-free asset is assumed to evolve according to

$$dB_t = rB_t\, dt \tag{2}$$

where $r \geq 0$. To avoid trivialities, we assume that $\mu_i > r$ for all $i = 1, \ldots, k$.

An investment policy is a (column) vector control process $f := \{f_t : t \geq 0\}$ in R^k with individual components $f_t^{(i)}, i = 1, \ldots, k$, where $f_t^{(i)}$ is the *proportion* of the investor's wealth invested in the risky stock i at time t, for $i = 1, \ldots, k$, with the remainder invested in the risk-free bond. It is assumed that $\{f_t, t \geq 0\}$ is a suitable, admissible $\mathscr{F}_t$-adapted control process, i.e., f_t is a nonanticipative function that satisfies $E \int_0^T f_t' f_t dt < \infty$, for every $T < \infty$. We place no other restrictions on f, for example, we allow $\sum_{i=1}^k f_t^{(i)} \geq 1$, whereby the investor is leveraged and has borrowed to purchase the stocks. (We also allow $f_t^{(i)} < 0$, whereby the investor is selling stock i short, however for $\mu_i > r$, this never happens in any of the problems considered here.)

Let X_t^f denote the *wealth* of the investor at time t, if it follows policy f, with $X_0 = x$. Since any amount not invested in the risky stock is held in the bond, this process then evolves as

$$\begin{aligned} dX_t^f &= X_t^f \left(\sum_{i=1}^k f_t^{(i)} \frac{dS_t^{(i)}}{S_t^{(i)}} \right) + X_t^f \sum_{i=1}^k (1 - f_t^{(i)}) \frac{dB_t}{B_t} \\ &= X_t^f \left(r + \sum_{i=1}^k f_t^{(i)} (\mu_i - r) \right) dt + X_t^f \sum_{i=1}^k \sum_{j=1}^k f_t^{(i)} \sigma_{ij} \, dW_t^{(j)} \end{aligned} \tag{3}$$

upon substituting from (1) and (2). This is the wealth equation first studied by Merton (1971).

If we introduce now the matrix $\sigma = (\sigma)_{ij}$ and the column vectors $\mu = (\mu_1, \ldots, \mu_k)'$, $\mathbf{1} = (1, \ldots, 1)'$, and $W_t^\bullet = \left(W_t^{(1)}, \ldots, W_t^{(k)} \right)'$, we can rewrite the wealth process of (3) as

$$dX_t^f = X_t^f \left[\left(r + f_t' (\mu - r\mathbf{1}) \right) dt + f_t' \sigma \, dW_t^\bullet \right] . \tag{4}$$

(Observe that the vector $W_t^\bullet$ does not contain the Brownian motion $W_t^{(k+1)}$.) For the sequel, we will also need the matrix $\Sigma := \sigma\sigma'$. It is assumed for the sequel that the square matrix σ is of full rank and invertible, hence σ^{-1} (and Σ^{-1}) exists.

2.1 The benchmark

As described above, our interest lies in determining investment strategies that are optimal relative to the performance of a benchmark. The benchmark we work with here is another stochastic process, Y, which is assumed to evolve according to

$$\begin{aligned} dY_t &= \alpha Y_t dt + \sum_{i=1}^k b_i \, Y_t \, dW_t^{(i)} + \beta \, Y_t \, dW_t^{(k+1)} \\ &= Y_t \left(\alpha dt + b \, dW_t^\bullet + \beta \, dW_t^{(k+1)} \right) , \end{aligned} \tag{5}$$

where we have introduced the (constant) vector $b = (b_1, \ldots, b_k)'$, and where $W_t^{(k+1)}$ is the remaining (standard) Brownian motion. Thus the benchmark, Y_t, is a geometric Brownian motion that is only partially correlated with the wealth process X_t^f, inasmuch as we allow $\beta \neq 0$ for complete generality.

For example, the benchmark might be an inflation rate, or an exchange rate, both of which have previously been modeled in the literature as geometric Brownian motion. It might also represent the value process of a non-traded asset. In all cases we might expect $\beta \neq 0$. Alternatively, the benchmark might be the wealth process from a different portfolio strategy, which if it invests solely in the same stocks $\left\{S_t^{(i)}, i = 1, \ldots, k\right\}$, would imply indeed that $\beta = 0$. This last example can be illustrated by the following: suppose indeed that Y_t corresponds to the wealth of another investor at time t, where investor Y follows the constant portfolio policy $\pi = (\pi_1, \ldots, \pi_k)'$ in investing also on the same stocks, $\left\{S^{(i)}, i = 1, \ldots, k\right\}$ of (1). Then similarly to (4), we would have

$$dY_t = Y_t \left[\left(r + \pi'(\mu - r\mathbf{1})\right) dt + \pi' \sigma \, dW_t^\bullet\right] \tag{6}$$

and so $\beta = 0$ for this example, and then $\alpha = r + \pi'(\mu - r\mathbf{1}), b = \sigma' \pi$. However, for the objectives considered in this paper, this last example is much less interesting than the more general case treated here. (For a finite-horizon active portfolio management problem related to this last case, see Browne (1996).)

2.2 Active portfolio management

While there are many possible objectives related to outperforming a benchmark, here we consider problems related to the achievement of relative performance goals and shortfalls. Specifically, for numbers l, u with $lY_0 < X_0 < uY_0$, we say that performance goal u is reached if $X_t^f = uY_t$, for some $t > 0$ and that performance shortfall level l occurs if $X_t^f = lY_t$ for some $t > 0$. The active portfolio management problems we consider in the sequel are: (i) Maximizing the probability performance goal u is reached before shortfall l occurs; (ii) Minimizing the expected time until the performance goal u is reached; (iii) Maximizing the expected time until shortfall l is reached; (iv) Maximizing the expected discounted reward obtained upon achieving goal u; (v) Minimizing the expected discounted penalty paid upon falling to shortfall level l. It is clear that for all these problems, the ratio of the wealth process to the benchmark is a sufficient statistic. Among other scenarios, these objectives are relevant to institutional money managers, whose performance is typically judged by the return on their managed portfolio relative to the return of a benchmark.

