REACHING GOALS BY A DEADLINE: DIGITAL OPTIONS AND CONTINUOUS-TIME ACTIVE PORTFOLIO MANAGEMENT

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Abstract
We study a variety of optimal investment problems for objectives related to attaining goals by a fixed terminal time. We start by finding the policy that maximizes the probability of reaching a given wealth level by a given fixed terminal time, for the case where an investor can allocate his wealth at any time between \( n + 1 \) investment opportunities: \( n \) risky stocks, as well as a risk-free asset that has a positive return. This generalizes results recently obtained by Kulldorff and Heath for the case of a single investment opportunity. We then use this to solve related problems for cases where the investor has an external source of income, and where the investor is interested solely in beating the return of a given stochastic benchmark, as is sometimes the case in institutional money management. One of the benchmarks we consider for this last problem is that of the return of the optimal growth policy, for which the resulting controlled process is a supermartingale. Nevertheless, we still find an optimal strategy. For the general case, we provide a thorough analysis of the optimal strategy, and obtain new insights into the behavior of the optimal policy. For one special case, namely that of a single stock with constant coefficients, the optimal policy is independent of the underlying drift. We explain this by exhibiting a correspondence between the probability maximizing results and the pricing and hedging of a particular derivative security, known as a digital or binary option. In fact, we show that for this case, the optimal policy to maximize the probability of reaching a given value of wealth by a predetermined time is equivalent to simply buying a European digital option with a particular strike price and payoff. A similar result holds for the general case, but with the stock replaced by a particular (index) portfolio, namely the optimal growth or log-optimal portfolio.

Keywords: Optimal gambling; stochastic control; portfolio theory; martingales; option pricing; hedging strategies; digital options

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1. Introduction
There are various approaches to the problem of determining optimal dynamic investment policies, depending on the objectives of the investor. In continuous time, which is the setting in this paper, optimal dynamic investment policies for the objective of maximizing expected utility derived from terminal wealth or consumption over a finite horizon as well as discounted utility of consumption over the infinite horizon are derived in the pioneering work of Merton [15]. Generalizations of the utility maximizing approach that incorporate bankruptcies as received 11 June 1997; revision received 16 March 1998.

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well as more general price processes than those considered earlier are surveyed in [16] and [7]. These results also have substantial implication for the pricing and hedging of contingent claims, see for example [5].

However, there are many investment scenarios where approaches alternative to that of utility maximization might be preferable. In particular, many actual investment objectives are related solely to the achievement of specific goals. For example, in institutional money management, the practice of benchmarking is quite prevalent. In this scenario, a portfolio manager is judged solely by how his portfolio performs relative to that of another benchmark portfolio, or index. The Standard and Poor's (S&P) 500 index is a typical example of a benchmark. There is a distinction made between passive portfolio management, and active portfolio management (see, e.g., [20]). A passive portfolio manager is simply interested in tracking the index, while an active portfolio manager is interested in beating the return of the predetermined given benchmark or index. From the viewpoint taken here, the passive portfolio manager's investment decision is uninteresting, since we assume that for all intents and purposes, a passive portfolio manager can simply invest directly in the benchmark. The active portfolio manager faces an interesting problem however, since he is investing in order to beat a 'goal'. The goal the active portfolio manager is trying to beat is the stochastic return of the benchmark. In this paper we treat this among other goal problems. The active portfolio management problem is also relevant to the hedging of contingent claims, where in that setting, the benchmark portfolio would be taken to be the replicating strategy of the contingent claim.

Optimal investment policies for objectives relating solely to the achievement of goals have been studied previously, although perhaps not to the extent that utility maximization has. For illustration, suppose the investor starts off with initial capital $0 < X_0 < b$. Then, some classical problems include determining an investment policy that (if appropriate) maximizes the probability of reaching $b$ before $0$, or (if appropriate) minimizes the expected time to the upper goal $b$. We can refer to these, respectively, as the survival problem, and the growth problem. In discrete time, and over an infinite horizon, the survival problem is the centerpiece of the classical work of Dubins and Savage [6], and the growth problem was studied in [2] and [9]. In continuous time, the survival problem over an infinite horizon for diffusion processes was studied in [17, 18]. Survival problems related to various generalizations of the portfolio problem of Merton [15] were studied in [3] and [4] (see also [14]). The growth problem in continuous time was studied in a general framework in [11], [16, Chapter 6], and generalized to a model that incorporated liabilities in [4]. However, all these results are specific to the case of an infinite horizon. Since the performance of money managers is not judged over the infinite horizon, but rather over a finite (sometimes quite short) horizon, these studies are not directly applicable to the problem of active portfolio management. Similarly, in pension fund management, the horizon is typically finite. With a finite horizon, the distinction between a survival problem and a growth problem tends to blur, since in both cases they relate to maximizing probabilities: the survival problem would be to maximize the probability that the lower goal is not hit before the horizon, while the growth problem would be to maximize the probability that the upper goal is hit before the deadline.

A finite-horizon goal problem was studied recently by Kulldorff [13], for a model with a single risky favorable investment opportunity. Kulldorff obtained the optimal investment policy for the objective of maximizing the probability that wealth attains a given constant goal by a fixed terminal time. Both continuous and discrete time problems were considered there. For the continuous time version of the problem, the return of the single risky asset was modeled as a Brownian motion with a time-dependent drift coefficient and a constant diffusion.
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coefficient. The goal as well as the constant diffusion coefficient were both normalized to 1. Heath, in [10], considered the same model, but with constant drift as well and took a somewhat different approach to establish the same results as in [13]. In both cases, there was no risk-free asset available other than cash, which had a zero return. One of the interesting features of their policy was the fact that for the case of a constant drift, the optimal policy was independent of this underlying drift. No explanation of this rather remarkable fact was given.

In this paper, we address a variety of more general goal problems, all with probability maximizing objectives. To that end we first generalize the important results of [10] and [13] in a few fundamental ways: first, we expand the investment opportunity set to include a risk-free asset that has a (positive) time-dependent rate of return, as well as multiple risky assets with time-varying covariance structure. We also obtain new representations and analysis of the resulting optimal policy. We then extend our general results to treat cases where the investor earns income from an exogenous source, and where the investor's objective is to beat the (stochastic) return of a given benchmark portfolio.