Since X_t^f is a controlled geometric Brownian motion, and Y_t is another geometric Brownian motion, it follows directly that the ratio process, Z^f, where $Z_t^f := X_t^f / Y_t$, is also a controlled geometric Brownian motion. Specifically, a direct application of Ito's formula gives

Proposition 1 *For X_t^f, Y_t defined by (4) and (5), let Z_t^f be defined by $Z_t^f := X_t^f / Y_t$. Then*

$$dZ_t^f = Z_t^f \left(\hat{r} + f_t^{'} \hat{\mu}\right) dt + Z_t^f \left(f_t^{'} \sigma - b^{'}\right) dW_t^{\bullet} - Z_t^f \beta \, dW_t^{(k+1)} , \tag{7}$$

where the constant $\hat{r}$ and vector $\hat{\mu}$ are defined by

$$\hat{r} := r - \alpha + b^{'} b + \beta^2 , \quad \textit{and} \quad \hat{\mu} := \mu - r\mathbf{1} - \sigma b . \tag{8}$$

In the next section we provide a general theorem in stochastic optimal control for the process $\left\{Z_t^f, t \geq 0\right\}$ of (7) that covers all the problems described above as special cases. In a later section we consider the more standard problem of maximizing the expected discounted terminal utility of the ratio.

3 Optimal control and a verification theorem

Most of the investment problems considered in this paper are all special cases of optimal control problems of the following (Dirichlet-type) form: For the process $\{Z_t^f, t \geq 0\}$ given by (7), let

$$\tau_x^f := \inf\{t > 0 : Z_t^f = x\} \tag{9}$$

denote the first hitting time to the point x under a specific policy $f = \{f_t, t \geq 0\}$. For given numbers l, u, with $l < Z_0 < u$, let $\tau^f := \min\{\tau_l^f, \tau_u^f\}$ denote first escape time from the interval (l, u), under this policy f.

For a given nonnegative function $\lambda(z) \geq 0$, a given real bounded continuous function $g(z)$, and a function $h(z)$ given for $z = l, z = u$, with $h(u) < \infty$, let $\nu^f(z)$ be the reward function under policy f, defined by

$$\begin{aligned} \nu^f(z) \quad = \quad & E_z \left(\int_0^{\tau^f} g(Z_t^f) \exp\left\{ - \int_0^t \lambda(Z_s^f) ds \right\} dt \right. \\ & \left. + h\left(Z_{\tau^f}^f\right) \exp\left\{ - \int_0^{\tau^f} \lambda(Z_s^f) ds \right\} \right) \end{aligned} \tag{10}$$

with

$$\nu(z) = \sup_{f \in \mathscr{G}} \nu^f(z) , \quad \text{and} \quad f_\nu^*(z) = \arg \sup_{f \in \mathscr{G}} \nu^f(z) \tag{11}$$

denoting respectively the optimal value function and associated optimal control function, where $\mathscr{G}$ denotes the set of admissible controls. (Here and in the sequel, we use the notations $P_z(\cdot)$ and $E_z(\cdot)$ as shorthand for $P(\cdot | Z_0 = z)$ and $E(\cdot | Z_0 = z)$.) We note at the outset that we are only interested in controls (and initial values z) for which $\nu^f(z) < \infty$.

Remark 3.1: Observe that the reward functional in (11) is sufficiently general to cover all the cases mentioned previously as a special case. For example, the probability of beating the benchmark before being beaten by it, following a given strategy $\{f_t\}$, i.e., $P_z\left(\tau_u^f < \tau_l^f\right)$, is a special case with $g(\cdot) = \lambda(\cdot) = 0$, $h(u) = 1$ and $h(l) = 0$. Similarly, by taking $\lambda(\cdot) = 0, g(\cdot) = 1$, and $h(u) = 0 = h(l)$, we obtain $E_z\left(\tau^f\right)$, and for $g(\cdot) = 0, \lambda(\cdot) = \lambda$, with $h(u) = 1$ we get $E_z\left(e^{-\lambda\tau_u^f}\right)$. Related optimal control problems have been treated previously in various forms for a variety of models. In particular see Pestien and Sudderth (1985), Heath et al. (1987), Orey et al. (1987), Majumdar and Radner (1991), Dutta (1994) and Browne (1995, 1997).

As a matter of notation, we note first that here, and throughout the remainder of the paper, the parameter γ will be defined by

$$\gamma := \hat{\mu}' \Sigma^{-1} \hat{\mu}/2 \equiv \left[(\mu - r\mathbf{1})' \Sigma^{-1} (\mu - r\mathbf{1}) + b'b - 2b'\sigma^{-1}(\mu - r\mathbf{1})\right]/2\,, \tag{12}$$

and the parameter δ will be defined by

$$\delta := \hat{r} + b'\sigma'\Sigma^{-1}\hat{\mu} = r - \alpha + \beta^2 + b'\sigma^{-1}(\mu - r\mathbf{1})\,. \tag{13}$$

In the following theorem, we show that the optimal value function is the solution to a particular nonlinear ordinary differential equation with Dirichlet boundary conditions, and that the optimal policy is given in terms of the first two derivatives of this solution. The proof, presented in the appendix, generalizes a related argument in Browne (1997) for a different model.

Theorem 1 *Suppose that $w(z) \in \mathscr{C}^2 : (l,u) \to (-\infty,\infty)$ is the concave increasing (i.e., $w_z > 0$ and $w_{zz} < 0$) solution to the nonlinear Dirichlet problem*

$$-\gamma\frac{w_z^2(z)}{w_{zz}(z)} + \delta z w_z(z) + \frac{1}{2}\beta^2 z^2 w_{zz}(z) + g(z) - \lambda(z)w(z) = 0\,, \quad \textit{for} \;\; l < z < u \tag{14}$$

with

$$w(l) = h(l)\,, \quad \textit{and} \;\; w(u) = h(u)\,, \tag{15}$$

and satisfies the following three conditions:

(i) $\frac{w_z^2(z)}{w_{zz}(z)}$ *is bounded for all z in (l,u);*
(ii) *either* (a) *there exists an integrable random variable X such that for all $t \geq 0$, $w(Z_t^f) \geq X$;*
or (b) *for every $t \geq 0$, and every admissible policy f, we have*

$$E\int_0^t \left(Z_s^f w_z\left(Z_s^f\right)\right)^2 \left(\left(f_s\sigma' - b\right)'\left(f_s\sigma' - b\right) + \beta^2\right) ds < \infty\,; \tag{16}$$

(iii) $\frac{w_z(z)}{w_{zz}(z)}$ *is locally Lipschitz continuous.*

Then $w(z)$ is the optimal value function, i.e., $w(z) = \nu(z)$, and moreover the optimal control vector, f_ν^*, can then be written as

$$f_\nu^*(z) = -\Sigma^{-1}\hat{\mu}\left(\frac{w_z(z)}{z\,w_{zz}(z)}\right) + \left(\sigma^{-1}\right)' b\,. \tag{17}$$

As highlighted earlier, the utility of Theorem 1 for our purposes is that for various choices of the functions $g(\cdot)$, $h(\cdot)$ and $\lambda(\cdot)$, it addresses all of the objective problems highlighted earlier. Moreover, it shows that for each of these problems, all we need do is solve the ordinary differential equation (14) and then take the appropriate derivatives to determine the optimal control by (17). The conditions (i) and (iii) are quite easy to check. Condition (ii) seems like the hard one, but for all cases considered here, it can be easily checked, as demonstrated below.

Remark 3.2: Observe that the last term in (17), $\left(\sigma^{-1}\right)' b$, is the vector portfolio strategy that minimizes the local variance of the ratio at every point. This can be seen by observing that Z_t^f is distributionally equivalent to the controlled diffusion process $dZ_t^f = Z_t^f \left[m(f_t)dt + v(f_t)d\tilde{W}_t\right]$, where $m(f) = \hat{r} + f'\hat{\mu}$, $v^2(f) = \left(f\sigma' - b\right)'\left(f\sigma' - b\right) + \beta^2$, and $\tilde{W}$ is a standard Brownian motion in R^1. It follows that v^2 has minimizer $f_* = (\sigma^{-1})' b$, with minimal local variance $v^2(f_*) = \beta^2$. Observe that $m(f_*) = \delta$, where δ is defined earlier in (13). The scalar in the first term in (17) is the inverse of the Arrow-Pratt measure of relative risk aversion of the optimal value function.

For the sequel, we will consider several specific applications of Theorem 1. In the next section, we consider the probability maximizing problem. For all relevant parameter values, this problem has a solution. This stands in contrast to some of the other problems we consider later. Specifically, we also consider the problem of *minimizing the expected time* to beat the benchmark by a predetermined amount. This problem has a solution if and only if a particular *favorability* parameter is positive. If the favorability parameter is negative, then the same policy is optimal for the objective of *maximizing the expected time until ruin*, where ruin is defined as falling below the benchmark by a predetermined amount.

4 Maximizing the probability of beating the benchmark

In this section we consider the case where the investor is trying to maximize the probability of beating the benchmark by some predetermined percentage, before going below it by some other predetermined percentage. This objective is relevant to the case of an institutional money manager who stands to receive a bonus upon hitting a performance measure, such as beating the return of the S&P 500 by a certain percentage, but whose job would be lost upon underperforming the benchmark by a predetermined percentage.

To formalize this problem, let $V^*(z)$ denote the maximal possible probability of beating the benchmark before being beaten by it, when starting from the state z; i.e., let $Z_0 = z$, and let u, l be given constants with $l < z < u$, then $V^*(z) = \sup_f P_z\left(\tau_u^f < \tau_l^f\right)$. Theorem 1 now applies to this problem by taking $\lambda = g = 0$, and setting $h(u) = 1$ and $h(l) = 0$. Specifically, by Theorem 1 , $V^*(z)$ must be the concave increasing solution to

$$-\gamma\frac{V_z^2(z)}{V_{zz}(z)}+\delta z V_z(z)+\frac{1}{2}\beta^2 z^2 V_{zz}(z)=0\,,\quad \text{for}\quad l<z<u \tag{18}$$

with $V^*(l)=0, V^*(u)=1$ (take $\lambda=g=0$ in (14)).

The solution to the nonlinear Dirichlet problem of (18), subject to the boundary conditions $V(l)=0, V(u)=1$ is seen to be $V(z)=\left(z^{c+1}-l^{c+1}\right)/\left(u^{c+1}-l^{c+1}\right)$, where c is a root to the quadratic equation $q(c)=0$, where $q(\cdot)$ is defined by

$$q(c):=\frac{\beta^2}{2}c^2+\delta c-\gamma\,. \tag{19}$$

This quadratic admits two real roots, c^+, c^-, with

$$c^\pm=\frac{-\delta\pm\sqrt{\delta^2+2\gamma\beta^2}}{\beta^2}\,. \tag{20}$$

Some direct manipulations establish further that $c^-<0<c^+$. For both roots, we have $V_z>0$, however it is only for the *negative* root, c^-, that we have $V_{zz}<0$. Moreover, for this function it is readily verified that conditions (i), (ii) and (iii) hold. (Condition (iia) obviously holds since $V^*(z)$ is bounded above and below by 1 and 0). As such the optimal value function is seen to be

$$V^*(z)=\frac{z^{c^-+1}-l^{c^-+1}}{u^{c^-+1}-l^{c^-+1}}\,,\quad \text{for}\quad l<z<u\,. \tag{21}$$

Furthermore, the optimal investment policy for this objective, call it $f_V^*(z)$, can be now obtained by substituting this value function (21) into (17) of Theorem 1. We summarize this in