The explanation as to why the optimal policy is independent of the underlying drift for the single stock case with constant coefficients is provided as a byproduct of our analysis for the more general case. It turns out that when there are multiple risky stocks, the optimal policy, for maximizing the probability of attaining a preset level of wealth by a finite deadline, is no longer independent of the drift parameters, even for the case of constant coefficients. The resulting policy is quite interesting and we provide a new analysis that allows for a complete quantitative assessment of the risk-taking behavior of an investor following such an objective. Furthermore, we obtain a new representation of the optimal wealth process. This representation, together with the addition of a risk-free asset, allows us to exhibit a remarkable correspondence between the probability maximizing policy and the hedging strategy of a digital option for the single-stock constant coefficients case. In particular we show that the optimal dynamic investment strategy for the objective of maximizing the probability of reaching a given goal by a fixed terminal time is completely equivalent to the (static) investment strategy which simply purchases a European digital call option on the underlying stock, with a particular strike price and payoff. This result is of independent interest since it provides an example where a policy which is optimal for an objective stated on wealth, is equivalent to the purchase of an option on the underlying stock. Moreover, it also implicitly contains the explanation as to why the optimal policy is independent of the drift in the constant coefficients single-stock case. For the general case, we are able to show a similar result, however with the single stock replaced by the return of a particular portfolio policy: the optimal growth, or log-optimal portfolio.

Specifically, we will show that the probability maximizing policy is completely equivalent to purchasing a European digital option on the return of the log-optimal portfolio.

A summary and outline of the remainder of the paper is as follows: In the next section, we introduce the basic model with multiple stocks and a risk-free asset with positive return. In Section 3 we provide the optimal policy for the problem of maximizing the probability of reaching the goal by terminal time $T < \infty$, as well as the new representation of the optimal wealth under this policy (Corollary 3.2 below). The proof is delayed until Section 6. In Section 4, we then use this representation to show the correspondence between the single stock case with constant coefficients and a digital option on the stock, as well as the correspondence with the digital option on the log-optimal portfolio in the general case.

In Section 5, we analyse the optimal policy for the general case. We first show that the optimal policy can be interpreted as a linear function of wealth, where the coefficient decomposes into the product of two distinct factors: (1) a purely time-dependent risk factor,
which is determined solely by the risk premiums of the stocks and the time remaining until the deadline; and (2) a purely state-dependent function which is parameterized solely by the current percentage of the distance to the (in our case, time-dependent) goal achieved. As intuition would suggest, the former function typically increases as the horizon decreases, while the latter function decreases as the percentage increases. The optimal policy is therefore a dynamic portfolio strategy that continuously rebalances the portfolio weights depending upon how much time remains to the deadline as well as how close the current wealth is to the goal. The interplay between these two factors is analysed to a fairly explicit extent next when we analyse the region where borrowing takes place. It turns out that this region is determined by a single equation involving the 'risk-adjusted' remaining time, and the percentage of the goal achieved to that point.

In Section 7 we consider the case where the investor earns income from an external source other than trading gains. We show that contrary to utility maximizing strategies—where an investor uses the exogenous income to take a more risky position in stocks than he would otherwise—a 'probability maximizing' investor relies on this exogenous income to be more cautious. In particular, we show that external income causes the investor to incorporate a performance bound: if the performance of the stocks is such that wealth ever falls to the level that could have been achieved by simply investing all the previous income into the risk-free asset, then all investment in the risky stocks ceases.

In Section 8, we consider the case where the investor's goal is to beat the return of a given (stochastic) benchmark by a prespecified amount by a predetermined time. We also find the related policy that allows the investor to control for the downside risk. When the stochastic benchmark is given by the optimal growth policy, or equivalently, the policy that maximizes logarithmic utility, which is sometimes referred to as the market portfolio in continuous-time finance, then certain complications arise. Specifically, it is well known that the ratio of the return from any arbitrary portfolio strategy to the return generated by the optimal growth strategy is a nonnegative local martingale, hence a supermartingale. Nevertheless, we find a policy that does achieve the theoretical upper bound on the probability of beating the return of the optimal growth by a predetermined amount, and is hence an optimal policy.

2. The model

The model under consideration here is that of a complete market as in [7, 15, 16] and others, wherein there are \( n \) (correlated) risky assets generated by \( n \) independent Brownian motions. The prices of these stocks are assumed to evolve as

\[
dS_i(t) = S_i(t) \left[ \mu_i(t) dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_j(t) \right], \quad i = 1, \ldots, n, \quad (2.1)
\]

where \( \mu_i(t), \sigma_{ij}(t) \) are deterministic functions, for \( i, j = 1, \ldots, n \), and \( t \geq 0 \), and where \( W_t := (W_1(t), \ldots, W_n(t))' \) is a standard \( n \)-dimensional Brownian motion, defined on the complete probability space \((\Omega, F, P)\), where \( \{F_t, t \geq 0\} \) is the \( P \)-augmentation of the natural filtration.

There is also a riskless asset whose price, \( B_t \), evolves according to

\[
dB_t = r(t) B_t dt \quad (2.2)
\]
plays a pivotal role here since then according to the Girsanov theorem (cf. [19]) the vector
interpretation of
is invertible for all \( 0 \leq f \). For Markov control processes \( \{s, t\} \),

\[
\int_0^f \theta s d\sigma t \leq \int_0^f \theta s d\sigma 1
\]

wealth \( \Psi_1(\theta, \ldots, 1) = \mu t \) under an investment policy

\[
\int \Psi_1(t, \cdots, 1) = \int \Psi_1(t, \cdots, 1)
\]

and functions

\[ W t = \int \Psi_1(t, \cdots, 1) \]

\[ t \] is a non-anticipating, \( t \)-adapted process that satisfies

\[
\int t \sigma t d\sigma t i j
\]

are all uniformly

\[
\int t \sigma t d\sigma 1
\]

and

\[ t \] is assumed to exist

\[ t \] is t is invertible for all 0 \leq f .
If in Theorem 3.1 we take $r = 1$, and correspondingly, (3.2) shows that we would in that case have

$$
\max \{ x(t) \} = x(T) = x^* = \sup x(T) = \sup x(T) \cdot \Phi_1(t) \cdot \Phi_1(t)
$$

where $\Phi_1(t)$ is the associated cumulative distribution function (c.d.f.).