Corollary 1 *For $l<z<u$, let $V^*(z):=\sup_f P_z\left(\tau_u^f<\tau_l^f\right)$ with associated optimal control vector $f_V^*(z)$. Then, for c^- as defined in (20), $V^*(z)$ is given in (21) and*

$$f_V^*(z)=-\Sigma^{-1}\hat{\mu}\left(\frac{V_z^*(z)}{zV_{zz}^*(z)}\right)+\left(\sigma^{-1}\right)'b\equiv-\left(\frac{1}{c^-}\right)\Sigma^{-1}\hat{\mu}+\left(\sigma^{-1}\right)'b\,. \tag{22}$$

As (22) displays, the optimal control vector $f_V^*(z)$ a constant proportion policy: regardless of the level of wealth or the value of the benchmark (or their ratio), the proportion of wealth invested in each of the the risky stocks is held constant, with the remainder held in the riskless bond. Moreover, these constants are independent of the levels l and u. However, this constant does depend on all the parameters of the model. (See Browne (1998) for further properties of constant proportion allocation strategies.)

We will have more to say about certain characteristics of the policy f_V^* later, when we analyze the optimal solution to other related problems (see Remark 5.2 below).

Remark 4.1: Observe that so long as $\beta\neq 0$, the probability maximizing objective has a unique optimal solution (in the absence of any other constraints), regardless of the signs and values of the other parameters. This occurs because regardless of how favorable the investment opportunities are relative to the benchmark, there is still no way to ensure that the controlled wealth will beat the benchmark with certainty. This ceases to be true if $\beta = 0$, i.e., if there is no uncontrollable source of randomness in the model. Specifically, for the case where $\beta = 0$, the solution to the correspondingly modified nonlinear Dirichlet problem (18) is still of the form $\left(z^{c+1} - l^{c+1}\right) / \left(u^{c+1} - l^{c+1}\right)$, however now $c = \gamma/\delta$, where γ and δ are modified accordingly. However, for the resulting value function to be concave (and hence *optimal*), we would require that $\delta < 0$. For $\beta = 0$, this translates into the requirement that

$$r + b^{'}\sigma^{-1}(\mu - r\mathbf{1}) < \alpha \tag{23}$$

i.e., that the appreciation rate of the benchmark, α, dominate the appreciation rate of the minimum variance portfolio, $r + b^{'}\sigma^{-1}(\mu - r\mathbf{1})$ (i.e., when $f_t = (\sigma^{-1})^{'} b$ for all $t > 0$ in (4), see Remark 3.2). Thus if (23) holds, then the optimal investment policy to maximize the probability of beating the benchmark (before being beaten by it) is again a constant rebalancing, with $f^* = \left(\sigma^{-1}\right)^{'} b - \Sigma^{-1}\hat{\mu}(\delta/\gamma)$.

However if the converse to (23) holds, then the probability maximizing problem is uninteresting, since in that case it is easy to see that the policy that takes $f_t = \left(\sigma^{-1}\right)^{'} b$, for all $t \geq 0$, results in a riskless ratio process that has *positive* drift, and so achieves the upper goal before the lower goal with probability one. Specifically, if $\beta = 0$, and if we take $f_t^{'} = b^{'}\sigma^{-1}$ for all $t \geq 0$ in (7), then Z_t^f is an increasing exponential function,since in that case (7) reduces to the deterministic equation

$$dZ_t^f = Z_t^f \left[r - \alpha + b^{'}\sigma^{-1}(\mu - r\mathbf{1})\right] dt\,.$$

Observe of course that for the case where $\beta = 0$ and Y is a *traded* asset, then there is an arbitrage unless $r + b^{'}\sigma^{-1}(\mu - r\mathbf{1})/\sigma = \alpha$ (see e.g. Duffie 1996), and so the previous discussion is moot.

5 Minimizing or maximizing the expected time

Rather than just maximizing the probability of beating the benchmark, some investment objectives relate to *minimizing the expected time* until the benchmark is beaten by some given multiple. Interestingly enough, this objective does not have an optimal solution for all values of the parameters. In fact, the existence of a solution depends upon whether the "favorability" parameter θ, defined by

$$\theta := \delta + \gamma - \frac{1}{2}\beta^2 \equiv r - \alpha + \frac{1}{2}\left[\beta^2 + b^{'}b + (\mu - r\mathbf{1})^{'}\,\Sigma^{-1}(\mu - r\mathbf{1})\right] \tag{24}$$

is positive or negative.

Let π^* be the constant vector defined by

$$\pi^* := \Sigma^{-1}(\mu - r\mathbf{1})\,. \tag{25}$$

For an investor whose wealth evolves according to (4), and who is not concerned with performance relative to any benchmark, the policy $f_t = \pi^*$, for all t, is called the *optimal growth*, or *log-optimal* policy (see Merton 1990). The reason for this is that (i) π^* maximizes the expected logarithm of wealth, for any fixed terminal time T, hence (ii) π^* maximizes the (expected) rate at which wealth compounds. More interesting perhaps, and certainly more relevant to our concerns here is the property (iii) π^* *minimizes the expected time* until any given level of wealth is achieved (needless to say, so long as that level is greater than the initial wealth). Merton (1990, Chapter 6) contains a comprehensive review of these properties (see also Heath et al. 1987 and Browne 1997).

It turns out that this policy has certain optimality properties in our setting as well, where performance is judged solely relative to a benchmark. Specifically, our main result in this section is the following: If $\theta > 0$, then $f_t^* = \pi^*$, for all $t > 0$, minimizes the expected time to beat the benchmark, while if $\theta < 0$, then $f_t^* = \pi^*$, for all $t > 0$, maximizes the expected time until the investor is beaten by the benchmark. Thus if the investment opportunity is favorable relative to the benchmark, then the ordinary optimal growth policy (which is independent of any benchmark parameters) also maximizes growth relative to a benchmark. However if the investment opportunity is unfavorable relative to the benchmark, then the ordinary optimal growth policy still has some optimality properties in this setting, namely, it *maximizes the expected time until ruin*.

We state this formally in the following corollary to Theorem 1.

Corollary 2 *Let* $G^*(z) := \inf_f E_z\left(\tau_u^f\right)$ *with optimizer* $f^*(z) = \arg\ \inf_f E_z\left(\tau_u^f\right)$, *and let* $G_*(z) := \sup_f E_z\left(\tau_l^f\right)$ *with optimizer* $f_*(z) = \arg\ \sup_f E_z\left(\tau_l^f\right)$.

If $\theta > 0$, *then*

$$G^*(z) = \frac{1}{\theta}\ln\left(\frac{u}{z}\right)\,, \quad \textit{with} \quad f^*(z) = \pi^*,\ \textit{for all}\ z \le u\,. \tag{26}$$

If $\theta < 0$, *then*

$$G_*(z) = \frac{1}{|\theta|}\ln\left(\frac{z}{l}\right)\,, \quad \textit{with} \quad f_*(z) = \pi^*,\ \textit{for all}\ z \ge l\,. \tag{27}$$

Proof: Observe first that while Theorem 1 is stated in terms of a maximization problem, it obviously contains the minimization case, as we can apply Theorem 1 to $\tilde{G}(z) := \sup_f\left\{-E_z\left(\tau_u^f\right)\right\}$, and then recognize that $G^* = -\tilde{G}$. As such, Theorem 1 applied with $g(z) = 1, \lambda(z) = 0$, shows that both G^* and G_* must solve the ordinary differential equation

$$-\gamma\frac{G_z^2(z)}{G_{zz}(z)} + \delta z G_z(z) + \frac{1}{2}\beta^2(1-\rho^2)z^2 G_{zz}(z) + 1 = 0\,, \tag{28}$$

however we must have $G^*(u) = 0$ and $G_*(l) = 0$. Moreover, G^* must be *convex decreasing* (since it is the solution to a minimization problem) while G_* must be concave increasing. It is now easy to substitute the claimed values from

(26) and (27) to verify that that in fact is the case, under the stated conditions on θ. Furthermore, for both cases we have $G_z/zG_{zz} = -1$, and as such (17) of Theorem 1 shows that the optimal control for both cases is then given by $\Sigma^{-1}\hat{\mu} + \left(\sigma^{-1}\right)' b$, which reduces to π^*.

It remains to verify whether the conditions (i) (ii) and (iii) of Theorem 1 hold: it is clear that (i) and (iii) hold. Condition (iib) is seen to hold for this case since in both cases we have $dG^*(z)/dz = -1/(z\theta)$, and $dG_*(z)/dz = 1/\left(z|\theta|\right) = -1/(z\theta)$. As such requirement (16) becomes

$$\int_0^t \theta^{-2} E\left(\left(f_s\sigma' - b\right)'\left(f_s\sigma' - b\right) + \beta^2\right) ds < \infty\,, \qquad \text{for every} \quad t > 0\,.$$

But this requirement *must* hold by definition of *admissibility*. □

Remark 5.1: Observe that π^* maximizes logarithmic utility for the investor. This follows directly since $\sup_f \left\{E\left[\ln\left(Z_T^f\right)\right]\right\} = \sup_f \left\{E\left[\ln\left(X_T^f\right)\right]\right\} - E\left[\ln(Y_T)\right]$. While the connection between log-utility and the minimal expected time to beat the benchmark problem (or the maximal expected time to being beaten by it) is now obvious in light of the logarithmic value functions of (26) and (27), this was by no means obvious or intuitive *apriori*.

To interpret the favorability condition properly, observe that if Z^* denotes the ratio process under policy π^*, then Z_t^* is the geometric Brownian motion given by

$$Z_t^* = Z_0 \exp\left\{\theta t + \left(\sigma^{-1}(\mu - r\mathbf{1})\right)' W_t^{\bullet} - \beta W_t^{(k+1)}\right\}$$

and so we have $\frac{1}{t}E\left[\ln\left(Z_t^*/Z_0\right)\right] = \theta$. Thus, standard results on Brownian motion imply that for $\theta > 0$, $Z_t^* \to \infty$, a.s., while for $\theta < 0$, $Z_t^* \to 0$, a.s. This in turn implies that $\tau_u^* < \infty$ for $\theta > 0$, while $\tau_l^* < \infty$ for $\theta < 0$, where τ_x^* denotes the first hitting time to x for the process Z^*.

Moreover, for any admissible policy $\{f_t\}$, and for $l < z < u$, we have the following:

Corollary 3 *If* $\theta < 0$, *then* $\inf_f E_z\left(\tau_u^f\right) = \infty$, *while if* $\theta > 0$, *then* $\sup_f E_z\left(\tau_l^f\right) = \infty$.

This corollary states that for the unfavorable case, $\theta < 0$, there is no admissible policy for which the expected time to beat the benchmark is finite, while for the favorable case, an admissible strategy can always be found whose expected time to ruin is infinite.

Remark 5.2: The favorability parameter, θ, helps in interpreting the portfolio policy f_V^* of (22) obtained earlier for the probability maximizing objective. Specifically, some direct manipulations will verify that for c^- of (20) being the negative root to the quadratic of (19), the following holds:

$$c^- \begin{cases} < -1 & \text{if } \theta > 0 \\ > -1 & \text{if } \theta < 0 \\ = -1 & \text{if } \theta = 0\,. \end{cases} \tag{29}$$

Consider now the optimal policy for maximizing the probability of beating the benchmark, before being beaten by it, f_V^* of (22). Observe that (29) is equivalent to saying that $f_V^* > \pi^*$ for $\theta < 0$, while $f_V^* < \pi^*$ for $\theta > 0$ (vector inequalities here are to be interpreted componentwise). Thus in an unfavorable situation, the strategy that minimizes the probability of ruin is *more aggressive* than the strategy that maximizes the expected time to ruin. So in a sense, and in the picturesque language of Dubins and Savage (1975), we observe that for an unfavorable game, *bold play* maximizes the probability of success while *timid play* maximizes the expected playing time. Conversely, for a relatively favorable game, timid play will minimize the probability of ruin while a bolder strategy will minimize the expected time to achieve a goal.