For the sequel, we will assume that $\theta \equiv 0$, and $\tilde{\theta} \equiv 1$. In the following theorem, we give maximising the normal variate, and $\max \{ x(t) \}$, the set of admissible controls for an investor whose wealth process evolves from the state $x(0)$,

Then ensuring that terminal wealth at time $T$ exceeds the fixed level $b$ which would in turn beat the goal of recovering the results of [13, Theorem 7].

For Theorem 3.1.

Remark on notation.

In this section, we present the optimal value function and optimal investment policy for the problem of maximizing the

The measure $\tilde{\theta} \equiv 1$, is sometimes referred to as the risk-neutral (or equivalent martingale) measure,

This is due, of course, to the fact that when there is a risk-free asset available for investment

and the vector $\mathbf{V}_t$ be the optimal policy, for $t \leq T$.

In Section 3, we will give the optimal policy for the problem of maximizing the

value function, as well as the optimal policy, for $t \leq T$.

In the next section, we will give the optimal policy for the problem of maximizing the

from the state $x(t) = x(0)$, whereby all the investor's wealth will be invested in the riskless asset.

Theorem 3.1.

Remark 3.1.
If there is only one risky stock, then $\theta$ and the optimal policy, for $t$ to $T$, can be directly obtained from the fact that for any $a$,

$$
\text{equation}
$$

Corollary 3.2. The optimal wealth process, $X_t$, is

$$
\text{strategy for a particular type of derivative security known as a digital option.}
$$

we will show in the next section, the policy of (3.5) is intimately connected to the hedging strategy. For the case of constant coefficients, i.e., when $a_t = 0$, treated earlier in [13] and [10]. This follows from (3.5). In this case, the optimal control $\theta_t$ reduces, for $t < T$, to

$$
(3.4)
$$

$$
(3.3)
$$

$$
(3.5)
$$

It is important to note that it is only for this single-stock case that the optimal control is in fact $\theta_t = 0$, independent of the underlying drift parameter $\mu$. This is consistent with the results of Theorem 3.1, which shows that for any portfolio of risky assets, the optimal control $\theta_t$ is non-zero. The optimal control $\theta_t$ is determined by the relationship between the risk-free rate $r$ and the risky asset's expected return $\mu$. For the case of constant coefficients, $\theta_t$ is a constant vector. Note that for this case, when $r = \mu$, we may or may not be true, depending on the relationship between $\mu$ and $\sigma$.
This representation provides the link between the probability maximizing objective and the digital, or binary, option, and contains the explanation as to why it is only in the single stock constant coefficient case that the policy is independent of the underlying drift. We discuss this directly in the next section. After that, we return to analyse the optimal policy in explicit detail. The proof of Theorem 3.1 will then be provided in the following section. After that we examine cases that include income and the problem of beating an index.

4. Connections with digital options

4.1. Constant coefficients

Consider a Black–Scholes world with a single stock whose price, \( S_t \), follows the stochastic differential equation

\[
\mathrm{d}S_t = \mu S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t
\]

(4.1)

as well as a risk-free asset with constant return \( r \).

A digital (or binary) option on this stock, with strike price \( K \) and payoff \( B \), is a contract that pays $\( B \) at time \( T \) if \( S_T \geq K \). Thus a digital option amounts to a straight bet on the terminal price of the underlying stock.

Let \( C(t, S_t) \) denote the current rational price of such an option. Then, a standard Black–Scholes pricing argument shows that

\[
C(t, S_t) = B e^{-r(T-t)} \phi_1 \left( \ln \left( \frac{S_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right)(T-t) \right) \sigma \sqrt{T-t}
\]

(4.2)

The underwriter of such an option (i.e., the party that agrees to pay $\( B \) at \( T \) if \( S_T > K \)) is interested in hedging its risk. A dynamic hedging strategy for the writer of such an option is a dynamic investment policy, say \( \{\Delta_1 t, t \leq T\} \), which holds \( \Delta_1 t \) shares of the underlying stock at time \( t \) so as to ensure that the underwriter's position is riskless at all times. It is also well known that this hedging, or replicating, strategy is given by

\[
\Delta_1 t = C_2(t, x) = \frac{\partial C}{\partial x}
\]

It is easy to see that the hedging strategy for the digital option is simply

\[
\Delta_1 t = B e^{-r(T-t)} \phi \left( \ln \left( \frac{S_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right)(T-t) \right) \sigma \sqrt{T-t}
\]

(4.3)

Observe that since \( \Delta_1 t \) is the number of shares of the underlying stock the investor holds at time \( t \), the actual amount of money invested in the stock at time \( t \) is \( \Delta_1 t \cdot S_t \).

A general treatment of options that discusses pricing and hedging of various options, including the digital and the derivation of (4.2) and (4.3) can be found in such basic texts as [12] and [21]. A valuable source for more theoretical issues is [7].

To see the connection with our problem, consider an investor who at time \( t \) has sold this digital option for the Black–Scholes price of \( C(t, S_t) \), and suppose the investor will then invest the proceeds in such a manner as to maximize the probability that he can pay off the claim of this option at time \( T \), i.e., for all intents and purposes, the investor's 'wealth' at time \( t \) is \( C(t, S_t) \), and the investor will then invest this wealth so as to maximize the probability that the terminal fortune from this strategy is equal to \( B \). Our previous results show that the optimal policy is at time \( t \) is given by \( f^*_t = (3.5) \) with \( x = C(t, S_t) \) and \( b = B \), i.e., by \( f^*_t(C(t, S_t), B) \).

\[
\textit{Example:}\ \phi \left( \ln \left( \frac{S_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right)(T-t) \right) \sigma \sqrt{T-t}
\]

\[
\textit{Example:}\ \phi \left( \ln \left( \frac{S_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right)(T-t) \right) \sigma \sqrt{T-t}
\]
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But placing $C(t, S_t)$ of (4.2) into (3.5) with $b = B$ and simplifying gives

$$f^* t(C(t, S_t); B) = Be^{-r(T-t)}\phi\left(\frac{\ln(S_t/K)}{\Phi1} + \frac{(r-\frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}\right)$$

where $\Phi1$ is given by (4.3). Thus, in this case, $f^* t$ is equivalent to the hedging strategy of the digital option.