6 Minimizing or maximizing expected discounted rewards

In this section we consider the problem of either maximizing $E_z\left(e^{-\lambda\tau_u^f}\right)$, or minimizing $E_z\left(e^{-\lambda\tau_l^f}\right)$, for a given discount rate $\lambda > 0$. The former objective is relevant when there is some (positive) reward of achieving the upper goal, while the latter is relevant when there is some cost to hitting the lower goal. Thus the former is related to growth while the latter is related to survival.

To address this formally, let $F^*(z) = \sup_f E_z\left(e^{-\lambda\tau_u^f}\right)$, with associated optimal control vector $f^*(z;\lambda)$, and let $F_*(z) = \inf_f E_z\left(e^{-\lambda\tau_l^f}\right)$, with associated optimal control vector $f_*(z;\lambda)$.

From Theorem 1 (with $g = 0$) we see that for both cases, the value function must satisfy the nonlinear problem

$$-\gamma\frac{F_z^2(z)}{F_{zz}(z)} + \delta z F_z(z) + \frac{1}{2}\beta^2 z^2 F_{zz}(z) - \lambda F(z) = 0\,, \tag{30}$$

however the concavity/convexity properties and the boundary conditions differ. Specifically, we require F^* to be concave increasing with $F^*(u) = 1$, while we require F_* to be convex decreasing with $F_*(l) = 1$.

Solutions to (30) are of the form $F(z) = Kz^{\eta+1}$, where K is a constant to be determined by the boundary condition, and η is a root to the cubic equation $C(\eta;\lambda) = 0$, where

$$C(\eta;\lambda) := \eta^3\beta^2/2 + \eta^2\left[\beta^2/2 + \delta\right] + \eta\left[\delta - \gamma - \lambda\right] - \gamma\,. \tag{31}$$

For $\lambda > 0$, this cubic equation admits three distinct and real roots, call them η_1, η_2, η_3, with

$$\eta_1(\lambda) < -1\,, \quad -1 < \eta_2(\lambda) < 0\,, \quad \eta_3(\lambda) > 0\,.$$

By Theorem 1, for the maximization problem we require $F_z^* > 0$, with $F_{zz}^* < 0$, which translates into the simultaneous requirements that $\eta + 1 > 0$, and $\eta(\eta + 1) < 0$. Thus, $\eta_2(\lambda)$ is the appropriate root, and we then have

$$F^*(z) = \left(\frac{z}{u}\right)^{\eta_2+1}, \quad \text{for} \quad z \le u.$$

It then follows from (22) that for this case we have the optimal control vector

$$f^*(z;\lambda) = -\Sigma^{-1}\hat{\mu}\left(\frac{1}{\eta_2(\lambda)}\right) + \left(\sigma^{-1}\right)' b\,.$$

Similarly, for the minimization problem, we require F_* to be convex decreasing, which translates into the simultaneous requirements that $\eta + 1 < 0$ and $\eta(\eta+1) > 0$. Clearly, it is now $\eta_1(\lambda)$ which is the appropriate root here. As such, we have

$$F_*(z) = \left(\frac{z}{l}\right)^{\eta_1+1}, \quad \text{for} \quad z \ge l,$$

with the associated optimal control vector

$$f_*(z;\lambda) = -\Sigma^{-1}\hat{\mu}\left(\frac{1}{\eta_1(\lambda)}\right) + \left(\sigma^{-1}\right)' b\,.$$

Observe that these optimal controls are again constant proportions, and for $\hat{\mu} > 0$ we have $f^* > f_*$.

Observe further that we can write the cubic function $C(\eta;\lambda)$ of (31) as $C(\eta;\lambda) = (\eta+1)\left[q(\eta) - \lambda\right] + \lambda$, where $q(\cdot)$ is the quadratic defined earlier by (19). Thus as $\lambda \downarrow 0$, $C(\eta;0) = (\eta+1)q(\eta)$, which implies that as $\lambda \downarrow 0$ the three roots to the cubic equation must converge to the two quadratic roots c^+, c^- of (20), and the new root, -1. Since $c^- < 0 < c^+$, it must be that $\eta_3(\lambda)$ converges to c^+, however, it is by no means clear to which values the other 2 converge. In fact we will now show below that this depends on the sign of the favorability parameter, θ.

Recall first some facts about Laplace transforms of continuous nonnegative random variables: If $H(\lambda) = E\left(e^{-\lambda\tau}\right)$, where τ is a nonnegative valued random variable and $\lambda \ge 0$, then $H(0) := \lim_{\lambda\downarrow 0} H(\lambda) = P(\tau < \infty)$. Thus if τ is a defective random variable, then the defect is $1 - H(0)$. It is of course necessary that $H(0) = 1$ for $E(\tau) < \infty$. Moreover, if $E(\tau) < \infty$, then

$$E(\tau) = \lim_{\lambda\downarrow 0}\frac{1 - H(\lambda)}{\lambda}\,.$$

Consider now the negative root c^-. As observed previously in (29), we have $c^- < (\ge) - 1$ if $\theta > (\le) 0$. This fact, combined with Corollary 1, now gives the following relationship.

Proposition 2 *As $\lambda \downarrow 0$, we have the following:*

I. If $\theta > 0$, then $\eta_2(\lambda) \downarrow \eta_2(0) = -1$, and $\eta_1(0) = c^-$.
II. If $\theta < 0$, then $\eta_1(\lambda) \uparrow \eta_1(0) = -1$, and $\eta_2(0) = c^-$.

7 Utility maximization

So far, we have considered objectives related solely to the achievement of (arbitrary) goals. In this section we consider the case where the investor is interested in maximizing expected discounted terminal utility of the ratio at some *fixed* terminal time $T < \infty$. Specifically, for a given (concave increasing) utility function $U(z)$, let $\Psi(t,z) = \sup_f E\left(e^{-\lambda(T-t)}U\left(Z_T^f\right)\right)$.