Moreover, if we specialize the representation of the optimal wealth process given in Corollary 3.2, i.e. $X^* t$ of (3.7), to the single stock case with constant coefficients, we find that

$$X^* t = Be^{-r(T-t)}\phi\left(\frac{\ln(S_t/S_0)}{\Phi1} + \frac{(r-\frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}\right)$$

is the wealth of an investor at time $t$ who started off with initial wealth $X_0$ and is investing so as to maximize the probability of his terminal time $T$ wealth being equal to $B$.

Observe now that by (4.1) we have

$$S_t = S_0 \exp\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\}$$

from which we may infer that

$$\sigma W_t + (\mu - r)t \equiv \ln\left(\frac{S_t}{S_0}\right) - \left(r - \frac{1}{2}\sigma^2\right)t.$$

When this is placed back into (4.5), it yields the following representation of the optimal wealth process in terms of the underlying stock price $X^* t = C(t, S_t)$, where $C(t, S_t)$ is given by (4.2), i.e., the optimal wealth process under policy $f^* t$ is just the Black–Scholes price for a digital option with payoff $B$.

It is interesting to note that the Black–Scholes value (4.2) and its resulting hedging strategy (4.3) are both calculated and determined by the risk-neutral probability measure (under which $\mu$ is replaced by $r$), while the optimal strategy for maximizing the probability of terminal wealth being greater than $B$ was determined under the regular measure.

The analysis above can be inverted to show the following rather interesting fact:
of a corresponding simple event determined by the terminal value $ST$.

The corresponding specific portfolio policy

4.2. The general case: optimal growth and digital options

with payoff $B$ and strike price $K^*$ is equivalent to the

$X$ the equivalence with the digital option, whose price must be independent of

Consider an investor, with initial wealth $X_0$. Proposition 4.1.

Let $\pi_1$ denote the corresponding 'wealth' under policy $\pi$ in (2.4) shows that

$\pi_1$ is referred to as the optimal growth policy $\pi_1$ in terms of the wealth process,

which when compared with (4.2) shows that

$X_0 = \Phi_1 \equiv \sqrt{\frac{2}{\sigma^2}} \left(1 + e^{-rT}\right)$

$\pi_1$ is determined completely by the

$\Phi_1$ satisfies the linear stochastic differential equation

for minimizing the expected time to reach any specific level of wealth (in an infinite horizon

see the discussion in [16, Chapter 6]. For the case of constant coefficients, it is also optimal

that maximizes logarithmic utility of wealth and also maximizes the growth rate of wealth,

Proof. The representation of the optimal wealth process under

$\pi_0$ is simply the Black–Scholes price at time

$\pi_0$ satisfies the linear stochastic

differential equation

$W$ for $\pi_0$. Then

$\pi_0$ in (2.4) shows that

$\pi_0$ is referred to as the optimal growth policy $\pi_1$ in terms of the wealth process,

which when compared with (4.2) shows that
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and as such we have

\[ t = \frac{1}{\pi_1} \exp \left\{ \int_0^t \left[ r(s) + \frac{1}{2} \theta(s) \right] ds + \int_0^t \theta(s) dW_s \right\}. \]

(4.9)

We can also represent the wealth process under the optimal growth policy in terms of the change of measure of (2.8) by

\[ t = \frac{1}{\pi_1} R(t, T) \mathbb{E} \left( \frac{dP}{\tilde{P}} \mid F_t \right). \]

The main result of this section is that the optimal policy of Theorem 3.2 is equivalent to the hedging strategy for a digital option on \( \frac{1}{\pi_1} T \), and so the optimal wealth process of Corollary 3.2 is therefore equivalent to the Black–Scholes price on this option. Before we state this formally, recognize that for every \( t \geq 0 \), the optimal growth portfolio, \( \frac{1}{\pi_1} t \), is equivalent in distribution to the process \( \hat{t} \), where

\[ \hat{t} = \sqrt{\frac{1}{\pi_1} t} \left[ r(t) + \sigma^2(t) \right] dt + \sqrt{\frac{1}{\pi_1} t} \theta(t) d\hat{W}_t. \]

(4.10)

As such, \( \hat{t} \) is the volatility of the optimal growth portfolio.

Remark 4.2. Note that in terms of the optimal growth policy, we may write the optimal policy, \( \hat{f} \), of (3.2) in Theorem 3.1 as

\[ \hat{f}(x; b) = \pi_1 \sqrt{\int_T^t \sigma^2(s) ds} b R(t, T) \phi(x - \pi_1) \]

(4.12)

Proposition 4.2. Consider an investor whose wealth, \( \{X_f \} \), satisfies (2.4), and whose objective is to maximize \( P(X_f(T) \geq b) \). Then the optimal policy for this objective, given by \( \hat{f} \) of (3.2) in Theorem 3.1, is completely equivalent to the static policy which purchases a European digital option on \( \frac{1}{\pi_1} T \) with payoff \( b \) and strike price \( K^{\ast\ast} \), where

\[ K^{\ast\ast} = \frac{1}{\pi_1} 0 \exp \left\{ \int_0^T \left[ r(s) - \frac{1}{2} \sigma^2(s) \right] ds - \sqrt{\int_0^T \sigma^2(s) ds} \right\} \left\{ \frac{x}{b R(t, T)} - 1 \right\} \].