For the special case $U(z) = \ln(z)$, our previous results show that the ordinary log-optimal policy, π^* is optimal once again, with $\Psi(t,z) = \ln(z)+(\theta-\lambda)(T-t)$.

For the general case, an analysis similar to that of Theorem 1 will show that (under suitable regularity conditions), the optimal value function Ψ must satisfy the nonlinear partial differential equation

$$\Psi_t - \gamma\frac{\Psi_z^2(z)}{\Psi_{zz}(z)} + \delta z\Psi_z(z) + \frac{1}{2}\beta^2 z^2\Psi_{zz}(z) - \lambda\Psi(z) = 0\,, \tag{32}$$

subject to the boundary condition $\Psi(T,z) = U(z)$. (For the special case $\beta = 0$ this reduces to a problem treated in Merton (1971).) The optimal control function is then given by

$$f^*(t,z) = -\Sigma^{-1}\hat{\mu}\left(\frac{\Psi_z}{z\Psi_{zz}}\right) + \left(\sigma^{-1}\right)' b\,. \tag{33}$$

For the special case where the utility function is of the form $U(z) = z^{\kappa+1}$, the explicit solution to the nonlinear Cauchy problem (32) is

$$\Psi(t,z) = z^{\kappa+1}\exp\left\{-\frac{1}{\kappa}C(\kappa;\lambda)(T-t)\right\} \tag{34}$$

where $C(\bullet;\lambda)$ is the cubic function defined earlier in (31). This utility function has constant relative risk aversion $-\kappa$ (for concavity, we would require that $-1 < \kappa < 0$). Substitution of (34) into (33) shows that for this case the optimal control is again a constant proportional strategy, with

$$f^* = -\Sigma^{-1}\hat{\mu}\left(\frac{1}{\kappa}\right) + \left(\sigma^{-1}\right)' b\,.$$

Comparison with earlier results will then show that just as there is a connection between maximizing logarithmic utility and the objective criteria of minimizing or maximizing the expected time to a goal, so too is there a connection between maximizing expected utility of terminal wealth for a power utility function, and the objective criteria of maximizing the probability of reaching a goal, or maximizing or minimizing the expected discounted reward of reaching certain goals. (In particular, by taking $\kappa = c^-, \eta_1, \eta_2$.)

This extends earlier connections obtained for the standard portfolio problem since it shows that power utility (Constant Relative Risk Aversion) functions relate to *survival* as well as *growth* objectives.

8 Conclusions

We have studied a variety of objective goal-related problems for the problem of outperforming a stochastic benchmark. The presence of a benchmark that need not be perfectly correlated with the investment opportunities allows a richer risk/reward framework than the standard model. We have identified a particular parameter whose sign determines whether we are in the favorable case or not. Regardless of the sign of this favorability parameter, we have determined the optimal policy for maximizing the probability of reaching a given performance goal before falling to a performance shortfall. For the favorable case, we have shown that the ordinary log-optimal policy is also the optimal policy for minimizing the expected time to the performance goal, while for the unfavorable case, the log-optimal policy maximizes the expected time to the performance shortfall. The discounted version of these problems has been solved as well. We have related all these goal problems to a more standard expected utility maximizing problem for power utility functions. Our results provide extended optimality properties of constant proportion policies.

A Appendix

A.1 Proof of Theorem 1

Observe first that for Markov control processes $\{f_t, t \geq 0\}$, and functions $\Psi(t,z) \in \mathscr{C}^{1,2}$, the generator of the ratio process Z_t^f of (7) is

$$\mathscr{A}^f \Psi(t,z) = \Psi_t + \left(\hat{r} + f_t' \hat{\mu}\right) z \Psi_z + \frac{1}{2}\left(\left(f_t\sigma' - b\right)'\left(f_t\sigma' - b\right) + \beta^2\right) z^2 \Psi_{zz} \,. \quad (35)$$

The Hamilton-Jacobi-Bellman optimality equation of dynamic programming (see Krylov (1980, Theorem 1.4.5) , or Fleming and Soner (1993, Sect. IV.5)) for maximizing $\nu^f(z)$ of (10) over control policies f_t, to be solved for a function ν is $\sup_f\{\mathscr{A}^f \nu + g - \lambda\nu\} = 0$, subject to the Dirichlet boundary conditions $\nu(l) = h(l)$ and $\nu(u) = h(u)$. Since $\nu(z)$ is independent of time, the generator of (35) shows that this is equivalent to

$$\max_f \left\{ \left(\hat{r} + f'\hat{\mu}\right) z\nu_z + \frac{1}{2}\left(\left(f\sigma' - b\right)'\left(f\sigma' - b\right) + \beta^2\right) z^2 \nu_{zz} + g - \lambda\nu \right\} = 0 \,. \quad (36)$$

Assuming now that (36) admits a classical solution with $\nu_x > 0$ and $\nu_{xx} < 0$, we may then use standard calculus to optimize with respect to f in (36) to obtain the optimal control function $f_\nu^*(x)$ of (17), with $\nu = w$. When (17) is then substituted back into (36) and simplified, we obtain the nonlinear Dirichlet problem of (14) (with $\nu = w$).

It remains only to verify that the policy f_ν^* is indeed optimal. Since the conditions required in the aforementioned results in Krylov (1980) and Fleming and Soner (1993) are not all necessarily met in this case, we will use the martingale optimality principle directly (cf Rogers and Williams 1987, or Davis and Norman 1990), which entails finding an appropriate functional which is a uniformly integrable martingale under the (candidate) optimal policy, but a *supermartingale* under any other admissible policy, with respect to the filtration $\mathscr{F}_t$.

To that end, let $\Lambda^f(s,t) := \int_s^t \lambda(Z_v^f)dv$, and define the process

$$M(t, Z_t^f) := e^{-\Lambda^f(0,t)} w(Z_t^f) + \int_0^t e^{-\Lambda^f(0,s)} g(Z_s^f) ds \,, \quad \text{for} \quad t \geq 0 \,, \tag{37}$$

where w is the concave increasing solution to (14).

Optimality of f_ν^* of (17) is then a direct consequence of the following lemma.

Lemma 1 *For any admissible policy f, and $M(t,\cdot)$ as defined in (37), we have*

$$E\left(M\left(t \wedge \tau^f, Z^f_{t\wedge\tau^f}\right)\right) \leq M(0, Z_0) = w(z)\,, \quad \textit{for} \quad t \geq 0 \tag{38}$$

with equality holding if and only if $f = f_\nu^*$, where f_ν^* is the policy given in (17). Moreover, under policy f_ν^*, the process $\left\{M\left(t \wedge \tau^f, Z^*_{t\wedge\tau^f}\right), t \geq 0\right\}$ is a uniformly integrable martingale.

Proof: Applying Ito's formula to $M(t, Z_t^f)$ of (37) using (7) shows that for $0 \leq s \leq t \leq \tau^f$

$$\begin{aligned} M(t, Z_t^f) &= M(s, Z_s^f) + \int_s^t e^{-\Lambda^f(s,v)} Q(f_v; Z_v^f) dv \\ &\quad + \int_s^t e^{-\Lambda^f(s,v)} Z_v^f w_z(Z_v^f) \left[\left(f_v'\sigma - b'\right) dW_v^{\bullet} - \beta dW_v^{(k+1)}\right] \end{aligned} \tag{39}$$

where $Q(f;z)$ denotes the function defined by

$$\begin{aligned} Q(f\,;\,z) &:= \left(\frac{1}{2}\right)\left(\left(f\sigma' - b\right)'\left(f\sigma' - b\right) + \beta^2\right) z^2 w_{zz}(z) \\ &\quad + z(\hat{r} + f'\hat{\mu}) w_z(z) + g(z) - \lambda(z) w(z)\,. \end{aligned} \tag{40}$$

Recognize now that for a fixed z, Q is a quadratic form in the vector f with $Q_{ff}(f;z) = \Sigma w_{zz}(z)$. Since Σ is symmetric positive definite and w is concave in z, it is easily verified that we always have $Q(f;z) \leq 0$, and that the maximum is achieved at the value

$$f^*(z) := -\Sigma^{-1}\hat{\mu}\left(\frac{w_z(z)}{z\, w_{zz}(z)}\right) + \left(\sigma^{-1}\right)' b$$

with corresponding maximal value

$$Q(f^*; z) = -\gamma \frac{w_z^2(z)}{w_{zz}(z)} + \delta z\, w_z(z) + \frac{1}{2}\beta^2 z^2 w_{zz}(z) + g(z) - \lambda(z) w(z) \equiv 0 \tag{41}$$

where the final equality follows from (14). Therefore the second term in the rhs of (39) is always less than or equal to 0. Moreover (39) shows that we have

$$\begin{aligned}\int_0^{t\wedge\tau^f} & e^{-\Lambda^f(0,v)} Z_v^f w_z(Z_v^f)\left[\left(f_v^{'}\sigma - b^{'}\right) dW_v^{\bullet} - \beta dW_v^{(2)}\right] \\ &= M\left(t\wedge\tau^f, Z_{t\wedge\tau^f}^f\right) - w(z) - \int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)} Q(f_v; Z_v^f) dv \quad (42)\\ &\geq M\left(t\wedge\tau^f, Z_{t\wedge\tau^f}^f\right) - w(z)\,. \quad (43)\end{aligned}$$

Thus, by (ii) we see that the stochastic integral term in (39) is a local martingale that is in fact a *supermartingale*. (If (iia) holds, then the RHS of (43) is bounded below by an integrable random variable, which is sufficient to make the stochastic integral term a supermartingale. If (iib) holds, then the stochastic integral is a martingale directly, and hence it is also a supermartingale.)

Hence, taking expectations on (39) therefore shows that

$$\begin{aligned} E\left(M(t\wedge\tau^f, Z_{t\wedge\tau^f}^f)\right) &\leq w(z) + E\left(\int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)} Q(f_v; Z_v^f) dv\right) \quad (44)\\ &\leq w(z) + E\left(\int_0^{t\wedge\tau^f} e^{-\Lambda^f(0,v)}\left[\sup_{f_u} Q(f_v; Z_v^f)\right] dv\right) \quad (45)\\ &= w(z) \quad (46)\end{aligned}$$

with the equality in (46) being achieved at the policy f_ν^*.

Thus we have established (38).

Note that under the policy f_v^*, the process Z^* satisfies the stochastic differential equation

$$\begin{aligned} dZ_t^* = & \left[\left(\left(\hat{r} + b^{'}\sigma^{-1}\hat{\mu}\right) Z_t^* - 2\gamma\frac{w_z(Z_t^*)}{w_{zz}(Z_t^*)}\right) dt \right.\\ & \left. - \left(\frac{w_z(Z_t^*)}{w_{zz}(Z_t^*)}\right)\left(\sigma^{-1}\hat{\mu}\right)^{'} dW_t^{\bullet} - \beta Z_t^* dW_t^{(k+1)}\right] I_{\{t\leq\tau^*\}} \quad (47)\end{aligned}$$

where $\tau^* := \tau^{f_\nu^*}$. By (iii) this equation admits a unique strong solution.

Furthermore note that under the (optimal) policy, f_ν^*, we have

$$\begin{aligned} M(t, Z_t^*) = M(s, Z_s^*) - & \int_s^t \exp\left\{-\int_s^v \lambda(Z_\rho^*) d\rho\right\} \\ & \times\left[\left(\frac{w_z^2(Z_v^*)}{w_{zz}(Z_v^*)}\right)\left(\sigma^{-1}\hat{\mu}\right)^{'} dW_v^{\bullet} + \beta Z_v^* w_z(Z_v^*) dW_v^{(k+1)}\right] \quad (48)\end{aligned}$$

which by (i) above is seen to be a uniformly integrable martingale. This completes the proof of the theorem and hence f_ν^* is indeed optimal. □

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