(4.13)

Equivalently, under the optimal policy \( \hat{f} \), the optimal wealth \( \{X_\ast \} \) is equivalent to the (no-arbitrage) Black–Scholes price of this digital option on \( \frac{1}{\pi_1} T \), i.e.,

\[ X_\ast(t) = \frac{b R(t, T)}{\Phi(x - \pi_1 \left( \frac{x}{b R(t, T)} - 1 \right) \} \}

(4.14)
stocks since then the denominator is also going to 0.

little time remaining, i.e., 'the borrowing region'.) However, should the investor get near bankruptcy, i.e., though he does have the ability to borrow an unlimited amount. See the discussion below on the investor does not even borrow an excessive amount when his wealth is close to 0, even risky stocks. (It is interesting to note that if there is enough time remaining until the deadline, should be patient and wait for his wealth to grow a bit before taking active positions in the

and start taking aggressive positions in the stocks to get away from 0, but rather the investor

if wealth is close to 0 with enough time remaining on the clock, the investor does not 'panic'

Remark 4.3.

5. Analysis of the optimal policy in the general case

In this section we examine the investment policy

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To see this, recall that if

We now move on to analyse the optimal investment policy,

We can use (4.9) to write the representation of the optimal wealth

Proof.

It remains to show that (4.14) is indeed the Black–Scholes price of a digital option on

where

Denote the expectation taken under the risk-neutral measure

where

In terms of the Brownian motion

We now move on to analyse the optimal investment policy,
Then we may write the optimal policy as

(Note that this is not the percentage of the former quantity is not as clear to check since we have allowed all parameters to be time dependent. For the special case of

One manifestation of risk-taking is in the amount of borrowing required by a policy. Here, we examine this dimension of the behavior required by

Since \( \nu \) can be characterized more explicitly by using (5.2). In particular, (5.2) shows that the borrowing component is the vector of purely time-dependent elements,

\[
\nu = \nu(t)
\]

and accordingly

\[
\nu \phi(\nu) / \Phi(1, \nu)
\]

is the total amount of money invested in the risky stocks. This region can be characterized more explicitly by using (5.2). In particular, (5.2) shows that the borrowing component

x = (x \theta(t))

is equivalent to the region

x \geq \sigma \left( \int_{-\infty}^{t} \sqrt{s \phi(1, s)} ds \right)

where the variable \( q \) is a monotonically increasing function of \( x \theta(t) \). We will examine the tradeoff

x \geq \sigma \left( \int_{-\infty}^{t} \sqrt{s \phi(1, s)} ds \right)

\( \theta(t) \) is the total wealth at time \( t \), gets closer to the (effective) goal, \( \theta(T) \) is the total amount of money invested in the risky stocks. This region can

where the (scalar) function

\[
q(t) = \sqrt{\int_{-\infty}^{t} \phi(1, s) ds}
\]

is the total amount of money invested in the risky stocks. This region can
For the case of constant coefficients, \( q(t, T) \) reduces to \( q(T - t) = \xi \cdot \sqrt{T - t} \), where the constant \( \xi \) is defined by

\[
\xi = \sqrt{\frac{\mu - r_1}{\Sigma_1} - \frac{1}{\Sigma_1} \left( \mu - r_1 \right)_{1}^{'}} \equiv \sqrt{\pi * \Sigma_1}
\]

which in the single-stock case reduces to \( \xi = \sigma \).

If we set \( \sqrt{T - t} = q(T - t) = \xi \sqrt{T - t} \), then we can consider \( T - t \) to be the risk adjusted time to play, since it is in fact just the time to play \( (T - t) \) multiplied by a risk factor \( \xi^2 \).

The borrowing region is then equivalent to

\[
/\Gamma_1(\nu, \tau) := \left\{ \nu : \phi(\nu) / \Phi_1(\nu) \geq \sqrt{\tau} \right\}
\]

In order to analyse the boundary of this region, let \( \nu^*(\tau) \) denote the root to the equation

\[
\phi(\nu) / \Phi_1(\nu) - \sqrt{\tau} = 0,
\]

i.e., \( \nu^*(\tau) \) is the unique number such that \( \phi(\nu^*) / \Phi_1(\nu^*) = \sqrt{\tau} \).

Since the left hand side is decreasing in \( \nu \) while the right hand side is increasing in \( \tau \), it is clear that there is a unique root. Furthermore, it is easy to establish that these roots are decreasing in the remaining time, \( \tau \), i.e., \( \nu^*(\tau) > \nu^*(\tau + \delta) \) for all \( \delta > 0 \).

The borrowing region is thus as follows: borrowing occurs when \( \nu(\tau) < \nu^*(\tau) \), but not if \( \nu(\tau) > \nu^*(\tau) \). Equivalently, since \( /\Phi_1(\cdot) \) is an increasing invertible function, we see that borrowing occurs only when \( /\Phi_1(\nu(\tau)) < /\Phi_1(\nu^*(\tau)) \), i.e., when \( z(\tau) < z^*(\tau) \), where \( z(\tau) \) has the interpretation of being the proportion of the goal obtained with \( \tau \) (risk-adjusted) time units left to go. A graph of \( z^*(\tau) \) calculated and drawn in MAPLE is given in FIGURE 1.

Some select values of \( \tau, \nu^*(\tau) \) and \( z^*(\tau) \) are given in Table 1. As Table 1 shows, if there is 0.05 time units left until the deadline, the investor must borrow under policy unless the investor's wealth is already 88% of the distance to the goal. As the time to go increases, the investor needs to borrow only at lower percentages. For example, if there is \( \tau = 1 \) unit of time left to go, then the investor will need to borrow unless his wealth at that time is at least 38% of the way to the effective goal.
Reaching goals by a deadline

TABLE 1: Borrowing region.

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \nu )</th>
<th>( z )</th>
<th>( \nu^\ast )</th>
<th>( z^\ast )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2.26</td>
<td>0.99</td>
<td>0.50</td>
<td>0.15</td>
</tr>
<tr>
<td>0.15</td>
<td>0.56</td>
<td>1.00</td>
<td>0.30</td>
<td>0.38</td>
</tr>
<tr>
<td>0.20</td>
<td>0.63</td>
<td>0.88</td>
<td>0.55</td>
<td>0.09</td>
</tr>
<tr>
<td>0.25</td>
<td>0.70</td>
<td>0.77</td>
<td>0.60</td>
<td>0.04</td>
</tr>
<tr>
<td>0.30</td>
<td>0.73</td>
<td>0.67</td>
<td>0.65</td>
<td>0.01</td>
</tr>
<tr>
<td>0.35</td>
<td>0.80</td>
<td>0.58</td>
<td>0.70</td>
<td>0.06</td>
</tr>
<tr>
<td>0.40</td>
<td>0.85</td>
<td>0.52</td>
<td>0.75</td>
<td>0.11</td>
</tr>
<tr>
<td>0.45</td>
<td>0.90</td>
<td>0.43</td>
<td>0.80</td>
<td>0.15</td>
</tr>
<tr>
<td>0.50</td>
<td>0.95</td>
<td>0.35</td>
<td>0.85</td>
<td>0.20</td>
</tr>
</tbody>
</table>

It is important to note that increasing the risk factor, \( \xi^2 \), has the same effect as increasing the actual time left to play, \( T - t \). Therefore, for a higher risk factor, one would borrow less, in the hopes of reaching the goal later.

Remark 5.1. Asymptotics, near the barriers.

When wealth, \( x \), is close to the barriers, \( 0 \) or \( b_R(t, T) \), then \( z \) is correspondingly near 0 or 1, and \( \nu \) is correspondingly near \(-\infty\) or \(+\infty\). It is interesting to examine what happens to the state-dependent factor, \( \phi(\nu)/\Phi_1(\nu) \), near these boundaries. Note that since \( \phi \) is symmetric, the behavior of \( \phi(\nu)/\Phi_1(\nu) \) as \( \nu \downarrow -\infty \) is equivalent to the behavior of \( \phi(\nu)/(1 - \phi_1(\nu)) \) as \( \nu \uparrow +\infty \).

Therefore the following two asymptotic results are immediate:

\[
\lim_{z \to 1} \frac{\phi(\nu)}{\Phi_1(\nu)} = 0 = \lim_{\nu \uparrow +\infty} \frac{\phi(\nu)}{\Phi_1(\nu)}
\]

\[
\lim_{z \to 0} \frac{\phi(\nu)}{\Phi_1(\nu)} = \lim_{\nu \downarrow -\infty} \frac{\phi(\nu)}{\Phi_1(\nu)} = +\infty
\]

As expected, we see that as wealth approaches the goal, the state dependent factor goes to 0, but as wealth approaches 0, the state dependent factor increases without bound even though, as we observed previously, total investment in the risky stocks actually decreases to 0 in this case.

5.2. Comparison with utility maximizing policies

It is interesting to compare this behavior with that of an investor whose objective is to maximize terminal utility from wealth. For example, consider the case where the investor wants to maximize \( E[u(X_f T)] \), where \( u(x) = \delta x^{1-\delta}/d \), for \( x > 0, \delta > 0 \).

(Note that this includes logarithmic utility, when \( \delta = 1 \).) This power utility function has constant relative risk aversion \( 1/\delta \). The optimal policy for this case, call it \( \{f_\delta t, \ 0 \leq t \leq T\} \), is the vector (cf. [7], [15])

\[
f_\delta t(x) = \delta \pi^\ast_t \cdot x,
\]

where \( \pi^\ast_t \) is the optimal growth policy discussed earlier. The utility maximizing investor invests more heavily in the risky stocks, relative to the...
probability maximizing investor, when

\[ f_\delta(t)(x) > f^*(t)(x; b) \]

and vice versa. It is easily seen that this occurs for values \((x, t)\) for which

\[ \phi(\nu) / \Phi_1(\nu) \leq \delta \sqrt{\int_0^T \theta(s) \prime \theta(s) \, ds} \]

and vice versa. Thus the dynamics of this comparison reduce essentially to that described above by the borrowing region, modified by the risk aversion parameter \(\delta\).

6. Proof of Theorem 3.1

In this section, we provide the proof of Theorem 3.1. We will first show that the function \(V\) satisfies the appropriate Hamilton–Jacobi–Bellman (HJB) equations of stochastic control theory and then employ a martingale argument to verify optimality. This will prove the Theorem as well as Corollary 3.2. We then show how we obtained the candidate value function by extending the elegant argument of [10] to our case.

6.1. Verification of optimality

Standard arguments in control theory (see e.g. [8, Example 2, p. 161]) show that the appropriate HJB optimality equation for \(V\) is

\[ \sup f \{ A f V(t, x) \} = 0 \quad (6.1) \]

subject to the boundary conditions

\[ V(t, x; b) = \begin{cases} \text{1 for } x \geq bR(t, T), t \leq T, \\ \text{0 for } x = 0, t \leq T. \end{cases} \quad (6.2) \]

The generator of (2.5) shows that the HJB optimality equation (6.1) is

\[ \sup f \{ V_t + \left( f_t(\mu(t) - r(t)) + r(t) x V_x \right) + \frac{1}{2} f_t(\sigma^2(t)) f_x x V_{xx} \} = 0 \quad (6.3) \]

Assuming that a classical solution to (6.3), say \(V\), exists and satisfies \(V_x > 0, V_{xx} < 0\) for \(0 < x < bR(t, T)\), we may then optimize with respect to \(f_t\) in (6.3) to obtain the maximizer

\[ f^*(t)(x; b) = -\sigma^{-1}(t) \left( \mu(t) - r(t) \right) V_x V_{xx} \equiv -\sigma^{-1}(t) \theta(t) V_x V_{xx}. \quad (6.4) \]

When (6.4) is then placed back into (6.3) and the resulting equation simplified, we find that (6.3) is equivalent to the nonlinear partial differential equation

\[ V_t + r(t) x V_x - \frac{1}{2} \theta(t) \prime \theta(t) V^2_{xx} = 0 \quad (6.5) \]

subject to the (discontinuous) boundary condition (6.2).

Recalling now the basic facts about the normal p.d.f. and c.d.f.:

\[ d/\Phi_1(y) dy = \phi(y); \quad d/\Phi_1^{-1}(y) dy = 1/\phi(\Phi_1^{-1}(y)); \quad d\phi(y) du = -y \phi(y); \quad (6.6) \]
so the optimal control is shown that such a discontinuity is acceptable, provided that the optimal wealth process, problem (for 'probability-maximizing objectives') is discussed in [8, Example 2, p. 161], where by the stochastic differential equation in (3.6). An application of Itô calculus involves the solution to (6.10) is, for \( H_t \), will show directly that this condition is in fact met here. However, all this is valid only for the values 0 and 1, but it still remains to show that \( \sqrt{\theta t} \) is bounded with 0 as \( \theta \to \infty \). For \( \theta = 1 \) (i.e., the wealth under the control \( \phi(\theta) \)) \( \sqrt{\theta t} \) can be verified that for the function \( \theta \), \( \phi(\theta) = \hat{\phi}(1, \theta) \) is a c.d.f., and hence \( b \in [0,1] \). The proof is given in [5]. The family of functions \( \{\phi(t)\}_{t \in [0,T]} \) is continuous in \( t \) and \( \theta \). Let \( \theta = 1 \). Then for any \( \theta > 1 \), \( \sqrt{\theta t} \) is bounded with 0 as \( \theta \to \infty \).
the last equality following from the fact that we were able to guess by extending an argument of Heath [10], as we now show. We obtained a candidate solution to (6.5) by reducing the problem to a form whose optimal value function (6.12) shows that

\[ \xi(t^*) = \xi(t) + \int_{t}^{t^*} \frac{\partial \xi}{\partial y} dy \]

where \( \xi \) and \( \hat{\xi} \) are equivalent, we must also have

\[ \hat{\xi}(t^*) = \hat{\xi}(t) + \int_{t}^{t^*} \frac{\partial \hat{\xi}}{\partial y} dy \]

Thus, we may now conclude from (6.11) that

\[ \hat{\xi}(t^*) = \hat{\xi}(t) + \int_{t}^{t^*} \left( \frac{\partial \hat{\xi}}{\partial y} - \frac{\partial \xi}{\partial y} \right) dy \]

and therefore converges at \( t^* \) now provides the representation

\[ HT(t, s, \omega) = \hat{H}(t, \omega) + \int_{t}^{s} \left( \frac{\partial \hat{H}}{\partial y} - \frac{\partial H}{\partial y} \right) dy \]

Furthermore, the martingale representation theorem (e.g. [19, 20]) now provides the representation

\[ HT(t, s, \omega) = \hat{H}(t, \omega) + \int_{t}^{s} \left( \frac{\partial \hat{H}}{\partial y} - \frac{\partial H}{\partial y} \right) dy \]

Observe now that by properties of stochastic integrals,

\[ \int_{t}^{s} dW(r) = \int_{t}^{s} \sum_{i=1}^{n} \Phi_i(r) \xi_i(r) \]

Since

\[ \phi_i(t, \omega) = \Phi_i(t, \omega) \xi_i(t, \omega) \]

has a continuous extension to \( t \) and variance

\[ \int_{t}^{\infty} \left( \Phi_i(r, \omega) \right)^2 \sigma_i^2(r, \omega) dr \]

we have shown that \( \phi_i(t, \omega) \) is surely path continuous. Observe now that by properties of stochastic integrals,

\[ \int_{t}^{s} dW(r) = \int_{t}^{s} \sum_{i=1}^{n} \Phi_i(r) \xi_i(r) \]

Substituting now for

\[ \int_{t}^{s} \theta(r) dW(r) = \int_{t}^{s} \sum_{i=1}^{n} \Phi_i(r) \xi_i(r) \]

we have shown that

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\[ \int_{t}^{\infty} \left( \Phi_i(r, \omega) \right)^2 \sigma_i^2(r, \omega) dr \]

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Substituting now for

\[ \int_{t}^{s} \theta(r) dW(r) = \int_{t}^{s} \sum_{i=1}^{n} \Phi_i(r) \xi_i(r) \]

we have shown that

\[ \phi_i(t, \omega) = \Phi_i(t, \omega) \xi_i(t, \omega) \]

has a continuous extension to \( t \) and variance

\[ \int_{t}^{\infty} \left( \Phi_i(r, \omega) \right)^2 \sigma_i^2(r, \omega) dr \]
It is clear, by the admissibility assumption on $\tilde{g}$, that we may conclude that $\sup_{\omega \leq T} \{ \mathbf{r} \} + \{ \mathbf{Uy} \}$. Hence, letting $\mathbf{A} g := \{ \exp \mathbf{b} \} \exp \mathbf{t} \{ \mathbf{x} \} \{ \mathbf{y} \}$ is in fact a martingale under the measure $\mathbf{P}$. Following Heath [10], reproduced here for the sake of completeness in the Appendix, then shows that the terms of the problem studied in [13] and [10]. Following Heath [10], we note that we can rewrite (6.14) in this is independent of the policy $\mathbf{g}$.

All we need do now is recognize that $\mathbf{r} = \{ \exp \mathbf{b} \} \exp \mathbf{t} \{ \mathbf{x} \} \{ \mathbf{y} \}$ is defined by (6.17). Of course, this argument still needs a rigorous verification, which is the content of the paper.

An HJB argument then shows that $\mathbf{r} = \{ \exp \mathbf{b} \} \exp \mathbf{t} \{ \mathbf{x} \} \{ \mathbf{y} \}$ is the function given by (6.16), and by differen-

\[
\mathbf{r} \left( \{ \mathbf{x} \} (\{ \mathbf{y} \}) \right) = \sum_{\mathbf{ij} = 1}^{\mathbf{n}} \left( \{ \exp \mathbf{b} \} \exp \mathbf{t} \{ \mathbf{s} \} \{ \mathbf{d} \} \{ \mathbf{1} \} \{ \mathbf{1} \} \right)
\]

...
The optimal value function and optimal policy for maximizing the probability that the investor achieves the wealth level $\Lambda_1$ are given by

$$V^*(x) = \max_{\pi \in \Pi} \mathbb{E}[\sum_{t=0}^{T-1} \beta^t r_t + \beta^T \min \{x_T, \Lambda_1\} | x_0 = x, \pi],$$

and the associated optimal investment policy is

$$\pi^*(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \leq \theta, \\
\frac{\lambda^2}{2\sigma^2} & \text{if } \theta < x < \Lambda_1, \\
1 & \text{if } x \geq \Lambda_1. \end{array} \right.$$
where wealth at time $t$ is the level of wealth that would have been achieved at time $t$ with optimizer $g$. Reaching goals by a deadline can never go bankrupt under policy $g$, which is the same as (6.13) and (6.14). Thus we observe immediately that if defined now by (till earlier, wealth is effectively bounded away from 0, and so the investor with positive income wealth into the risk-free asset. Therefore, as opposed to the case without income treated...
where $\kappa(x, t)$ is given by (7.2). Thus we may invert this to deduce that in terms of a goal problem for $X f_t$ given by (7.1), we have

$$\sup_{P \in \Omega} \left( X f_t \geq b \mid X_t = x \right) = U(t, \kappa(x, t); \Lambda_1),$$

as well as

$$f(l_t)(x; b) = \exp \left\{ \int_0^t r(u) \, d\! u \right\} g^* t(\kappa(x, t); \Lambda_1),$$

where the function $U(t, \cdot; \cdot)$ is defined by (6.16), $g^* t(\cdot; \cdot)$ is defined by (6.17), $\kappa(x, t)$ by (7.2) and $\Lambda_1$ by (7.3). Direct substitution then shows that $U(t, \kappa(x, t); \Lambda_1) = \Psi_1(t, x; b, T)$ of (7.4) and that the policy given in (7.5) is indeed optimal for the problem of maximizing the probability of attaining the goal $b$ by time $T$.

Remark 7.2. It is tempting to extend the analysis here to include the case of liabilities as well, however for this case the optimal policy allows wealth to become negative. This raises a variety of new problems and issues which will be discussed elsewhere.

8. Beating another portfolio

In this section we apply our earlier results to derive optimal portfolio strategies for an investor, such as a fund manager, who is interested solely in beating a given stochastic benchmark. The benchmark is most typically an index, such as the S&P 500. In particular, it is just one specific portfolio strategy. We consider first the problem of beating the benchmark portfolio by a given percentage, which is the focal point of active portfolio management. We then find the related policy which corrects for the downside risk. When the benchmark portfolio strategy is the optimal growth policy, then the ratio of the wealth from any arbitrary portfolio strategy to the benchmark is in fact a supermartingale, and as such problems arise. Nevertheless, we are able to find a strategy which does achieve the maximal possible probability of beating the optimal growth policy by a prespecified percentage by a fixed deadline.

8.1. Beating the benchmark portfolio by a given percentage

Here we consider the objective of ensuring that $X f_T$ exceeds the terminal value of a benchmark portfolio, $Q_T$, by a preset percentage. To that end, we will find it more convenient to use a geometric, or proportional parameterization of the model, i.e., instead of parameterizing the problem by taking $f_t$ to be the vector of absolute amounts invested in the stocks, we instead— for the remainder of this section—let $f(i)_t$ denote the proportion of wealth invested in stock $i$ at time $t$, with $f_t := (f(1)_t, \ldots, f(n)_t)^\prime$ now denoting the corresponding column vector of proportions. Then under this parameterization, it is clear that the wealth process evolves as (compare with (2.3))

$$dX f_t = X f_t \left( \sum_{i=1}^n f(i)_t dS_i(t) S_i(t) + X f_t (1 - \sum_{i=1}^n f(i)_t) dB_t B_t \right) = X f_t \left[ r(t) + \sum_{i=1}^n f(i)_t (\mu_i(t) - r(t)) \right] dt + X f_t \sum_{i=1}^n \sum_{j=1}^n f(i)_t \sigma_{ij}(t) dW(j)_t,$$

(8.1)
where the function $\gamma(\cdot)$ depends solely on the current value of the ratio of the two portfolios, $Z_t$. The optimal policy is given by

$$\pi_t = \operatorname{arg sup}_{\pi} \int_t^T \mathbb{E} \left[ r_t \mid \pi_t \right] \, dt$$

and the optimal value function satisfies

$$f_t = \mathbb{E} \left[ f_T \right] - \mathbb{E} \left[ \int_t^T r_s \, ds \mid \pi_t \right].$$

Similarly, let the benchmark portfolio, $\{Z_t^\lambda \mid t \geq T\}$, be defined as

$$Z_t^\lambda := \frac{\lambda}{\Phi_1} Z_t,$$

where $\Phi_1$ is the (column) vector of portfolio weights in the benchmark process. Let $\lambda$ be a predetermined percentage. Denote the resulting optimal policy by $\pi_t^\lambda$. Theorem 8.1 states that

$$\pi_t^\lambda = \frac{\lambda}{\Phi_1} \pi_t,$$

where $\pi_t^\lambda$ is the (column) vector of portfolio weights in the benchmark process. Similarly, let the benchmark portfolio, $\{Z^\lambda \mid t \geq T\}$, be defined as

$$Z^\lambda := \frac{\lambda}{\Phi_1} Z_t.$$
Denote this optimal value function by \( \hat{g} \). Instead, consider then the following objective: the portfolio manager's goal is to beat the prescribed benchmark by replacing \( \mu \) with \( \lambda \) in \eqref{eq:9.1}. This probably entails too much risk-taking for most applications. But this last term is precisely the problem considered earlier in Theorem 9.1, where the non-negativity condition was implicit. As such, we know that the ratio at the terminal time might end up at 0, with which the value function is given by \( \hat{g} \). Then it is clear that we have the downside risk control in \eqref{eq:8.5}. The previous development gives a policy under which the maximal probability of beat-
8.3. Can we beat the optimal growth policy?

Policies also sometimes referred to as the market portfolio. It is interesting to note that this policy invests as a martingale, although in general we would need to restrict our attention to policies that indeed a martingale, although in general we would need to restrict our attention to policies that are admissible. Reaching goals by a deadline...
a policy that achieves this upper bound. To see that

The solution to this stochastic differential equation is

Note that the linear form of the value function in (8.14) precludes us from using

Then the optimal value function is

Proof.

which, by virtue of the fact that
\[
\int \theta^s \frac{d}{d\theta} \ln \theta \tag{6.16}
\]
and hence the corresponding
\[
\tilde{\theta}
\]
and since under
\[
\text{Reaching goals by a deadline}
\]
\[
(\int \theta^s \frac{d}{d\theta}) \text{ of (6.16). Dynamic programming then applies for arbitrary}
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